Slope Matrices

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The purpose of this paper is to indicate a simple connection between matrix functions and slopes (divided differences) and thus to suggest the use of slopes as computational quantities in their own right, since a very simple calculus holds for them. Traditional slope notation is incomplete, to be sure, and makes it difficult to exploit the possibilities completely. Consequently, a useful generalization will be proposed. The Author first used slope matrices in lectures two years ago; more recently, Descloux [1] has introduced for other purposes a matrix W which reduces to a slope matrix in special cases.

Let G be a region in the complex x-plane, $f(x)$ a complex-valued function on G, and X a system of $n+1$ pairwise distinct nodes x_r in G, $r = 0, 1, \ldots, n$, its *support*.

A *slope matrix* is defined to be the matrix function

$$S[f;X] = f(A_X), \tag{1}$$

where A_X is the (diagonal-like) *support matrix*

$$A_X = \begin{pmatrix} x_0 & 1 & 0 & \ldots & 0 \\ 0 & x_1 & 1 & \ldots & 0 \\ 0 & 0 & x_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & x_n \end{pmatrix}; \tag{2}$$

for information about matrix functions see, for example, [7], p. 262f. Since the eigenvalues of $S[f;X]$ are the function values $f(x_r)$ and its eigenvectors are the same as those of A_X, the slope matrix is easy to represent. For Q_X, the matrix of eigenvectors of A_X there holds

$$Q_X = (q_{jk}), \quad q_{jk} = \begin{cases} \prod_{\nu=j}^{k-1}(x_k - x_\nu)^{-1} & \text{for } j < k, \\ 1 & \text{for } j = k, \\ 0 & \text{for } j > k. \end{cases} \tag{3}$$

Q_X^{-1} is obtained simply as the matrix of the so-called left eigenvectors of A_X,

$$Q_X^{-1} = (\bar{q}_{jk}), \quad \bar{q}_{jk} = \begin{cases} \prod_{\nu=j+1}^{k}(x_j - x_\nu)^{-1} & \text{for } j < k, \\ 1 & \text{for } j = k, \\ 0 & \text{for } j > k. \end{cases} \tag{4}$$

If F_X is the *nodal value matrix*, the diagonal matrix $\langle f(x_0), f(x_1), \ldots, f(x_n)\rangle$, then there holds the representation

$$S[f;X] = Q_X F_X Q_X^{-1}. \tag{5}$$

The elements of the slope matrix are found to be

$$S[f;X] = (s_{jk}), \quad s_{jk} = \begin{cases} \sum_{r=j}^{k} q_{jr}\bar{q}_{rk} f(x_r) & \text{for } j \le k, \\ 0 & \text{for } j > k. \end{cases} \tag{6}$$

This formula shows that the elements of $S[f;X]$ for $j < k$ are exactly the slopes constructed with the function $f(x)$ and the nodes of X according to the usual definition (cf. [2], p. 134). The following notation is proposed for slopes:

$$[f(x); x_j, x_{j+1}, \ldots, x_k]_x = s_{jk}, \quad s_{jk} \text{ from (6)}. \tag{7}$$

The ordinary notation is thus augmented by a function part — $f(x)$ — and the *slope variable* — x — is adjoined as an index; these additions will be omitted if not necessary for clarity.

The most important properties of the slope matrices follow at once from (5), namely

$$S[f;X] = c_1 S[f_1;X] + c_2 S[f_2;X], \quad f(x) = c_1 f_1(x) + c_2 f_2(x); \tag{8}$$

$$S[f;X] = S[f_1;X]S[f_2;X], \quad f(x) = f_1(x) f_2(x). \tag{9}$$

Thus, for a fixed support X there holds:

(a) The construction of the slope matrix (and thus the slopes) is a linear operation.

(b) The slope matrices form a commutative ring, and by means of the construction of S the ring of all functions over G is mapped homomorphically into the ring of slope matrices.

Confluent slope matrices in the case that the nodes "coalesce" with given multiplicity are best explained by requiring that the slope matrices defined by (1) are $G \times \cdots \times G$ continuous functions of the nodes; $f(x)$ must accordingly be differentiable sufficiently often over G. A_X is no longer diagonal-like in the passage to the limit; consequently (5) and (7) lose their validity for the confluent form, but not the final forms (8) and (9).

Several examples of applications will be indicated below.

1. The known computational rules for slopes are obtained in a simple *deductive* way. It follows from (7) that

$$[f; x_j] = f(x_j), \quad j = 0, 1, \ldots, n, \tag{10a}$$

and from the commutability of $S[f;X]$ with A_X, the *recursion formula*

$$[f; x_j, \ldots, x_k] = \frac{[f; x_{j-1}, \ldots, x_k] - [f; x_j, \ldots, x_{k-1}]}{x_k - x_j}, \quad j < k, \tag{10b}$$

follows by elementwise comparison. From (9) follows at once the important *product formula*,

$$[f_1(x) f_2(x); x_j, \ldots, x_k]_x = \sum_{r=j}^{k} [f_1(x); x_j, \ldots, x_r]_x [f_2(x); x_r, \ldots, x_k]_x, \tag{11}$$

in which f_1 and f_2 can be interchanged; this formula is indeed due to Steffensen.

