Computer Sciences Department

Privacy Skyline: Privacy with Multidimensional Adversarial Knowledge

Bee-Chung Chen
Kristen LeFevre
Raghu Ramakrishnan

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Bee-Chung Chen* Kristen LeFevre† Raghu Ramakrishnan
University of Wisconsin – Madison, USA Yahoo! Research
{beechung, lefevre}@cs.wisc.edu ramakris@yahoo-inc.com

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ABSTRACT

Privacy is an important issue in data publishing. Many organizations distribute non-aggregate personal data for research, and they must take steps to ensure that an adversary cannot predict sensitive information pertaining to individuals with high confidence. This problem is further complicated by the fact that, in addition to the published data, the adversary may also have access to other resources (e.g., public records and social networks relating individuals), which we call *external knowledge*. A robust privacy criterion should take this external knowledge into consideration.

In this paper, we first describe a general framework for reasoning about privacy in the presence of external knowledge. Within this framework, we propose a novel multidimensional approach to quantifying an adversary's external knowledge. This approach allows the publishing organization to investigate privacy threats and enforce privacy requirements in the presence of various types and amounts of external knowledge. Our main technical contributions include a multidimensional privacy criterion that is more intuitive and flexible than previous approaches to modeling background knowledge. In addition, we provide algorithms for measuring disclosure and sanitizing data that improve computational efficiency several orders of magnitude over the best known techniques.

1. INTRODUCTION

A number of recent high-profile attacks have illustrated the importance of protecting individuals' privacy when publishing or distributing sensitive personal data. For example, by combining a public voter registration list and a released database of health insurance information, Sweeney was able to identify the medical record of the governor of Massachusetts [16].

In the context of data publishing, it is intuitive to think of privacy as a game between a data owner, who wants to release data for research, and an adversary, who wants to discover sensitive information about the individuals in the database. Following most of the previous literature, we take a constrained optimization approach. That is, the data owner seeks to find the "snapshot" (*release candidate*) of her original dataset that simultaneously satisfies the given privacy criterion and maximizes some utility measure. Note that the privacy criterion determines the safety of the released data, and the utility measure is an orthogonal issue.

The focus of this paper is developing a practical privacy criterion that captures the problem of attribute disclosure in the presence of external knowledge. Specifically, we consider the case where the data owner has a table of data (denoted by **D**), in which each row is a record pertaining to some individual. The attributes of this table consist of (1) a set of *identifier* (ID) attributes which will be removed from the released dataset, (2) a set of *quasi-identifier* (QI) attributes that together can potentially be used to re-identify individuals, and (3) a *sensitive* attribute (denoted by *S*), which is possibly set-valued. For example, consider the original data in Figure 1. In this example, *Name* is the ID attribute. *Age*, *Gender*, *Zipcode* are the QI attributes, and *Disease* is the sensitive attribute.

After applying an "anonymization" procedure, the data owner publishes the resulting release candidate $\mathbf{D}^*$. In this paper, we consider two approaches to generating $\mathbf{D}^*$. The first approach generalizes the QI attribute values to obtain a *generalized table* (as in [6, 7, 16]). Figure 2 shows an example. The second approach partitions the individuals into disjoint groups, producing a *bucketized dataset*, and releases the multiset (or bag) of sensitive values for each group (as in [13, 17]), e.g., Figure 3.

Now consider an adversary whose goal is to predict whether a target individual *t* has a target sensitive value *s*. In making this prediction, he has access to the released dataset $\mathbf{D}^*$, as well as his own external knowledge *K*. This external knowledge may include similar datasets released by other organizations, social networks relating individuals, and other instance-level information. A robust privacy criterion should place an upper bound on the adversary's confidence in predicting that any individual *t* has sensitive value *s*. In other words, the criterion should guarantee that $\Pr(t \text{ has } s \mid K, \mathbf{D}^*)$ is sufficiently small.

Returning to the example in Figure 3, assume that each individual has only one disease in the original dataset. In the absence of external knowledge, intuitively the adversary can predict Tom to have AIDS with confidence $\Pr(\text{Tom has AIDS} \mid \mathbf{D}^*) = 1/4$ because there are four individuals in group 2, only one of whom has AIDS. However, the adversary can improve his confidence based on external knowledge:

- The adversary knows Tom personally, and is sure he does not have Cancer. After removing the record with Cancer, the probability of Tom having AIDS becomes 1/3.
- From another dataset, the adversary determines that Gary has Flu. By further removing Gary's Flu record, the probability of Tom having AIDS becomes 1/2.
- From public records, the adversary knows that Ann is Tom's wife. Thus, it is likely that if Ann has AIDS, then Tom does

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Name	Age	Gender	Zipcode	Disease
Ann	20	F	12345	AIDS
Bob	24	M	12342	Flu
Cary	23	F	12344	Flu
Dick	27	M	12343	AIDS
Ed	35	M	12412	Flu
Frank	34	M	12433	Cancer
Gary	31	M	12453	Flu
Tom	38	M	12455	AIDS

Figure 1. Original dataset

	Age	Gender	Zipcode	Disease
(Ann)	2*	*	1234*	AIDS
(Bob)				Flu
(Cary)				Flu
(Dick)				AIDS
(Ed)	3*	M	124**	Flu
(Frank)				Cancer
(Gary)				Flu
(Tom)				AIDS

Figure 2. Generalized table

	Age	Gender	Zipcode	Group
(Ann)	20	F	12345	1
(Bob)	24	M	12342	
(Cary)	23	F	12344	
(Dick)	27	M	12343	
(Ed)	35	M	12412	2
(Frank)	34	M	12433	
(Gary)	31	M	12453	
(Tom)	38	M	12455	

Group	Disease
1	AIDS
	Flu
	Flu
	AIDS
2	Flu
	Cancer
	Flu
	AIDS

Figure 3. Bucketized dataset

as well. We will return to this example later in the paper.

In designing a privacy criterion incorporating adversarial knowledge, we must address two key problems. First, we must provide the data owner with the means to specify adversarial knowledge K. Second, we must compute Pr(t has s | K, $\mathbf{D}^*$). Unfortunately, the first problem is further complicated by the fact that, in general, the data owner does not know precisely what knowledge an adversary has. In fact, when data is published on the worldwide web, there may be many different adversaries, each with different external knowledge.

Martin et al. provide the first formal treatment of adversarial external knowledge in attribute disclosure [13]. Their framework provides a language for expressing such knowledge. Because it is nearly impossible for the data owner to anticipate specific adversarial knowledge, they instead propose quantifying the external knowledge, and releasing data that is resilient to a certain *amount* of knowledge (in the worst case, regardless of the specific content of this knowledge). Unfortunately, the way that they quantify external knowledge (the maximum number k of implications that an adversary may know) is not intuitive. In practice, this makes it difficult for the data owner to set an appropriate k value. One of our main goals is to provide intuitive, and hence more usable, quantification of external knowledge.

1.1 Contributions & Organization

In Section 2, we describe a theoretical framework for computing the breach probability Pr(t has s | K, $\mathbf{D}^*$). This is related to several Bayesian interpretations of privacy in data publishing [11, 13, 18]. In addition, we extend the study of attribute disclosure under adversarial knowledge to set-valued sensitive attributes, which has not previously been studied.

In Section 3, we describe our desiderata for the design of a practical privacy criterion. Following these desiderata, in Section 4, we develop a novel multidimensional approach to quantifying adversarial knowledge, creating a multidimensional knowledge space for data privacy, which has not been studied before.

Using this multidimensional approach, we make several important technical contributions: (1) In Section 4.2, we define a novel skyline privacy criterion, which provides the data owner a flexible way to enforce her privacy policy. (2) In Section 4.3, we propose a novel skyline exploratory tool, which allows the data owner to investigate the multidimensional knowledge space and understand whether a particular release candidate is safe in the presence of various types and amounts of adversarial knowledge. Using this tool, we show (in Section 7.3) that an ℓ-diverse [11] release candidate can be unsafe under certain types of external knowledge. (3) In Sections 5 and 6, we develop efficient and scalable algorithms for measuring disclosure and sanitizing data (using an advanced multidimensional generalization technique [7]) in the presence of external knowledge. Each of these algorithms is based on an important "congregation" property, and as shown in Section 7, the algorithms improve computational efficiency several orders of magnitude over the best known technique ([13]).

2. THEORETICAL FRAMEWORK

In this section, we give an overview of the theoretical framework for computing the probability of a target statement E about an original dataset $\mathbf{D}$ (e.g., individual t has sensitive value s in $\mathbf{D}$) given a release candidate $\mathbf{D}^*$ derived from $\mathbf{D}$ and external knowledge K about $\mathbf{D}$, where $\mathbf{D}$ is not observed. The framework is depicted diagrammatically in Figure 4.

2.1 Formalism

Like [11, 13], we conservatively assume that whenever the adversary has knowledge about an individual, he always knows the individual's QI values, or *full identification information* (e.g., from public records). Under this assumption, we model the original dataset as a set of individuals, each with a set or multiset of associated sensitive values.

Original dataset: An original dataset is of the following form:

$$\mathbf{D} = \{(u_1, S_1), \ldots, (u_n, S_n)\},$$

where $u_1, \ldots, u_n$ are n distinct individuals, and $S_1, \ldots, S_n$ are sets or multisets of sensitive values. We say t has s (denoted by $s \in t[S]$) in $\mathbf{D}$ iff $(t, t[S]) \in \mathbf{D}$ and $s \in t[S]$.

Integrity Constraints: Integrity constraints may be defined on the original dataset. In this paper, we consider the following cases:

- **SVPI** (single value per individual): Each individual has exactly one sensitive value in $\mathbf{D}$. That is, $|S_i| = 1$, for all i. Note that the case where some individuals do not have any sensitive values can be handled by including a special sensitive value meaning "no sensitive value." Many studies of data privacy only consider the SVPI case.
- **MVPI** (multiple values per individual): Each individual can have multiple sensitive values in $\mathbf{D}$. We further distinguish two sub-cases. In the **MVPI-Set** case, each S_i is a (possibly empty) set. In the **MVPI-Multiset** case, each S_i is a (possibly empty) duplicate-preserving multiset.

In the rest of the paper, we will treat these three cases (SVPI, MVPI-Set, and MVPI-Multiset) separately, whenever necessary.

Release candidate: An anonymization procedure takes the original dataset as input, and produces a release candidate. We model a release candidate as a set of disjoint groups, each of which contains a set of individuals and their respective sensitive values. Formally, a release candidate for original dataset $\mathbf{D}$ is of the form:

$$\mathbf{D}^* = \{(G_1, X_1), \ldots, (G_B, X_B)\},$$

such that $\cup_i G_i = \{u_1, \ldots, u_n\}$, $G_i \cap G_j = \varnothing$ for $i \neq j$, and X_i is the multiset containing all occurrences of all sensitive values for all the individuals in G_i. We call each (G_i, X_i) a **QI-group**. Notice that generalized tables (Figure 2) and bucketized datasets (Figure

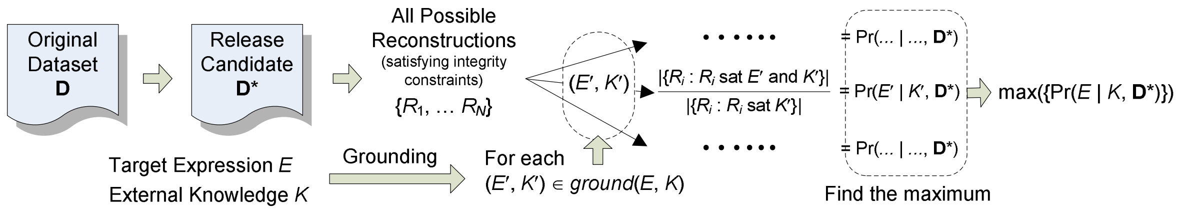


Figure 4. Theoretical framework

3) can be modeled in this way. For example, the bucketized data in Figure 3 is represented as follows: $\mathbf{D}^* =$

$\{(G_1=\{\text{Ann, Bob, Cary, Dick}\}, X_1=\{\text{AIDS, AIDS, Flu, Flu}\}),$
$(G_2=\{\text{Ed, Frank, Gary, Tom}\}, X_2=\{\text{AIDS, Cancer, Flu, Flu}\})\}.$

Reconstruction: After observing $\mathbf{D}^*$, the adversary tries to reconstruct the original dataset. A reconstruction R is an assignment that matches each occurrence of each sensitive value in X_i with some individual in G_i, such that the result satisfies the integrity constraints defined on the original dataset. We use $R(\mathbf{D}^*)$ to denote the result, which is a possible original dataset. For example, consider the bucketization in Figure 3; the following is one of many reconstructions in the MVPI-Multiset case:

$$R(\mathbf{D}^*) = \{(\text{Ann}, \{\text{Flu, Flu}\}), (\text{Bob}, \{\text{AIDS}\}), (\text{Cary}, \{\text{AIDS}\}), (\text{Ed}, \{\text{Cancer, Flu}\}), (\text{Frank}, \{\text{AIDS, Flu}\})\}.$$

Notice that the above $R(\mathbf{D}^*)$ is not a reconstruction in the SVPI or MVPI-Set case because it does not satisfy the corresponding integrity constraints. In addition to integrity constraints, the adversary may be able to eliminate certain reconstructions based on his external knowledge.

External Knowledge: The adversary may also have access to some external knowledge. In a very general sense, we can model this external knowledge using a logical expression, possibly containing variables. We say that an expression is **ground** if it contains no variables. A ground expression can be evaluated on a possible original dataset, and it returns true or false. We say that reconstruction R satisfies expression E iff E is true on $R(\mathbf{D}^*)$.

The precise syntax of expressions is application dependent and need not be logic sentences. In this paper, we call an expression of the form $s\in t[S]$ or $s\notin t[S]$ a **literal**. An example of a ground logic expression is $(\text{Flu}\in \text{Ann}[S] \wedge \text{Flu}\in \text{Bob}[S])$. The above example reconstruction does not satisfy this expression. Suppose t_1 and t_2 are variables ranging over individuals. In this case, $(\text{Flu}\in t_1[S] \rightarrow \text{Flu}\in t_2[S])$ is an expression with variables. The substitution of variables with actual individuals or sensitive values is called **grounding**. One grounding of the above example substitutes t_1 and t_2 with Ann and Bob, respectively. We use *ground*(E, K) to denote the set of all pairs of ground expressions that can be derived from a pair (E, K) of expressions.

Worst-Case Disclosure: Given a release candidate $\mathbf{D}^*$, a known set of integrity constraints, and an external knowledge expression K, our goal is to compute (and ultimately bound) the probability of a target expression E. Because we want to provide worst-case safety, when K or E has variables, we compute

$$\max\,\{\Pr(E' \mid K', \mathbf{D}^*) : (E', K') \in \mathit{ground}(E, K)\}.$$

For ease of exposition, we use the following notation.

$$\{\Pr(E \mid K, \mathbf{D}^*)\} \equiv \{\Pr(E' \mid K', \mathbf{D}^*) : (E', K') \in \mathit{ground}(E, K)\}.$$

For example, the data owner may believe that an adversary has the ability to obtain a sensitive value for each of k individuals. Thus, let $K = (\wedge_{i\in[1,k]}\ s_i\in t_i[S])$, where t_i and s_i are variables. The data owner wants to guarantee that, regardless of which k individuals and sensitive values the adversary knows, the probability that the adversary can determine that another individual t (a variable) has sensitive value s (a variable) is lower than threshold c. Formally, this is stated as follows:

$$\max\,\{\Pr(s\in t[S] \mid (\wedge_{i\in[1,k]}\ s_i\in t_i[S]), \mathbf{D}^*)\} < c.$$

The max function gives the variables the "for all" semantics; for all groundings of the variables, the criterion must hold.

Probability Computation: When computing probabilities, we make the standard random worlds assumption, following [2, 13, 18]. Let E and K be two ground expressions. Let $\{R_1, \ldots, R_N\}$ denote the set of all reconstructions of $\mathbf{D}^*$. In the absence of any information in addition to $\mathbf{D}^*$, we assume each reconstruction is equally likely. Under this assumption,

$$\Pr(E \mid \mathbf{D}^*) = |\{R_i : R_i \text{ satisfies } E\}| \,/\, N.$$

By the definition of conditional probability,

$$\Pr(E \mid K, \mathbf{D}^*) = |\{R_i : R_i \text{ satisfies both } E \text{ and } K\}| \,/\, |\{R_i : R_i \text{ satisfies } K\}|.$$

Note that the above formula defines the answer to $\Pr(E \mid K, \mathbf{D}^*)$, but to find the answer, it is not always necessary to enumerate the reconstructions of $\mathbf{D}^*$. Finally, let ε be a special expression, meaning empty. For pedantic reasons, we define $\Pr(\varepsilon \mid K, \mathbf{D}^*) = 1$.

2.2 Conjunctions of Literals

One important class of expressions, considered throughout this paper, consists of expressions that are conjunctions of literals. In this section, we briefly describe two propositions that will be used later in the paper. The basic idea is that, for conjunctions of literals, the probability computation for each QI-group is independent.

Let E_g and K_g denote two ground conjunctions of literals that only involve individuals in QI-group g (i.e., individuals in G_g), for $g = 1, \ldots, B$. For example, $E_1 = (\text{Flu}\in \text{Ann}[S] \wedge \text{AIDS}\notin \text{Bob}[S])$, where Ann and Bob are in QI-group 1.

Proposition 1.

$$\Pr(\wedge_{g\in[1,B]}\, E_g \mid \wedge_{g\in[1,B]}\, K_g, \mathbf{D}^*) = \textstyle\prod_{g\in[1,B]} \Pr(E_g \mid K_g, \mathbf{D}^*).$$

Let $E_{g,x}$ and $K_{g,x}$ denote two ground conjunctions of literals that only involve individuals in G_g and sensitive value $x \in X_g$, for $g = 1, \ldots, B$. For example, $E_{1,\text{Flu}} = (\text{Flu}\in \text{Ann}[S] \wedge \text{Flu}\notin \text{Bob}[S])$.

Proposition 2. *In the MVPI (either Set or Multiset) case,*

$$\Pr(\wedge_{g\in[1,B],\, x\in X_g}\, E_{g,x} \mid \wedge_{g\in[1,B],\, x\in X_g}\, K_{g,x}, \mathbf{D}^*) = \textstyle\prod_{g\in[1,B]} \prod_{x\in X_g} \Pr(E_{g,x} \mid K_{g,x}, \mathbf{D}^*).$$

The proofs are in Appendix A1. Note that E_g, K_g, $E_{g,x}$ and $K_{g,x}$ can be ε (the empty expression), and "$x\in X_g$" in the subscript means "for each distinct $x \in X_g$." Also note that Proposition 1 applies to both the SVPI and MVPI cases. If E and K are two conjunctions of literals, then, to compute $\Pr(E \mid K, \mathbf{D}^*)$, we first rewrite E and K as $\wedge_{g\in[1,B]}\ E_g$ and $\wedge_{g\in[1,B]}\ K_g$ and then compute $\Pr(E_g \mid K_g, \mathbf{D}^*)$

independently. Similarly, Proposition 2 says, in the MVPI case, each distinct sensitive value in each QI-group is reconstructed independently.

2.3 Research Direction

In general, computing $\Pr(E \mid K, \mathbf{D}^*)$ is NP-hard, even if E and K are ground. Martin et al. [13] showed that, if K is ground and of the form $(\wedge_{i\in[1,k]}\ (x_i \in t_i[S] \leftrightarrow y_i \in u_i[S]))$, it is NP-complete to decide whether $\Pr(K \mid \mathbf{D}^*) > 0$ and #P-complete to compute $\Pr(s \in t[S] \mid K, \mathbf{D}^*)$. We can also prove that even if $\mathbf{D}^*$ consists of only one QI-group (i.e., $\mathbf{D}^* = \{(G_1, X_1)\}$), it is still NP-complete to decide whether $\Pr(K \mid \mathbf{D}^*) > 0$ (see Theorem 6 in Appendix A5).

Because of the hardness results, developing a general technique to compute $\Pr(s \in t[S] \mid K, \mathbf{D}^*)$ is not a practical goal. Broadly speaking, the interesting research questions involve finding classes of expressions that are of practical interest and efficiently solvable. The work in [13] shows a special case that is polynomial-time solvable, but does not correspond well to natural real-world scenarios. In this paper, we identify three types of expressions representing external knowledge that arise naturally in practice. We show in Sections 5 and 6 that expressions that combine these types of knowledge can be handled very efficiently. Assume the adversary wants to discover Tom's sensitive value. We consider:

- **Knowledge about the target individual:** An interesting class of instance-level knowledge involves information that the adversary may know about the target individual. For example, Tom does not have cancer.
- **Knowledge about others:** Similarly, the adversary may have information about individuals other than the target. For example, Gary has flu.
- **Knowledge about same-value families:** We think the most intuitive kind of knowledge relating different individuals is the knowledge that a group (or family) of individuals have the same sensitive value. For example, {Ann, Cary, Tom} could be a same-value family, meaning if any one of them has a sensitive value (e.g., Flu), all the others tend also to have the same sensitive value.

While the technical contributions of this paper focus on these classes of expressions, these are by no means the only interesting knowledge expressions. In Section 8, we describe several other natural expression types that should be considered in future work.

3. DESIDERATA & RELATED WORK

In this section, we outline a number of characteristics we consider crucial to the design of a practical privacy criterion. At the same time, we review the literature, indicating how previous work does not match our desired characteristics.

From our perspective, a practical privacy criterion should display the following characteristics:

1. **Intuitive:** The data owner (usually not a computer scientist) should be able to understand the privacy criterion in order to set the appropriate parameters.
2. **Efficiently checkable:** Whether a release candidate satisfies the privacy criterion should be efficiently checkable.
3. **Flexible:** In data publishing, the data owner often considers a tradeoff between disclosure risk and data utility. A practical privacy criterion should provide this flexibility.
4. **External knowledge:** The privacy criterion should guarantee safety in the presence of common types of external knowledge.
5. **Value-centric:** Often, different sensitive values have different degrees of sensitivity (e.g., AIDS is more sensitive than flu). Thus, a practical privacy criterion should have the flexibility to provide different levels of protection for different sensitive values, not just uniform protection for all the values in the sensitive attribute. We call the latter *attribute-centric*. An attribute-centric criterion tends to over-protect the data. For example, to protect individuals having AIDS, the data owner must set the strongest level of protection, which is not necessary for individuals having flu. Instead, we take the more flexible *value-centric* approach.
6. **Set-valued sensitive attributes:** In many real-world scenarios, an individual may have several sensitive values, e.g., diseases.

