# Detecting Program Components With Equivalent Behaviors

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The execution behavior of a program component is defined as the sequence of values produced at the component during program execution. This paper presents an efficient algorithm for detecting program components – in *one or more* programs – that exhibit identical execution behaviors. The algorithm operates on a new graph representation for programs that combines features of static-single-assignment forms and program dependence graphs. The result provides insight into the relationship between execution behaviors and (control and flow) dependences in the program. The algorithm, called the *Sequence-Congruence Algorithm*, is applicable to programs written in a language that includes scalar variables and constants, assignment statements, conditional statements, and while-loops. The Sequence-Congruence Algorithm can be used as the basis for an algorithm for integrating program variants.

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## 1. INTRODUCTION

This paper defines a graph structure for representing programs and presents an efficient algorithm for detecting program components (assignment statements and predicates) – in *one or more* programs – that exhibit identical execution behaviors. By the execution behavior of a program component, we mean the sequence of values produced at the component during program execution. For an assignment statement, this means the sequence of values assigned to the target variable; for a predicate, this means the sequence of boolean values produced by the successive evaluations of the predicate.

Although the problem of deciding whether two components have the same execution behaviors is, in general, undecidable, a *safe* approximation can be achieved with reasonable effort. An algorithm for deciding whether two components have the same behaviors is safe if the equivalence classes of components determined by the algorithm are a refinement of the actual equivalence classes of components with the same behavior. For example, in [12,13], it has been shown that two components must have the same

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execution behavior if the program slices[1] taken with respect to the two components are isomorphic. Even if the two components happen to be in *different* programs, the result still holds, as long as the programs are run on identical – or actually just sufficiently similar – initial states.

This paper presents a new algorithm, called the *Sequence-Congruence Algorithm*, for detecting program components that exhibit identical execution behaviors. As with the equivalence-detection algorithm based on comparing slices, the Sequence-Congruence Algorithm can detect components with identical execution behaviors even if the components are in *different* programs. Two such components that are in the same equivalence class have identical execution behaviors whenever the two programs are run on identical – or sufficiently similar – initial states. The Sequence-Congruence Algorithm is strictly stronger than the method based on comparing slices in that all pairs of components with isomorphic slices are found by the Sequence-Congruence Algorithm as well, but not *vice versa*.

The Sequence-Congruence Algorithm operates on a graph representation of programs. These graphs are called *program representation graphs* in this paper to distinguish them from other related representations. As explained in Section 3, program representation graphs combine features of static-single-assignment forms [15,4,5,14] and program dependence graphs [11,6,9].

The Sequence-Congruence Algorithm is based on an idea of [4] for finding equivalence classes of program components by first optimistically grouping possibly equivalent components in an initial partition and then finding the coarsest partition consistent with the initial partition. However, in refining the initial partition, the algorithm of [4] considers only flow dependences among program components. It is shown in [4] that components in the same partition produce the same values at *certain moments* during execution. In contrast, the Sequence-Congruence Algorithm given in this paper considers control dependences as well as flow dependences and can detect components whose *execution behaviors* are identical.

A further point of contrast between our work and that of [4] concerns the idea of applying partitioning to more than one program simultaneously. This idea was not studied in [4], and the semantic property proved there concerning congruent vertices does not characterize the result of applying the algorithm to multiple programs simultaneously. The algorithm from [4] is essentially the first phase of our Sequence-Congruence Algorithm, and our first result (the Data-Congruence Lemma) establishes a semantic property of vertices in the same equivalence class when the algorithm is applied to multiple programs. We then go on to show that with an additional partitioning phase, it is possible to detect program components that have identical execution behaviors even though they occur in different programs.

The reason for our interest in detecting components of different programs that have identical execution behaviors is that such information is fundamental to our algorithm for automatic program integration [9]. Given a program *Base* and two variants *A* and *B*, each created by editing separate copies of *Base*, our program integration algorithm determines whether the changes made to *Base* to produce *A* and *B* interfere; if there is no interference, the algorithm produces a merged program *M* that incorporates the changed behavior of *A* with respect to *Base*, the changed behavior of *B* with respect to *Base*, and the unchanged behavior common to *Base*, *A*, and *B*.

One of the key issues in program integration is how to determine whether a component of a variant has the same execution behavior as the corresponding component of *Base*. The integration algorithm of [9] compares program slices as a safe way of determining whether two program components have identical

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[1] The slice of a program with respect to a program component *c* is (roughly) all the statements and predicates in the program that can potentially affect the values produced at *c* during program execution.

execution behaviors. The Sequence-Congruence Algorithm provides an alternative method for determining whether two components have the same execution behaviors. It can be shown that the Sequence-Congruence Algorithm finds larger equivalence classes than the method based on comparing program slices. A new algorithm for automatic program integration that makes use of the Sequence-Congruence Algorithm is described in [16].

The remainder of this paper is organized into four sections, as follows: Section 2 describes the programming language under consideration in this paper. Section 3 defines program representation graphs. Section 4 presents the Sequence-Congruence Algorithm and justifies the algorithm. Section 5 discusses how the work reported here relates to previous work.

## 2. THE PROGRAMMING LANGUAGE UNDER CONSIDERATION

We are concerned with a programming language with the following characteristics: expressions contain only scalar variables and constants; statements are either assignment statements, conditional statements, while-loops, or *end* statements. An *end* statement, which can only appear at the end of a program, names zero or more of the variables used in the program. An example program is shown in the upper left-hand corner of Figure 1.

Our discussion of the language's semantics is in terms of the following informal model of execution. We assume a standard operational semantics for sequential execution; the statements and predicates of a program are executed in the order specified by the program's control flow graph; at any moment there is a single locus of control; the execution of each assignment statement or predicate passes control to a single successor; the execution of each assignment statement changes a global execution state. An execution of the program on an initial state yields a (possibly infinite) sequence of values for each predicate and assignment statement in the program; the $i^{th}$ element in the sequence for program component $c$ consists of the value computed when $c$ is executed for the $i^{th}$ time. The variables named in the *end* statement are those whose final values are of interest to the programmer; when execution terminates, the final state is defined on only those variables in the *end* statement.

## 3. PROGRAM REPRESENTATION GRAPH

Program representation graphs (PRGs) combine features of static-single-assignment forms (SSA forms) [15,4,5,14] and program dependence graphs [11,6,9]. In the SSA form of a program, special assignment statements ($\phi$ assignments) are inserted so that exactly one assignment to a variable $x$, either an assignment from the original program or a $\phi$ assignment, can reach a use of $x$ from the original program. The $\phi$ statements assign the value of a variable to itself; at most two assignments to a variable $x$ can reach the use of $x$ in a $\phi$ statement. For instance, consider the following example program fragments:

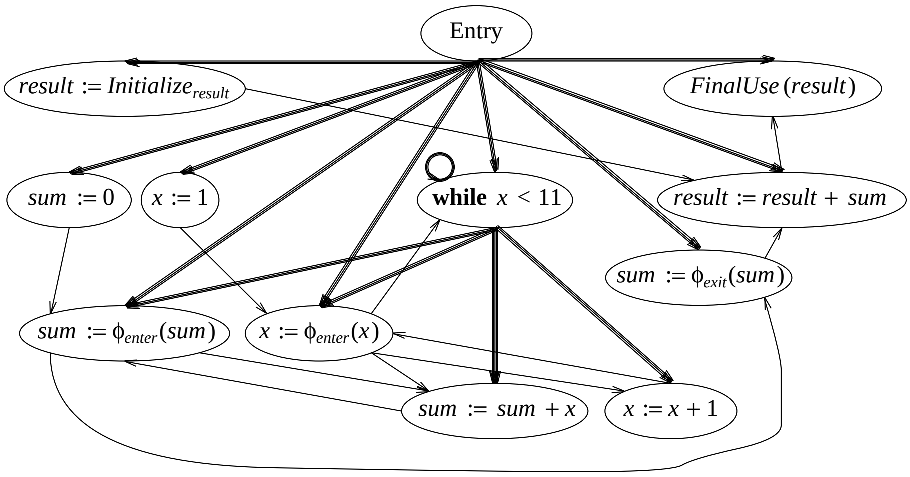
| | | | | |
|---|---|---|---|---|
| $L_1$ | $x := 1$ | | $L_1$ | $x := 1$ |
| | **if** $p$ **then** | | | **if** $p$ **then** |
| $L_2$ | $\quad x := 2$ | | $L_2$ | $\quad x := 2$ |
| | **fi** | | | **fi** |
| $L_4$ | $y := x + 3$ | | $L_3$ | $x := \phi_{if}(x)$ |
| | | | $L_4$ | $y := x + 3$ |

In the source program (on the left), both assignments to $x$ at $L_1$ and $L_2$ can reach the use of $x$ at $L_4$; after the insertion of "$x := \phi_{if}(x)$" at $L_3$ (on the right), only the $\phi$ assignment to $x$ can reach the use of $x$ at $L_4$. Both assignments to $x$ at $L_1$ and $L_2$ can reach the use of $x$ at $L_3$.

Different definitions of program dependence graphs have been given, depending on the intended application; nevertheless, they are all variations on a theme introduced in [10], and share the common feature of having an explicit representation of data dependences. The program dependence graph defined in [6] introduced the additional feature of an explicit representation for control dependences. The program representation graph, defined below, has edges that represent control dependences and one kind of data dependence, called flow dependence.