2. The usefulness of the notation introduced in (7) can be realized by the representation of certain relationships without the need of further elaboration:

(a) For a slope whose slope variables are themselves nodes of another slope, one can for example immediately write

$$[[f(u); x_j, \ldots, x_r, x]_u; x_{r+1}, \ldots, x_k]_x = [f; x_j, \ldots, x_k], \quad r = j, \ldots, k-1. \tag{12}$$

(b) The polynomial $g_{jk}(x)$ of degree at most $k-j$ which approximates $f(x)$ according to the interpolation condition $f(x_r) = g_{jk}(x_r)$, $r = j, \ldots, k$ can be represented in the compact form

$$g_{jk}(x) = (x - x_j) \cdots (x - x_k)[f(u)/(x-u); x_j, \ldots, x_k]_u, \tag{13}$$

where (13) only needs to be supplemented by the requirement of continuity at $x = x_r$. This formula permits a rapid transformation of the Newton expansion of $g_{0n}(x)$ to the recursive representation of Aitken ([2], p. 156ff.) in that one applies to (13) the product formula (11) once and the recursion formula (10b) once.

3. The properties of slope matrices described by (8) and (9) at once permit developing slope matrices of rational combinations of functions. For nonsingular $S[f; X]$ with $f(x_r)\bar{f}(x_r) = 1$, $r = 0, 1, \ldots, n$, one has immediately

$$S[\bar{f}; x] = (S[f; X])^{-1}. \tag{14}$$

From this follows the explicit representation for the slopes,

$$[1/f(x); x_j, \ldots, x_k]_x = (-1)^{k-j} \cdot \det \Delta^0_{jk} \Big/ \prod_{r=j}^{k} f(x_r), \qquad j < k, \tag{15}$$

where $f(x_r) \neq 0$, $r = j, j+1, \ldots, k$ and Δ^0_{jk} is the (contiguous in S) $(k-j)$-rowed submatrix of $S[f; X]$ with $[f; x_j, \ldots, x_k]$ as its upper right element. A formula exactly as simple holds for the slope of the quotient of two functions. (15) is a generalization of the representation of the derivatives of $1/f(x)$ in terms of derivatives of $f(x)$, see [5], p. 181.

4. For a sufficiently large number of nodes, the slope matrices of polynomials are band matrices. This leads to a simplification of computation in the case of interpolation by rational functions. Let $Z(x)$ and $N(x)$ be polynomials of degree at most l and m, respectively $(l + m = n)$ and the interpolation condition be $f(x_r) = Z(x_r)/N(x_r)$, $r = 0, 1, \ldots, n$, then according to (9) there holds: $S[Z; X] = S[N; X]S[f; X]$. The first rows of these matrices give the equations

$$[Z; x_0, \ldots, x_k] = \sum_{r=0}^{\kappa} [N; x_0, \ldots, x_r][f; x_r, \ldots, x_k], \quad k = 0, 1, \ldots, l, \tag{16a}$$

$$\begin{aligned} 0 &= \sum_{r=1}^{\kappa} [N; x_0, \ldots, x_r][f; x_r, \ldots, x_k], \quad k = l+1, \ldots, n, \\ \kappa &= \min(k, m). \end{aligned} \tag{16b}$$

The homogeneous system (16b) furnishes the slopes $[N; x_0, \ldots, x_r]$, $r = 0, 1, \ldots, m$, thus the denominator polynomial. The slopes of $Z(x)$ and thus the numerator polynomial is obtained from (16a) — without solution of equations. If $l < m$ then one can work with the function $1/f(x)$ and formulate the systems corresponding to (16a,b):

$$[N; x_0, \ldots, x_k] = \sum_{r=0}^{\bar{\kappa}} [Z; x_0, \ldots, x_r][1/f; x_r, \ldots, x_k], \quad k = 0, 1, \ldots, m, \tag{17a}$$

$$\begin{aligned} 0 &= \sum_{r=1}^{\bar{\kappa}} [Z; x_0, \ldots, x_r][1/f; x_r, \ldots, x_k], \quad k = m+1, \ldots, n, \\ \bar{\kappa} &= \min(k, l). \end{aligned} \tag{17b}$$

The order of the system of equations which one has to solve for interpolation with $n + 1$ nodes thus never exceeds the number $n/2$. Also, the method proposed by Werner [6], by means of which approximations to rational Tschebyscheff-approximation can be found, can be carried out by means of the system (16b). This has the advantage one only needs to calculate the slopes of the functions denoted by f and g in [6].

Furthermore, (17b) has already been employed with success by the Author ([3],[4]) for the determination of zeros of $f(x)$ by using the l zeros of $Z(x)$ — l can be prescribed arbitrarily — as approximations to l zeros of $f(x)$ and improved stepwise. This method presents the possibility to approximate several "closely packed" zeros simultaneously without needing $f(x)$ itself to be a polynomial; one thus circumvents the difficulties which usually arise in such cases. Other known procedures are found to be special cases of this method.

5. If one forms a matrix A_X^* in which the units in the upper secondary diagonal by numbers a_1 through a_n, then there holds for the elements s_{jk}^* of $S^*[f; X] = f(A_X^*)$:

$$s_{jk}^* = \begin{cases} a_{j+1} a_{j+2} \cdots a_k \cdot s_{jk} & \text{for } j < k, \\ s_{jk} & \text{for } j \geq k. \end{cases} \tag{18}$$

By choice of the values a_r, one has a far-reaching ability to specify a desired order of magnitude for the numerical evaluation of the *modified slopes* s_{jk}^* without essentially complicating the calculation of the s_{jk}^* by its use.

References

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