No existing privacy criterion fully satisfies our desiderata. The most closely-related work is that of Martin et al. [13], which considers adversarial knowledge $\mathscr{L}_{basic}(k)$ to be a conjunction of k basic implications. Each basic implication is of the form $((\wedge_{i\in[1,m]}\ x_i \in u_i[S]) \rightarrow (\vee_{j\in[1,n]}\ y_j \in v_j[S]))$, where $m > 0$, $n > 0$, and x_i, u_i, y_j and v_j are all variables. A release candidate $\mathbf{D}^*$ is (c,k)-safe if $\max\{\Pr(s \in t[S] \mid K, \mathbf{D}^*)\} < c$, where s and t are also variables. The authors showed that the probability is maximized when K is of a simpler form $\mathscr{L}_{simple}(k) = \wedge_{i\in[1,k]}\ (z_i \in w_i[S] \rightarrow s \in t[S])$, and developed a polynomial time algorithm to solve

$$\max\{\Pr(s \in t[S] \mid \wedge_{i\in[1,k]}\ (z_i \in w_i[S] \rightarrow s \in t[S]), \mathbf{D}^*)\},$$

where all t, s, w_i, z_i are variables.

While groundbreaking in the treatment of external knowledge, the approach has several important shortcomings:

- The knowledge quantification is not intuitive. It is hard to understand the practical meaning of k implications.
- Martin et al. showed that their language can express any logic-based expression of external knowledge, when the number k of basic implications is unbounded. However, their language cannot *practically* express some important types of knowledge, e.g., simply Flu∈Bob[S] (a very common kind of knowledge that the adversary may obtain from a similar dataset). Expressing such knowledge in their language requires ($|S|-1$) basic implications, where $|S|$ is the number of sensitive values. However, with this number of basic implications, no release candidate can possibly be safe. Thus, Flu∈Bob[S] will never be used in their criterion. A formal study of *practical expressibility* is in Appendix A8.
- The privacy criterion is attribute-centric, and there is no straightforward extension of the proposed algorithm to the more flexible value-centric case. The reason is that the algorithm can only compute $\max\{\Pr(s \in t[S] \mid K, \mathbf{D}^*)\}$ for the sensitive value s that is most frequent in at least one QI-group. However, the sensitive values that need the most protection (e.g., AIDS) are usually infrequent ones.
- Each individual is assumed to have only one sensitive value.

Our work builds upon [13] and addresses the above issues. Note that our language can express some knowledge (e.g., Flu∈Bob[S]) that cannot be *practically* expressed in their language, and vice versa. For details, see Section 4.4.

In other related work, k-Anonymity and ℓ-diversity are privacy criteria that attempt to capture adversarial knowledge in a less formal way. k-Anonymity requires that no individual be identifiable from a group of k individuals[16]. ℓ-Diversity requires that each QI-group contain at least ℓ "well-represented" sensitive values [11]. In Section 4.4, we show these two criteria are special cases of our basic privacy criterion.

Query-view privacy was studied in [3, 4, 12, 14]. Given a set of public views of a database, the goal is to check whether they reveal any information about a private view of the same database, where views are defined by conjunctive queries. Views can be used to express adversarial knowledge. However, each of [4, 12, 14] uses an extremely strong definition of privacy, requiring the sensitive information to be completely independent of the released data. This approach does not provide flexibility to tradeoff privacy for utility. Dalvi et al. relax the strong requirement [3], but describe a privacy criterion based on asymptotic probabilities when the domain size goes to infinity, which is not intuitive. Checking query-view safety in the general setting is NP-hard [4, 14]. Polynomial time algorithms for some special cases were given in [3, 12]. Other studies of data privacy in multiple (project-only or select-project) views of a single original table are [5, 19].

Several other recent works have considered probabilistic disclosure, but have not incorporated adversarial knowledge, including [10, 18] and others. Ignoring external knowledge can be dangerous. Consider the following QI-group:

({Ann, Bob, Cary, Dick, Ed}, {Flu, Flu, Flu, Flu, AIDS}).

In the SVPI case, the probability that any one has AIDS is 0.2, which may be sufficiently low. However, by an investigation of only 4 individuals (i.e., knowing 4 individuals not having AIDS), one can conclude that the other one has AIDS. In this sense, this QI-group does not preserve privacy as well as a QI-group containing 100 individuals, 20 of whom have AIDS, despite the fact that the disclosure probability is the same in both cases (0.2).

Finally, though not specifically concerned with data privacy, the framework described in Section 2 is closely related to the framework for reasoning about uncertainty (the "random worlds approach") in the presence of specific logical and probabilistic knowledge that was introduced by Bacchus et al. [2].

4. MULTIDIMENSIONAL PRIVACY

We now define our privacy criterion. To incorporate external knowledge, the data owner needs to specify the knowledge that an adversary may have. Because it is nearly impossible for the data owner to anticipate the specific knowledge available to an adversary, we take the approach of [13], and propose a new mechanism for "quantifying" external knowledge. In this approach, the privacy criterion must guarantee safety when the adversary has up to a certain "amount" of knowledge, regardless of the specific things that are known.

As discussed in Section 2.3, in general, it is NP-hard to check safety of a release candidate. Thus, our goal is to find special cases that are both useful in practice and efficiently solvable.

In the rest of this section, we propose an intuitive and usable approach to quantifying adversarial knowledge. The key idea is to break down quantification into several meaningful components, rather than a single number as in [13]. We then define a skyline privacy criterion and a skyline exploratory tool.

4.1 Three-Dimensional Knowledge

Consider an adversary who wants to determine whether **target individual** t (a variable) has **target sensitive value** σ (a specific value, e.g., AIDS). Note that t is a variable because the target can be anyone, while σ is not because we want to provide a possibly different safety guarantee for each unique sensitive value σ. Intuitively, we consider the following three types of knowledge: (note the subscripts, where σ denotes the target sensitive value)

- $K_{\sigma t}$: Knowledge about the target individual t.
- $K_{\sigma u}$: Knowledge about individuals ($u_1, \ldots, u_k$) other than t.
- $K_{\sigma v,t}$: Knowledge about the relationship between t and other individuals ($v_1, \ldots, v_m$).

We note that knowledge about relationships is the most interesting type of knowledge. In this paper, we focus on same-value families, which we consider to be the most natural form of relationship in attribute disclosure. In general, relationships may be expressed using graphs, which is future work.

We use the following convention throughout the paper.

- σ is the target sensitive value (a specific value, not a variable).
- t is the target individual (a variable).
- u_i, v_i are variables ranging over individuals.
- x_i, y_i are variables ranging over sensitive values.
- f, g are (the indices of) QI-groups.

Because the SVPI case and MVPI case have very different characteristics, we discuss these two cases separately.

4.1.1 *Case of Single Value per Individual*

We use (ℓ, k, m) to quantify the three types of knowledge, respectively. Specifically, this indicates that the adversary knows: (1) ℓ sensitive values that target individual t does not have, (2) the sensitive values of k other individuals, and (3) m members in t's same-value family (a group of people who tend to have the same sensitive values). Note that the precise meaning of the third dimension is "m individuals such that if any one of them has σ, then t also has σ." Consider t = Tom, σ = AIDS, and (ℓ, k, m) = (2, 3, 1). An example of adversary's knowledge is the conjunction of the following three expressions:

- Flu$\notin$Tom[S] $\wedge$ Cancer$\notin$Tom[S] (obtained from Tom's friends).
- Flu$\in$Bob[S] $\wedge$ Flu$\in$Cary[S] $\wedge$ Cancer$\in$Frank[S] (obtained from another hospital's medical records)
- AIDS$\in$Ann[S] $\rightarrow$ AIDS$\in$Tom[S] (because Ann is Tom's wife).

Definition: $\mathscr{L}_{t,\sigma}^{\mathrm{SVPI}}(\ell, k, m)$. *Formally, an adversary's knowledge is a parameterized expression* $\mathscr{L}_{t,\sigma}^{\mathrm{SVPI}}(\ell, k, m) = K_{\sigma t}(\ell) \wedge K_{\sigma u}(k) \wedge K_{\sigma v,t}(m)$, *where*

- $K_{\sigma t}(\ell) = (\wedge_{i\in[1,\ell]}\ x_i \notin t[S])$ *indicates that the adversary knows ℓ sensitive values (the x_i's) that the target t does not have.*
- $K_{\sigma u}(k) = (\wedge_{i\in[1,k]}\ y_i \in u_i[S])$ *where $u_i \neq t$, indicates that the adversary knows the sensitive values (the y_i's) of k individuals (the u_i's) other than the target t.*
- $K_{\sigma v,t}(m) = (\wedge_{i\in[1,m]}\ (\sigma \in v_i[S] \rightarrow \sigma \in t[S]))$ *where $v_i \neq u_j$ and $v_i \neq t$, indicates that the adversary knows m individuals such that if any one of them has σ, then t also has σ.*

Note that $u_i \neq t$, $v_i \neq t$ and $v_i \neq u_j$ specify the constraints on variable grounding, meaning when we substitute the variables with actual individuals, we cannot assign the same individual to u_i and t, and so on. The reason is that if $u_i = t$, the adversary knows t's sensitive value without the released dataset. Similarly, if $v_i = u_j$, the adversary also knows t's sensitive value without the released dataset because $(\sigma \in v_i[S]) \wedge (\sigma \in v_i[S] \rightarrow \sigma \in t[S])$ implies $\sigma \in t[S]$.

Also note that the subscript of $\mathscr{L}_{t,\sigma}^{\mathrm{SVPI}}(\ell, k, m)$ indicates that the target individual is variable t and the target sensitive value is σ.

4.1.2 *Case of Multiple Values per Individual*

The types of knowledge considered in the MVPI case are different from those in the SVPI case. Consider two different sensitive values σ_1 and σ_2. We first note that a special case of proposition 2 is $\Pr(\sigma_1 \in t[S] \mid \sigma_2 \in u[S], \mathbf{D}^*) =$

$$\Pr(\sigma_1 \in t[S] \mid \varepsilon, \mathbf{D}^*) \cdot \Pr(\varepsilon \mid \sigma_2 \in u[S], \mathbf{D}^*) = \Pr(\sigma_1 \in t[S] \mid \mathbf{D}^*),$$

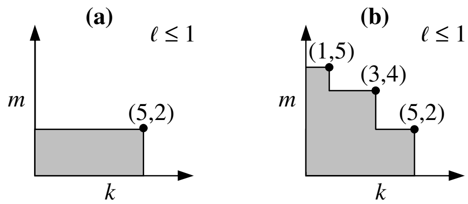


Figure 5. Example of privacy skylines

where ε is the empty expression. This means $\sigma_1 \in t[S]$ is independent of $\sigma_2 \in u[S]$ (also $\sigma_2 \notin u[S]$) as long as $\sigma_1 \neq \sigma_2$, regardless of whether $t = u$. Thus, the first two forms of knowledge in the SVPI case are useless to the adversary in determining whether t has σ.

Instead, in the MVPI case, we use (ℓ, k, m) to indicate that the adversary knows: (1) ℓ sensitive values that co-occur with target value σ for target individual t, (2) k other individuals who do not have σ, and (3) m members in t's same-value family. Consider t=Tom, σ=AIDS, and $(\ell, k, m) = (1, 3, 1)$, examples of the three types of knowledge in the MVPI case are:

- Cancer$\in$Tom[S] $\rightarrow$ AIDS$\in$Tom[S] (obtained from a hypothetical medical study).
- AIDS$\notin$Bob[S] $\wedge$ AIDS$\notin$Cary[S] $\wedge$ AIDS$\notin$Frank[S] (obtained from another hospital's medical records)
- AIDS$\in$Ann[S] $\rightarrow$ AIDS$\in$Tom[S] (because Ann is Tom's wife).

Definition: $\mathscr{L}_{t,\sigma}^{\text{MVPI}}(\ell, k, m)$. *Formally, an adversary's knowledge is expression* $\mathscr{L}_{t,\sigma}^{\text{MVPI}}(\ell, k, m) = K_{\sigma\backslash t}(\ell) \wedge K_{\sigma\backslash u}(k) \wedge K_{\sigma\backslash v,t}(m)$, *where*

- $K_{\sigma\backslash t}(\ell) = (\wedge_{i \in [1, \ell]} (x_i \in t[S] \rightarrow \sigma \in t[S]))$ *indicates that the adversary knows ℓ sensitive values (the x_i's) that co-occur with target value σ for target individual t. Thus, if t has any x_i, t should also have σ.*
- $K_{\sigma\backslash u}(k) = (\wedge_{i \in [1,k]} \sigma \notin u_i[S])$ *where $u_i \neq t$, indicates that the adversary knows k individuals (the u_i's) who do not have sensitive value σ.*
- $K_{\sigma\backslash v,t}(m) = (\wedge_{i \in [1,m]} (\sigma \in v_i[S] \rightarrow \sigma \in t[S]))$ *where $v_i \neq u_j$ and $v_i \neq t$. This is the same as the $K_{\sigma\backslash v,t}(m)$ in the SVPI case.*

For ease of exposition, we use $K_{\sigma\backslash t}(\ell)$ and $K_{\sigma\backslash u}(k)$ to denote the first two dimensions in both the SVPI and the MVPI cases, even though the actual expressions are different in the two cases. If we want to distinguish the two cases, we will say so explicitly.

4.2 Privacy Criterion

In the rest of this paper, we use $\mathscr{L}_{t,\sigma}(\ell, k, m)$ to denote both $\mathscr{L}_{t,\sigma}^{\text{SVPI}}(\ell, k, m)$ and $\mathscr{L}_{t,\sigma}^{\text{MVPI}}(\ell, k, m)$. Also, if (ℓ, k, m) is not important in our discussion, we just write $\mathscr{L}_{t,\sigma}^{\text{SVPI}}$ and $\mathscr{L}_{t,\sigma}^{\text{MVPI}}$.

Given a release candidate $\mathbf{D}^*$, for a particular grounding of the variables, $\Pr(\sigma \in t[S] \mid \mathscr{L}_{t,\sigma}(\ell, k, m), \mathbf{D}^*)$ is the adversary's confidence that individual t has sensitive value σ given external knowledge. A privacy criterion should provide a worst-case guarantee. That is, no matter how we substitute variables with the actual individuals and sensitive values, the adversary's confidence should not exceed a given threshold value c. This leads to the following definition.

Definition: Basic 3D privacy criterion. *Given knowledge threshold (ℓ, k, m) and confidence threshold c, release candidate $\mathbf{D}^*$ is safe for sensitive value σ iff*

$$\max \{\Pr(\sigma \in t[S] \mid \mathscr{L}_{t,\sigma}(\ell, k, m), \mathbf{D}^*)\} < c.$$

We call $\max\{\Pr(\sigma \in t[S] \mid \mathscr{L}_{t,\sigma}(\ell, k, m), \mathbf{D}^*)\}$ *the **breach probability**.*

For example, in the SVPI case, suppose that the data owner specifies $(\ell, k, m) = (1, 5, 2)$ and $c = 50\%$ for sensitive value AIDS. The privacy criterion guarantees that the adversary cannot predict any individual t to have AIDS with confidence $\geq 50\%$ if the following conditions hold: (1) The adversary knows $\ell \leq 1$ sensitive values that target individual t does not have, (2) the adversary knows the sensitive values of $k \leq 5$ other individuals, and (3) the adversary knows $m \leq 2$ members in t's same-value family. It is easy to see that the breach probability increases with increasing amounts of adversarial knowledge. Thus, if $\mathbf{D}^*$ is safe under (1, 5, 2), it is also safe under any (ℓ, k, m) such that $\ell \leq 1$, $k \leq 5$ and $m \leq 2$, which is the shaded region of Figure 5 (a). For simplicity, we only show a two-dimensional plot.

The basic privacy criterion is useful and intuitive, but it may not be sufficient for expressing the data owner's desired level of privacy. For example, the threshold (1,5,2) provides no protection guarantee for (1,3,4) because (1,3,4) is not in the shaded region of Figure 5 (a). To provide more precise and flexible control, we extend the basic privacy criterion to allow the data owner to specify a set of *incomparable* points called a *skyline* (e.g., as shown in Figure 5 (b), the skyline is {(1,1,5), (1,3,4), (1,5,2)}) such that release candidate $\mathbf{D}^*$ is safe if the breach probability is less than the confidence threshold (e.g., 50%) given any adversary's knowledge with amount beneath the skyline (e.g., the shaded area in Figure 5 (b)).

We can also include the confidence threshold c in the skyline. We say (ℓ_1, k_1, m_1, c_1) dominates (ℓ_2, k_2, m_2, c_2) if $\ell_1 \geq \ell_2$, $k_1 \geq k_2$, $m_1 \geq m_2$ and $c_1 \leq c_2$. It can be easily seen that if $\mathbf{D}^*$ is safe under (ℓ_1, k_1, m_1, c_1), it is also safe under (ℓ_2, k_2, m_2, c_2). A set of points is a skyline if no point dominates another.

Definition: Skyline privacy criterion. *Given a skyline* $\{(\ell_1, k_1, m_1, c_1), \ldots, (\ell_r, k_r, m_r, c_r)\}$, *release candidate $\mathbf{D}^*$ is safe for sensitive value σ iff, for $i = 1$ to r,*

$$\max \{\Pr(\sigma \in t[S] \mid \mathscr{L}_{t,\sigma}(\ell_i, k_i, m_i), \mathbf{D}^*)\} < c_i.$$

In practice, the data owner specifies a skyline for each sensitive value. The skyline privacy criterion is attractive because it allows the data owner to enforce privacy requirements for different situations separately. Although a skyline involves many parameter values, it is much more intuitive for the data owner to specify a skyline (in a case-by-case manner) than to figure out a way to combine many considerations into a single threshold value. Also, the data owner can set default parameter values for common cases and only fine-tune some special cases.

4.3 Skyline Exploratory Tool

In the skyline privacy criterion, the user specifies a skyline, and the system checks whether a release candidate is safe under the skyline. However, the skyline itself may be a useful exploratory tool, providing valuable information to the data owner in considering a particular release candidate.

In the following, we say $(\ell, k, m) > (\ell_i, k_i, m_i)$ if $\ell \geq \ell_i$, $k \geq k_i$, $m \geq m_i$ and at least one inequality holds.

Definition: Knowledge Skyline. *The knowledge skyline of release candidate $\mathbf{D}^*$ at confidence threshold c for sensitive value σ is the set* $\{(\ell_1, k_1, m_1), \ldots, (\ell_r, k_r, m_r)\}$ *of all points such that $\mathbf{D}^*$ is safe for σ under (ℓ_i, k_i, m_i) at confidence threshold c, but not safe for any $(\ell, k, m) > (\ell_i, k_i, m_i)$, for all i.*

For a given release candidate, the knowledge skyline separates the multidimensional knowledge space into two regions. Intuitively, the release candidate is resilient to adversarial knowledge below or on the skyline, but not to knowledge above the skyline.

Knowledge skylines are a useful exploratory tool. Regardless of whether the released data is generated based on our privacy criterion, before the data is actually released, it is always good for the data owner to check the knowledge skyline of the release candidate, and see whether the dataset is safe or not under various amounts and types of adversarial external knowledge.

4.4 Comparisons

We first compare $\mathscr{L}_{t,\sigma}^{\text{SVPI}}$ with $\mathscr{L}_{t,\sigma}^{\text{MVPI}}$, and then compare $\mathscr{L}_{t,\sigma}^{\text{SVPI}}$ with k-anonymity [16], ℓ-diversity [11] and $\mathscr{L}_{\text{basic}}$ [13].

As described in [13], in the SVPI case, $(\wedge_{i\in[1,\ell]}(x_i\in t[S]\rightarrow \sigma\in t[S]))$ is actually equivalent to $(\wedge_{i\in[1,\ell]} x_i\notin t[S])$, because t can only have one sensitive value. Thus, the $K_{\sigma|t}(\ell)$ in the SVPI case actually has the same form as the $K_{\sigma|t}(\ell)$ in the MVPI case, although they have different interpretations. Now, the only difference between the two cases is in $K_{\sigma|u}(k)$, which represents knowledge about individuals other than the target. We think $(\wedge_{i\in[1,k]} y_i\in u_i[S])$ is the most natural knowledge about individuals. Thus, we use it in the SVPI case. However, in the MVPI case, $y_i\in u_i[S]$ is independent of $\sigma\in t[S]$ if $y_i \neq \sigma$. Even if $y_i = \sigma$, the knowledge of $\sigma\in u_i[S]$ cannot help the adversary increase his confidence. Thus, in the MVPI case, we choose $(\wedge_{i\in[1,k]}\ \sigma\notin u_i[S])$ because it is still easily interpretable and is also useful for the adversary.

We now compare $\mathscr{L}_{t,\sigma}^{\text{SVPI}}$ with k-anonymity [16], ℓ-diversity [11] and $\mathscr{L}_{\text{basic}}$ [13], which are all in the SVPI case. For proofs, see Appendix A1.

Proposition 3. *k-anonymity (in our framework, defined as each QI-group having at least k individuals) is a special case of the basic 3D privacy criterion when the sensitive values are the identities of the individuals, the knowledge threshold is* $(0, k-2, 0)$ *and the confidence threshold is* 1, *for all sensitive values* σ.

Proposition 4. *(c,ℓ)-diversity is a special case of the basic 3D privacy criterion when the knowledge threshold is* $(\ell-2, 0, 0)$ *and the confidence threshold is* $c/(c+1)$, *for all sensitive values* σ.

Basically, k-anonymity considers knowledge of form $K_{\sigma|u}(k)$ and ℓ-diversity considers knowledge of form $K_{\sigma|t}(\ell)$ in the SVPI case. For the comparison of $\mathscr{L}_{t,\sigma}^{\text{SVPI}}$ and $\mathscr{L}_{\text{basic}}$, no one is more general than the other, because $\mathscr{L}_{t,\sigma}^{\text{SVPI}}$ cannot express, say, (Flu$\in$Bob[S] $\rightarrow$ AIDS$\in$Tom[S]), and $\mathscr{L}_{\text{basic}}$ cannot *practically* express, say, Flu$\in$Bob[S] (as discussed in Section 3). However, our $\mathscr{L}_{t,\sigma}^{\text{SVPI}}$ is more intuitive and quantifies knowledge more precisely than $\mathscr{L}_{\text{basic}}$. A formal comparison between $\mathscr{L}_{t,\sigma}^{\text{SVPI}}$ and $\mathscr{L}_{\text{basic}}$ is in Appendix A8.

5. EFFICIENT, SCALABLE ALGORITHMS

In this section, we develop algorithms: **SkylineCheck** for checking whether a release candidate is safe and **SkylineAnonymize** for generating a safe and useful release candidate. The algorithm for finding the knowledge skyline of a release candidate is in Appendix A7.

Our algorithms rely critically upon a proposed *congregation* property. Because we carefully design our knowledge quantification to satisfy this property, our algorithms are very efficient when the number of distinct sensitive values is a constant. In contrast, the knowledge quantification of Martin et al. [13] does not satisfy this property. Although both algorithms run in polynomial time, there is a big difference in efficiency between their algorithm and ours.

In this section, we describe a general computation framework that works for the three cases (SVPI, MVPI-Set and MVPI-Multiset). In Section 6, we provide the formulas for the probability computation specific to each case.

5.1 SkylineCheck Algorithm

SkylineCheck algorithm checks whether a release candidate satisfies a skyline criterion for every sensitive value. The main ideas behind SkylineCheck are as follows:

1. Convert implication-based knowledge into literals (so that we can use Propositions 1 and 2).
2. Show that the breach probability is maximized when all the individuals (involved in adversarial knowledge) *congregate* in no more than two QI-groups.