The program representation graph of a program *P*, denoted by $R_P$, is constructed in two steps. First an augmented control flow graph is built and then the program representation graph is constructed from the augmented control flow graph. An example program, its augmented control flow graph, and its program representation graph are shown in Figure 1.

```
program Main
   sum := 0
   x := 1
   while x < 11 do
      sum := sum + x
      x := x + 1
   od
   result := result + sum
end(result)
```



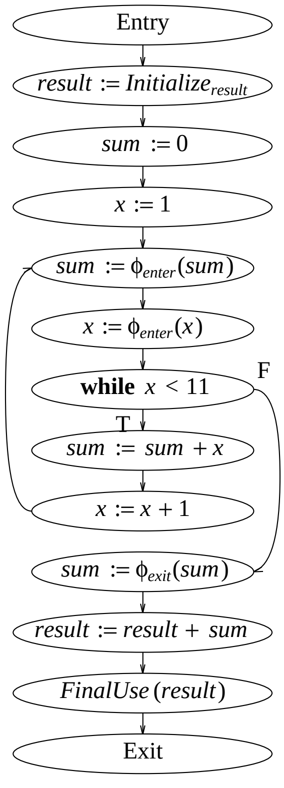




**Figure 1.** An example program is shown on the top left. This example sums the integers 1 to 10 and adds the sum to the variable *result*. On the right is the augmented control flow graph for the program. Note the absence of *Initialize* and *FinalUse* vertices for *sum* and *x* and of a $\phi_{exit}$ vertex for *x*. On the bottom left is the program representation graph for the program. Note that there is a control dependence edge from the *while* predicate $x < 11$ to itself. The boldface arrows represent control dependence edges; thin arrows represent flow dependence edges. The label on each control dependence edge – *true* or *false* – has been omitted.

*Step 1***:**

The control flow graph[2] of program $P$ is augmented by adding *Initialize*, *FinalUse*, $\phi_{if}$, $\phi_{enter}$, and $\phi_{exit}$ vertices, as follows:

(1) A vertex labeled "$x :=$ *Initialize*$_x$" is added at the beginning of the control flow graph for each variable $x$ that may be used before being defined in the program. If there are many *Initialize* vertices for a program, their relative order is not important as long as they come immediately after the *Entry* vertex.

(2) A vertex labeled "*FinalUse* $(x)$" is added at the end of the control flow graph for each variable $x$ that appears in the *end* statement of the program. If there are many *FinalUse* vertices for a program, their relative order is not important as long as they come immediately before the *Exit* vertex.

(3) For every variable $x$ that is defined within an *if* statement, and that may be used before being redefined after the *if* statement, a vertex labeled "$x := \phi_{if}(x)$" is added immediately after the *if* statement. If there are many $\phi_{if}$ vertices for an *if* statement, their relative order is not important as long as they come immediately after the *if* statement.

(4) For every variable $x$ that is defined inside a loop, and that may be used before being redefined inside the loop or may be used before being redefined after the loop, a vertex labeled "$x := \phi_{enter}(x)$" is added immediately before the predicate of the loop. If there are many $\phi_{enter}$ vertices for a loop, their relative order is not important as long as they come immediately before the loop predicate. After the insertion of $\phi_{enter}$ vertices, the first $\phi_{enter}$ vertex of a loop becomes the entry point of the loop.

(5) For every variable $x$ that is defined inside a loop, and that may be used before being redefined after the loop, a vertex labeled "$x := \phi_{exit}(x)$" is added immediately after the loop. If there are many $\phi_{exit}$ vertices for a loop, their relative order is not important as long as they come immediately after the loop.

Note that $\phi_{enter}$ vertices are placed inside of loops, but $\phi_{exit}$ vertices are placed outside of loops.

*Step 2***:**

Next, the program representation graph is constructed from the augmented control flow graph. The vertices of the program representation graph are those in the augmented control flow graph (except the *Exit* vertex). Edges are of two kinds: control dependence edges and flow dependence edges.

A control dependence edge from a vertex $u$ to a vertex $v$, denoted by $u \longrightarrow_c v$, means that, during execution, whenever the predicate represented by $u$ is evaluated and its value matches the label – *true* or *false* – on the edge to $v$, then the program component represented by $v$ will eventually be executed if the program terminates. The source of a control dependence edge is the *Entry* vertex or a predicate vertex.

- There is a control dependence edge from *Entry* to a vertex $v$ if $v$ occurs on every path from *Entry* to *Exit* in the augmented control flow graph. This control dependence edge is labeled *true*.
- There is a control dependence edge from a predicate vertex $u$ to a vertex $v$ if, in the augmented control flow graph, $v$ occurs on every path from $u$ to *Exit* along one branch out of $u$ but not the other. This control dependence edge is labeled by the truth value of the branch in which $v$ always occurs.

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[2]In control flow graphs, vertices represent the program's assignment statements and predicates; in addition, there are two additional vertices, *Entry* and *Exit*, which represent the beginning and the end of the program.

Note that there is a control dependence edge from a *while* predicate to itself.[3] Methods for determining control dependence edges for programs with unrestricted flow of control are given in [6,5]; however, for our restricted language, control dependence edges can be determined in a simpler fashion: Except for the extra control dependence edge incident on a $\phi_{enter}$ vertex, the control dependence edges merely reflect the nesting structure of the program (see [9].)

A flow dependence edge from a vertex $u$ to a vertex $v$, denoted by $u \longrightarrow_f v$, means that the value produced at $u$ may be used at $v$. There is a flow dependence edge $u \longrightarrow_f v$ if there is a variable $x$ that is assigned a value at $u$ and used at $v$, and there is an $x$-definition-free path from $u$ to $v$ in the augmented control flow graph. The flow dependence edges of a program representation graph are computed using dataflow analysis. For the restricted language considered in this paper, the necessary computations can be defined in a syntax-directed manner (see [8]).

The *imported variables* of a program $P$, denoted by $Imp_P$, are the variables that might be used before being defined in $P$, *i.e.*, the variables for which there are *Initialize* vertices in the *PRG* of $P$.

Textually different programs may have isomorphic program representation graphs. However, we have shown that if two programs have isomorphic program representation graphs, then the programs are semantically equivalent [17]:

*THEOREM.* (EQUIVALENCE THEOREM FOR PROGRAM REPRESENTATION GRAPHS). *Suppose P and Q are programs for which $R_P$ is isomorphic to $R_Q$. If $\sigma$ is a state on which P halts, then for any state $\sigma'$ that agrees with $\sigma$ on the imported variables of P, (1) Q halts on $\sigma'$, (2) P and Q compute the same sequence of values at each corresponding program component, and (3) the final states of P and Q agree on all variables for which there are final-use vertices in $R_P$ and $R_Q$.*

## 4. THE SEQUENCE-CONGRUENCE ALGORITHM AND THE SEQUENCE-CONGRUENCE THEOREM

In this section we present the Sequence-Congruence Algorithm and a theorem about the equivalence classes of vertices it produces. The Sequence-Congruence Algorithm divides vertices of one or more program representation graphs into disjoint equivalence classes. The theorem states that program components designated by vertices in the same class produce the same sequence of values when the programs are run on sufficiently similar initial states.

This section is divided into three subsections: Section 4.1 presents the Sequence-Congruence Algorithm. Section 4.2 proves the Sequence-Congruence Theorem. Section 4.3 describes three simple enhancements to the Sequence-Congruence Algorithm.

### 4.1. The Sequence-Congruence Algorithm

The execution behavior of a program component is, by definition, the sequence of values produced at the component during program execution. A component's execution behavior depends on three factors: the operator in the component, the operands available when the operator is applied, and the predicates that control the execution of the operation. It is not unreasonable to expect that vertices with the same operators, equivalent operands, and equivalent controlling predicates will have the same execution behaviors. This expectation is confirmed by the Sequence-Congruence Theorem, which is proved in Section 4.2.

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[3]In the program dependence graphs of [9], the control dependence edge from a *while* predicate to itself is omitted. However, such edges are needed for the Sequence-Congruence Algorithm, so they are included in program representation graphs.

The Sequence-Congruence Algorithm consists of two passes. The initial partition puts vertices with the same operators into the same classes. Flow dependence edges (and some additional edges) are used in the first pass to refine the initial partition; in the second pass, control dependence edges are used to further refine the partition obtained from the first pass. Both passes make use of the same partitioning algorithm to refine the partition of the graph's vertices; only the starting partition and the edges considered in the two passes are different.

The operator in a statement or a predicate vertex is determined from the expression part of the vertex. For instance, a statement like "$x := a + b * c$" has the same operator as a statement like "$y := d + e * f$" but a different operator than a statement like "$z := g * h$"; that is, the structure of the expression in the vertex defines the operator. An expression like "$a + b * c$" is viewed as an operator that takes three arguments $a$, $b$, and $c$, and returns the value of "$a + b * c$".

A predicate is *simple* if it consists of a single boolean variable; an assignment statement is simple if its right-hand-side expression consists of a single variable. Both vertices that represent simple predicates and vertices that represent simple assignments are referred to as simple vertices. The operator in a simple vertex is the *identity* operator, that is, an operator that takes one argument and returns the value of the argument. An assignment or a predicate vertex is a *constant vertex* if its expression consists of a single constant. The operator in a constant vertex is the *constant* operator that takes no argument and always returns the value of the constant.

Two vertices that are the same kind of $\phi$ vertex (*i.e.*, $\phi_{enter}$, $\phi_{exit}$, or $\phi_{if}$) or that have the same operators must have the same number of incoming control and flow dependence edges in the *PRG*s. Thus, we can speak of the "corresponding" flow (or control) predecessors of the two vertices. To be more specific, we assign *types* to edges in the *PRG*s; the notion of corresponding flow (or control) predecessors of two vertices is then defined in terms of the types of edges. (Note that the numbers for the edge types, specified below, are chosen arbitrarily; these numbers are used only to distinguish different types of edges.)