We first focus on checking whether release candidate $\mathbf{D}^*$ is safe for a single sensitive value σ, and then extend to all σ's. Note that we have abstracted the knowledge expressions in both the SVPI and the MVPI cases in the same form: $(K_{\sigma|t}(\ell) \wedge K_{\sigma|u}(k) \wedge K_{\sigma|v,t}(m))$. As described in Section 4.4, in the SVPI case, $(\wedge_{i\in[1,\ell]} (x_i\in t[S] \rightarrow \sigma\in t[S]))$ is equivalent to $K_{\sigma|t}(\ell) = (\wedge_{i\in[1,\ell]}\ x_i\notin t[S])$ because t can have only one sensitive value. Thus, we use $K_{\sigma|t}(\ell) = (\wedge_{i\in[1,\ell]}\ x_i\in t[S] \rightarrow \sigma\in t[S]))$, for both the SVPI and the MVPI cases. Now, the only difference between the two cases is in $K_{\sigma|u}(k)$.

Given knowledge threshold (ℓ, k, m) and confidence threshold c, $\mathbf{D}^* = \{(G_1, X_1), \ldots, (G_B, X_B)\}$ is safe for σ if the breach probability is less than c, where the breach probability (BP) is

$$BP_\sigma(\ell, k, m) = \max\{\Pr(\sigma\in t[S] \mid K_{\sigma|t}(\ell) \wedge K_{\sigma|u}(k) \wedge K_{\sigma|v,t}(m), \mathbf{D}^*)\}.$$

The above maximization is over the following variables:

- Individuals: t (in $K_{\sigma|t}(\ell)$), $u_1, \ldots, u_k$ (in $K_{\sigma|u}(k)$) and $v_1, \ldots, v_m$ (in $K_{\sigma|v,t}(m)$).
- Sensitive values: $x_1, \ldots, x_\ell$ (in $K_{\sigma|t}(\ell)$), $y_1, \ldots, y_k$ (in $K_{\sigma|u}(k)$).

Note that we sometimes directly call t, u_i's and v_i's individuals.

Now our goal is to compute $BP_\sigma(\ell, k, m)$. Note that $K_{\sigma|t}(\ell)$ and $K_{\sigma|v,t}(m)$ involve implications. Probability computation under implication-based knowledge is not easy. Thus, we use Lemma 1 (which is Lemma 12 in [13]) to convert implications into literals.

Lemma 1. $\Pr(\sigma\in t[S] \mid K_{\sigma|t}(\ell) \wedge K_{\sigma|u}(k) \wedge K_{\sigma|v,t}(m), \mathbf{D}^*) = 1 / (NR + 1)$, *where*

$$NR = \frac{\Pr(\sigma\notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \wedge (\wedge_{i\in[1,m]} \sigma\notin v_i[S]) \mid K_{\sigma|u}(k), \mathbf{D}^*)}{\Pr(\sigma\in t[S] \mid K_{\sigma|u}(k), \mathbf{D}^*)}$$

We call NR the ***negated ratio***. (*For the proof, see Appendix A2.*)

Note that Lemma 1 is true for both the SVPI and the MVPI cases. Also note that, because $K_{\sigma|u}(k)$ is a conjunction of k literals, NR only involves conjunctions of literals.

Based on Lemma 1, to maximize the breach probability is to minimize the negated ratio. Thus, we define:

$$minNR_\sigma(\ell, k, m) = \min_{t,\, v_i,\, x_i,\, K_{\sigma|u}(k)} NR.$$

Since $BP_\sigma(\ell, k, m) = 1 / (minNR_\sigma(\ell, k, m) + 1)$, our goal now is to compute $minNR_\sigma(\ell, k, m)$, which only involves literals.

In general, minimizing the negated ratio is not easy. In principle, we need to try all possible groundings of the variables and find the one that gives the minimum. In each grounding, we need to set variables t, $u_1, \ldots, u_k$ and $v_1, \ldots, v_m$ to individuals in possibly different QI-groups of $\mathbf{D}^*$. After fixing the QI-groups of the individuals, the minimum negated ratio (over variables $x_1, \ldots, x_\ell$, $y_1, \ldots, y_k$ for sensitive values) can be computed using the formulas in Section 6. In this section, we focus on how to distribute the individuals (t, u_i's and v_i's) into QI-groups in order to minimize the negated ratio.

To find the minimum negated ratio, we may need to try all possible ways of distributing those individuals into the QI-groups in $\mathbf{D}^*$. A dynamic-programming technique [13] can find the minimum in polynomial time, but computational efficiency is still an issue. Thus, the following *congregation* property is extremely useful. Intuitively, we say that $K_{\sigma|u}(k)$ (or $K_{\sigma|v,t}(m)$) is 1-group congregated iff the breach probability is maximized (i.e., the negated ratio is minimized) when all the individuals except t (which we do not care about) involved in $K_{\sigma|u}(k)$ (or $K_{\sigma|v,t}(m)$) are in one QI-group. If $K_{\sigma|u}(k)$ and $K_{\sigma|v,t}(m)$ are both 1-group-congregated, then a much more simple and efficient algorithm is possible.

Definition: Congregation. *Let $K = K_1 \wedge \ldots \wedge K_n$ be an expression with variables. K_i is 1-group congregated in K iff there exists a grounding maximizing* $\Pr(\sigma \in t[S] \mid K, \mathbf{D}^*)$ *such that, in the grounding, all the variables other than t (the target, which we do not care about) that represent individuals involved in K_i are set to individuals in one QI-group.*

Theorem 1. *$K_{\sigma|u}(k)$ and $K_{\sigma|v,t}(m)$ are both 1-group congregated, in all the three cases (SVPI, MVPI-Set and MVPI-Multiset).*

We defer the proof to Section 6, or see Appendix A5 for details.

We now discuss how to use Theorem 1 to develop an efficient algorithm. First, recall that $K_{\sigma|t}(\ell)$ only involves individual t (the target), $K_{\sigma|u}(k)$ only involves individuals $u_1, \ldots, u_k$, and $K_{\sigma|v,t}(m)$ only involves individuals $v_1, \ldots, v_m$ and t. By Theorem 1, the negated ratio is minimized when all $u_1, \ldots, u_k$ are in one QI-group and all $v_1, \ldots, v_m$ are in one QI-group.

Without loss of generality, we assume the negated ratio is minimized when [1]

t is in QI-group g and $v_1, \ldots, v_m$ are in QI-group f.

Proposition 5. *The negated ratio is minimized when all the u_i's (in $K_{\sigma|u}(k)$) are either in QI-group g or QI-group f.*

Rationale: By Proposition 1, if u_i is not in QI-group g or f, then $y_i \in u_i[S]$ (in $K_{\sigma|u}(k)$ for the SVPI case) and $\sigma \notin u_i[S]$ (in $K_{\sigma|u}(k)$ for the MVPI case) are independent of the negated ratio; i.e., they will not affect the value of the negated ratio. Thus, to minimize the negated ratio, all the u_i's should be in QI-group g or f. For details, see Appendix A1. ❑

By Proposition 5, the negated ratio is minimized when all the individuals (in the adversary's knowledge) are in QI-group g or f. If $g = f$, we define the following.

Definition: $minNR_\sigma(g, \ell, k, m)$.

$$minNR_\sigma(g, \ell, k, m) = \min_{t,\, v_i,\, x_i,\, K_{\sigma|u}(k)} NR,$$

such that t, $v_1, \ldots, v_m$ and $u_1, \ldots, u_k$ (in $K_{\sigma|u}(k)$) are in QI-group g, where NR is the negated ratio defined in Lemma 1.

Thus, if g=f, then $minNR_\sigma(g, \ell, k, m)$ is the minimum negated ratio.

Now consider $g \neq f$. We define the following.

Definition: $T_\sigma(g, \ell, k)$ and $V_\sigma(f, m, k)$.

$$T_\sigma(g, \ell, k) = \min_{t, x_i, K_{\sigma|u}(k)} \frac{\Pr(\sigma \notin t[S] \wedge (\wedge_{i \in [1,\ell]} x_i \notin t[S]) \mid K_{\sigma|u}(k), \mathbf{D}^*)}{\Pr(\sigma \in t[S] \mid K_{\sigma|u}(k), \mathbf{D}^*)},$$

such that t and $u_1, \ldots, u_k$ are in QI-group g.

$$V_\sigma(f, m, k) = \min_{v_i,\, K_{\sigma|u}(k)} \Pr(\wedge_{i \in [1,m]} \sigma \notin v_i[S] \mid K_{\sigma|u}(k), \mathbf{D}^*),$$

such that $v_1, \ldots, v_m$ and $u_1, \ldots, u_k$ (in $K_{\sigma|u}(k)$) are in QI-group f.

Consider the following situation: $(0 \le h \le k)$

- QI-group g contains t and $u_1, \ldots, u_h$.
- QI-group f contains $v_1, \ldots, v_m$ and the rest $(k–h)$ of the u_i's.

If $g \neq f$, by Proposition 1, the literals in NR that involve t and $u_1, \ldots, u_h$ are independent of the literals that involve $v_1, \ldots, v_m$ and the rest $(k–h)$ of the u_i's. Thus, the minimum negated ratio becomes

$$\min_{t,\, v_i,\, x_i,\, K_{\sigma|u}(k)} NR = T_\sigma(g, \ell, h) \cdot V_\sigma(f, m, k–h),$$

by applying Proposition 1 to both the numerator and denominator of NR. (For detailed derivation, see Derivation 1 in Appendix A4.)

By Theorem 1, we know all the u_i's are in one QI-group; i.e., h is either 0 or k. The computation of $minNR_\sigma(g, \ell, k, m)$, $T_\sigma(g, \ell, k)$ and $V_\sigma(f, m, k)$ is case-specific and will be discussed in Section 6.

Theorem 2. *The minimum negated ratio $minNR_\sigma(\ell, k, m)$ on release candidate $\mathbf{D}^*$ is the minimum of the following three:*

- $\min_{g \in \mathbf{D}^*} minNR_\sigma(g, \ell, k, m)$,
- $(\min_{g \in \mathbf{D}^*} T_\sigma(g, \ell, 0)) \cdot (\min_{f \in \mathbf{D}^*} V_\sigma(f, m, k))$,
- $(\min_{g \in \mathbf{D}^*} T_\sigma(g, \ell, k)) \cdot (\min_{f \in \mathbf{D}^*} V_\sigma(f, m, 0))$,

where "$g \in \mathbf{D}^*$" means "for each QI-group g in $\mathbf{D}^*$."

Proof: By Theorem 1, we only need to consider the situations in which all the u_i's are in one QI-group and all the v_i's are in one QI-group. If t, the u_i's and the v_i's are all in one QI-group, then the first case above gives the minimum negated ratio. Otherwise, let t be in group g and all the v_i's be in group f, where $g \neq f$. By Proposition 5, all the u_i's are either in g or f. If all the u_i's are in f, then the minimum negated ratio is

$$\min_{g,f} T_\sigma(g, \ell, 0) \cdot V_\sigma(f, m, k) = (\min_{g \in \mathbf{D}^*} T_\sigma(g, \ell, 0)) \cdot (\min_{f \in \mathbf{D}^*} V_\sigma(f, m, k)),$$

which gives the second case. Note that if the above is minimized at $g = f$ (i.e., all t, u_i's, v_i's are in one QI-group), then the first case will be even smaller because, as can be seen from the computation formulas in Section 6,

$$minNR_\sigma(g, \ell, k, m) = T_\sigma(g, \ell, k) \cdot V_\sigma(g, m, k+1) \le T_\sigma(g, \ell, 0) \cdot V_\sigma(g, m, k),$$

for all g. Thus, the first case will be the minimum and give the correct answer.

Similarly, if all the u_i's are in g, we obtain the third case. ❑

Sufficient Statistics: Given release candidate $\mathbf{D}^*$ and knowledge threshold (ℓ, k, m) for sensitive value σ, the five minimum quantities in Theorem 2 are sufficient for computing the minimum negated ratio, thus the breach probability. We call them the *sufficient statistics* for (ℓ, k, m) on $\mathbf{D}^*$, and use the following notation:

$$SS1_{\sigma(\ell, k, m)}(\mathbf{D}^*) = \min_{g \in \mathbf{D}^*} minNR_\sigma(g, \ell, k, m).$$
$$SS2_{\sigma(\ell, k, m)}(\mathbf{D}^*) = \min_{g \in \mathbf{D}^*} T_\sigma(g, \ell, 0).$$
$$SS3_{\sigma(\ell, k, m)}(\mathbf{D}^*) = \min_{g \in \mathbf{D}^*} T_\sigma(g, \ell, k).$$
$$SS4_{\sigma(\ell, k, m)}(\mathbf{D}^*) = \min_{g \in \mathbf{D}^*} V_\sigma(g, m, 0).$$
$$SS5_{\sigma(\ell, k, m)}(\mathbf{D}^*) = \min_{g \in \mathbf{D}^*} V_\sigma(g, m, k).$$

Note that, to compute $minNR_\sigma(g, \ell, k, m)$, $T_\sigma(g, \ell, \cdot)$ and $V_\sigma(g, m, \cdot)$, we only need data in a single QI-group g.

SkylineCheck algorithm: Given release candidate $\mathbf{D}^*$, in which the QI-groups are clustered (i.e., all the data in a QI-group is stored on disk consecutively), and a skyline $\{(\ell_1, k_1, m_1, c_1), \ldots, (\ell_r, k_r, m_r, c_r)\}$, our goal is to check whether $\mathbf{D}^*$ is safe for sensitive value σ; i.e., $1 / (minNR_\sigma(\ell_i, k_i, m_i) + 1) < c_i$, for all i. The algorithm is simple. We scan $\mathbf{D}^*$ once, during which, for each QI-

[1] We assume that t, $u_1, \ldots, u_k$ and $v_1, \ldots, v_m$ can fit in each QI-group of $\mathbf{D}^*$ that contains σ. Otherwise, the breach probability is simply one.

```
Input: Original dataset as QI-group g0, privacy parameters (ℓ, k, m) and c
Output: A minimal release candidate safe under (ℓ, k, m) and c
Global variables: Sufficient statistics SS1, SS2, SS3, SS4, SS5.
anonymize(g0, ℓ, k, m, c)
    // Initialize the global sufficient statistics
    SS1 = minNR_σ(g0, ℓ, k, m);  SS2 = T_σ(g0, ℓ, 0);  SS3 = T_σ(g0, ℓ, k);
    SS4 = V_σ(g0, m, 0);  SS5 = V_σ(g0, m, k);
    // Greedily partition (split) the data and maintain the statistics
    D* = empty;
    queue.pushBack(g0);
    while(queue is not empty)
        g = queue.popFront();
        if ({g1, …, gn} = safeSplit(g, ℓ, k, m, c) is not empty)
            for (i = 1 to n)
                queue.pushBack(gi);
                SS1 = min{ SS1, minNR_σ(gi, ℓ, k, m) };
                SS2 = min{ SS2, T_σ(gi, ℓ, 0) };    SS3 = min{ SS3, T_σ(gi, ℓ, k) };
                SS4 = min{ SS4, V_σ(gi, m, 0) };  SS5 = min{ SS4, V_σ(gi, m, k)};
        else D*.pushBack(g);
    return D*;
subroutine safeSplit(g, ℓ, k, m, c)
    sort candidate splits of g by priority;  // application-specific ordering
    // Check safety for each candidate split
    for each candidate split that splits g into {g1, …, gn}
        A1 = SS1;  A2 = SS2;  A3 = SS3;  A4 = SS4;  A5 = SS5;
        for (i = 1 to n)
            A1 = min{ A1, minNR_σ(gi, ℓ, k, m) };
            A2 = min{ A2, T_σ(gi, ℓ, 0) };    A3 = min{ A3, T_σ(gi, ℓ, k) };
            A4 = min{ A4, V_σ(gi, m, 0) };  A5 = min{ A4, V_σ(gi, m, k)};
        NR = min{ A1,  A2*A5,  A3*A4 };
        BP = 1 / (NR + 1);
        if (BP < c) return {g1, …, gn};
    return empty;
```

Figure 6. SkylineAnonymize algorithm

group, we maintain the sufficient statistics for each (ℓ_i, k_i, m_i). Finally, we check whether $1 / (minNR_\sigma(\ell_i, k_i, m_i) + 1) < c_i$, for all i.

Theorem 3. *The above algorithm correctly checks whether* $\mathbf{D}^*$ *is safe for sensitive value* σ *under a skyline of r points by a single scan over* $\mathbf{D}^*$ *using memory* $O(r)$ *to keep the sufficient statistics.*

It can be easily seen that the above algorithm also works for checking safety for all the sensitive values. Now, r becomes the total number of skyline points in all the skylines, each of which is for a sensitive value.

5.2 SkylineAnonymize Algorithm

In this section, we describe a simple and efficient algorithm using multidimensional generalization [7] to find a *minimal* safe release candidate based on the congregation property, which allows us to use just five global sufficient statistics to check safety for a skyline point. It has been shown in [7, 8] that multidimensional generalization techniques produce more useful data than single-dimensional generalization techniques [6]. Thus, we only develop an algorithm based on the former. An algorithm based on the latter is straightforward. For ease of exposition, we describe the algorithm for a single skyline point (ℓ, k, m, c), but the extension to multiple skyline points for each sensitive value is straightforward. The algorithm is based on an adaptation of a partitioning scheme originally proposed for k-anonymity in [7].

Intuitively, a release candidate is *minimal* if it is safe and no QI-group can be safely divided. Formally, we define a partial ordering over all the release candidates of an original dataset $\mathbf{D}$ as follows. Let $\mathbf{D}^*_1$ and $\mathbf{D}^*_2$ be release candidates of $\mathbf{D}$, we say $\mathbf{D}^*_1 \preccurlyeq \mathbf{D}^*_2$ iff, for each QI-group $(G_g, X_g) \in \mathbf{D}^*_1$, there exists a QI-group $(G_f, X_f) \in \mathbf{D}^*_2$ such that $G_g \subseteq G_f$. That is, each QI-group in $\mathbf{D}^*_2$ is the union of one or more QI-groups in $\mathbf{D}^*_1$.

Definition: Minimal Release Candidate. *Release candidate* $\mathbf{D}^*$ *is said to be minimal iff it is safe and there does not exist any other safe release candidate* $\mathbf{D}^*_1$ *such that and* $\mathbf{D}^*_1 \preccurlyeq \mathbf{D}^*$.

To find a minimal release candidate, we use the following properties. We say that QI-groups $g_1, \ldots, g_n$ partition QI-group g if they are disjoint and the union of them is g.

Theorem 4. *If QI-groups* $g_1, \ldots, g_n$ *partition QI-group* g, *then in the SVPI case, for any fixed* (ℓ, k, m), *the following hold*:

- $T_\sigma(g, \ell, k) \geq \min_{1 \leq i \leq n} T_\sigma(g_i, \ell, k)$,
- $V_\sigma(g, m, k) \geq \min_{1 \leq i \leq n} V_\sigma(g_i, m, k)$,
- $minNR_\sigma(g, \ell, k, m) \geq$ *the minimum of*:
  (a) $\min_{1 \leq i \leq n} minNR_\sigma(g_i, \ell, k, m)$,
  (b) $(\min_{1 \leq i \leq n} T_\sigma(g_i, \ell, k)) \cdot (\min_{1 \leq i \leq n} V_\sigma(g_i, m, 0))$.

Definition: Monotonicity. *Let* $\mathbf{D}^*_1$ *and* $\mathbf{D}^*_2$ *be release candidates of* $\mathbf{D}$ *such that* $\mathbf{D}^*_1 \preccurlyeq \mathbf{D}^*_2$. *A privacy criterion is monotonic iff the fact that* $\mathbf{D}^*_1$ *is safe under the criterion implies that* $\mathbf{D}^*_2$ *is also safe.*

Corollary. *In the SVPI case, the basic 3D privacy criterion and the skyline privacy criterion are monotonic.*

The proofs of Theorem 4 and its corollary are in Appendix A5. We note that Theorem 4 and its corollary do not apply to the MVPI case. We discuss the implication later.

Our algorithm works as follows. Starting from a single QI-group, which is the original dataset, we recursively partition (or split) each QI-group in a "greedy" manner as long as it is still safe to do so. In each step, if there are several ways to partition a QI-group, we choose the one that is expected to generate the most useful release candidate based on an application-specific split criterion (e.g., [8]). The algorithm maintains the five global sufficient statistics (across all the QI-groups in the current partitioning). Using only these statistics, we are able to check whether or not splitting a QI-group increases the breach probability beyond the specified confidence threshold c. It is important to note that we do not need to look at the entire dataset in order to determine whether it is safe to split a particular group g. Instead, this determination can be made using only the global statistics and the data in g. The pseudo-code for the algorithm is given in Figure 6. In the safeSplit subroutine, candidate splits for QI-group g can be selected and prioritized using any application-specific criteria (e.g., [8]).

Theorem 5. *The anonymization algorithm produces a safe release candidate. In the SVPI case, the release candidate is minimal.*

Proof sketch: The BP computed in the safeSplit subroutine is always greater than or equal to the breach probability on the current $\mathbf{D}^*$ with QI-group g replaced by $g_1, \ldots, g_n$. Thus, if BP < c, the breach probability must be less than c; i.e., it is safe to split g into $g_1, \ldots, g_n$. In the SVPI case, by Theorem 4, BP is actually equal to the breach probability, and by the corollary, the returned release candidate is minimal. The detailed proof is in Appendix A5. ❑

Scalability: The anonymization algorithm can be implemented in a scalable way using the *Rothko-Tree* approach described in [9]. Specifically, candidate splits can be chosen and evaluated based on the set of (*unique attribute value, unique sensitive value, count*) triples, which is often much smaller than the size of the full input dataset and usually fits in memory.

Discussion: Our algorithm is guaranteed to produce a minimal release candidate in the SVPI case. In the MVPI case, it is guaranteed to produce a safe release candidate, but the candidate may not be minimal. We have done a simulation study, which shows that the chances that Theorem 4 holds in the MVPI case are very high (only 100 counterexamples in 7,778,625,148 randomly generated partitionings). Thus, we think, in practice, our algorithm will generate nearly minimal release candidates in the MVPI case.

Comparison: The efficiency and scalability of the anonymization algorithm come from the congregation property. Because of this property, we are able to use just five global variables (for each skyline point) to check safety. We note that if we were to adapt the same partitioning scheme to the privacy criterion of Martin et al. [13], the resulting algorithm would be complex, less efficient and not scalable because their knowledge expression does not satisfy the congregation property. Intuitively, the resulting algorithm may need to go through all QI-groups once for each candidate split (in the safeSplit subroutine). When the dataset is large, the QI-groups may not fit in memory.

6. CASE-SPECIFIC FORMULAS & PROOF

We will show the computation formulas for $minNR_\sigma(g, \ell, k, m)$, $T_\sigma(g, \ell, k)$ and $V_\sigma(g, m, k)$ defined in Section 5.1, and discuss the proof of Theorem 1. For detailed explanations, see Appendix A3.

We use the following notation:

- n_g denotes the number of distinct individuals in QI-group g.
- $\#\sigma_g$ denotes the number of the occurrences of σ (the target sensitive value) in QI-group g.
- $s_{g(1)}, \ldots, s_{g(\ell)}$ denote the ℓ most frequent sensitive values in QI-group g with σ removed (i.e., $\sigma \neq s_{g(i)}$, for all i).
- $\#s_{g(1..\ell)}$ is shorthand for $\sum_{i\in[1,\ell]} \#s_{g(i)}$.
- $\Pr(E \mid K, g)$ is shorthand for $\Pr(E \mid K, \mathbf{D}^*)$, such that all the individuals in expressions E and K are in QI-group g.

6.1 Computation Formulas

In all three cases, $minNR_\sigma(g, \ell, k, m) = T_\sigma(g, \ell, k) \cdot V_\sigma(g, m, k+1)$.

In the SVPI case:

- $T_\sigma(g, \ell, k) = (n_g - \#\sigma_g - \#s_{g(1..\ell)} - k) / \#\sigma_g$
- $V_\sigma(g, m, k) = \prod_{i\in[0,m-1]} ((n_g - \#\sigma_g - k - i) / (n_g - k - i))$

In the MVPI-Set case:

- $T_\sigma(g, \ell, k) = [(n_g - \#\sigma_g - k) / \#\sigma_g]\cdot[\prod_{i\in[1,\ell]} ((n_g - \#s_{g(i)}) / n_g)]$
- $V_\sigma(g, m, k) = \prod_{i\in[0,m-1]} ((n_g - \#\sigma_g - k - i) / (n_g - k - i))$

In the MVPI-Multiset case:

- $T_\sigma(g,\ell,k) = \dfrac{[(n_g-k-1)/(n_g-k)]^{\#\sigma_g}}{1-[(n_g-k-1)/(n_g-k)]^{\#\sigma_g}} \cdot [(n_g-1)/n_g]^{\#s_{g(1..\ell)}}$
- $V_\sigma(g,m,k) = [(n_g-k-m)/(n_g-k)]^{\#\sigma_g}$

If the numerator of any of the above fractions becomes negative, then the corresponding formula is set to be 0. For detailed explanations, see Appendix A3.

6.2 Proof of Theorem 1

We will use the following four propositions (proven in Appendix A1).

Proposition 6. *Let $\alpha_1 \geq \ldots \geq \alpha_m \geq 0$ and $\beta_1 \geq \ldots \geq \beta_m \geq 0$ be two non-increasing series of numbers. Then, $(\prod_{i\in[1,h]} \alpha_i)\cdot(\prod_{i\in[1,m-h]} \beta_i)$, for $0 \leq h \leq m$, is minimized when $h = 0$ or m.*

Proposition 7. *Let a, b, c, d, m be positive numbers, such that $m \leq \min\{a, c\}$. Then, the following formula, for $0 \leq h \leq m$, is minimized when $h = 0$ or m.*

$$\left(\frac{a-h}{a}\right)^b \left(\frac{c-(m-h)}{c}\right)^d \tag{1}$$

Proposition 8. *Let a, b, c, d, k and m be positive numbers such that $c < d$* and $k \leq \min\{a, c-(m-1)\}$. *Then, the following formula, for $0 \leq p \leq k$, is minimized when $p = 0$ or k.*

$$\frac{a-p}{b} \cdot \prod_{i\in[0,m-1]} \frac{c-i-(k-p)}{d-i-(k-p)} \tag{2}$$

Proposition 9. *Let a, b, c, d, e, k and n be positive numbers such that $c < d$ and $k \leq \min\{n-1, c\}$. Then, the following formula, for $0 \leq p \leq k$, is minimized when $p = 0$ or k.*

$$\frac{[(n-p-1)/(n-p)]^a}{1-[(n-p-1)/(n-p)]^a} \cdot b \cdot \left(\frac{c-(k-p)}{d-(k-p)}\right)^e \tag{3}$$

Theorem 1 states that the breach probability is maximized when $u_1, \ldots, u_k$ (in $K_{\sigma u}(k)$) are in a single QI-group and $v_1, \ldots, v_m$ (in $K_{\sigma v}(m)$) are in a single QI-group. By Lemma 1, it is equivalent to show that the negated ratio (NR) is minimized in this situation. Basically, we consider how to distribute $t, u_1, \ldots, u_k$ and $v_1, \ldots, v_m$ into QI-groups in order to minimize the negated ratio.

In the following proof, we assume the minimum negated ratio is greater than 0. The proof for the boundary case is straightforward.

We prove Theorem 1 by induction on the number B of QI-groups.

Base case: When $B = 1$, our claim trivially holds. Thus, we consider $B = 2$. The two QI-groups are QI-group g and QI-group f. Without loss of generality, assume that when the negated ratio is minimized, the following two hold:

- QI-group g contains $t, u_1, \ldots, u_p$ and $v_1, \ldots, v_h$.
- QI-group f contains the rest $(k-p)$ of u_i's and $(m-h)$ of v_i's.

Our goal is to prove $h = 0$ or m (i.e., all the v_i's are in a single group), and $p = 0$ or k (i.e., all the u_i's are in a single group).

By Proposition 1, the literals in NR (defined in Lemma 1) that involve $t, u_1, \ldots, u_p$ and $v_1, \ldots, v_h$ are independent of the literals that involve the rest $(k-p)$ of the u_i's and $(m-h)$ of the v_i's. Thus, the minimum negated ratio becomes

$$\begin{aligned}\min_{t,\, v_i,\, x_i,\, K_{\sigma u}(k)} NR &= minNR_\sigma(g, \ell, p, h) \cdot V_\sigma(f, m-h, k-p)\\ &= T_\sigma(g, \ell, p) \cdot V_\sigma(g, h, p+1) \cdot V_\sigma(f, m-h, k-p).\end{aligned}$$

(For detailed derivation, see Derivation 1 in Appendix A4.)

Congregation of the v_i's: We now show NR is minimized when all the v_i's are in one QI-group; i.e., $h = 0$ or m. Since $T_\sigma(g, \ell, p)$ does not involve any v_i by definition, we only need to prove the following formula (4) is minimized when $h = 0$ or m.

$$V_\sigma(g, h, p+1) \cdot V_\sigma(f, m-h, k-p). \tag{4}$$

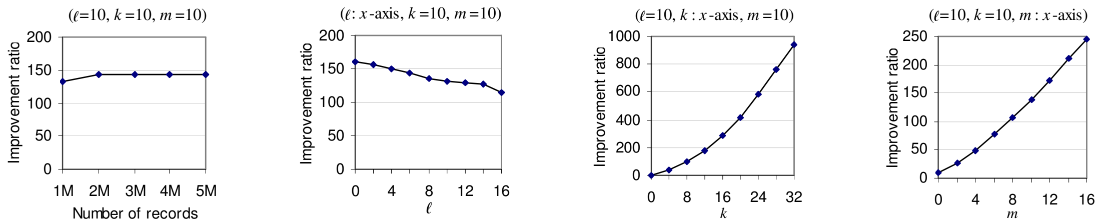
In the following, the proof is case-specific.

- In the SVPI and MVPI-Set cases, if we let $\alpha_i = (n_g-\#\sigma_g-p-i) / (n_g-p-i)$ and $\beta_i = (n_f-\#\sigma_f-(k-p)-i+1) / (n_f-(k-p)-i+1)$, we can rewrite formula (4) as $(\prod_{i\in[1,h]} \alpha_i)\cdot(\prod_{i\in[1,m-h]} \beta_i)$. Note that here i start from 1, not 0. Then, by Proposition 6, formula (4) is minimized when $h = 0$ or m.
- In the MVPI-Multiset case, we can rewrite formula (4) as formula (1) by setting $a = n_g-(p+1)$, $b = \#\sigma_g$, $c = n_f-(k-p)$, and $d = \#\sigma_f$. Then, by Proposition 7, formula (4) is minimized when $h = 0$ or m.

Since NR is minimized when all the v_i's are in one QI-group, $K_{\sigma v,t}(m)$ is 1-group congregated.

Congregation of the u_i's: We now show NR is minimized when all the u_i's are in one QI-group; i.e., $p = 0$ or k. If all the v_i's are in QI-group g (i.e., $h = m$), the minimum negated ratio becomes

$$T_\sigma(g, \ell, p) \cdot V_\sigma(g, m, p+1),$$



(a) **(b)** **(c)** **(d)**

Figure 7. Improvement over the dynamic programming technique [13]

because $V_{\sigma}(f, 0, k–p) = 1$. It is easy to see that $p = k$ maximizes the above formula. Thus all the u_i's are in one QI-group.

Now, if all the v_i's are in QI-group f (i.e., $h = 0$), the minimum negated ratio becomes the following formula (5).

$$T_{\sigma}(g, \ell, p) \cdot V_{\sigma}(f, m, k–p). \quad (5)$$

We need to show formula (5) is minimized when $p = 0$ or k.

- In the SVPI and MVPI-Set cases, we can rewrite formula (5) as formula (2) by appropriately setting a, b, c, d, k, m. Thus, by Proposition 8, formula (5) is minimized at $p = 0$ or k.
- In the MVPI-Multiset case, we can rewrite formula (5) as formula (3) by appropriately setting a, b, c, d, e, k, n. Thus, by Proposition 9, formula (5) is minimized when $p = 0$ or k.

Since NR is minimized when all the u_i's are in one QI-group, $K_{\sigma|u}(k)$ is 1-group congregated.

Induction argument: Now assume Theorem 1 holds for (B−1) QI-groups. We show that it also holds for B QI-groups. We first consider the v_i's. Without loss of generality, assume the negated ratio is minimized when $v_1, \ldots, v_h$ are in the first (B−1) QI-groups and the rest (m–h) are in the Bth QI-group. By the induction assumption, $v_1, \ldots, v_h$ are in one QI-group, say g. Now, the v_i's can only be in two QI-groups. Similar to the argument in the base case, $h = 0$ or m. Thus, all the v_i's are in one QI-group; i.e., $K_{\sigma|v,t}(m)$ is 1-group congregated.

By a similar argument, it is easy to show that all the u_i's are in one QI-group; i.e., $K_{\sigma|u}(k)$ is 1-group congregated. ❑

7. EXPERIMENTS

In this section, we describe a set of experiments intended to address the following three high-level questions. First, recall that in Section 5.1 we developed an efficient algorithm for checking the safety of a release candidate in the presence of three-dimensional external knowledge, based on the congregation property. In Section 7.1, we show that this algorithm improves performance several orders of magnitude over the best existing technique [13]. Second, we describe (in Section 7.2) an experiment demonstrating the efficiency and scalability of the anonymization algorithm described in Section 5.2. Finally, in Section 7.3, we present an interesting case study, which demonstrates how the skyline exploratory tool can be used in a practical setting.

7.1 Efficiency Comparison

Our algorithms rely heavily on the *congregation* property. In this experiment, we show the importance of this property. Recall that, to check whether a release candidate is safe, we maximize the breach probability. Without the *congregation* property, the best known technique for maximizing the breach probability is the dynamic-programming technique developed in [13]. Although the technique was originally developed for computing the breach probability under a knowledge expression different from ours, it can be adapted to ours easily. In addition, we use a simple technique to remove recursive calls to make the dynamic-programming algorithm faster. For details, see Appendix A6.

We generate release candidates synthetically. There are 20 distinct uniformly distributed values in the sensitive attribute. We fix the size of each QI-group to be 100 individuals. By varying the number of QI-groups in a release candidate, we generate release candidates with sizes from one million records to five million records. We define the **improvement ratio** to be the CPU time of the dynamic-programming algorithm over the CPU time of the *SkylineCheck* algorithm (described in Section 5.1) when they applied to a same release candidate. Both algorithms have the same IO time and always output the same answer. The experiment was run on a Windows XP machine with a 2.0 GHz dual-core processor and 2 GB memory. The breach probabilities were computed for the SVPI case.

Figure 7 shows the experimental results. Each point in the plots is an average improvement ratio over five runs. In Figure 7 (a), we set the knowledge threshold to be (ℓ, k, m) = (10, 10, 10) and vary the size of the release candidate. In this setting, our algorithm is about 140 times faster than the dynamic programming algorithm. In Figure 7 (b), we vary ℓ from 0 to 16. The improvement decreases as ℓ increases, because both algorithms have roughly the same computational dependency on the ℓ value. As the ℓ value increases, it gradually dominates the running time. Thus, the difference between the two algorithms becomes smaller. In Figure 7 (c), we vary k from 0 to 32 and observe that the improvement increases as k increases. At $k = 32$, our algorithm is about 1,000 times faster than the dynamic-programming algorithm. Note that, in practice, the k value may be even larger. Finally, in Figure 7 (d), we vary m from 0 to 16, and also observe that the improvement increases as m increases.

Note that in this experiment, we compare the two algorithms for checking whether a release candidate is safe. The algorithm for generating a safe release candidate is more complex than that for checking safety. Although we did not show experimental results comparing our technique with the dynamic-programming technique for generating a safe release candidate, it can be easily seen that the improvement will be larger.

7.2 Scalability

We also conducted an experiment that demonstrates the scalability of the *SkylineAnonymize* algorithm (in Section 5.2) using the *Rothko-Tree* approach described in [9]. The scale-up experiment was run on a single-processor 2.4 GHz Linux machine with 512 MB of memory. We used a synthetic data set similar to that described in [1], and each data tuple was a fixed 44 bytes. Hypothetically, we set *Zipcode* (9 distinct values) to be the sensitive attribute. Figure 8 shows our results for two different privacy settings. In each case, the scale-up performance is well-

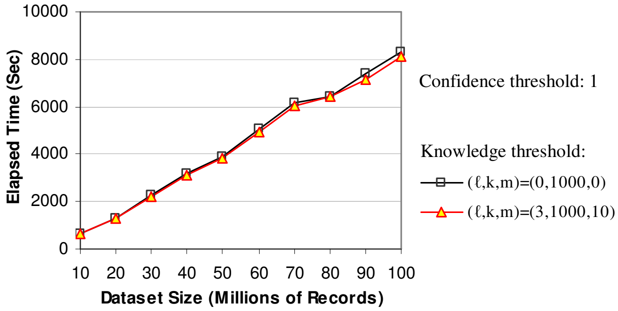


Figure 8. Scalability experimental result

behaved for datasets substantially larger than main memory. The case of (ℓ, *k*, *m*) = (0, 1000, 0) roughly corresponds to generating a *k*-anonymous dataset with *k* = 1000. The case of (ℓ, *k*, *m*) = (3, 1000, 10), we think, is a more reasonable privacy setting. Because the number of sensitive value is just 9, the ℓ value cannot be large. Also, considering that the adversary knows *m*=10 members in the target individual's same-value family is usually sufficient. We set *k* to be a much larger number, because *k* represents that the adversary obtains a list of *k* individuals from other datasets, which can be large.

7.3 Case Study: Adult Dataset

The adult dataset from the UCI Machine Learning Repository (http://www.ics.uci.edu/~mlearn/MLRepository.html) has been used in a number of privacy-related studies (e.g., [7, 11, 13]). In this section, we describe a case study, using the skyline exploratory tool to investigate the safety of release candidates. In particular, we find that an ℓ-diverse [11] release candidate can be unsafe in the presence of certain kinds of adversarial knowledge. Based on the experiment in [13], ℓ-diversity has similar behavior to (*c*, *k*)-safety [13]. Thus, our case study also suggests that a (*c*, *k*)-safe release candidate may also be unsafe in the presence of certain external knowledge.

The adult dataset has 45,222 records after removing records with missing values. Following [11, 13], we treat *Occupation* (14 distinct values) as the sensitive attribute. Each individual has exactly one sensitive value (i.e., the SVPI case). Suppose the data owner wants to publish a safe version of the adult dataset using ℓ-diversity. She first generates a (*c*=3, ℓ =6)-diverse release candidate, where (*c*=3, ℓ=6) is a common setting in [11, 13]. Note that (*c*=3, ℓ=6)-diversity is actually equivalent to our basic 3D privacy criterion by setting (ℓ, *k*, *m*) = (4, 0, 0) and confidence threshold to be 75%, for all sensitive values. Thus, we use our anonymization algorithm to generate such a release candidate.

Before publishing the release candidate, the data owner investigates how safe the release candidate is under various amounts and types of external knowledge using the knowledge skyline. The following are the resulting skyline points for sensitive value "Exec-managerial" at confidence threshold 95%:

ℓ k m	ℓ k m	ℓ k m	ℓ k m
(0, 4, 0),	(1, 3, 1),	(2, 2, 2),	(3, 1, 2),
(2, 1, 3),	(4, 0, 3),	(3, 0, 4).	

When the number of points on the skyline is large, we can show these points in a 3D visualization interface. The release candidate is safe if and only if the adversary has knowledge with amount below or on the skyline points. Thus, the first point (0, 4, 0) tells us that, in the worst case, if the adversary knows the sensitive values of only 5 individuals (and nothing else), then he would be able to successfully predict a target individual to be an executive manager with confidence at least 95%. This is a privacy breach. One may say that it is unlikely to be the worst case. However, our exploratory tool can also identify the five individuals that cause the worst case (by looking at the grounding of the variables that maximizes the breach probability). Thus, after the release candidate is published, the adversary can also use our tool to identify those five individuals and, by a small-scale investigation of five people, he can achieve 95% confidence. This demonstrates that an ℓ-diverse release candidate can be quite unsafe.

As another example, consider the skyline point (2, 1, 3). This point tells us that the adversary cannot succeed if he knows ≤ 2 sensitive values that the target individual does not have, the sensitive value of ≤ 1 other individual, and ≤ 3 other members of the target individual's same-value family. However, if the adversary has any knowledge more than this amount, in the worst case, he could succeed.

8. CONCLUSIONS & FUTURE WORK

In this paper, we first described a clean theoretical framework for reasoning about attribute disclosure in the presence of external knowledge. In general, the problem of measuring disclosure is NP-hard when external knowledge is involved. For this reason, the interesting research direction is to find special forms of external knowledge that both arise naturally in practice and can be efficiently handled. Previous work [13] identified a special form that can be handled in polynomial time but is not very natural.

Thus, we defined a privacy criterion based on a combination of three special forms of knowledge that arise naturally in practice, and developed efficient and scalable algorithms for checking safety and generating safe release candidates. We showed that our checking algorithm improves efficiency several orders of magnitude over the best known technique [13], and our anonymization algorithm is well-behaved on datasets much larger than main memory. Based on the three special forms, we also proposed a three-dimensional skyline exploratory tool that is useful for investigating the safety of a dataset to be released.

In the future, an important research direction is identifying other classes of background knowledge that are both natural and can be handled efficiently. In particular, there are several types of external knowledge that we find especially compelling:

- **Graphs:** It is natural to express relationships among individuals using graphs, in which nodes are properties of individuals and edges represent relationships. What kinds of graphs are both useful and efficiently solvable is an open problem.
- **Other release candidates:** The adversary may have access to other release candidates (e.g., an anonymized dataset from another organization). How to express this kind of knowledge and what special cases are efficiently solvable are wide open.
- **Probabilistic external knowledge:** In Section 2, we described a theoretical framework based on deterministic external knowledge. An interesting extension to this framework would allow external knowledge to be probabilistic. In particular, when we evaluate an expression *E* on a possible original dataset $R(\mathbf{D}^*)$, instead of returning either true or false, we return $\Pr(E \mid R(\mathbf{D}^*))$. In this extension, assuming that each reconstruction *R* is equally likely in the absence of any external knowledge, we obtain

$$\Pr(E \mid K, \mathbf{D}^*) = \sum_R \Pr(E \wedge K \mid R(\mathbf{D}^*)) \;/\; \sum_R \Pr(K \mid R(\mathbf{D}^*)),$$

This extension is closely related to the language of (sometimes uncertain) knowledge bases described in [2].

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A. APPENDIX

In Sections A1 and A2, we first prove all the propositions and lemmas, and then explain the correctness of the computation formulas for $minNR_\sigma(g, \ell, k, m)$, $T_\sigma(g, \ell, k)$ and $V_\sigma(g, m, k)$ in Section A3. In Section A4, we show a formula derivation. In Section A5, we prove the theorems. In Section A6, we describe an algorithm for checking whether a release candidate is safe based on a dynamic-programming technique (which is originally developed in [13]). In Section A7, we describe an algorithm for finding the knowledge skyline of a release candidate. Finally, we formally discuss *expressibility* and compare our knowledge expressions with the language proposed in [13] in Section A8.

A1. Proofs of the Propositions

In this section, we prove the propositions.

Let $\mathbf{D}^* = \{(G_1, X_1), \ldots, (G_B, X_B)\}$ be a release candidate with B QI-groups.

Proposition 1. *Let $E_1, \ldots, E_B$ be B conjunctions of atoms such that E_g only involves individuals in QI-group g. Also, let $K_1, \ldots, K_B$ be another B conjunctions of atoms such that K_g only involves individuals in QI-group g. Then,*

$$\Pr(\wedge_{g\in[1,B]} E_g \mid \wedge_{g\in[1,B]} K_g, \mathbf{D}^*) = \textstyle\prod_{g\in[1,B]} \Pr(E_g \mid K_g, \mathbf{D}^*).$$

Proof: Let $n_g(E_g)$ denote the number of reconstructions of QI-group g that satisfies expression E_g. The number of reconstructions of $\mathbf{D}^*$ that satisfies $(\wedge_{g\in[1,B]} E_g)\wedge(\wedge_{g\in[1,B]} K_g) = \wedge_{g\in[1,B]} (E_g\wedge K_g)$ is $\prod_{g\in[1,B]} n_g(E_g\wedge K_g)$. Similarly, the number of reconstructions of $\mathbf{D}^*$ that satisfy $\wedge_{g\in[1,B]} K_g$ is $\prod_{g\in[1,B]} n_g(K_g)$. Thus, by the assumption that every reconstruction has an equal probability,

$\Pr(\wedge_{g\in[1,B]} E_g \mid \wedge_{g\in[1,B]} K_g, \mathbf{D}^*)$
$= \Pr((\wedge_{g\in[1,B]} E_g)\wedge(\wedge_{g\in[1,B]} K_g) \mid \mathbf{D}^*) / \Pr(\wedge_{g\in[1,B]} K_g \mid \mathbf{D}^*)$
$= (\prod_{g\in[1,B]} n_g(E_g\wedge K_g)) / (\prod_{g\in[1,B]} n_g(K_g))$
$= \prod_{g\in[1,B]} (n_g(E_g\wedge K_g) / n_g(K_g))$
$= \prod_{g\in[1,B]} \Pr(E_g \mid K_g, \mathbf{D}^*)$. ❑

Note that each E_g or K_g can be an empty expression.