Due to the presence of $\phi$ vertices in *PRG*s, each use of a variable in a non-$\phi$ vertex is reached by exactly one definition (either one of the original assignment statements or one of the $\phi$ assignments). Therefore, if the operator in a non-$\phi$ vertex is an $n$-ary operator, there are exactly $n$ incoming flow dependence edges for this vertex. These flow dependence edges are assigned types 1, 2, . . ., $n$, one for each operand. Edge-type numbers for other kinds of edges in a *PRG* start at $m + 1$, where $m$ is the greatest number of flow edges incident on some non-$\phi$ vertex. In what follows, we will assume that $m = 3$, and start numbering other edges at 4.

A vertex $u$ labeled "$x := \phi_{if}(x)$" has two incoming flow dependence edges: one represents the value that flows to $u$ from or around the *true* branch of the associated *if* statement; the other represents the value that flows to $u$ from or around the *false* branch. The flow dependence edges incident on a $\phi_{if}$ vertex are assigned types 4 and 5, respectively. For instance, consider the following program fragment:

$L_1$ $\quad x := 1$
$\quad$ **if** $p$ **then**
$L_2$ $\quad\quad x := 2$
$\quad$ **else**
$\quad\quad$ **skip**
$\quad$ **fi**
$L_3$ $\quad x := \phi_{if}(x)$

The definition at $L_1$ reaches $L_3$ around the *false* branch of the *if* statement, so the flow dependence edge from $L_1$ to $L_3$ has type 5. The definition at $L_2$ reaches $L_3$ from the *true* branch, so the flow dependence edge from $L_2$ to $L_3$ has type 4.

A vertex $u$ labeled "$x := \phi_{enter}(x)$" has two incoming flow dependence edges: one represents the value that flows to $u$ from outside the associated loop (due to an assignment to $x$ before the loop); the other represents the value that flows to $u$ from inside the loop. These flow dependence edges are assigned types 6 and 7, respectively.

A vertex $u$ labeled "$x := \phi_{exit}(x)$" has one incoming flow dependence edge; the source of this flow dependence edge is the associated $\phi_{enter}$ vertex. The flow dependence edge incident on a $\phi_{exit}$ vertex is assigned type 8.

All vertices except $\phi_{enter}$ and *while* predicate vertices have exactly one incoming control dependence edge. The control dependence edges that form self-loops on *while* predicates are assigned type 9. The incoming control dependence edge of a $\phi_{enter}$ vertex $u$ whose source is *not* the associated *while* predicate for $u$ is assigned type 10 or 11 depending on whether the label on the control dependence edge is *true* or *false*. All other control dependence edges are assigned type 12 or 13 depending on whether the label on the control dependence edge is *true* or *false*.

The corresponding flow (or control) predecessors of two vertices $u_1$ and $u_2$ are two vertices $v_1$ and $v_2$ such that the flow (or control, respectively) dependence edges $u_1 \longrightarrow v_1$ and $u_2 \longrightarrow v_2$ have the same type.

The partitioning algorithm in Figure 2 is adapted from [4,1], which is based on an algorithm of [7] for minimizing a finite state machine. The $m$-successors of a vertex $u$ are the vertices $v$ such that there is an edge $u \longrightarrow v$ of type $m$. The partitioning algorithm finds the coarsest partition of a graph that is consistent with a given initial partition of the graph's vertices; it guarantees that two vertices $v$ and $v'$ are in the same class after partitioning if and only if they are in the same class before partitioning and for any predecessor $u$ of $v$ there is a corresponding predecessor $u'$ of $v'$ such that $u$ and $u'$ are in the same class after partitioning.

Figure 3 presents the Sequence-Congruence Algorithm, which operates on one or more program representation graphs. When the algorithm operates on more than one program's *PRG*, the multiple *PRG*s are treated as one graph; thus, when we refer below to "the graph," we mean the collection of *PRG*s.

*Pass 1*:

For the first pass, some additional edges are added to the graph: an edge from every *if* predicate to each associated $\phi_{if}$ vertex and an edge from every *while* predicate to each associated $\phi_{exit}$ vertex are added to the *PRG*s. These added edges are assigned types 14 and 15, respectively. The initial partition is based on the operators in the vertices. Initially, there is a class for all the non-$\phi$ vertices that have the same operators. There is a class for all the *Entry* vertices; for each variable $x$ there is a class for all the $Initialize_x$ vertices; for each nesting level of *while* loops, there is a class for all the $\phi_{enter}$ vertices at this nesting level; there is a class for all the $\phi_{exit}$ vertices; there is a class for all the $\phi_{if}$ vertices. The initial partition is refined by the partitioning algorithm; however, all control dependence edges are ignored in the first pass. (The edges added in the beginning of the first pass − those of types 14 and 15 − are discarded at the end of the first pass.)

*Pass 2*:

The second pass considers only control dependence edges, and applies the partitioning algorithm again to refine the partition obtained from the first pass.

*Definition*.[4] Vertices are *data-congruent* if they are in the same class after the first pass of partitioning.

---

```
The initial partition is B[1], B[2], ..., B[p]
WAITING := { 1, 2,..., p }
q := p
while WAITING ≠ ∅ do
        select and delete an integer i from WAITING
        for each type m of edge do
                FOLLOWER := ∅
                for each vertex u in B[i] do
                        FOLLOWER := FOLLOWER ∪ m-successor(u)
                od
                for each j such that B[j] ∩ FOLLOWER ≠ ∅ and B[j] ⊈ FOLLOWER do
                        q := q + 1
                        create a new class B[q]
                        B[q] := B[j] ∩ FOLLOWER
                        B[j]  := B[j] - B[q]
                        if j ∈ WAITING
                                then add q to WAITING
                                else if size(B[j]) ≤ size(B[q])
                                        then add j to WAITING
                                        else  add q to WAITING
                                    fi
                        fi
                od
        od
od
```

**Figure 2.** The partitioning algorithm. This algorithm, which is adapted from [Alpern88, Aho74], finds the coarsest partition of a graph that is consistent with a given initial partition of the graph's vertices. The algorithm guarantees that two vertices $v$ and $v'$ are in the same class after partitioning if and only if they are in the same class before partitioning and for any predecessor $u$ of $v$ there is a corresponding predecessor $u'$ of $v'$ such that $u$ and $u'$ are in the same class after partitioning.

Vertices are *sequence-congruent* if they are in the same class after the second pass of partitioning.

In the worst case, the data-congruence classes can be determined in $O(E_1 \log E_1)$ where $E_1$ is the number of flow dependence edges plus the number of $\phi_{if}$ and $\phi_{exit}$ vertices. The sequence-congruence classes can be determined in $O(E_1 \log E_1 + E_2 \log E_2)$ where $E_1$ is as above and $E_2$ is the number of control dependence edges in the graph.

*Example*. Figure 4 shows an example of partitioning. The initial partition is <$A0$, $B0$>, <$A1$, $B1$>, <$A2$, $B2$>, <$A3$, $B3$>, and <$A4$, $B4$>. This partition remains unchanged after the first pass of partitioning. After the second pass of partitioning, the final partition is <A0, B0>, <A1>, <B1>, <A2, B2>, <A3, B3>, and <A4, B4>. Note that A1 and B1 are no longer in the same class; thus, $A1$ and $B1$ are data-

[4]Our terminology differs from that of [4]: our concept of *data-congruence* is similar to that of *congruence* in [4]; our concept of *sequence-congruence* is a new concept that does not appear in [4].

Pass 1: Add an *if*-edge from every *if* predicate to each associated $\phi_{if}$ vertex.
Add a *while*-edge from every *while* predicate to each associated $\phi_{exit}$ vertex.
The starting partition is based on the operators in the vertices as stated in the text.
Apply the partitioning algorithm to refine the initial partition, ignoring all control dependence edges.

Pass 2: The starting partition is the partition obtained from the first pass.
Apply the partitioning algorithm, using only control dependence edges, to further refine the partition.

**Figure 3.** The Sequence-Congruence Algorithm. The Sequence-Congruence Algorithm consists of two passes. Both passes make use of the partitioning algorithm presented in Figure 2; only the starting partition and the edges considered in the two passes are different.

congruent but not sequence-congruent. Note also that vertices A4 and B4 are sequence-congruent even though their slices are not isomorphic (in this example, the slices with respect to A4 and B4 consist of the respective program fragments in their entirety).

### 4.2. The Sequence-Congruence Theorem

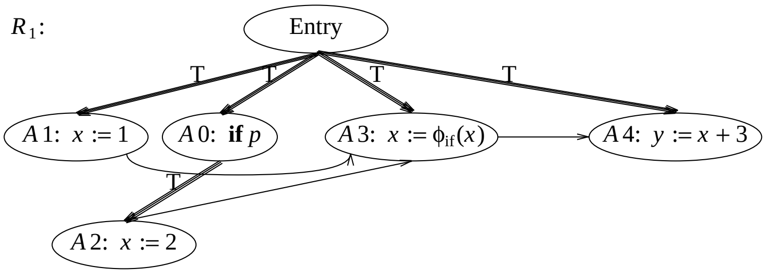
The Sequence-Congruence Algorithm operates on program representation graphs; the Sequence-Congruence Theorem relates the partitioning operation to the execution behaviors of program components. The Theorem asserts that components designated by sequence-congruent vertices produce the same sequences of values when the programs are run on sufficiently similar initial states. (Since a vertex in a *PRG* designates a statement or a predicate in the program, in what follows, "a vertex" is used as a synonym for "a statement or a predicate.")