Proposition 2. *Let $E_{g,x}$ and $K_{g,x}$ denote two conjunctions of atoms that only involves individuals in G_g and sensitive value $x \in X_g$, for g* = 1 *to B. Then, in the MVPI (no matter Set or Multiset) case,*

$$\Pr(\wedge_{g\in[1,B],\, x\in X_g} E_{g,x} \mid \wedge_{g\in[1,B],\, x\in X_g} K_{g,x}, \mathbf{D}^*) = \textstyle\prod_{g\in[1,B]} \prod_{x\in X_g} \Pr(E_{g,x} \mid K_{g,x}, \mathbf{D}^*).$$

Proof: Let $n_{g,x}(E_{g,x})$ denote the number of possible assignments, each of which "assigns an individual in G_g to an occurrence of sensitive value $x \in X_g$, for all the occurrences of x," that satisfy expression $E_{g,x}$; i.e., $n_{g,x}(E_{g,x})$ is the number of reconstructions of the group of individuals having sensitive value x in QI-group g that satisfies expression $E_{g,x}$. The number of reconstructions of $\mathbf{D}^*$ that satisfies $(\wedge_{g\in[1,B],x\in X_g} E_{g,x})\wedge(\wedge_{g\in[1,B],x\in X_g} K_{g,x}) = \wedge_{g\in[1,B],x\in X_g} (E_{g,x}\wedge K_{g,x})$ is $\prod_{g\in[1,B],x\in X_g} n_{g,x}(E_{g,x}\wedge K_{g,x})$. Similarly, the number of reconstructions of $\mathbf{D}^*$ that satisfy $\wedge_{g\in[1,B],x\in X_g} K_{g,x}$ is $\prod_{g\in[1,B],x\in X_g} n_{g,x}(K_{g,x})$. Thus, by the assumption that every reconstruction has an equal probability,

$\Pr(\wedge_{g\in[1,B],x\in X_g} E_{g,x} \mid \wedge_{g\in[1,B],x\in X_g} K_{g,x}, \mathbf{D}^*)$
$= \Pr((\wedge_{g\in[1,B],x\in X_g} E_{g,x})\wedge(\wedge_{g\in[1,B],x\in X_g} K_{g,x}) \mid \mathbf{D}^*) / \Pr(\wedge_{g\in[1,B],x\in X_g} K_{g,x} \mid \mathbf{D}^*)$
$= (\prod_{g\in[1,B],x\in X_g} n_{g,x}(E_{g,x}\wedge K_{g,x})) / (\prod_{g\in[1,B],x\in X_g} n_{g,x}(K_{g,x}))$
$= \prod_{g\in[1,B],x\in X_g} (n_{g,x}(E_{g,x}\wedge K_{g,x}) / n_{g,x}(K_{g,x}))$
$= \prod_{g\in[1,B],x\in X_g} \Pr(E_{g,x} \mid K_{g,x}, \mathbf{D}^*)$. ❑

Proposition 3. *In the SVPI case, k-anonymity* [16] *is a special case of the basic 3D privacy criterion when the sensitive values are the identities of the individuals, the knowledge threshold is* $(0, k–2, 0)$ *and the confidence threshold is* 1, *for all sensitive values* σ.

Proof: Note that here the use of sensitive value is special. Each user has a unique sensitive value, which is his/her identity. In this instantiation, the privacy criterion states that $\mathbf{D}^*$ is safe if for each user t,

$$\Pr(t \text{ can be identified} \mid \text{the identities of at most } k–2 \text{ other individuals}) < 1.$$

It can be easily seen, if t is in a QI-group has fewer than k individuals, then we can identify t exactly. Thus, each QI-group must have at least k individuals. This is the protection provided by k-anonymity. ❑

Proposition 4. *In the SVPI case*, (c,ℓ)*-diversity* [11] *is a special case of the basic 3D privacy criterion when the knowledge threshold is* $(\ell-2, 0, 0)$ *and the confidence threshold is* $c/(c+1)$, *for all sensitive values* σ.

Proof: In Appendix E of [13], Martin et al. proved that (c,ℓ)-diversity is equivalent to $(c/(c+1), \ell-2)$-safety, which is the instantiation of the generic basic privacy criterion using a conjunction of at most $(\ell-2)$ negated atoms with confidence threshold $c/(c+1)$. It can be easily seen that the breach probability is maximized when the conjunction of negated atoms has the following form: $\wedge_{i\in[1,\ell-2]}\ t[S]\neq x_i$, which is $K_{\sigma|t}(\ell-2)$. Thus, it is equivalent to $\mathscr{L}_{t,\sigma}^{\mathrm{SVPI}}(\ell-2, 0, 0)$. ❑

Proposition 5. *If the negated ratio is minimized when t is in QI-group g and* $v_1, \ldots, v_m$ *are in QI-group f, then, at the minimum, all the* u_i's *(in* $K_{\sigma|u}(k)$*) are either in QI-group g or QI-group f.*

Proof: Assume that k_j of the u_i's are in QI-group j such that $\sum_j k_j = k$ and $k_j \geq 0$. Our goal is to prove that, at the minimum negated ratio, $k_g + k_f = k$. Let there be B QI-groups. Now, the negated ratio is

$$\frac{\Pr(\sigma\notin t[S]\wedge(\wedge_{i\in[1,\ell]}x_i\notin t[S])\wedge(\wedge_{i\in[1,m]}\sigma\notin v_i[S])\mid\wedge_{j\in[1,B]}K_{\sigma|u}(k_j),\mathbf{D}^*,(t\in g,v_i\in f))}{\Pr(\sigma\in t[S]\mid\wedge_{j\in[1,B]}K_{\sigma|u}(k_j),\mathbf{D}^*,t\in g)}.$$

Note that $K_{\sigma|u}(k_j)$ only involves individuals in QI-group j.

If $g = f$, by Proposition 1, the minimum negated ratio becomes

$$\min\frac{\Pr(\sigma\notin t[S]\wedge(\wedge_{i\in[1,\ell]}x_i\notin t[S])\wedge(\wedge_{i\in[1,m]}\sigma\notin v_i[S])\mid K_{\sigma|u}(k_g),\mathbf{D}^*,(t\in g,v_i\in g))\cdot[\prod_{j\neq g}\Pr(\varepsilon\mid K_{\sigma|u}(k_j),\mathbf{D}^*)]}{\Pr(\sigma\in t[S]\mid K_{\sigma|u}(k_g),\mathbf{D}^*,t\in g)\cdot[\prod_{j\neq g}\Pr(\varepsilon\mid K_{\sigma|u}(k_j),\mathbf{D}^*)]}$$

$$=\min\frac{\Pr(\sigma\notin t[S]\wedge(\wedge_{i\in[1,\ell]}x_i\notin t[S])\wedge(\wedge_{i\in[1,m]}\sigma\notin v_i[S])\mid K_{\sigma|u}(k_g),\mathbf{D}^*,(t\in g,v_i\in g))}{\Pr(\sigma\in t[S]\mid K_{\sigma|u}(k_g),\mathbf{D}^*,t\in g)}$$

$= minNR_\sigma(g, \ell, k_g, m)$.

It can be easily seen that the above is minimized when $k_g = k$, by using the formula in Section 6.1.

If $g \neq f$, by Proposition 1, the negated ratio becomes

$$\min\frac{\Pr(\sigma\notin t[S]\wedge(\wedge_{i\in[1,\ell]}x_i\notin t[S])\mid K_{\sigma|u}(k_g),\mathbf{D}^*,t\in g)\cdot\Pr(\wedge_{i\in[1,m]}\sigma\notin v_i[S]\mid K_{\sigma|u}(k_f),\mathbf{D}^*,v_i\in f)\cdot[\prod_{j\neq g,j\neq f}\Pr(\varepsilon\mid K_{\sigma|u}(k_j),\mathbf{D}^*)]}{\Pr(\sigma\in t[S]\mid K_{\sigma|u}(k_g),\mathbf{D}^*,t\in g)\cdot[\prod_{j\neq g}\Pr(\varepsilon\mid K_{\sigma|u}(k_j),\mathbf{D}^*)]}$$

$$=\min\frac{\Pr(\sigma\notin t[S]\wedge(\wedge_{i\in[1,\ell]}x_i\notin t[S])\mid K_{\sigma|u}(k_g),\mathbf{D}^*,t\in g)\cdot\Pr(\wedge_{i\in[1,m]}\sigma\notin v_i[S]\mid K_{\sigma|u}(k_f),\mathbf{D}^*,v_i\in f)}{\Pr(\sigma\in t[S]\mid K_{\sigma|u}(k_g),\mathbf{D}^*,t\in g)}$$

$$=\left(\min\frac{\Pr(\sigma\notin t[S]\wedge(\wedge_{i\in[1,\ell]}x_i\notin t[S])\mid K_{\sigma|u}(k_g),\mathbf{D}^*,t\in g)}{\Pr(\sigma\in t[S]\mid K_{\sigma|u}(k_g),\mathbf{D}^*,t\in g)}\right)\cdot\left(\min\ \Pr(\wedge_{i\in[1,m]}\sigma\notin v_i[S]\mid K_{\sigma|u}(k_f),\mathbf{D}^*,v_i\in f)\right)$$

$= T_\sigma(g, \ell, k_g) \cdot V_\sigma(f, m, k_f)$.

It can be easily seen that the above is minimized when $k_g+k_f = k$, by using the formulas in Section 6.1. ❑

Proposition 6. *Let* $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_m \geq 0$ *and* $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_m \geq 0$ *be two non-increasing series of numbers. Then,* $(\prod_{i\in[1,h]} \alpha_i)\cdot(\prod_{i\in[1,m-h]} \beta_i)$*, for* $0 \leq h \leq m$*, is minimized when* $h = 0$ *or* m.

Proof: Without loss of generality, we assume $\prod_{i\in[1,m]} \alpha_i \leq \prod_{i\in[1,m]} \beta_i$. Our goal is to show that

$$\prod_{i\in[1,m]} \alpha_i \leq (\prod_{i\in[1,h]} \alpha_i)\cdot(\prod_{i\in[1,m-h]} \beta_i),\ \text{for any } 0 \leq h \leq m.$$

We will prove this by contradiction. Assume $\prod_{i\in[1,m]} \alpha_i > (\prod_{i\in[1,h]} \alpha_i)\cdot(\prod_{i\in[1,m-h]} \beta_i)$, for a particular h such that $1 \leq h \leq m-1$. Since $\prod_{i\in[1,m]} \alpha_i = (\prod_{i\in[1,h]} \alpha_i)\cdot(\prod_{i\in[h+1,m]} \alpha_i)$, we conclude that $\prod_{i\in[h+1,m]} \alpha_i > \prod_{i\in[1,m-h]} \beta_i$. This implies that $\alpha_{h+1} > \beta_{m-h}$. Otherwise, $\beta_1 \geq \ldots \geq \beta_{m-h} \geq \alpha_{h+1} \geq \ldots \geq \alpha_m$, which implies that $\prod_{i\in[1,m-h]} \beta_i \geq \prod_{i\in[h+1,m]} \alpha_i$. Because $\alpha_{h+1} > \beta_{m-h}$, we obtain $\alpha_1 \geq \ldots \geq \alpha_h \geq \alpha_{h+1} > \beta_{m-h} \geq \beta_{m-h+1} \geq \ldots \geq \beta_m$, which implies that $\prod_{i\in[1,h]} \alpha_i > \prod_{i\in[m-h+1, m]} \beta_i$. Finally, we obtain the following contradiction.

$$\prod_{i\in[1,m]} \alpha_i > \left(\prod_{i\in[1,h]} \alpha_i\right)\cdot\left(\prod_{i\in[1,m-h]} \beta_i\right) > \left(\prod_{i\in[m-h+1,\,m]} \beta_i\right)\cdot\left(\prod_{i\in[1,m-h]} \beta_i\right)\cdot = \prod_{i\in[1,m]} \beta_i.$$ ❑

Proposition 7. *Let a, b, c, d, m be positive numbers, such that $m \le \min\{a, c\}$. Then, the following formula, for $0 \le h \le m$, is minimized when $h = 0$ or m.*

$$\left(\frac{a-h}{a}\right)^b \left(\frac{c-(m-h)}{c}\right)^d .$$

Proof: If $m = \min\{a, c\}$, then it is easy to see that the above formula is minimized (returning 0) when $h = 0$ or m. Now, we consider $m < \min\{a, c\}$. Now the above formula is always a positive number. Thus, the h value that minimizes the log of the above formula also minimizes the formula itself. We then take log.

$$\text{Let } L(h) = b\log\frac{a-h}{a} + d\log\frac{c-(m-h)}{c} .$$

Consider h to be a real number. If $L(h)$ is minimized when $h = 0$ or m, then it is also true when h only takes integer values. Now, we claim that $L(h)$ is concave (i.e., $L''(h) < 0$, which is the second derivative of $L(h)$). Then, $L(h)$ is minimized at the boundary, which is either $h = 0$ or $h = m$.

We now show that $L''(h) < 0$.

$$L'(h) = -\frac{b}{a-h} + \frac{d}{c-m+h} \quad\text{and}\quad L''(h) = -\frac{b}{(a-h)^2} - \frac{d}{(c-m+h)^2} < 0$$ ❑

Proposition 8. *Let a, b, c, d, k and m be positive numbers such that $c < d$, $k \le \min\{a, c-(m-1)\}$. Then, the following formula, for $0 \le p \le k$, is minimized when $p = 0$ or k.*

$$\frac{a-p}{b}\cdot\prod_{i\in[0,m-1]}\frac{c-i-(k-p)}{d-i-(k-p)} .$$

Proof: If $k = \min\{a, c-(m-1)\}$, then it can be easily seen that the above formula is minimized (returning 0) when $p = 0$ or $p = k$. Now, we consider $k < \min\{a, c-(m-1)\}$. Now, the above formula is always a positive number. Thus, the p value that minimizes the log of the above formula also minimizes the formula itself. We then take log.

$$\text{Let } L(p) = \log(a-p) - \log b + \sum_{i\in[0,m-1]}\left[\log(c-i-k+p) - \log(d-i-k+p)\right].$$

Consider p to be a real number. If $L(p)$ is minimized when $p = 0$ or $p = k$, then it is also true when p only takes integer values. Now, we claim that $L(p)$ is concave (i.e., $L''(p) < 0$, which is the second derivative of $L(p)$). Then, $L(p)$ is minimized at the boundary, which is either $p = 0$ or $p = k$.

We now show that $L''(p) < 0$.

$$L'(p) = \frac{-1}{a-p} + \sum_{i\in[0,m-1]}\left[\frac{1}{c-i-k+p} - \frac{1}{d-i-k+p}\right]$$

$$L''(p) = \frac{-1}{(a-p)^2} + \sum_{i\in[0,m-1]}\left[\frac{-1}{(c-i-k+p)^2} + \frac{1}{(d-i-k+p)^2}\right] < 0$$

$L''(p) < 0$ because $c < d$. ❑

Proposition 9. *Let a, b, c, d, e, k and n be positive numbers such that $c < d$ and $k \le \min\{n-1, c\}$. Then, the following formula, for $0 \le p \le k$, is minimized when $p = 0$ or k.*

$$\frac{[(n-p-1)/(n-p)]^a}{1-[(n-p-1)/(n-p)]^a}\cdot b\cdot\left(\frac{c-(k-p)}{d-(k-p)}\right)^e .$$

Proof: If $k = \min\{n-1, c\}$, then it can be easily seen that the above formula is minimized (returning 0) when $p = 0$ or $p = k$. Now, we consider $k < \min\{n-1, c\}$. Now, the above formula is always a positive number. Thus, the p value that minimizes the log of the above formula also minimizes the above formula. We first rewrite the above formula as

$$\frac{1}{[(n-p)/(n-p-1)]^a-1}\cdot b\cdot\left(\frac{c-(k-p)}{d-(k-p)}\right)^e.$$

We then take log.

$$\text{Let } L(p)=-F(p)+\log b+e\cdot\left[\log(c-k+p)-\log(d-k+p)\right].$$

where $F(p) = \log([(n-p) / (n-p-1)]^a - 1)$. Consider p to be a real number. If $L(p)$ is minimized when $p = 0$ or $p = k$, then it is also true when p only takes integer values. Now, we claim that $L(p)$ is concave (i.e., $L''(p) < 0$, which is the second derivative of $L(p)$). Then, $L(p)$ is minimized at the boundary, which is either $p = 0$ or $p = k$. We now show that $L''(p) < 0$.

$$L'(p)=-F'(p)+e\cdot\left[\frac{1}{c-k+p}-\frac{1}{d-k+p}\right]$$

$$L''(p)=-F''(p)+e\cdot\left[\frac{-1}{(c-k+p)^2}+\frac{1}{(d-k+p)^2}\right]$$

If $F''(p) \geq 0$, then $L''(p) < 0$ because $c < d$ and e is positive.

We now show $F''(p) \geq 0$. We first focus on $G(p) = [(n-p) / (n-p-1)]^a$. Let $H(p) = \log G(p)$. Then, $H(p) = a\cdot[\log(n-p) - \log(n-p-1)]$.

$$H'(p)=a\cdot\left[\frac{1}{n-p-1}-\frac{1}{n-p}\right]=\frac{a}{(n-p-1)(n-p)}=\tfrac{d}{dp}\log G(p)=\frac{G'(p)}{G(p)}>0.$$

$$H''(p)=a\cdot\left[\frac{1}{(n-p-1)^2}-\frac{1}{(n-p)^2}\right]=\tfrac{d}{dp^2}\log G(p)=\frac{G''(p)}{G(p)}-\frac{[G'(p)]^2}{[G(p)]^2}>0.$$

Note that $x^2 - y^2 = (x - y)\cdot(x + y)$. Thus, we rewrite $H''(p)$ as follows.

$$H''(p)=a\cdot\left[\frac{1}{n-p-1}-\frac{1}{n-p}\right]\cdot\left[\frac{1}{n-p-1}+\frac{1}{n-p}\right]=\frac{G'(p)}{G(p)}\cdot\left[\frac{2(n-p)-1}{(n-p-1)(n-p)}\right]=\frac{[G'(p)]^2}{[G(p)]^2}\cdot\frac{2(n-p)-1}{a}.$$

By equating the above two formulas of $H''(p)$, we obtain

$$\frac{G''(p)}{G'(p)}-\frac{G'(p)}{G(p)}=\frac{G'(p)}{G(p)}\cdot\frac{2(n-p)-1}{a}.$$

Note that $F(p) = \log(G(p) - 1)$. We obtain

$$F''(p)=\frac{G''(p)}{G(p)-1}-\frac{[G'(p)]^2}{[G(p)-1]^2}=\frac{G'(p)}{[G(p)-1]}\cdot\left[\frac{G''(p)}{G'(p)}-\frac{G'(p)}{G(p)-1}\right]=\frac{G'(p)}{[G(p)-1]}\cdot\left(\left[\frac{G''(p)}{G'(p)}-\frac{G'(p)}{G(p)}\right]+\left[\frac{G'(p)}{G(p)}-\frac{G'(p)}{G(p)-1}\right]\right)$$

$$=\frac{G'(p)}{[G(p)-1]}\cdot\left(\frac{G'(p)}{G(p)}\cdot\frac{2(n-p)-1}{a}-\frac{G'(p)}{G(p)}\cdot\frac{1}{G(p)-1}\right)=\frac{[G'(p)]^2}{a\cdot[G(p)-1]^2\cdot G(p)}\cdot\left([2(n-p)-1]\cdot[G(p)-1]-a\right).$$

Note that $[G'(p)]^2 > 0$, $a > 0$, $[G(p)-1]^2 > 0$ and $G(p) > 0$. We claim $([2(n-p)-1]\cdot[G(p)-1] - a) \geq 0$. Thus, $F''(p) \geq 0$.

Finally, we prove $([2(n-p)-1]\cdot[G(p)-1] - a) \geq 0$. Let $x = n - p$. Note that $x \geq 2$ because $p \leq k < n-1$. We rewrite the formula in terms of x and a:

$$E(x,a)=(2x-1)\cdot[G(n-x)-1]-a=(2x-1)\cdot([x/(x-1)]^a-1)-a.$$

Then, the goal is to prove $E(x, a) \geq 0$, for any $x \geq 2$ and $a \geq 1$. We first show that $E(x, 1) \geq 0$.

$$E(x, 1) = (2x-1)\cdot([x/(x-1)] - 1) - 1 = x/(x-1) \geq 0, \text{ for any } x \geq 2.$$

We now show that $E(x, a)$ is an increasing function in a, for $a \geq 1$.

$$\tfrac{d}{da}E(x,a)=(2x-1)\cdot[x/(x-1)]^a\cdot[\log(x)-\log(x-1)]-1\geq(2x-2)\cdot[\log(x)-\log(x-1)]-1.$$

Let $D(y) = 2y\cdot[\log(y+1) - \log(y)]$. Note that $\tfrac{d}{da}E(x,a)\geq D(x-1)-1$. Thus, to show $E(x, a)$ is increasing, we show $D(y) > 1$, for $y \geq 1$. Note that $D(1) = 2 \log 2 > 1$. We now focus on $y \geq 2$.

$$D'(y)=2\cdot[\log(y+1)-\log(y)-1/(y+1)]$$

$D''(y) = -2 \cdot /[y \cdot (y+1)^2] < 0$, for $y \ge 1$; i.e., $D'(y)$ is decreasing.

Because $D'(2) \cong -0.3145 < 0$ and $D'(y)$ is decreasing, we obtain $D'(y) < 0$, for $y \ge 2$; i.e., $D(y)$ is decreasing, for $y \ge 2$. Thus, the minimum value of $D(y)$ is when $y \to \infty$.

$$\lim_{y\to\infty} D(y) = 2 \cdot \frac{\lim_{y\to\infty}[\log(y+1) - \log(y)]}{\lim_{y\to\infty}(1/y)}$$

Note that both the numerator and denominator goes to 0 when $y \to \infty$. Thus, we apply the L'Hospital rule to the above formula by replacing both the numerator and denominator with their derivatives.

$$\lim_{y\to\infty} D(y) = 2 \cdot \frac{\lim_{y\to\infty}[\log(y+1) - \log(y)]}{\lim_{y\to\infty} 1/y} = 2 \cdot \frac{\lim_{y\to\infty} -1/[y \cdot (y-1)]}{\lim_{y\to\infty} -1/y^2} = 2 \cdot \lim_{y\to\infty}[1 + 1/(y-1)] = 2 .$$

Thus, $D(y) \ge 2$, for any $y \ge 1$ This implies $\frac{d}{da}E(x,a) \ge D(x-1) - 1 \ge 1$, for any $x \ge 2$ and $a \ge 1$.

Because, for any $x \ge 2$, $E(x, 1) \ge 0$ and $E(x, a)$ is increasing in a, for $a \ge 1$, we obtain $E(x, a) \ge 0$. We now complete the proof. ❑

Proposition 10. *Let a_1, a_2, b_1, b_2 be positive numbers. Then,*

$$\min\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}\right\} \le \frac{a_1 + a_2}{b_1 + b_2} .$$

Proof: Without loss of generality, assume $a_1/b_1 \le a_2/b_2$; i.e., $a_1b_2 \le a_2b_1$. Then, we obtain

$$\frac{a_1 + a_2}{b_1 + b_2} - \frac{a_1}{b_1} = \frac{a_2b_1 - a_1b_2}{b_1(b_1 + b_2)} \ge 0$$ ❑

Proposition 11. *Let a, b, c, d be positive numbers such that $a/b \le c/d < 1$ and $b \le d$. Then,*

1. $(a-k)/b \le (c-k)/d$, *for* $0 \le k \le \min\{a, c\}$, *and*
2. $(a-k)/(b-k) \le (c-k)/(d-k)$, *for* $0 \le k < \min\{a, b, c, d\}$.

Proof: Case 1: $(a-k)/b = a/b - k/b \le c/d - k/b \le c/d - k/d = (c-k)/d$.

Case 2: First, note that, $a/b \le c/d$ implies that $(b-a)/b \ge (d-c)/d$. Then, we obtain

$$\frac{a-k}{b-k} = \frac{a}{b} - \frac{k(b-a)}{b(b-k)} \le \frac{c}{d} - \frac{k(b-a)}{b(b-k)} \le \frac{c}{d} - \frac{k(d-c)}{d(b-k)} \le \frac{c}{d} - \frac{k(d-c)}{d(d-k)} = \frac{c-k}{d-k}$$ ❑

A2. Proofs of the Lemmas

Lemma 1. $\Pr(t[S]=\sigma \mid K_{\sigma|t}(\ell) \wedge K_{\sigma|u}(k) \wedge K_{\sigma|v,t}(m), \mathbf{D}^*) = 1 / (NR + 1)$, *where*

$$NR = \frac{\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \wedge (\wedge_{i\in[1,m]} \sigma \notin v_i[S]) \mid K_{\sigma|u}(k), \mathbf{D}^*)}{\Pr(\sigma \in t[S] \mid K_{\sigma|u}(k), \mathbf{D}^*)} .$$

Proof: Note that $K_{\sigma|t}(\ell) = (\wedge_{i\in[1,\ell]} (x_i \in t[S] \to \sigma \in t[S])$ and $K_{\sigma|v,t}(m) = (\wedge_{i\in[1,m]} (\sigma \in v_i[S] \to \sigma \in t[S])$. Let A denote $\sigma \in t[S]$; $A_1, \ldots, A_\ell$ denote $x_1 \in t[S], \ldots, x_\ell \in t[S]$; and $A_{\ell+1}, \ldots, A_{\ell+m}$, denote $\sigma \in v_1[S], \ldots, \sigma \in v_m[S]$.

$$\begin{aligned}
&\Pr(\sigma \in t[S] \mid K_{\sigma|t}(\ell) \wedge K_{\sigma|u}(k) \wedge K_{\sigma|v,t}(m), \mathbf{D}^*) \\
&= \Pr(A \mid (\wedge_{i\in[1,\ell+m]} (A_i \to A)) \wedge K_{\sigma|u}(k), \mathbf{D}^*) \\
&= \frac{\Pr(A \wedge (\wedge_{i\in[1,\ell+m]}(\neg A_i \vee A)) \mid K_{\sigma|u}(k), \mathbf{D}^*)}{\Pr(\wedge_{i\in[1,\ell+m]}(\neg A_i \vee A) \mid K_{\sigma|u}(k), \mathbf{D}^*)} \qquad (A_i \to A \text{ is equivalent to } \neg A_i \vee A) \\
&= \frac{\Pr(A \mid K_{\sigma|u}(k), \mathbf{D}^*)}{\Pr((\neg A \vee A) \wedge (\wedge_{i\in[1,\ell+m]}(\neg A_i \vee A)) \mid K_{\sigma|u}(k), \mathbf{D}^*)}
\end{aligned}$$

$$= \frac{\Pr(A \mid K_{\sigma|u}(k), \mathbf{D}^*)}{\Pr((\neg A \wedge (\wedge_{i\in[1,\ell+m]} \neg A_i)) \vee A) \mid K_{\sigma|u}(k), \mathbf{D}^*)} \qquad \text{(distributive law)}$$

$$= \frac{\Pr(A \mid K_{\sigma|u}(k), \mathbf{D}^*)}{\Pr(\neg A \wedge (\wedge_{i\in[1,\ell+m]} \neg A_i) \mid K_{\sigma|u}(k), \mathbf{D}^*) + \Pr(A \mid K_{\sigma|u}(k), \mathbf{D}^*)} \qquad (A \text{ and } (\neg A \wedge B) \text{ do not overlap})$$

$$= \frac{1}{\dfrac{\Pr(\neg A \wedge (\wedge_{i\in[1,\ell+m]} \neg A_i) \mid K_{\sigma|u}(k), \mathbf{D}^*)}{\Pr(A \mid K_{\sigma|u}(k), \mathbf{D}^*)} + 1}$$

Note that the derivation does not depend on whether we consider the SVPI case or the MVPI case. ❑

A3. Correctness of the Computation Formulas

In this section, we provide intuition on how the computation formulas are derived and also show the correctness of the formula. We use the following convention and notation:

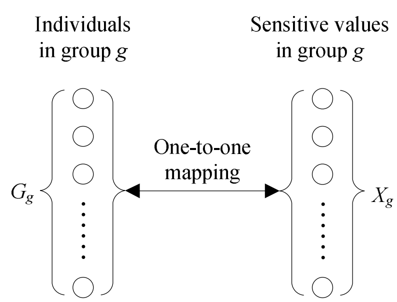
- σ is the target sensitive value (a specific value, not a variable).
- t is the target individual (a variable).
- u_i, v_i are variables ranging over individuals.
- x_i, y_i are variables ranging over sensitive values.
- f, g are (the indices of) QI-groups.
- n_g denotes the number of distinct individuals in QI-group g.
- $\#\sigma_g$ denotes the number of the occurrences of σ (the target sensitive value) in QI-group g.
- $s_{g(1)}, \ldots, s_{g(\ell)}$ denote the ℓ most frequent sensitive values in QI-group g with σ removed (i.e., $\sigma \neq s_{g(i)}$, for all i). $\#s_{g(i)}$ denotes the number of occurrences of $s_{g(i)}$ in QI-group g.
- $\#s_{g(1..\ell)}$ is shorthand for $\sum_{i\in[1,\ell]} \#s_{g(i)}$.
- $\Pr(E \mid K, g)$ is shorthand for $\Pr(E \mid K, \mathbf{D}^*)$ such that all the individuals in expressions E and K are in QI-group g.

Consider release candidate $\mathbf{D}^* = \{(G_1, X_1), \ldots (G_B, X_B)\}$. We assume each QI-group that contains σ is large enough to contain $t, u_1, \ldots, u_k$ and $v_1, \ldots, v_m$. Otherwise, the breach probability is simply 1, which is a straightforward boundary case.

A3.1 Case of Single Value per Individual

In the SVPI case, each individual has exactly one sensitive value in the original dataset.

Intuition: We now describe how QI-group g is reconstructed. As shown in the following figure, a reconstruction of QI-group g is a one-to-one mapping between G_g and X_g. Intuitively, a reconstruction can be thought of as drawing balls from a bag of n_g balls (or sensitive values), in which $\#\sigma_g$ balls are labeled σ and $\#s_{g(i)}$ balls are labeled $s_{g(i)}$. We now pick a ball for individual t. It can be easily seen that $\Pr(\sigma \in t[S] \mid g) = \#\sigma_g / n_g$, which is the probability that the chosen ball has label σ in one draw.



***$T_\sigma(g, \ell, k)$*:** Recall that

$$T_\sigma(g,\ell,k) = \min_{t,x_i,u_i,y_i} \frac{\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \mid \wedge_{i\in[1,k]} y_i \in u_i[S], g)}{\Pr(\sigma \in t[S] \mid \wedge_{i\in[1,k]} y_i \in u_i[S], g)}.$$

The minimization is about how to set $x_1, \ldots, x_\ell$ and $y_1, \ldots, y_k$. The setting of t and $u_1, \ldots, u_k$ does not affect the above probabilities as long as $t, u_1, \ldots, u_k$ are distinct individuals. To minimize $T_\sigma(g, \ell, k)$, we set $t, u_1, \ldots, u_k$ to be distinct individuals, set $x_1, \ldots, x_\ell$ to $s_{g(1)}, \ldots, s_{g(\ell)}$ (the ℓ

most frequent sensitive values other than σ), and set $y_1, \ldots, y_k$ to any sensitive values other than $\sigma, s_{g(1)}, \ldots, s_{g(\ell)}$. We discuss why this gives the minimum later. Under this setting, the denominator in the definition of $T_\sigma(g, \ell, k)$ is

$$\Pr(\sigma \in t[S] \mid \wedge_{i\in[1,k]}\, y_i \in u_i[S], g) = \#\sigma_g \,/\, (n_g - k),$$

which is the probability of choosing a ball with label σ from a bag of $(n_g - k)$ balls, in which $\#\sigma_g$ are labeled σ. Because $y_1 \in u_1[S], \ldots, y_k \in u_k[S]$ are given and $y_i \neq \sigma$, we removed k balls not with label σ from the bag. The numerator of $T_\sigma(g, \ell, k)$ is

$$\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]}\, s_{g(i)} \notin t[S]) \mid \wedge_{i\in[1,k]}\, y_i \in u_i[S], g) = (n_g - \#\sigma_g - \#s_{g(1..\ell)} - k) \,/\, (n_g - k),$$

which is the probability of choosing a ball with a label $\notin \{\sigma, s_{g(1)}, \ldots, s_{g(\ell)}\}$ (because of $\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]}\, s_{g(i)} \notin t[S])$) from a bag of $(n_g - k)$ balls, in which $(n_g - \#\sigma_g - \#s_{g(1..\ell)} - k)$ have the acceptable labels. Note that k balls have been removed because of the knowledge about $u_1, \ldots, u_k$. Before removing the k balls, the number of acceptable balls is $(n_g - \#\sigma_g - \#s_{g(1..\ell)})$. Since $u_1, \ldots, u_k$ all have sensitive values with the acceptable labels, after removing the k balls (representing the sensitive values for $u_1, \ldots, u_k$), the number of acceptable balls become $(n_g - \#\sigma_g - \#s_{g(1..\ell)} - k)$.

It is easy to see that our setting minimizes the numerator and maximizes the denominator of $T_\sigma(g, \ell, k)$. If we change any x_i to be a less frequent sensitive value, then the numerator will increase. If we change any y_i to be in $\{\sigma, s_{g(1)}, \ldots, s_{g(\ell)}\}$, the numerator will increase. If $u_1, \ldots, u_k$ are not distinct, the numerator will increase and the denominator will decrease. Thus, we obtain

$$T_\sigma(g, \ell, k) = \frac{n_g - \#\sigma_g - \#s_{g(1..\ell)} - k}{\#\sigma_g}.$$

***$V_\sigma(g, m, k)$*:** Recall that

$$V_\sigma(g, m, k) = \min_{v_i, u_i, y_i} \Pr(\wedge_{i\in[1,m]}\, \sigma \notin v_i[S] \mid \wedge_{i\in[1,k]}\, y_i \in u_i[S], g).$$

The minimization is about how to set $y_1, \ldots, y_k$. The setting of $v_1, \ldots, v_m$ and $u_1, \ldots, u_k$ does not affect the above probability as long as $v_1, \ldots, v_m, u_1, \ldots, u_k$ are distinct individuals. Note that, by definition, $v_i \neq u_j$ for any i and j. To minimize $V_\sigma(g, m, k)$, we set $v_1, \ldots, v_m, u_1, \ldots, u_k$ to be distinct individuals, and set $y_1, \ldots, y_k$ to have any sensitive values other than σ. We discuss why this gives the minimum later. Then, by the definition of conditional probability, $\Pr(\alpha \wedge \beta \mid \gamma) = \Pr(\alpha \mid \gamma)\cdot\Pr(\beta \mid \alpha \wedge \gamma)$. Thus, $\Pr(\wedge_{i\in[1,m]}\, \alpha_i \mid \gamma) = \prod_{i\in[1,m]} \Pr(\alpha_i \mid (\wedge_{j\in[1,i-1]}\, \alpha_j) \wedge \gamma)$. We apply this to $V_\sigma(g, m, k)$, and obtain

$$\Pr(\wedge_{i\in[1,m]}\, \sigma \notin v_i[S] \mid \wedge_{i\in[1,k]}\, y_i \in u_i[S], g) = \prod_{i\in[1,m]} \Pr(\sigma \notin v_i[S] \mid (\wedge_{j\in[1,i-1]}\, \sigma \notin v_j[S]) \wedge (\wedge_{i\in[1,k]}\, y_i \in u_i[S]), g).$$

Thus, $V_\sigma(g, m, k) = \prod_{i\in[0,m-1]} \dfrac{n_g - \#\sigma_g - k - i}{n_g - k - i}$,

which is the probability of choosing m balls with labels $\neq \sigma$ from a bag of $(n_g - k)$ balls, in which $(n_g - \#\sigma_g - k)$ are not labeled σ. The bag has $(n_g - k)$ balls with $(n_g - \#\sigma_g - k)$ not labeled σ because of $(\wedge_{i\in[1,k]}\, u_i[S] = y_i)$, where $y_i \neq \sigma$. Thus, $\Pr(\sigma \notin v_1[S] \mid (\wedge_{i\in[1,k]}\, y_i \in u_i[S]), g) = (n_g - \#\sigma_g - k)/(n_g - k)$, which is the probability that the first chosen ball is not labeled σ. Similarly, $\Pr(\sigma \notin v_2[S] \mid (\sigma \notin v_1[S]) \wedge (\wedge_{i\in[1,k]}\, y_i \in u_i[S]), g) = (n_g - \#\sigma_g - k - 1)/(n_g - k - 1)$, which is the probability that the second chosen ball is not labeled σ given the fact that the first ball is not labeled σ. If we keep doing so, we obtain the above formula for $V_\sigma(g, m, k)$.

It can be easily seen that our setting gives the minimum. If we change any y_i to σ, then $V_\sigma(g, m, k)$ will increase.

***$minNR_\sigma(g, \ell, k, m)$*:** Recall that $minNR_\sigma(g, \ell, k, m) = \min_{t, v_i, x_i, u_i, y_i} NR$ subject to that $t, u_1, \ldots, u_k$ and $v_1, \ldots, v_m$ are all in QI-group g, where

$$NR = \frac{\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]}\, x_i \notin t[S]) \wedge (\wedge_{i\in[1,m]}\, \sigma \notin v_i[S]) \mid \wedge_{i\in[1,k]}\, y_i \in u_i[S], g)}{\Pr(\sigma \in t[S] \mid \wedge_{i\in[1,k]}\, y_i \in u_i[S], g)}$$

By the definition of conditional probability, $\Pr(\alpha \wedge \beta \mid \gamma) = \Pr(\alpha \mid \gamma)\cdot\Pr(\beta \mid \alpha \wedge \gamma)$. By applying this to the numerator of NR, we obtain $NR = A\cdot B$, where

$$A = \frac{\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]}\, x_i \notin t[S]) \mid \wedge_{i\in[1,k]}\, y_i \in u_i[S], g)}{\Pr(\sigma \in t[S] \mid \wedge_{i\in[1,k]}\, y_i \in u_i[S], g)},$$

$$B = \Pr(\wedge_{i\in[1,m]}\, \sigma \notin v_i[S]) \mid \sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]}\, x_i \notin t[S]) \wedge (\wedge_{i\in[1,k]}\, y_i \in u_i[S]), g).$$

The minimization is about how to set $x_1, \ldots, x_\ell$ and $y_1, \ldots, y_k$. The setting of $t, u_1, \ldots, u_k$ and $v_1, \ldots, v_m$ does not affect the probabilities as long as t, the u_i's and the v_i's are distinct individuals. To minimize NR, we set $t, u_1, \ldots, u_k$, and $v_1, \ldots, v_m$ to be distinct individuals, set $x_1, \ldots, x_\ell$ to $s_{g(1)}, \ldots, s_{g(\ell)}$, and set $y_1, \ldots, y_k$ to any sensitive values other than $\sigma, s_{g(1)}, \ldots, s_{g(\ell)}$. Note that, in this setting, A is the same as $T_\sigma(g, \ell, k)$. Thus, A is minimized. Now, consider B. Note that, in this setting, we can rewrite B as

$$B = \Pr(\wedge_{i\in[1,m]}\, \sigma \notin v_i[S]) \mid (t[S], u_1[S], \ldots, u_k[S] \notin \{\sigma, s_{g(1)}, \ldots, s_{g(\ell)}\}), g).$$

Thus, similar to the discussion of $V_\sigma(g, m, k)$,

$$B = \prod_{i\in[0,m-1]} \frac{n_g - \#\sigma_g - k - 1 - i}{n_g - k - 1 - i} = V_\sigma(g, m, k+1),$$

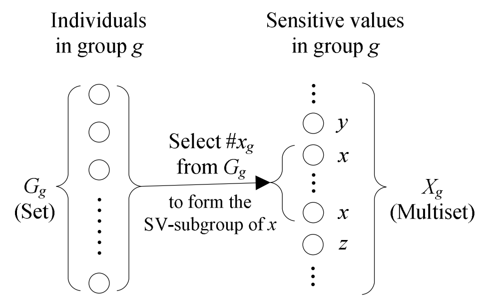
which is the probability of choosing m balls with labels $\neq \sigma$ from a bag of (n_g-k-1) balls, in which $(n_g-\#\sigma_g-k-1)$ are not labeled σ. Note that because of the knowledge of k+1 individuals (t, u_1, …, u_k), k+1 balls have been removed from the bag. The removed balls are not labeled σ in our setting. It can be easily seen that our setting minimizes B. Since our setting minimizes both A and B, we obtain

$$minNR_\sigma(g, \ell, k, m) = T_\sigma(g, \ell, k) \cdot V_\sigma(g, m, k+1).$$

A3.2 Case of Multiple Value per Individual − Set Semantics

In the MVPI-Set case, each individual has a set of sensitive values in the original dataset.

Intuition: We now describe how QI-group g is reconstructed. By Proposition 2, within each QI-group g, for each distinct sensitive value $x \in X_g$, we reconstruct the set of the individuals having sensitive value x independently. As shown in the following figure, $\#x_g$ denote the number of occurrences of x in X_g. We select $\#x_g$ individuals from G_g *without replacement*; i.e., each individual can only be selected once. We call the set of the individuals selected to have sensitive value x in QI-group g the "sensitive value subgroup" (or SV-subgroup) of x in QI-group g. We can reconstruct each SV-subgroup independently because the fact that individual u has value x does not prevent u from having other sensitive values (this is not true in the SVPI case). It can be easily seen that $\Pr(\sigma \in t[S] \mid g) = \#\sigma_g / n_g$, which is the probability that t is selected in the process of selecting $\#\sigma_g$ individuals from n_g to have sensitive value σ.



***T$_\sigma$(g, ℓ, k)*:** Recall that

$$T_\sigma(g,\ell,k) = \min_{t,x_i,u_i} \frac{\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g)}{\Pr(\sigma \in t[S] \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g)}.$$

The minimization is about how to set x_1, …, x_ℓ. The setting of t and u_1, …, u_k does not affect the above probabilities as long as t, u_1, …, u_k are distinct individuals. To minimize $T_\sigma(g, \ell, k)$, we set t, u_1, …, u_k to be distinct individuals, set x_1, …, x_ℓ to $s_{g(1)}$, …, $s_{g(\ell)}$ (the ℓ most frequent sensitive values other than σ). We discuss why this gives the minimum later. Under this setting, the denominator above becomes

$$\Pr(\sigma \in t[S] \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g) = \#\sigma_g / (n_g - k),$$

which is the probability that t is selected in the process of selecting $\#\sigma_g$ individuals from (n_g-k) individuals to have sensitive value σ. Because u_1, …, u_k are known not to have σ, they have been removed. By Proposition 2, the numerator of $T_\sigma(g, \ell, k)$ is

$$\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} s_{g(i)} \notin t[S]) \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g) = \Pr(\sigma \notin t[S] \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g) \cdot \prod_{i\in[1,\ell]} \Pr(s_{g(i)} \notin t[S] \mid g)$$

$$= [1 - \#\sigma_g/(n_g-k)] \cdot \prod_{i\in[1,\ell]} (1 - \#s_{g(i)}/n_g)$$

$$= [(n_g-\#\sigma_g-k)/(n_g-k)] \cdot \prod_{i\in[1,\ell]} (n_g-\#s_{g(i)})/n_g.$$

It is easy to see that our setting minimizes the numerator and maximizes the denominator of $T_\sigma(g, \ell, k)$. If we change any x_i to be a less frequent sensitive value, then the numerator will increase. If u_1, …, u_k are not distinct, the numerator will increase and the denominator will decrease. Thus, we obtain

$$T_\sigma(g, \ell, k) = \frac{n_g - \#\sigma_g - k}{\#\sigma_g} \cdot \prod_{i\in[1,\ell]} \frac{n_g - \#s_{g(i)}}{n_g}.$$

***V$_\sigma$(g, m, k)*:** Recall that

$$V_\sigma(g, m, k) = \min_{v_i, u_i} \Pr(\wedge_{i\in[1,m]} \sigma \notin v_i[S] \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g).$$

To minimize $V_\sigma(g, m, k)$, we just set $v_1, \ldots, v_m, u_1, \ldots, u_k$ to be distinct individuals. By the definition of conditional probability, $\Pr(\alpha \wedge \beta \mid \gamma) = \Pr(\alpha \mid \gamma) \cdot \Pr(\beta \mid \alpha \wedge \gamma)$. Thus, $\Pr(\wedge_{i\in[1,m]} \alpha_i \mid \gamma) = \prod_{i\in[1,m]} \Pr(\alpha_i \mid (\wedge_{j\in[1,i-1]} \alpha_j) \wedge \gamma)$. We apply this to $V_\sigma(g, m, k)$, and obtain

$$\Pr(\wedge_{i\in[1,m]} \sigma\notin v_i[S] \mid \wedge_{i\in[1,k]} \sigma\notin u_i[S], g) = \prod_{i\in[1,m]} \Pr(\sigma\notin v_i[S] \mid (\wedge_{i\in[1,j-1]} \sigma\notin v_j[S]) \wedge (\wedge_{i\in[1,k]} \sigma\notin u_i[S]), g).$$

Thus, $V_\sigma(g, m, k) = \prod_{i\in[0,m-1]} \frac{n_g - \#\sigma_g - k - i}{n_g - k - i}$,

which is the probability that $v_1, \ldots, v_m$ are selected in the process of selecting $(n_g - \#\sigma_g - k)$ individuals from $(n_g - k)$ individuals to not have sensitive value σ. Because $u_1, \ldots, u_k$ are known not to have σ, they have been removed. It can be easily seen that our setting gives the minimum.