*THEOREM.* (SEQUENCE-CONGRUENCE THEOREM). *Let $P_1$ and $P_2$ be two (not necessarily distinct) programs with imported variables $Imp_1$ and $Imp_2$, respectively. Let $\sigma_1$ and $\sigma_2$ be two states that agree on $(Imp_1 \cap Imp_2)$. Let $x_1$ and $x_2$ be two vertices in $P_1$ and $P_2$, respectively, that are sequence-congruent. Then*
*(1) If $P_1$ and $P_2$ halt on $\sigma_1$ and $\sigma_2$, respectively, then the sequences of values produced at $x_1$ and $x_2$, respectively, are the same.*
*(2) If $P_1$ halts on $\sigma_1$ but $P_2$ does not halt on $\sigma_2$, then the sequence of values produced at $x_2$ is an initial segment of the sequence of values produced at $x_1$.*
*(3) If $P_1$ does not halt on $\sigma_1$ but $P_2$ halts on $\sigma_2$, then the sequence of values produced at $x_1$ is an initial segment of the sequence of values produced at $x_2$.*
*(4) If neither $P_1$ nor $P_2$ halts on $\sigma_1$ and $\sigma_2$, respectively, then either (a) the sequences of values produced at $x_1$ and $x_2$, respectively, are identical infinite sequences, or (b) the sequence of values produced at $x_1$ is finite and is an initial segment of the sequence of values produced at $x_2$, or* vice versa.

Since the theorem concerns execution behaviors of program components, we define explicitly the notion of a moment immediately before (or after) an execution step. Statements and predicates of a program *P* are executed in the order specified by the augmented control flow graph of *P*. $\phi$ assignments are considered as statements during program execution. A *moment* immediately before (or after) the execution of a vertex *u* denotes the time when *u* is about to start executing (or, respectively, has just finished). There is a subtle distinction between a moment immediately after the execution of a vertex and a moment immediately

$A1$: $x := 1$
$A0$: **if** $p$
$A2$: **then** $x := 2$
**fi**
$A3$: $x := \phi_{if}(x)$
$A4$: $y := x + 3$

$R_1$:




$B0$: **if** $p$
$B2$: **then** $u := 2$
$B1$: **else** $u := 1$
**fi**
$B3$: $u := \phi_{if}(u)$
$B4$: $w := u + 3$

$R_2$:

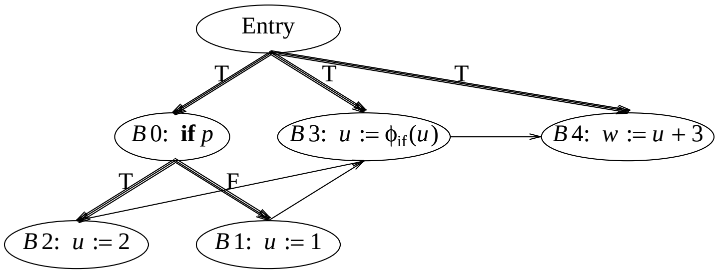



**Figure 4.** $R_1$ and $R_2$ are two fragments of program representation graphs. The incoming flow dependence edges of the two *if* predicates are omitted. The labels on the vertices are included for ease of reference only. The initial partition is <$A0$, $B0$>, <$A1$, $B1$>, <$A2$, $B2$>, <$A3$, $B3$>, and <$A4$, $B4$>. This partition remains unchanged after the first pass of partitioning. After the second pass of partitioning, the final partition is <A0, B0>, <A1>, <B1>, <A2, B2>, <A3, B3>, and <A4, B4>. Note that A1 and B1 are no longer in the same class; thus, $A1$ and $B1$ are data-congruent but not sequence-congruent.

before the execution of the following one. For instance, consider the following program fragment:

$L_1$ $x := 0$
$L_2$ **while** $p$ **do**
. . .
**od**

Suppose there are no $\phi_{enter}$ vertices for the *while* loop. The locus of control is *outside* the loop at the moment immediately after the execution of $L_1$, whereas the locus of control moves *inside* the loop at the moment immediately before the execution of $L_2$. Note also that the locus of control moves outside the loop at the moment immediately after the *while* predicate $L_2$ evaluates to *false*.

It is important to identify the loops that are executing at a moment during program execution. A loop $L$ is executing at a moment $t$ if the locus of control at $t$ is inside $L$. The *current loop predicate* at a moment $t$, written as $CLP(t)$, is the predicate of the innermost loop that is executing at $t$. If there is no such loop, $CLP(t)$ is the *Entry* vertex. In particular, if $t$ is the moment immediately before (or after) executing a $\phi_{enter}$ statement, the locus of control is inside the loop of the $\phi_{enter}$ statement at $t$; hence $CLP(t)$ is the predicate of the associated loop.

A vertex $u$ is an *ancestor* of another vertex $v$ if there is a control-dependence path from $u$ to $v$ in the *PRG*; thus, *while* predicates are ancestors of the associated $\phi_{enter}$ vertices. A loop *encloses* a vertex $v$ (or, equivalently, *v is enclosed in* the loop) if the predicate of the loop is an ancestor of $v$.

Since loops may be executed repeatedly, we distinguish *executions* and *iterations* of a loop. During program execution, there may be several *executions* of a loop; during each execution, there may be one or more *iterations*. There is at least one iteration during an execution of a loop:[5] the $\phi_{enter}$ vertices and the loop predicate must be executed at least once during an execution of the loop.

Since a vertex enclosed in a loop may be executed repeatedly, a vertex is *active* at a moment $t$ (defined below) if the "appropriate" value produced at the vertex is available for use at $t$ [4].

*Definition*. A vertex $u$ in a program representation graph is *active* at a moment $t$ during program execution if (1) $u$ is not enclosed in a loop and has already been executed at $t$, or (2) the innermost loop that encloses $u$ is executing at $t$ and $u$ has been executed during the current iteration.

According to the definition, $\phi_{enter}$ vertices and *while* predicates are active only when the locus of control is inside their loops.

In order to compare execution behaviors of components that may belong to *different* programs, it is necessary to relate two moments, $t_1$ and $t_2$, during the respective executions of the two programs. We say $t_1$ and $t_2$ are *concurrent* (defined below) if the executions of $P_1$ and $P_2$ at $t_1$ and $t_2$ are synchronized in the sense defined below.

*Definition*. Let $P_1$ and $P_2$ be two (not necessarily distinct) programs. Let $t_1$ and $t_2$ be two moments during the executions of $P_1$ and $P_2$, respectively. $t_1$ and $t_2$ are *concurrent* if (1) $CLP(t_1)$ and $CLP(t_2)$ are at the same loop nesting level, (2) corresponding *while* predicates on the control-dependence paths from *Entry* to $CLP(t_1)$ and $CLP(t_2)$, respectively, in the program representation graphs of $P_1$ and $P_2$ are data-congruent, and (3) corresponding *while* predicates have executed the same number of iterations during the current executions of the loops at $t_1$ and $t_2$, respectively.

Note that *if* predicates are ignored in the above definition. For instance, in Figure 4, if $t_1$ is the moment immediately after $A1$ executes and $t_2$ is the moment immediately after $B1$ executes, $t_1$ and $t_2$ are concurrent.

Since sequence-congruence is a refinement of data-congruence, the Sequence-Congruence Theorem (for sequence-congruence classes) is founded on the Data-Congruence Lemma (for data-congruence classes), which states that active, data-congruent vertices have the same values[6] at concurrent moments when the programs run on sufficiently similar initial states.

*LEMMA.* (DATA-CONGRUENCE LEMMA). *Let $P_1$ and $P_2$ be two (not necessarily distinct) programs with imported variables $Imp_1$ and $Imp_2$, respectively. Let $\sigma_1$ and $\sigma_2$ be two states that agree on $(Imp_1 \cap Imp_2)$. Let $t_1$ and $t_2$ be two moments during the executions of $P_1$ and $P_2$ on initial states $\sigma_1$ and $\sigma_2$, respectively. Let $x_1$ and $x_2$ be two vertices in $P_1$ and $P_2$, respectively. If (1) $t_1$ and $t_2$ are concurrent, (2) $x_1$ is active at $t_1$, (3) $x_2$ is active at $t_2$, and (4) $x_1$ and $x_2$ are data-congruent, then $x_1$ and $x_2$ have the same values at $t_1$ and $t_2$, respectively.*

[5]The number of iterations during an execution of a loop defined here differs from the traditional point of view: With our definition, the iteration count is one greater than normal. This convention makes the statement of the proof easier; it does not carry any semantic significance.

[6]The value of a vertex at a moment is the most recent value produced at the vertex before that moment.

*PROOF*. We prove this lemma by contradiction. Suppose the lemma is not correct. Then there exist $x_1$, $x_2$, $t_1$, and $t_2$ that satisfy (1), (2), (3), and (4) above but $x_1$ and $x_2$ have different values at $t_1$ and $t_2$, respectively. Let $t_1$ be the earliest moment during the execution of $P_1$ on initial state $\sigma_1$ such that there is a moment $t_2$ during the execution of $P_2$ on initial state $\sigma_2$ and there are two vertices $x_1$ and $x_2$ of $P_1$ and $P_2$, respectively, such that (1), (2), (3), and (4) hold but $x_1$ and $x_2$ have different values at $t_1$ and $t_2$, respectively. It is possible that, for a given $t_1$, there are many $t_2$, $x_1$, and $x_2$ that fit the above conditions. In this case, the ones with the earliest $t_2$ are chosen. It is also possible that, given $t_1$ and $t_2$, there are many $x_1$ and $x_2$ that fit the above conditions. In this case, the earliest $x_1$ (in terms of appearance in the augmented control flow graph of $P_1$) in $P_1$ is chosen. It is also possible that, given $t_1$, $t_2$, $x_1$, there are many $x_2$ that fit the above conditions. In this case, the earliest $x_2$ in $P_2$ is chosen.

Since $x_1$ and $x_2$ are data-congruent, they must either be $\phi$ statements of the same kind or they must have the same operators. Hence, they have the same incoming control and flow dependence edges. There are five cases depending on the type of vertex $x_1$. We will derive a contradiction in each case.

*Case 1*. Vertex $x_1$ is a *FinalUse* vertex, a non-$\phi$ assignment statement vertex, or a predicate vertex. If $x_1$ is a constant vertex (that is, the expression in $x_1$ is a constant), so is $x_2$ and they must be the same constant. In this case, $x_1$ and $x_2$ always have same values whenever they are active. So assume $x_1$ and $x_2$ are not constant vertices.

Since $x_1$ and $x_2$ have the same number of incoming flow dependence edges, let $y_1$ and $y_2$ be any corresponding flow predecessors of $x_1$ and $x_2$, respectively. Since $x_1$ and $x_2$ are data-congruent, $y_1$ and $y_2$ are also data-congruent. Because there is a flow edge $y_1 \longrightarrow_f x_1$ and $x_1$ is not a $\phi_{exit}$ vertex, any loop enclosing $y_1$ must also enclose $x_1$ (due to the $\phi_{exit}$ vertices in the graph, $y_1$ cannot be nested more deeply than $x_1$). Because $x_1$ is active at $t_1$, the innermost loop enclosing $y_1$, if any, must be executing at $t_1$ and $x_1$ must have been executed during the current iteration of that loop.

Note that flow dependence edges incident on any non-$\phi_{enter}$ vertex run from left to right. Because $x_1$ has been executed, $y_1$ must have already been executed during the current iteration of the innermost loop enclosing $y_1$ (if any); therefore, $y_1$ is active at $t_1$. Similarly, $y_2$ is active at $t_2$.

Note that $y_1$ and $y_2$ come before $x_1$ and $x_2$ in $P_1$ and $P_2$, respectively. Thus, $y_1$ and $y_2$ must have the same values at $t_1$ and $t_2$, respectively, for otherwise we would have chosen $y_1$ and $y_2$ instead of $x_1$ and $x_2$. Because corresponding operands of $x_1$ and $x_2$ have the same values at $t_1$ and $t_2$, respectively, $x_1$ and $x_2$ must have the same values at $t_1$ and $t_2$, respectively, which contradicts the previous assumption that $x_1$, $x_2$, $t_1$, and $t_2$ violate the lemma.

*Case 2*. Vertex $x_1$ is a $\phi_{if}$ vertex. Let $z_1$ and $z_2$ be the *if* predicates for $x_1$ and $x_2$, respectively. Since $x_1$ and $x_2$ are data-congruent, $z_1$ and $z_2$ are also data-congruent. Because $x_1$ is active at $t_1$, $z_1$ is also active at $t_1$. Similarly, $z_2$ is active at $t_2$. Note that $z_1$ and $z_2$ come before $x_1$ and $x_2$ in $P_1$ and $P_2$, respectively. Thus, $z_1$ and $z_2$ must have the same values at $t_1$ and $t_2$, respectively, for otherwise we would have chosen $z_1$ and $z_2$ instead of $x_1$ and $x_2$. Without loss of generality, assume the values of $z_1$ and $z_2$ at $t_1$ and $t_2$, respectively, are *true*.

Let $y_1 \longrightarrow_f x_1$ and $y_2 \longrightarrow_f x_2$ be the incoming flow dependence edges of $x_1$ and $x_2$ from (or around) the *true* branches of $z_1$ and $z_2$, respectively. Because there is a flow edge $y_1 \longrightarrow_f x_1$ and $x_1$ is active at $t_1$, by the same arguments as in Case 1, $y_1$ is active at $t_1$. Similarly, $y_2$ is active at $t_2$. Because $x_1$ and $x_2$ are data-congruent, $y_1$ and $y_2$ are also data-congruent. Thus, $y_1$ and $y_2$ must have the same values at $t_1$ and $t_2$, respectively, for otherwise we would have chosen $y_1$ and $y_2$ instead of $x_1$ and $x_2$. Since $y_1$ and $y_2$ have the same values at $t_1$ and $t_2$, $x_1$ and $x_2$ must have the same values at $t_1$ and $t_2$, respectively, which contradicts the previous assumption that $x_1$, $x_2$, $t_1$, and $t_2$ violate the lemma.

*Case 3.* Vertex $x_1$ is a $\phi_{enter}$ vertex. Let $z_1$ and $z_2$ be the *while* predicates associated with $x_1$ and $x_2$, respectively. Note that a $\phi_{enter}$ vertex is active only when the locus of control is inside the associated loop. Because $t_1$ and $t_2$ are concurrent, corresponding *while* predicates on the control-dependence paths from *Entry* to $CLP(t_1)$ and $CLP(t_2)$ are data-congruent and have executed the same number of iterations during the current executions of the loops at moments $t_1$ and $t_2$, respectively. Since $z_1$ and $z_2$ are at the same nesting level, $z_1$ and $z_2$ are corresponding *while* predicates on the control-dependence paths from *Entry* to $CLP(t_1)$ and $CLP(t_2)$. Hence $z_1$ and $z_2$ are data-congruent and have executed the same number of iterations during the current executions of the loops, at moments $t_1$ and $t_2$, respectively.

(1) Suppose, at $t_1$, it is the first iteration of the loop of $z_1$ during the current execution of the loop. It is also the first iteration of the loop of $z_2$ during the current execution of the loop at $t_2$. Therefore, the values of $x_1$ and $x_2$ at $t_1$ and $t_2$ come from outside the loops of $z_1$ and $z_2$, respectively.

    Let $y_1$ and $y_2$ be the flow predecessors of $x_1$ and $x_2$ from outside the loops of $z_1$ and $z_2$, respectively. Since $x_1$ and $x_2$ are data-congruent, $y_1$ and $y_2$ are also data-congruent. Since $x_1$ and $x_2$ are active at $t_1$ and $t_2$, respectively, $y_1$ and $y_2$ are also active at $t_1$ and $t_2$, respectively. Thus, $y_1$ and $y_2$ must have the same values at $t_1$ and $t_2$, respectively, for otherwise we would have chosen $y_1$ and $y_2$ instead of $x_1$ and $x_2$. Since $y_1$ and $y_2$ have the same values at $t_1$ and $t_2$, $x_1$ and $x_2$ must have the same values at $t_1$ and $t_2$, respectively, which contradicts the previous assumption that $x_1$, $x_2$, $t_1$, and $t_2$ violate the lemma.

(2) Suppose, at $t_1$, it is the $k^{th}$ iteration of the loop of $z_1$ during the current execution of the loop, for some $k > 1$. It is also the $k^{th}$ iteration of the loop of $z_2$ during the current execution of the loop at $t_2$. Therefore, the values of $x_1$ and $x_2$ at $t_1$ and $t_2$ come from inside the loops of $z_1$ and $z_2$, respectively (*i.e.*, the values are produced during the $k-1^{st}$ iterations).

    Let $y_1$ and $y_2$ be the flow predecessors of $x_1$ and $x_2$ from inside the loops of $z_1$ and $z_2$, respectively. Since $x_1$ and $x_2$ are data-congruent, $y_1$ and $y_2$ are also data-congruent. Let $t_1'$ be the moment immediately before the end of the $k-1^{st}$ iteration of the loop of $z_1$ and $t_2'$ be the moment immediately before the end of the $k-1^{st}$ iteration of the loop of $z_2$. Note that $y_1$ and $y_2$ are active at $t_1'$ and $t_2'$, respectively. Note also that $t_1'$ and $t_2'$ are earlier than $t_1$ and $t_2$, respectively, and $t_1'$ and $t_2'$ are concurrent. Thus, $y_1$ and $y_2$ must have the same values at $t_1'$ and $t_2'$, respectively, for otherwise we would have chosen $t_1'$, $t_2'$, $y_1$, and $y_2$ instead of $t_1$, $t_2$, $x_1$, and $x_2$. Since the value of $x_1$ at $t_1$ is the value of $y_1$ at $t_1'$ and the value of $x_2$ at $t_2$ is the value of $y_2$ at $t_2'$, $x_1$ and $x_2$ must have the same values at $t_1$ and $t_2$, respectively, which contradicts the previous assumption that $x_1$, $x_2$, $t_1$, and $t_2$ violate the lemma.

*Case 4.* Vertex $x_1$ is a $\phi_{exit}$ vertex. Let $z_1$ and $z_2$ be the *while* predicates associated with $x_1$ and $x_2$, respectively. Let $y_1$ and $y_2$ be the $\phi_{enter}$ vertices associated with $x_1$ and $x_2$, respectively. Since $x_1$ and $x_2$ are data-congruent, $z_1$ and $z_2$ are data-congruent and $y_1$ and $y_2$ are data-congruent. Because $y_1$ and $y_2$ are data-congruent, the respective loops of $z_1$ and $z_2$ are at the same nesting level.