***minNR*$_\sigma$*(g, ℓ, k, m)*:** Recall that $minNR_\sigma(g, \ell, k, m) = \min_{t,\, v_i,\, x_i,\, u_i} NR$ subject to that $t, u_1, \ldots, u_k$ and $v_1, \ldots, v_m$ are all in QI-group g, where

$$NR = \frac{\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \wedge (\wedge_{i\in[1,m]} \sigma \notin v_i[S]) \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g)}{\Pr(\sigma \in t[S] \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g)}$$

By the definition of conditional probability, $\Pr(\alpha \wedge \beta \mid \gamma) = \Pr(\alpha \mid \gamma) \cdot \Pr(\beta \mid \alpha \wedge \gamma)$. By applying this to the numerator of NR, we obtain $NR = A \cdot B$, where

$$A = \frac{\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g)}{\Pr(\sigma \in t[S] \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g)},$$

$$B = \Pr(\wedge_{i\in[1,m]} \sigma \notin v_i[S]) \mid \sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \wedge (\wedge_{i\in[1,k]} \sigma \notin u_i[S]), g)$$

$$= \Pr(\wedge_{i\in[1,m]} \sigma \notin v_i[S]) \mid \sigma \notin t[S] \wedge (\wedge_{i\in[1,k]} \sigma \notin u_i[S]), g)\text{ , by Proposition 2.}$$

The minimization is about how to set $x_1, \ldots, x_\ell$. The setting of $t, u_1, \ldots, u_k$ and $v_1, \ldots, v_m$ does not affect the probabilities as long as t, the u_i's and the v_i's are distinct individuals. To minimize NR, we set $t, u_1, \ldots, u_k$, and $v_1, \ldots, v_m$ to be distinct individuals, and set $x_1, \ldots, x_\ell$ to $s_{g(1)}, \ldots, s_{g(\ell)}$. Note that, in this setting, A is the same as $T_\sigma(g, \ell, k)$. Thus, A is minimized. Now, consider B. Note that, in this setting, we can rewrite B as

$$B = \Pr(\wedge_{i\in[1,m]} \sigma \notin v_i[S]) \mid (t[S], u_1[S], \ldots, u_k[S] \notin \{\sigma\}), g)\ .$$

Thus, similar to the discussion of $V_\sigma(g, m, k)$,

$$B = \prod_{i\in[0,m-1]} \frac{n_g - \#\sigma_g - k - 1 - i}{n_g - k - 1 - i} = V_\sigma(g, m, k+1),$$

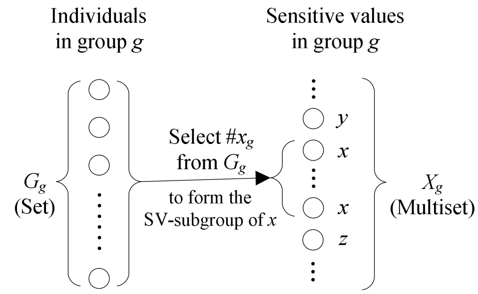
which is the probability that $v_1, \ldots, v_m$ are selected in the process of selecting $(n_g - \#\sigma_g - k - 1)$ individuals from $(n_g - k - 1)$ individuals to not have sensitive value σ. Because $t, u_1, \ldots, u_k$ are known not to have σ, they have been removed. It can be easily seen that our setting minimizes B. Since our setting minimizes both A and B, we obtain

$$minNR_\sigma(g, \ell, k, m) = T_\sigma(g, \ell, k) \cdot V_\sigma(g, m, k+1).$$

A3.3 Case of Multiple Value per Individual – Multiset Semantics

In the MVPI-Multiset case, each individual has a multiset of sensitive values in the original dataset.

Intuition: We now describe how QI-group g is reconstructed. By Proposition 2, within each QI-group g, for each distinct sensitive value $x \in X_g$, we reconstruct the multiset of the individuals having sensitive value x independently. As shown in the following figure, $\#x_g$ denote the number of occurrences of x in X_g. We select $\#x_g$ individuals from G_g *with replacement*; i.e., each individual can be selected many times. We call the multiset of the individuals selected to have sensitive value x in QI-group g the "sensitive value subgroup" (or SV-subgroup) of x in QI-group g. We can reconstruct each SV-subgroup independently because the fact that individual u has value x does not prevent u from having other sensitive values (this is not true in the SVPI case). It can be easily seen that $\Pr(\sigma\notin t[S] \mid g) = [(n_g - 1)/n_g]^{\#\sigma_g}$, which is the probability that t is not selected in the process of selecting an individual from n_g individuals, for $\#\sigma_g$ times. Thus, $\Pr(\sigma\in t[S] \mid g) = 1 - [(n_g - 1)/n_g]^{\#\sigma_g}$.



***$T_\sigma(g, \ell, k)$*:** Recall that

$$T_\sigma(g,\ell,k) = \min_{t,x_i,u_i} \frac{\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g)}{\Pr(\sigma \in t[S] \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g)}.$$

The minimization is about how to set $x_1, \ldots, x_\ell$. The setting of t and $u_1, \ldots, u_k$ does not affect the above probabilities as long as $t, u_1, \ldots, u_k$ are distinct individuals. To minimize $T_\sigma(g, \ell, k)$, we set $t, u_1, \ldots, u_k$ to be distinct individuals, set $x_1, \ldots, x_\ell$ to $s_{g(1)}, \ldots, s_{g(\ell)}$ (the ℓ most frequent sensitive values other than σ). We discuss why this gives the minimum later. Under this setting, the denominator above becomes

$$\Pr(\sigma \in t[S] \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g) = 1 - [(n_g - k - 1)/(n_g - k)]^{\#\sigma_g},$$

which is one minus the probability that t is not selected in the process of selecting an individual from (n_g-k) individuals, for $\#\sigma_g$ times. Because $u_1, \ldots, u_k$ are known not to have σ, they have been removed. By Proposition 2, the numerator of $T_\sigma(g, \ell, k)$ is

$$\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} s_{g(i)} \notin t[S]) \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g) = \Pr(\sigma \notin t[S] \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g) \cdot \prod_{i\in[1,\ell]} \Pr(s_{g(i)} \notin t[S] \mid g)$$

$$= [(n_g - k - 1)/(n_g - k)]^{\#\sigma_g} \cdot \prod_{i\in[1,\ell]} [(n_g - 1)/n_g]^{\#s_{g(i)}}$$

$$= [(n_g - k - 1)/(n_g - k)]^{\#\sigma_g} \cdot [(n_g - 1)/n_g]^{\#s_{g(1..\ell)}}$$

It is easy to see that our setting minimizes the numerator and maximizes the denominator of $T_\sigma(g, \ell, k)$. If we change any x_i to be a less frequent sensitive value, then the numerator will increase. If $u_1, \ldots, u_k$ are not distinct, the numerator will increase and the denominator will decrease. Thus, we obtain

$$T(g, \ell, k) = \frac{[(n_g - k - 1)/(n_g - k)]^{\#\sigma_g}}{1 - [(n_g - k - 1)/(n_g - k)]^{\#\sigma_g}} \cdot [(n_g - 1)/n_g]^{\#s_{g(1..\ell)}}.$$

***$V_\sigma(g, m, k)$*:** Recall that

$$V_\sigma(g, m, k) = \min_{v_i, u_i} \Pr(\wedge_{i\in[1,m]} \sigma \notin v_i[S] \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g).$$

To minimize $V_\sigma(g, m, k)$, we just set $v_1, \ldots, v_m, u_1, \ldots, u_k$ to be distinct individuals. Thus, we obtain

$$V_\sigma(g, m, k) = \left(\frac{n_g - k - m}{n_g - k}\right)^{\#\sigma_g},$$

which is the probability that all $v_1, \ldots, v_m$ are not selected in the process of selecting an individual from (n_g-k) individuals, for $\#\sigma_g$ times. Because $u_1, \ldots, u_k$ are known not to have σ, they have been removed. It can be easily seen that our setting gives the minimum.

***$minNR_\sigma(g, \ell, k, m)$*:** Recall that $minNR_\sigma(g, \ell, k, m) = \min_{t, v_i, x_i, u_i} NR$ subject to that $t, u_1, \ldots, u_k$, and $v_1, \ldots, v_m$ are all in QI-group g, where

$$NR = \frac{\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \wedge (\wedge_{i\in[1,m]} \sigma \notin v_i[S]) \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g)}{\Pr(\sigma \in t[S] \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g)}$$

By the definition of conditional probability, $\Pr(\alpha \wedge \beta \mid \gamma) = \Pr(\alpha \mid \gamma) \cdot \Pr(\beta \mid \alpha \wedge \gamma)$. By applying this to the numerator of NR, we obtain $NR = A \cdot B$, where

$$A = \frac{\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g)}{\Pr(\sigma \in t[S] \mid \wedge_{i\in[1,k]} \sigma \notin u_i[S], g)},$$

$$B = \Pr(\wedge_{i\in[1,m]} \sigma \notin v_i[S]) \mid \sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \wedge (\wedge_{i\in[1,k]} \sigma \notin u_i[S]), g)$$

$$= \Pr(\wedge_{i\in[1,m]} \sigma \notin v_i[S]) \mid \sigma \notin t[S] \wedge (\wedge_{i\in[1,k]} \sigma \notin u_i[S]), g)$$, by Proposition 2.

The minimization is about how to set $x_1, \ldots, x_\ell$. The setting of t, $u_1, \ldots, u_k$ and $v_1, \ldots, v_m$ does not affect the probabilities as long as t, the u_i's and the v_i's are distinct individuals. To minimize *NR*, we set t, $u_1, \ldots, u_k$, and $v_1, \ldots, v_m$ to be distinct individuals, and set $x_1, \ldots, x_\ell$ to $s_{g(1)}, \ldots, s_{g(\ell)}$. Note that, in this setting, A is the same as $T_\sigma(g, \ell, k)$. Thus, A is minimized. Now, consider B. Note that, in this setting, we can rewrite B as

$$B = \Pr(\wedge_{i\in[1,m]} \sigma \notin v_i[S]) \mid (t[S], u_1[S], \ldots, u_k[S] \notin \{\sigma\}), g) \ .$$

Thus, similar to the discussion of $V_\sigma(g, m, k)$,

$$B = \left(\frac{n_g - k - 1 - m}{n_g - k - 1}\right)^{\#\sigma_g} = V_\sigma(g, m, k+1),$$

which the probability that all $v_1, \ldots, v_m$ are not selected in the process of selecting an individual from $(n_g - k - 1)$ individuals, for $\#\sigma_g$ times. Because t, $u_1, \ldots, u_k$ are known not to have σ, they have been removed. It can be easily seen that our setting minimizes B. Since our setting minimizes both A and B, we obtain

$$minNR_\sigma(g, \ell, k, m) = T_\sigma(g, \ell, k) \cdot V_\sigma(g, m, k+1).$$

A4. Formula Derivation

Derivation 1. *Consider the following condition*:

- *QI-group g contains* t, $u_1, \ldots, u_p$ *and* $v_1, \ldots, v_h$, *and*
- *QI-group f contains the rest* $(k-p)$ *of the* u_i*'s and the rest* $(m-h)$ *of the* v_i*'s, and*
- $g \neq f$.

The minimum negated ratio subject to this condition is $T_\sigma(g, \ell, p)\cdot V_\sigma(g, h, p+1)\cdot V_\sigma(f, m-h, k-p)$. *As a special case, if* $h = 0$, *the minimum negated ratio subject to this condition is* $T_\sigma(g, \ell, p)\cdot V_\sigma(f, m, k-p)$.

Proof: Under this condition, the negated ratio is

$$\frac{\Pr([\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \wedge (\wedge_{i\in[1,h]} \sigma \notin v_i[S])] \wedge [\wedge_{i\in[1,m-h]} \sigma \notin v_i[S]] \mid K_{\sigma\backslash u}(p) \wedge K_{\sigma\backslash u}(k-p), \mathbf{D}^*)}{\Pr(\sigma \in t[S] \mid K_{\sigma\backslash u}(p) \wedge K_{\sigma\backslash u}(k-p), \mathbf{D}^*)},$$

where $[\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \wedge (\wedge_{i\in[1,h]} \sigma \notin v_i[S])]$, $K_{\sigma\backslash u}(p)$ are expressions only involving individuals in QI-group g, and $(\wedge_{i\in[1,m-h]} \sigma \notin v_i[S])$ and $K_{\sigma\backslash u}(k-p)$ are expressions only involving individuals in QI-group f.

Thus, by Proposition 1, the minimum negated ratio is

$$\min \frac{\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \wedge (\wedge_{i\in[1,h]} \sigma \notin v_i[S]) \mid K_{\sigma\backslash u}(p), g) \cdot \Pr(\wedge_{i\in[1,m-h]} \sigma \notin v_i[S] \mid K_{\sigma\backslash u}(k-p), f)}{\Pr(\sigma \in t[S] \mid K_{\sigma\backslash u}(p), g)}$$

$$= (\min \frac{\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \wedge (\wedge_{i\in[1,h]} \sigma \notin v_i[S]) \mid K_{\sigma\backslash u}(p), g)}{\Pr(\sigma \in t[S] \mid K_{\sigma\backslash u}(p), g)}) \cdot (\min \Pr(\wedge_{i\in[1,m-h]} \sigma \notin v_i[S] \mid K_{\sigma\backslash u}(k-p), f))$$

$= minNR_\sigma(g, \ell, p, h) \cdot V_\sigma(f, m-h, k-p)$

$= T_\sigma(g, \ell, p)\cdot V_\sigma(g, h, p+1)\cdot V_\sigma(f, m-h, k-p)$. ❑

A5. Proofs of the Theorems

Theorem 1. $K_{\sigma\backslash u}(k)$ *and* $K_{\sigma\backslash v,t}(m)$ *are both 1-group congregated, in all the three cases* (*SVPI, MVPI-Set and MVPI-Multiset*).

Proof: Theorem 1 states that the negated ratio (the *NR* in Lemma 1) is minimized when all the individuals $u_1, \ldots, u_k$ (in $K_{\sigma\backslash u}(k)$) are in one QI-group and all the individuals $v_1, \ldots, v_m$ (in $K_{\sigma\backslash v,t}(m)$) are in one QI-group.

We will prove Theorem 1 by induction on the number B of QI-groups. Basically, we consider how to distribute t, $u_1, \ldots, u_k$ and $v_1, \ldots, v_m$ into QI-groups in order to minimize the negated ratio.

In the following proof, we only consider the case where the breach probability is less than 1 (i.e., the minimum negated ratio is greater than 0). That is to assume each QI-group that contains σ is large enough to contain t, $u_1, \ldots, u_k$ and $v_1, \ldots, v_m$. The boundary case (i.e., the breach probability is 1) is straightforward.

Base case: When $B = 1$, our claim trivially holds. Thus, we consider $B = 2$ as the base case. Without loss of generality, assume that when the negated ratio is minimized, the following two hold:

- QI-group g contains t, $u_1, \ldots, u_p$ and $v_1, \ldots, v_h$.
- QI-group f contains the rest $(k–p)$ of u_i's and $(m–h)$ of v_i's.

Our goal is to prove $h = 0$ or m, and $p = 0$ or k. By proposition 1, the negated ratio is

$$\frac{\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \wedge (\wedge_{i\in[1,m]} \sigma \notin v_i[S]) \mid K_{\sigma|u}(k), g, f)}{\Pr(\sigma \in t[S] \mid K_{\sigma|u}(k), g, f)}$$

$$= \frac{\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \wedge (\wedge_{i\in[1,h]} \sigma \notin v_i[S]) \mid K_{\sigma|u}(p), g)}{\Pr(\sigma \in t[S] \mid K_{\sigma|u}(p), g)} \cdot \Pr(\wedge_{i\in[h+1,m]} \sigma \notin v_i[S] \mid K_{\sigma|u}(k-p), f)\ .$$

Note that QI-group f does not contain t. Thus, all the expression about t are removed from the second part of the above formula. Since the two QI-groups are independent, we can minimize them separately. By the definition of $minNR_\sigma(g, \ell, k, m)$ and $V_\sigma(g, m, k)$ and the discussion in Section A3, we obtain that the minimum negated ratio is

$$minNR_\sigma(g, \ell, p, h) \cdot V_\sigma(f, m–h, k–p) = T_\sigma(g, \ell, p) \cdot V_\sigma(g, h, p+1) \cdot V_\sigma(f, m–h, k–p).$$

Now, we consider the three cases separately.

Base case in the SVPI case: We first consider the v_i's. Note that $T_\sigma(\ldots)$ does not involve the v_i's by definition. Let $\alpha_i = (n_g–\#\sigma_g–p–i) / (n_g–p–i)$ and $\beta_i = (n_f–\#\sigma_f–(k–p)–i+1) / (n_f–(k–p)–i+1)$. Then, we obtain

$$V_\sigma(g, h, p+1) \cdot V_\sigma(f, m–h, k–p) = (\textstyle\prod_{i\in[1,h]} \alpha_i)\cdot(\prod_{i\in[1,m-h]} \beta_i).$$

Note that here i start from 1, not 0. By Proposition 6, the above formula is minimized when $h = 0$ or m. Thus, all the v_i's are in one QI-group; i.e., $K_{\sigma|v,t}(m)$ is 1-group congregated.

We now consider the u_i's. If all the v_i's are in QI-group g, the minimum negated ratio becomes

$$T_\sigma(g, \ell, p)\cdot V_\sigma(g, m, p+1),$$

which is minimized when $p = k$. If all the v_i's are in QI-group f, for an appropriate choice of a, b, c and d (i.e., $a = n_g–\#\sigma_g–\#s_{g(1..\ell)}$, $b = \#\sigma_g$, $c = n_f–\#\sigma_f$, and $d = n_f$), the negated ratio becomes

$$T_s(g, \ell, p) \cdot V_s(f, m, k-p) = \frac{a-p}{b} \cdot \prod_{i\in[0,m-1]} \frac{c-i-(k-p)}{d-i-(k-p)}$$

By Proposition 8, the above is minimized when $p = 0$ or k. Thus, all the u_i's are in one QI-group; i.e., $K_{\sigma|u}(k)$ is 1-group congregated.

Base case in the MVPI-Set case: We first consider the v_i's. Note that $T_\sigma(\ldots)$ does not involve the v_i's by definition. The computation formula for $V_\sigma(g, m, k)$ in the MVPI-Set case is the same as that in the SVPI case. Thus, by the same argument, we conclude that the negated ratio is minimized when all the v_i's are in one QI-group; i.e., $K_{\sigma|v,t}(m)$ is 1-group congregated.

We now consider the u_i's. If all the v_i's are in QI-group g, the minimum negated ratio becomes

$$T_\sigma(g, \ell, p)\cdot V_\sigma(g, m, p+1),$$

which is minimized when $p = k$. If all the v_i's are in QI-group f, for an appropriate choice of a, b, c and d (i.e., $a = n_g–\#\sigma_g$, $b = \#\sigma_g\cdot[\prod_{i\in[1,\ell]} n_g/(n_g–\#s_{g(i)})]$, $c = n_f–\#\sigma_f$, and $d = n_f$), the negated ratio becomes

$$T_s(g, \ell, p) \cdot V_s(f, m, k-p) = \frac{a-p}{b} \cdot \prod_{i\in[0,m-1]} \frac{c-i-(k-p)}{d-i-(k-p)}$$

By Proposition 8, the above is minimized when $p = 0$ or k. Thus, all the u_i's are in one QI-group; i.e., $K_{\sigma|u}(k)$ is 1-group congregated.

Base case in the MVPI-Multiset case: We first consider the v_i's. Note that $T_\sigma(\ldots)$ does not involve the v_i's by definition. By an appropriate choice of a, b, c and d (i.e., $a = n_g–(p+1)$, $b = \#\sigma_g$, $c = n_f–(k–p)$, and $d = \#\sigma_f$), we obtain

$$V_\sigma(g, h, p+1) \cdot V_\sigma(f, m–h, k–p) = \left(\frac{a-h}{a}\right)^b \left(\frac{c-(m-h)}{c}\right)^d\ .$$

By Proposition 7, the above formula is minimized when $h = 0$ or m. Thus, all the v_i's are in one QI-group; i.e., $K_{\sigma|v,t}(m)$ is 1-group congregated.

We now consider the u_i's. If all the v_i's are in QI-group g, the minimum negated ratio becomes

$$T_\sigma(g, \ell, p)\cdot V_\sigma(g, m, p+1),$$

which is minimized when $p = k$. If all the v_i's are in QI-group f, for an appropriate choice of a, b, c, d, e and n (i.e., $a = \#\sigma_g$, $b = [(n_g-1)/n_g]^{\#s_{g(1..\ell)}}$, $c = n_f-m$, $d = n_f$, $e = \#\sigma_f$, and $n = n_g$), the negated ratio becomes

$$T_\sigma(g, \ell, p)\cdot V_\sigma(f, m, k-p) = \frac{[(n-p-1)/(n-p)]^a}{1-[(n-p-1)/(n-p)]^a}\cdot b\cdot\left(\frac{c-(k-p)}{d-(k-p)}\right)^e.$$

By Proposition 9, the above is minimized when $p = 0$ or k. Thus, all the u_i's are in one QI-group; i.e., $K_{\sigma|u}(k)$ is 1-group congregated.

Induction argument: Now assume Theorem 1 holds for $(B-1)$ QI-groups. We claim that it also holds for B QI-groups. We first consider the v_i's. Without loss of generality, assume the negated ratio is minimized when $v_1, \ldots, v_h$ are in the first $(B-1)$ QI-groups and the rest $(m-h)$ are in the Bth QI-group. By the induction assumption, $v_1, \ldots, v_h$ are in one QI-group, say g. Now, the v_i's can only be in two QI-groups. Similar to the argument in the base case, it is easy to show that $h = 0$ or m. Thus, all the v_i's are in one QI-group; i.e., $K_{\sigma|v,t}(m)$ is 1-group congregated.

By a similar argument, it is easy to show that all the u_i's are in one QI-group; i.e., $K_{\sigma|u}(k)$ is 1-group congregated. ❑

Theorem 2. This theorem has been proven in Section 5.1.

Theorem 3. This theorem is a straightforward application of Theorem 2.

Theorem 4. *If QI-groups* $g_1, \ldots, g_n$ *partition QI-group* q *in release candidate* $\mathbf{D}^*$, *then in the SVPI case*, *for any fixed* (ℓ, k, m), *the following hold*:

- $T_\sigma(q, \ell, k) \geq \min_{1\leq i\leq n} T_\sigma(g_i, \ell, k)$,
- $V_\sigma(q, m, k) \geq \min_{1\leq i\leq n} V_\sigma(g_i, m, k)$,
- $minNR_\sigma(q, \ell, k, m) \geq$ *the minimum of*:
  - (c) $\min_{1\leq i\leq n} minNR_\sigma(g_i, \ell, k, m)$,
  - (d) $(\min_{1\leq i\leq n} T_\sigma(g_i, \ell, k)) \cdot (\min_{1\leq i\leq n} V_\sigma(g_i, m, 0))$.

Proof: We prove this theorem by considering q is partitioned into two QI-groups g and f. By a simple induction argument, it is easy to see that this theorem also holds when q is partitioned into n QI-groups.