Because $x_1$ is a $\phi_{exit}$ vertex of the loop of $z_1$ and it is active at $t_1$, the most recent execution of the loop of $z_1$ must have finished. Similarly, since $x_2$ is active at $t_2$, the most recent execution of the loop of $z_2$ must have finished. Let $n_1$ be the number of iterations the most recent execution of the loop of $z_1$ iterated. Let $n_2$ be the number of iterations the most recent execution of the loop of $z_2$ iterated.

We first show that $n_1 = n_2$. Suppose $n_1 \neq n_2$. First assume $n_1 < n_2$. Let $s_1$ be the moment immediately before the ${n_1}^{th}$ evaluation of $z_1$ during the most recent execution of the loop of $z_1$. Let $s_2$ be the moment immediately before the ${n_1}^{th}$ evaluation of $z_2$ during the most recent execution of the loop of $z_2$. Note that the loops of $z_1$ and $z_2$ are executing the ${n_1}^{th}$ iteration during the most recent executions at $s_1$ and $s_2$, respectively. Therefore, $s_1$ and $s_2$ are concurrent. Since the $n_1{}^{th}$ value produced at $z_1$ is *false* but the $n_1{}^{th}$ value produced at $z_2$ is *true*, at least one pair of corresponding operands of $z_1$ and $z_2$ must have different values at the two moments $s_1$ and $s_2$, respectively.

However, because $s_1$ and $s_2$ are concurrent and corresponding operands of $z_1$ and $z_2$ are data-congruent and are active at $s_1$ and $s_2$, respectively, corresponding operands must have the *same* values at $s_1$ and $s_2$, respectively, for otherwise we would have chosen $s_1$ and $s_2$ instead of $t_1$ and $t_2$. Because corresponding operands of $z_1$ and $z_2$ have the same values at $s_1$ and $s_2$, respectively, the $n_1{}^{th}$ values produced at $z_1$ and $z_2$, respectively, must be the same, which contradicts the assumption that $n_1 < n_2$. Hence $n_1$ *greater_or_equal* $n_2$. By the same argument we know $n_2$ *greater_or_equal* $n_1$. Therefore, $n_1 = n_2$. (Let $n$ be $n_1$ or, equivalently, $n_2$.)

Recall that $y_1$ and $y_2$ are the $\phi_{enter}$ vertices associated with $x_1$ and $x_2$, respectively. Let $r_1$ be the moment immediately before the $n^{th}$ evaluation of $z_1$ during the most recent execution of the loop of $z_1$. Let $r_2$ be the moment immediately before the $n^{th}$ evaluation of $z_2$ during the most recent execution of the loop of $z_2$. Note that the loops of $z_1$ and $z_2$ are executing the $n^{th}$ iteration during the most recent executions at $r_1$ and $r_2$, respectively. Therefore, $r_1$ and $r_2$ are concurrent. Since $y_1$ and $y_2$ are data-congruent, $y_1$ is active at $r_1$, $y_2$ is active at $r_2$, and $r_1$ and $r_2$ are concurrent, $y_1$ and $y_2$ must have the same values at $r_1$ and $r_2$, respectively, for otherwise we would have chosen $r_1$, $r_2$, $y_1$, and $y_2$ instead of $t_1$, $t_2$, $x_1$, and $x_2$. Because the value of $x_1$ at $t_1$ is the same as that of $y_1$ at $r_1$ and the value of $x_2$ at $t_2$ is the same as that of $y_2$ at $r_2$, $x_1$ and $x_2$ must have the same values at $t_1$ and $t_2$, respectively, which contradicts the previous assumption that $x_1$, $x_2$, $t_1$, and $t_2$ violate the lemma.

*Case 5*. Vertex $x_1$ is an *Initialize* vertex. Since $x_1$ and $x_2$ are data-congruent, they must be the *Initialize* vertices for the same variable. Since $x_1$ and $x_2$ are not in any loops, they are executed exactly once. Because $\sigma_1$ and $\sigma_2$ agree on $Imp_1 \cap Imp_2$, $x_1$ and $x_2$ must have the same values whenever they are active. This contradicts the previous assumption that $x_1$, $x_2$, $t_1$, and $t_2$ violate the lemma.

We have shown that each of the five cases leads to a contradiction. Therefore, it is impossible to find $t_1$, $t_2$, $x_1$, and $x_2$ such that (1), (2), (3), and (4) are satisfied but $x_1$ and $x_2$ have different values at $t_1$ and $t_2$. □

The Sequence-Congruence Theorem states that sequence-congruent vertices produce the same sequence of values when their programs are run on sufficiently similar initial states.

*THEOREM*. (SEQUENCE-CONGRUENCE THEOREM). *Let $P_1$ and $P_2$ be two (not necessarily distinct) programs with imported variables $Imp_1$ and $Imp_2$, respectively. Let $\sigma_1$ and $\sigma_2$ be two states that agree on ($Imp_1 \cap Imp_2$). Let $x_1$ and $x_2$ be two vertices in $P_1$ and $P_2$, respectively, that are sequence-congruent. Then*
*(1) If $P_1$ and $P_2$ halt on $\sigma_1$ and $\sigma_2$, respectively, then the sequences of values produced at $x_1$ and $x_2$, respectively, are the same.*
*(2) If $P_1$ halts on $\sigma_1$ but $P_2$ does not halt on $\sigma_2$, then the sequence of values produced at $x_2$ is an initial segment of the sequence of values produced at $x_1$.*
*(3) If $P_1$ does not halt on $\sigma_1$ but $P_2$ halts on $\sigma_2$, then the sequence of values produced at $x_1$ is an initial segment of the sequence of values produced at $x_2$.*
*(4) If neither $P_1$ nor $P_2$ halts on $\sigma_1$ and $\sigma_2$, respectively, then either (a) the sequences of values produced at $x_1$ and $x_2$, respectively, are identical infinite sequences, or (b) the sequence of values produced at $x_1$ is finite and is an initial segment of the sequence of values produced at $x_2$, or* vice versa.

*PROOF*. We prove the theorem by contradiction. We first show that if the theorem is not correct, then the following Proposition must hold:

*Proposition*. There are two sequence-congruent vertices $x_1$ and $x_2$ in $P_1$ and $P_2$, respectively, and a constant $k$ such that the $k^{th}$ value produced at $x_1$ is different from the $k^{th}$ value produced at $x_2$.

(The Proposition implies that if the Sequence-Congruence Theorem is not correct, then a counterexample occurs in a finite number of steps.) There are four cases to consider; each case corresponds to a clause in the Theorem. In each case, we will show that if the theorem is not correct, then the Proposition holds.

*Case 1*. Suppose $P_1$ and $P_2$ halt on $\sigma_1$ and $\sigma_2$, respectively. If the theorem is not correct, then the sequences of values produced at $x_1$ and $x_2$, respectively, are different. There are two ways in which the sequences of values produced at $x_1$ and $x_2$ could be different.

(1) There is a constant $k$ such that the $k^{th}$ value produced at $x_1$ is different from the $k^{th}$ value produced at $x_2$. Thus, the Proposition holds.

(2) The sequences of values produced at $x_1$ and $x_2$, respectively, are of different lengths and the shorter sequence is an initial segment of the longer one. We may examine the sequences of values produced at corresponding control ancestors of $x_1$ and $x_2$. Since the sequences of values produced at $x_1$ and $x_2$ are of different lengths and the shorter sequence is an initial segment of the longer one, it is impossible that each pair of corresponding control ancestors of $x_1$ and $x_2$ have produced the same sequence of values. Thus, there must be two corresponding control ancestors, $x_1'$ and $x_2'$, of $x_1$ and $x_2$, respectively, and a constant $k$ such that the $k^{th}$ value produced at $x_1'$ differs from the $k^{th}$ value produced at $x_2'$. Thus, the Proposition holds.

*Case 2*. Suppose $P_1$ halts on $\sigma_1$ but $P_2$ does not halt on $\sigma_2$. If the theorem is not correct, then the sequence of values produced at $x_2$ is not an initial segment of the sequence of values produced at $x_1$. There are two ways in which the sequence of values produced at $x_2$ might not be an initial segment of the sequence of values produced at $x_1$.

(1) There is a constant $k$ such that the $k^{th}$ value produced at $x_1$ is different from the $k^{th}$ value produced at $x_2$. Thus, the Proposition holds.

(2) The sequence of values produced at $x_1$ is a proper initial segment of the sequence of values produced at $x_2$. We may examine the sequences of values produced at corresponding control ancestors of $x_1$ and $x_2$. Since the sequence of values produced at $x_1$ is a proper initial segment of the sequence of values produced at $x_2$, it is impossible that each pair of corresponding control ancestors of $x_1$ and $x_2$ have produced the same sequence of values. Thus, there must be two corresponding control ancestors, $x_1'$ and $x_2'$, of $x_1$ and $x_2$, respectively, and a constant $k$ such that the $k^{th}$ value produced at $x_1'$ differs from the $k^{th}$ value produced at $x_2'$. Thus, the Proposition holds.

*Case 3*. Suppose $P_1$ does not halt on $\sigma_1$ but $P_2$ halts on $\sigma_2$. This case is similar to *Case 2*.

*Case 4*. Suppose neither $P_1$ nor and $P_2$ halts on $\sigma_1$ and $\sigma_2$, respectively. There are two cases to consider depending on whether the sequences of values produced at $x_1$ and $x_2$ are infinite.

(1) Suppose the sequences of values produced at $x_1$ and $x_2$, respectively, are infinite. If the theorem is not correct, the two infinite sequences are not identical. Thus, there must be a constant $k$ such that the $k^{th}$ value produced at $x_1$ differs from the $k^{th}$ value produced at $x_2$. Thus, the Proposition holds.

(2) Suppose at least one of the sequences of values produced at $x_1$ and $x_2$ is finite. If the theorem is not correct, the sequence of values produced at $x_1$ is not an initial segment of the sequence produced at $x_2$, or *vice versa*. Thus, there must be a constant $k$ such that the $k^{th}$ value produced at $x_1$ differs from the $k^{th}$ value produced at $x_2$. Thus, the Proposition holds.

We have shown that if the theorem is not correct, then the Proposition holds. Thus, to prove the theorem, it is sufficient to show that the Proposition leads to a contradiction.

Suppose the theorem is not correct. Then, by the argument given above, the Proposition holds. We can find two sequence-congruent vertices, $x_1$ and $x_2$, in $P_1$ and $P_2$, respectively, and a constant $k$ such that the $k^{th}$ value produced at $x_1$ is different from the $k^{th}$ value produced at $x_2$. Let $t_1$ be the moment immediately after the $k^{th}$ value of $x_1$ is produced and $t_2$ be the moment immediately after the $k^{th}$ value of $x_2$ is produced. There may be many $x_1$, $x_2$, and $k$ that satisfy the Proposition. In this case, the ones with the earliest $t_1$ and $t_2$ are chosen.

Because $x_1$ and $x_2$ are sequence-congruent, either both $x_1$ and $x_2$ are *while* predicates or neither is a *while* predicate. (Due to the control dependence edges that form self-loops on *while* predicate vertices, a *while* predicate can only be sequence-congruent to other *while* predicates.) There are two cases to consider. We will derive a contradiction in each case.

*Case 1*. Suppose neither $x_1$ nor $x_2$ is a *while* predicate. Because $x_1$ and $x_2$ are sequence-congruent, $CLP(t_1)$ and $CLP(t_2)$ are at the same loop nesting levels. Because $x_1$ and $x_2$ are sequence-congruent, $CLP(t_1)$ and $CLP(t_2)$ and each pair of corresponding control ancestors of $CLP(t_1)$ and $CLP(t_2)$ are sequence-congruent and hence data-congruent. Because $t_1$ and $t_2$ are the earliest moments when the theorem fails, $CLP(t_1)$ and $CLP(t_2)$ and each pair of corresponding control ancestors of $CLP(t_1)$ and $CLP(t_2)$ must have produced the same sequences of values during the executions of $P_1$ and $P_2$ from the beginning to $t_1$ and $t_2$, respectively. In particular, all corresponding *while* predicates have executed the same number of iterations during their *current* executions. Therefore, $t_1$ and $t_2$ are concurrent.

Because $x_1$ and $x_2$ are sequence-congruent, they are also data-congruent. Because $x_1$ and $x_2$ are not *while* predicates, $x_1$ and $x_2$ are always active immediately after they are executed. That is, $x_1$ is active at $t_1$ and $x_2$ is active at $t_2$. Thus, from the Data-Congruence Lemma, $x_1$ and $x_2$ have the same value at $t_1$ and $t_2$, respectively, which contradicts the assumption that $x_1$ and $x_2$ have different values at $t_1$ and $t_2$, respectively.

*Case 2*. Suppose $x_1$ and $x_2$ are *while* predicates. Let $t_1'$ and $t_2'$ be the moments immediately before the $k^{th}$ evaluations of $x_1$ and $x_2$, respectively. That is, $t_1'$ and $t_2'$ are the moments just one step earlier than $t_1$ and $t_2$, respectively. Note that $CLP(t_1')$ is $x_1$ and $CLP(t_2')$ is $x_2$. Because $x_1$ and $x_2$ are sequence-congruent, $x_1$ and $x_2$ must be at the same loop nesting levels. Because $x_1$ and $x_2$ are sequence-congruent, each pair of corresponding control ancestors of $x_1$ and $x_2$ are sequence-congruent and hence data-congruent. Because the theorem does not fail until moments $t_1$ and $t_2$, and because moments $t_1'$ and $t_2'$ are earlier than $t_1$ and $t_2$, respectively, we know that $x_1$ and $x_2$ and each pair of their corresponding control ancestors must have produced the same sequence of values from the beginning to $t_1'$ and $t_2'$, respectively. In particular, all corresponding *while* predicates have executed the same number of iterations during their *current* executions. Therefore, $t_1'$ and $t_2'$ are concurrent.

Let $y_1$ and $y_2$ be any corresponding flow predecessors of $x_1$ and $x_2$, respectively. Since $x_1$ and $x_2$ are sequence-congruent, $y_1$ and $y_2$ are data-congruent. Furthermore, $y_1$ is active at $t_1'$ and $y_2$ is active at $t_2'$. >From the Data-Congruence Lemma, $y_1$ and $y_2$ have the same value at $t_1'$ and $t_2'$, respectively. Because corresponding flow predecessors of $x_1$ and $x_2$ have the same values at $t_1'$ and $t_2'$, respectively, $x_1$ and $x_2$ must evaluate to the same values at $t_1$ and $t_2$, respectively, which contradicts the assumption that $x_1$ and $x_2$ have different values at $t_1$ and $t_2$, respectively.

In both cases we have shown a contradiction. Because the Proposition leads to a contradiction, the theorem is proved. □

### 4.3. Enhancements to the Sequence-Congruence Algorithm

In this subsection we consider three simple enhancements to the Sequence-Congruence Algorithm. The first is concerned with simple assignment statements and simple predicates. Due to the property of the *identity* operator in a simple vertex, a simple vertex is, in fact, always data-congruent to its (sole) flow predecessor although this would not be discovered by the Sequence-Congruence Algorithm as defined above. To permit the computation of larger classes of data-congruent vertices, we can merge a simple vertex $v$ with its flow predecessor $u$ before performing the first pass of partitioning. By "merging a vertex $v$ with another vertex $u$" we mean "replace every edge $v \longrightarrow x$ with an edge $u \longrightarrow x$, remove edge $u \longrightarrow v$, and remove vertex $v$." This merge operation is undone before the second pass, but vertices $u$ and $v$ are left in the same partition. Vertices $u$ and $v$ may or may not be put in different partitions during the second pass. For instance, consider the following example:

| | | | |
|---|---|---|---|
| $L_1$ | $a := 1$ | $L_3$ | $c := 1$ |
| $L_2$ | $b := a + 2$ | $L_4$ | $d := c$ |
| | | $L_5$ | $e := d + 2$ |

If we merge the simple assignment statement $L_4$ with its flow predecessor $L_3$ before performing the first pass of partitioning, we can discover that $L_2$ and $L_5$ are both data-congruent and sequence-congruent. The proofs in the previous section can be directly adapted to account for this change by extending the notion of "corresponding" flow predecessors of two vertices to take simple vertices into account.

In the Sequence-Congruence Algorithm, we assume that a statement like "$x := a + b * c$" has the same operator as a statement like "$y := d + e * f$" but a different operator than a statement like "$z := g * h$"; that is, the structure of the right-hand-side expression defines the operator. An expression like "$a + b * c$" is viewed as an operator that takes three arguments $a$, $b$, and $c$, and returns the value of "$a + b * c$". Thus, in the following program fragment, $L_1$ and $L_2$ are not sequence-congruent because they have different operators.

$L_1$ $\quad x := a + b * c$

$\quad\quad z := b * c$
$L_2$ $\quad y := a + z$

We can detect more sequence-congruent components if the program is transformed to three-address code before partitioning. For the above example, the assignment to $x$ is replaced by two statements when the program fragment is transformed to three-address code; $L_3$ and $L_4$ are found to be sequence-congruent by the Sequence-Congruence Algorithm.

$\quad\quad temp := b * c$
$L_3$ $\quad x := a + temp$

$\quad\quad z := b * c$
$L_4$ $\quad y := a + z$

Similarly, a constant inside an expression is tightly coupled with the operator. An expression like "$a + 1$" is viewed as a unary operator that takes an argument $a$ and returns the value of "$a + 1$". Therefore, in the following program fragment, $L_5$ and $L_6$ are not sequence-congruent because they have different operators (and different number of incoming flow dependence edges).

$L_5$ $x := a + 1$

$z := 1$
$L_6$ $y := a + z$

As before, a simple transformation can improve the result of partitioning: (1) for each constant $c$ that appears in the program, a new variable *Const_c* is created, (2) an assignment statement "*Const_c* := *c*" is added at the very beginning of the program, and (3) all references to $c$ in the program are changed to references to *Const_c*. This transformation does not change the execution behavior of a program; however, larger sequence-congruence classes will result from partitioning.

We close this section with an observation about how some additional enhancements to the Sequence-Congruence Algorithm can be made. Although the Sequence-Congruence Algorithm presented above uses the same partitioning algorithm—the one given in Figure 2—for both Pass 1 and Pass 2, this is not strictly necessary. The proof of the Sequence-Congruence Theorem depends only on the condition that the equivalence classes used at the start of Pass 2 have the properties listed in the Data-Congruence Lemma. Thus, any techniques applied during Pass 1 that result in larger equivalence classes with these properties will not affect the arguments we have given to establish the properties of the equivalence classes computed by Pass 2; the equivalence classes computed by Pass 2 will still have the properties listed in the Sequence-Congruence Theorem.

One kind of enhancement that may be worthwhile incorporating into Pass 1 is one that takes into account the mathematical properties of an expression's operator. For instance, consider the following example:

| | | | |
|---|---|---|---|
| $L_1$ | $a := 1$ | $L_5$ | $c := 2$ |
| $L_2$ | $b := 2$ | $L_6$ | $d := 1$ |
| $L_3$ | $x := a + b$ | $L_7$ | $u := c + d$ |
| $L_4$ | $y := x * 3$ | $L_8$ | $v := u * 3$ |

With the present algorithm for Pass 1, $L_3$ and $L_7$ are eventually placed in separate data-congruence classes, and hence $L_4$ and $L_8$ are also placed in separate data-congruence classes. However, because addition is commutative, $L_3$ and $L_7$ could be placed in a single equivalence class, which then also makes it possible for $L_4$ and $L_8$ to be members of a single equivalence class. The benefits of finding larger equivalence classes during Pass 1 carry over to Pass 2; in this example, $L_3$ and $L_7$ would be members of one sequence-congruence class, and $L_4$ and $L_8$ would be members of another.

## 5. RELATION TO PREVIOUS WORK

Program representation graphs combine features of both program dependence graphs [11,6,9] and static-single-assignment forms [4,14,5] (especially the value graph associated with an *SSA* form [4]). Although program dependence graphs contain no $\phi$ vertices they do contain additional data-dependence edges not found in program representation graphs. For example, the program dependence graphs used in [9] contain def-order dependence edges. We have shown elsewhere that *PRG*s and the program dependence graphs of [9] are equivalent program representations in the sense that two programs have isomorphic *PRG*s if and only if their program dependence graphs are isomorphic [17]. In essence, the $\phi$ vertices of program representation graphs introduce extra flow dependence edges that substitute for def-order dependence edges.

There are two main differences between our *PRG*s and (the value graphs of) *SSA* forms:

(1) *PRG*s contain control dependence edges, whereas *SSA* forms do not. Control dependence edges were added so that the Sequence-Congruence Algorithm could take control dependences into

account during partitioning.

(2) The $\phi$ statements in *PRG*s are slightly different from those in *SSA* forms. In the *SSA* forms defined in [4,14,5] a $\phi$ operator in a $\phi$ statement is a binary operator; that is, a $\phi$ statement is of the form "$x_1 := \phi(x_2, x_3)$." Variable occurrences are renamed (by adding subscripts, for example) so that each variable is assigned to exactly once in the program text (whence the name "static-*single-assignment* form"). Because variable renaming is not necessary for our purposes, we chose a simpler form of $\phi$ statement.[7]

Another difference between the $\phi$ statements in *PRG*s and those in *SSA* forms is that *PRG*s include only $\phi$ statements whose left-hand side variable is live (*i.e.* every $\phi$ statement has a flow successor). For instance, consider the following program fragment (which is augmented with $\phi$ statements for the $\phi$ vertices of its *PRG*):

```
      a := 0
L1    x := 1
      if p then
            x := 2
            a := x
      fi
      a := φ_if(a)
L2    x := 3 + a
```

If the *PRG* were to include non-live $\phi$ statements, there would be a $\phi$ statement "$x := \phi_{if}(x)$" immediately after the *if* statement; however, this $\phi_{if}$ statement is not included in the *PRG* for this program because the variable *x* is defined before being used after the *if* statement. The reasons why we excluded these extra $\phi$ statements from *PRG*s are not directly relevant to the questions addressed in this paper; however, to summarize briefly, our decisions were motivated by the following concerns:

(1) The exclusion of non-live $\phi$ statements permits larger sets of semantically equivalent programs to have the same *PRG*. For example, the same *PRG* represents not only the program shown above, but also a version of the program in which $L_1$ comes *after* the *if* statement (but before $L_2$). As shown elsewhere [17], *PRG*s and the program dependence graphs of [9] are equivalent program representations in the sense that two programs have isomorphic *PRG*s if and only if their program dependence graphs are isomorphic. The two representations would not be equivalent in this way if *PRG*s were to contain the additional $\phi$ statements of previous definitions.

(2) A useful operation on *PRG*s is that of *slicing*: The slice of a *PRG R* with respect to a set of non-$\phi$ vertices *S* is the subgraph of *R* induced by all vertices from which there is a path to an element of *S* via control and/or flow dependence edges in *R*. We wished to have the property that any slice of a *PRG* would be (isomorphic to) the *PRG* of some program. (This corresponds to a similar property that holds for slices of program dependence graphs — the Feasibility Lemma of [12].) For instance, if non-live $\phi$ vertices were required in *PRG*s, the example given above would be

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[7] It has been recognized by others that variable renaming is not necessary for all uses of SSA forms [3].

$a := 0$
$L_1$ $x := 1$
**if** $p$ **then**
$x := 2$
$a := x$
**fi**
$x := \phi_{if}(x)$
$a := \phi_{if}(a)$
$L_2$ $x := 3 + a$

Its slice with respect to $L_2$ would correspond to the fragment

$a := 0$
**if** $p$ **then**
$x := 2$
$a := x$
**fi**
$a := \phi_{if}(a)$
$L_2$ $x := 3 + a$

However, the slice of the *PRG* does not correspond to any program; the fragment shown above is not annotated properly with $\phi$ statements so as to correspond to any program. In particular, because it lacks a (non-live) $\phi$ vertex $x := \phi_{if}(x)$ just after the *if* statement, it does *not* correspond to the program

$a := 0$
**if** $p$ **then**
$x := 2$
$a := x$
**fi**
$L_2$ $x := 3 + a$

By excluding non-live $\phi$ statements from *PRG*s, infeasible slices do not arise; every slice of a *PRG* is (isomorphic to) the *PRG* of some program [17].

The Sequence-Congruence Algorithm is based on an idea of [4] for finding equivalence classes of program components by first optimistically grouping possibly equivalent components in an initial partition (of an *SSA* form's value graph) and then finding the coarsest partition of the graph's vertices that is consistent with the initial partition. The algorithm of [4] considers only flow dependences among program components. The property established in [4] shows that components of a *single* program in the same partition produce the same values at *certain moments* during execution. Stated using our terminology: "If two data-congruent components are both active at some moment, they have produced the same values."

In contrast to [4], our Sequence-Congruence Algorithm has two important properties: (1) By considering control dependences as well as data dependences, it is able to detect components with equivalent *execution behaviors*, and (2) it is able to do so even if the components are in *different* programs.

The algorithm of [4] is essentially the first pass of our Sequence-Congruence Algorithm. An important difference is that our algorithm can be applied to *one or more* programs. Our Data-Congruence Lemma establishes a semantic property for components that are in the same partition after the first pass: "If two data-congruent components are active at concurrent moments when the programs are run on sufficiently similar initial states, they have produced the same values." It is the concept of concurrent moments that makes it possible to compare the executions of more than one program.

The second pass of our Sequence-Congruence Algorithm uses control dependences to refine the partitioning produced by the first pass. Our Sequence-Congruence Theorem establishes a semantic property about the overall execution behaviors of congruent components (rather than about their behaviors at certain moments): "If two components are sequence-congruent, they have identical execution behaviors whenever the programs they are in are run on sufficiently similar initial states.”

Our work on program representation graphs and the Sequence-Congruence Algorithm has been motivated by the desire to improve our techniques for automatic program integration [9]. The goal of such work is to create a tool that can automatically determine whether changes that have been made in several variants of a base program interfere; if there is *no* interference, the tool should produce a new program that combines the changed behaviors of the variants as well as the behaviors common to the base program and all variants. A program integration tool is needed, for example, when a number of collaborators are collectively producing updates in a large programming project.

One of the key issues of program integration is to determine whether the execution behavior of a program component in one of the variants differs from that of the corresponding component in the base program. In the integration algorithm of [9], this is done by finding corresponding program components whose slices are not isomorphic. The justification for this approach is found in [12] where (for the same language considered in this paper) it is shown that program components with isomorphic slices have identical execution behaviors.

The Sequence-Congruence Algorithm given in this paper provides a different method for determining whether program components in two programs have different execution behaviors. Rather than comparing slices, the Sequence-Congruence Algorithm starts from an optimistic assumption about which program components may exhibit identical execution behaviors and then refines this assumption by considering flow and control dependences. The justification for this approach is provided by the Sequence-Congruence Theorem, which shows that sequence-congruent components have identical execution behaviors. It is easy to see that the equivalence classes found by the Sequence-Congruence Algorithm are strictly larger than those found by comparing slices.

We have devised a new algorithm for program integration, described in [16], that makes use of the Sequence-Congruence Algorithm. This algorithm has several advantages when compared with the integration algorithm described in [9]. One advantage concerns the ability of users to rename variables: because the algorithm from [9] detects changed execution behavior by comparing program slices, all program elements that depend on a variable whose name has been changed will be considered to have different execution behavior. This increases the likelihood that the integration algorithm will report that two users’ modifications are in conflict. In contrast, the Sequence-Congruence Algorithm will determine that such program elements will not have different execution behaviors. Thus, the integration algorithm that uses the Sequence-Congruence Algorithm is able to avoid reporting some of the spurious conflicts that arise when a variable’s name is changed.

Variable renaming is an example of one kind of meaning-preserving modification — one that does not introduce any structural changes to a program. In addition to being able to detect that two components have the same execution behavior in spite of variable renaming, the Sequence-Congruence Algorithm is also able to detect that two components have the same execution behavior for a limited class of meaning-preserving structural changes. Thus, a second advantage of the new integration algorithm is that it can accommodate a limited class of meaning-preserving structural changes and not report some of the spurious conflicts that would be reported by the algorithm from [9].

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