Because group g and group f partition group q, we have: (the notation is defined in Section A3)

- $n_q = n_g + n_f$ and $\#\sigma_q = \#\sigma_g + \#\sigma_f$.
- $\#s_{q(1..\ell)} \leq \#s_{g(1..\ell)} + \#s_{f(1..\ell)}$. The see this, let $n_g(s_{q(i)})$ and $n_f(s_{q(i)})$ denote the numbers of occurrences of sensitive value $s_{q(i)}$ in group g and group f, respectively. Thus, $\#s_{q(i)} = n_g(s_{q(i)}) + n_f(s_{q(i)})$. Then, $\#s_{q(1..\ell)} = \sum_{i\in[1,\ell]} \#s_{q(i)} = \sum_{i\in[1,\ell]} [n_g(s_{q(i)}) + n_f(s_{q(i)})] = \sum_{i\in[1,\ell]} n_g(s_{q(i)}) + \sum_{i\in[1,\ell]} n_f(s_{q(i)}) \leq \sum_{i\in[1,\ell]} \#s_{g(i)} + \sum_{i\in[1,\ell]} \#s_{f(i)} = \#s_{g(1..\ell)} + \#s_{f(1..\ell)}$, because with sensitive value σ removed, $s_{g(1)}, \ldots, s_{g(\ell)}$ (and $s_{f(1)}, \ldots, s_{f(\ell)}$) are the ℓ most frequent sensitive values in group g (and group f).

Consider part 1.

$$T_\sigma(q,\ell,k) = \frac{n_q - \#\sigma_q - \#s_{q(1..\ell)} - k}{\#\sigma_q} \geq \frac{n_g + n_f - (\#\sigma_g + \#\sigma_f) - (\#s_{g(1..\ell)} + \#s_{g(1..\ell)}) - k}{\#\sigma_g + \#\sigma_f}$$

Let $a_g = n_g-\#\sigma_g-\#s_{g(1..\ell)}$ and $a_f = n_g-\#\sigma_g-\#s_{g(1..\ell)}$. Then, by Proposition 10, we obtain

$$T_\sigma(q,\ell,k) \geq \frac{a_g + a_f - k}{\#\sigma_g + \#\sigma_f} = \frac{a_g + a_f}{\#\sigma_g + \#\sigma_f} - \frac{k}{\#\sigma_g + \#\sigma_f} \geq \min\left\{\frac{a_g}{\#\sigma_g}, \frac{a_f}{\#\sigma_f}\right\} - \frac{k}{\#\sigma_g + \#\sigma_f} \geq \min\left\{\frac{a_g - k}{\#\sigma_g}, \frac{a_f - k}{\#\sigma_f}\right\}$$

Note that $(a_g-k)/\#\sigma_g = T_\sigma(g, \ell, k)$ and $(a_f-k)/\#\sigma_f = T_\sigma(f, \ell, k)$. Thus, we complete the proof of part 1.

Consider part 2. Let $b_g = n_g-\#\sigma_g$ and $b_f = n_f - \#\sigma_f$.

$$V_\sigma(q,m,k) = \prod_{i\in[0,m-1]} \frac{n_q - \#\sigma_q - k - i}{n_q - k - i} = \prod_{i\in[0,m-1]} \frac{n_g + n_f - (\#\sigma_g + \#\sigma_f) - k - i}{n_g + n_f - k - i} = \prod_{i\in[0,m-1]} \frac{b_g + b_f - i - k}{n_g + n_f - i - k}.$$

By Proposition 10, we obtain

$$\frac{b_g + b_f}{n_g + n_f} \geq \min\left\{\frac{b_g}{n_g}, \frac{b_f}{n_f}\right\}.$$

Without loss of generality, assume $b_g/n_g \leq b_f/n_f$. Now, our goal is to prove $V_\sigma(q, m, k) \geq V_\sigma(g, m, k)$. By Proposition 11, we complete the proof of part 2.

$$\frac{b_g + b_f - k - i}{n_g + n_f - k - i} \geq \frac{b_g - k - i}{n_g - k - i} \text{ and}$$

$$V_\sigma(q,m,k) = \prod_{i\in[0,m-1]} \frac{b_g + b_f - i - k}{n_g + n_f - i - k} \geq \prod_{i\in[0,m-1]} \frac{b_g - i - k}{n_g - i - k} = V_\sigma(g,m,k)\ .$$

Consider part 3. $minNR_\sigma(q, \ell, k, m) = T_\sigma(q, \ell, k)\cdot V_\sigma(q, m, k+1)$. We use the previously defined a_g, a_f, b_g, b_f.

$$minNR_\sigma(q, \ell, k, m) = T_\sigma(q,\ell,k)\cdot \prod_{i\in[0,m-1]} \frac{b_g + b_f - i - k - 1}{n_g + n_f - i - k - 1}$$

From part 1, we know $T_\sigma(q, \ell, k) \geq \min\{T_\sigma(g, \ell, k), T_\sigma(f, \ell, k)\}$. Without loss of generality, we assume $T_\sigma(g, \ell, k) \leq T_\sigma(f, \ell, k)$. By Proposition 10, we obtain

$$\frac{(b_g - k - 1) + b_f}{(n_g - k - 1) + n_f} \geq \min\left\{\frac{b_g - k - 1}{n_g - k - 1}, \frac{b_f}{n_f}\right\}.$$

If $(b_g-k-1)/(n_g-k-1) \leq b_f/n_f$, then, by Proposition 11, we obtain

$$minNR_\sigma(q, \ell, k, m) \geq T_\sigma(g,\ell,k)\cdot \prod_{i\in[0,m-1]} \frac{(b_g - k - 1) + b_f - i}{(n_g - k - 1) + n_f - i} \geq T_\sigma(g,\ell,k)\cdot \prod_{i\in[0,m-1]} \frac{(b_g - k - 1) - i}{(n_g - k - 1) - i}\ .$$

The last part of the above is actually $T_\sigma(g, \ell, k)\cdot V_\sigma(g, m, k+1) = minNR_\sigma(g, \ell, k, m)$.

Now, if $b_f/n_f \leq (b_g-k-1)/(n_g-k-1)$, then by Proposition 11, we obtain

$$minNR_\sigma(q, \ell, k, m) \geq T_\sigma(g,\ell,k)\cdot \prod_{i\in[0,m-1]} \frac{(b_g - k - 1) + b_f - i}{(n_g - k - 1) + n_f - i} \geq T_\sigma(g,\ell,k)\cdot \prod_{i\in[0,m-1]} \frac{b_f - i}{n_f - i} = T_\sigma(g, \ell, k)\cdot V_\sigma(f, m, 0).$$ ❑

Corollary. *In the SVPI case, the basic 3D privacy criterion and the skyline privacy criterion are monotonic.*

Proof: Let $\mathbf{D}^*_1$ and $\mathbf{D}^*_2$ be two release candidates such that $\mathbf{D}^*_1 \preccurlyeq \mathbf{D}^*_2$. Consider skyline point (ℓ, k, m, c). Assume $\mathbf{D}^*_1$ is safe under (ℓ, k, m, c); i.e. $\max\{\Pr(\sigma\in t[S] \mid \mathscr{L}_{t,\sigma}(\ell, k, m), \mathbf{D}^*_1)\} < c$. Because each QI-group q in $\mathbf{D}^*_2$ is the union of a set $g_1, \ldots, g_n$ of QI-groups of $\mathbf{D}^*_1$ that partition QI-group q, by Theorem 4 and Theorem 2, we conclude that the negated ratio on $\mathbf{D}^*_1$ is smaller than or equal to that on $\mathbf{D}^*_2$. Thus,

$$c > \max\{\Pr(\sigma\in t[S] \mid \mathscr{L}_{t,\sigma}(\ell, k, m), \mathbf{D}^*_1)\} \geq \max\{\Pr(\sigma\in t[S] \mid \mathscr{L}_{t,\sigma}(\ell, k, m), \mathbf{D}^*_2)\},$$

which means $\mathbf{D}^*_2$ is also safe. Similarly, for a set of skyline points, that fact that $\mathbf{D}^*_1$ is safe implies that $\mathbf{D}^*_2$ is also safe. ❑

Theorem 5. *The SkylineAnonymize algorithm produces a safe release candidate. In the SVPI case, the release candidate is minimal.*

Proof: First, we assume that initial release candidate that takes the entire dataset as a single QI-group is safe. Otherwise, there is nothing that can be released.

Second, note that, in each iteration of the while loop in the **anonymize** function, we take a QI-group out from the *queue* and then either "partition this QI-group and put the new partitions into the *queue*" or "put the QI-group into $\mathbf{D}^*$ if the QI-group cannot be further partitioned". The union of $\mathbf{D}^*$ and the *queue* in the SkylineAnonymize algorithm is, in fact, the current release candidate, where $\mathbf{D}^*$ contains the QI-groups that cannot be further partitioned, and the *queue* contains the QI-groups that will later be checked for whether it can be further partitioned or not. We use $\mathbf{D}^+$ to denote the current release candidate (i.e., the union of $\mathbf{D}^*$ and the *queue*). We will show that $\mathbf{D}^+$ is safe at all time. Thus, when the algorithm returns $\mathbf{D}^*$, since the *queue* is empty, $\mathbf{D}^* = \mathbf{D}^+$ is safe.

Consider skyline point (ℓ, k, m, c) for sensitive value σ. Consider the end of each iteration of the while loop in the **anonymize** function. Let $\mathbf{Q}$ be the set of QI-groups that has been seen in the algorithm so far. Note that $\mathbf{D}^+ \subseteq \mathbf{Q}$. It is easy to see that

SS1 = $\min_{g\in \mathbf{Q}} minNR_{\sigma}(g, \ell, k, m) \le \min_{g\in \mathbf{D}^+} minNR_{\sigma}(g, \ell, k, m)$.
SS2 = $\min_{g\in \mathbf{Q}} T_{\sigma}(g, \ell, 0) \le \min_{g\in \mathbf{D}^+} T_{\sigma}(g, \ell, 0)$.
SS3 = $\min_{g\in \mathbf{Q}} T_{\sigma}(g, \ell, k) \le \min_{g\in \mathbf{D}^+} T_{\sigma}(g, \ell, k)$.
SS4 = $\min_{g\in \mathbf{Q}} V_{\sigma}(g, m, 0) \le \min_{g\in \mathbf{D}^+} V_{\sigma}(g, m, 0)$.
SS5 = $\min_{g\in \mathbf{Q}} V_{\sigma}(g, m, k) \le \min_{g\in \mathbf{D}^+} V_{\sigma}(g, m, k)$.

Note that SS1, …, SS5 are the five global variables in the algorithm.

Let NR^+ and BP^+ denote the minimum negated ratio and the breach probability on $\mathbf{D}^+$. Let NR^Q and BP^Q denote the minimum negated ratio and the breach probability computed based on SS1, …, SS5.

$$NR^Q = \min\{\text{SS1, SS2*SS5, SS3*SS4}\} \text{ and } BP^Q = 1/(NR^Q+1).$$

Note that the statement "BP < c" in the safeSplit subroutine guarantees that if QI-groups $g_1, \ldots, g_n$ are added into $\mathbf{D}^+$, then $BP^Q < c$.

It is easy to see that $NR^+ \ge NR^Q$, which means $BP^+ \le BP^Q < c$.

Thus, $\mathbf{D}^+$ is always safe anytime.

We now consider the SVPI case. By Theorem 4, the newly generated QI-groups always make the minimum negated ratio smaller; i.e.,

$$NR^Q = \min\{\text{SS1, SS2*SS5, SS3*SS4}\} = NR^+.$$

Thus, $BP^Q = BP^+$.

By the Corollary of Theorem 4, it can be easily seen that the returned $\mathbf{D}^*$ is minimal because no QI-group in $\mathbf{D}^*$ can be safely partitioned.

❑

Theorem 6. *Given a release candidate* $\mathbf{D}^* = \{(G_1, X_1)\}$ *that has only one QI-group, it is NP-complete to decide whether there exists a reconstruction that satisfies a ground expression of form* $(\wedge_{i\in[1,k]} (x_i\in t_i[S] \leftrightarrow x_i\in u_i[S]))$.

Proof: Given a reconstruction of $\mathbf{D}^*$, it is easily to check whether the reconstruction satisfies a ground expression of the above form. Thus, the problem is in NP.

We now reduce a strongly NP-complete problem, BIN PACKING [15], to this problem. Given integers $a_1, \ldots, a_N$, C and B, in BIN PACKING, we are asked whether $a_1, \ldots, a_N$ can be partitioned into B subsets, each of which has total sum at most C. Let n denote the length of the input to the BIN PACKING problem. Let $p_i(n)$, $p_C(n)$ and $p_B(n)$ denote length of a_i, C and B, respectively. Because BIN PACKING is strongly NP-complete, it is still NP-complete if $p_i(n)$, $p_C(n)$ and $p_B(n)$ are polynomial in n.

The reduction is easy. We consider the SVPI case.

- Let $\mathbf{D}^* = \{(G_1, X_1)\}$, where G_1 is a set of $C\cdot B$ individuals, and X_1 contains B distinct sensitive values $s_1, \ldots, s_B$, each of which has exactly C occurrences in X_1.
- We construction a ground expression K as follows. Initially, K is empty. For each a_j, we add $B\cdot(a_j-1)$ expressions of form $(x_i\in t_i[S] \leftrightarrow x_i\in u_i[S])$ into K. Specifically,

    K = empty;
    for $j = 1$ **to** N **do**
        Let $t_{j,1}, \ldots, t_{j,a_j}$ be any a_j individuals that do not appear in K, so far;
        for $h = 1$ **to** B **do**
            $K = K \wedge [\wedge_{p\in[2,\, a_j]} (s_h\in t_{j,1}[S] \leftrightarrow s_h\in t_{j,p}[S])]$;

Note that K constrains $t_{j,1}, \ldots, t_{j,a_j}$ to have the same sensitive value, for all j.

If there exists a reconstruction of $\mathbf{D}^*$ that satisfies K, then there exists a way to partition $a_1, \ldots, a_N$ into B subsets, each of which has total sum at most C. The B subsets are constructed as follows. For $j = 1$ to N, if individual $t_{j,1}, \ldots, t_{j,a_j}$ has sensitive value s_h in the reconstruction, then, we put a_j in the hth subset. Because we have exactly C occurrences of s_h in X_1, the hth subset will have total sum at most C, for all h.

If there exists a way to partition $a_1, \ldots, a_N$ into B subsets, each of which has total sum at most C, then there exists a reconstruction of $\mathbf{D}^*$ that satisfies K. We reconstruct $\mathbf{D}^*$ as follows. For $j = 0$ to N, if a_j is in the hth subset, then we assign each of $t_{j,1}, \ldots, t_{j,a_j}$ to have sensitive value s_h. It can be easily seen that this reconstruction will satisfy K.

Finally, we note that the length of K is $O(B\cdot(\sum_j a_j))$, which is polynomial in n (the input length of the BIN PACKING problem). Also, the length of $\mathbf{D}^*$ is $O(B\cdot C)$, which is also polynomial in n. Thus, we have successfully reduced the BIN PACKING problem to the problem of whether there exists a reconstruction that satisfies a ground expression of form $(\wedge_{i\in[1,k]} (x_i\in t_i[S] \leftrightarrow x_i\in u_i[S]))$. ❑

A6. Dynamic-Programming Algorithm for Checking Safety

We now describe an algorithm for checking whether a release candidate is safe based on a dynamic-programming algorithm (originally developed in [13] for a knowledge expression different from ours) without using the *congregation* property proposed in the privacy skyline paper. This is the best known algorithm that our algorithm (which uses the *congregation* property) is compared with in the experiment.

Given knowledge threshold (ℓ, k, m) and confidence threshold c, release candidate $\mathbf{D}^* = \{(G_1, X_1), \ldots, (G_B, X_B)\}$ is safe for σ if the breach probability (BP) is less than c, where the breach probability is defined as

$$BP_\sigma(\ell, k, m) = \max\{\Pr(\sigma \in t[S] \mid K_{\sigma|t}(\ell) \wedge K_{\sigma|u}(k) \wedge K_{\sigma|v,t}(m), \mathbf{D}^*)\}.$$

The above maximization is over the following variables:

- Individuals: t (in $K_{\sigma|t}(\ell)$), $u_1, \ldots, u_k$ (in $K_{\sigma|u}(k)$), $v_1, \ldots, v_m$ (in $K_{\sigma|v,t}(m)$).
- Sensitive values: $x_1, \ldots, x_\ell$ (in $K_{\sigma|t}(\ell)$), $y_1, \ldots, y_k$ (in $K_{\sigma|u}(k)$).

By Lemma 1, $BP_\sigma(\ell, k, m) = 1 / (minNR_\sigma(\ell, k, m) + 1)$, where

$$minNR_\sigma(\ell, k, m) = \min\left\{\frac{\Pr(\sigma \notin t[S] \wedge (\wedge_{i\in[1,\ell]} x_i \notin t[S]) \wedge (\wedge_{i\in[1,m]} \sigma \notin v_i[S]) \mid K_{\sigma|u}(k), \mathbf{D}^*)}{\Pr(\sigma \in t[S] \mid K_{\sigma|u}(k), \mathbf{D}^*)}\right\}.$$

Now our goal is to find the minimum negated ratio $minNR_\sigma(\ell, k, m)$ over all possible groundings of the variables. In particular, we consider how to distribute the individuals $t, u_1, \ldots, u_k$ and $v_1, \ldots, v_m$ into QI-groups of $\mathbf{D}^*$ in order to reach the minimum.

Assume that the negated ratio is minimized when the following hold:

- QI-group j contains k_j of the u_i's and m_j of the v_i's, for $j = 1$ to B, and
- t is in QI-group g,

where $\sum_j k_j = k$ and $\sum_j m_j = m$.

In this setting, by Proposition 1, the minimum negated ratio can be expressed as

$$V_\sigma(1, m_1, k_1)\cdots V_\sigma(g-1, m_{g-1}, k_{g-1}) \cdot [T_\sigma(g, \ell, k_g)\cdot V_\sigma(g, m_g, k_g+1)] \cdot V_\sigma(g+1, m_{g+1}, k_{g+1})\cdots V_\sigma(B, m_B, k_B).$$

We can think of $k_1, \ldots, k_B, m_1, \ldots, m_B$ and g as variables such that $k_j \geq 0$, $m_j \geq 0$, $\sum_j k_j = k$, $\sum_j m_j = m$ and $1 \leq g \leq B$. Thus, we obtain $minNR_\sigma(\ell, k, m) =$

$$\min\{\ V_\sigma(1, m_1, k_1)\cdots V_\sigma(g-1, m_{g-1}, k_{g-1}) \cdot [T_\sigma(g, \ell, k_g)\cdot V_\sigma(g, m_g, k_g+1)] \cdot V_\sigma(g+1, m_{g+1}, k_{g+1})\cdots V_\sigma(B, m_B, k_B)\ \}.$$

A dynamic program can be used to find the above minimum in polynomial time.

We first define the following:

- $NR_\sigma^{(with\ t)}(f, \ell, k, m) = minNR_\sigma(\ell, k, m)$ subject to that $t, u_1, \ldots, u_k, v_1, \ldots, v_m$ are all in the first f QI-groups.
- $NR_\sigma^{(without\ t)}(f, \ell, k, m) = minNR_\sigma(\ell, k, m)$ subject to that $u_1, \ldots, u_k, v_1, \ldots, v_m$ are all in the first f QI-groups and t is not in the first f QI-groups.

It can be easily seen that

- $NR_\sigma^{(with\ t)}(f, \ell, k, m)$ is the minimum of the following two:
  - $\min_{0 \leq i \leq k, 0 \leq j \leq m}\ NR_\sigma^{(with\ t)}(f-1, \ell, i, j)\cdot V_\sigma(f, m-j, k-i)$.
  - $\min_{0 \leq i \leq k, 0 \leq j \leq m}\ NR_\sigma^{(without\ t)}(f-1, \ell, i, j)\cdot[T_\sigma(f, \ell, k-i)\cdot V_\sigma(f, m-j, k-i+1)]$.
- $NR_\sigma^{(without\ t)}(f, \ell, k, m) = \min_{0 \leq i \leq k, 0 \leq j \leq m}\ NR_\sigma^{(without\ t)}(f-1, \ell, i, j)\cdot V_\sigma(f, m-j, k-i)$.
- $minNR_\sigma(\ell, k, m) = NR_\sigma^{(with\ t)}(B, \ell, k, m)$, which gives the final answer.

The above formulas together give the dynamic-programming algorithm. To implement the algorithm, we use two three-dimensional arrays. One is for $NR_\sigma^{(with\ t)}(f, \ell, i, j)$, and the other is for $NR_\sigma^{(without\ t)}(f, \ell, i, j)$, where ℓ is fixed, $1 \leq f \leq B$, $0 \leq i \leq k$ and $0 \leq j \leq m$. Then, for $f = 1$ to B, we fill in the two arrays by using the above formulas. Note that, if the QI-groups in release candidate $\mathbf{D}^*$ are clustered (i.e., all the data in a QI-group is stored on disk consecutively), then this algorithm can output the answer by scanning the dataset once, assuming the main memory size is at least $O(k\cdot m)$. Note that it is not necessary to fit the two entire three-dimensional arrays in memory. To compute $NR_\sigma^{(with\ t)}(f, \ell, i, j)$ and $NR_\sigma^{(without\ t)}(f, \ell, i, j)$, we only need $NR_\sigma^{(with\ t)}(f-1, \ell, i, j)$ and $NR_\sigma^{(without\ t)}(f-1, \ell, i, j)$. Thus, the memory requirement is only $O(k\cdot m)$.

A7. Algorithm for Finding Knowledge Skylines

In this section, we describe a simple algorithm for finding the knowledge skyline of a release candidate. The algorithm is based a binary search.

The inputs to the algorithm are as follows.

- Release candidate $\mathbf{D}^*$
- Confidence threshold c
- Target sensitive value σ

Let $BP_\sigma(\ell, k, m)$ be a function that returns $\max\{\Pr(\sigma \in t[S] \mid \mathscr{L}_{t,\sigma}(\ell, k, m), \mathbf{D}^*)\}$, computed using the algorithm described in Section 5.1. The algorithm for finding the knowledge skyline of $\mathbf{D}^*$ is as follows.

PointList = empty;
for $\ell = 0$ **to** infinity
  if $BP_\sigma(\ell, 0, 0) > c$ **then break**; // go out of the loop for ℓ.
  for $m = 0$ **to** infinity
    if $BP_\sigma(\ell, 0, m) > c$ **then break**; // go out of the loop for m.
    Binary search for the k value such that $BP_\sigma(\ell, k, m) < c$ and $BP_\sigma(\ell, k+1, m) \geq c$;
    Add (ℓ, k, m) into *PointList*;
Cleanup *PointList* by removing all the points that are dominated by other points in *PointList*;

A more efficient and scalable algorithm is future work.

A8. Expressibility

In this section, we formally compare our knowledge expression $\mathscr{L}_{t,\sigma}^{\text{SVPI}}(\ell, k, m)$ with the knowledge language $\mathscr{L}_{basic}(k)$ proposed in [13] (where Martin et al. use $\mathscr{L}_{basic}{}^k$ to denote this language) and discuss the theory behind the comparison.

Let $\mathfrak{I}$ denote a set of individuals, and $\boldsymbol{S}$ denote a set of sensitive values. All expressions, datasets and release candidates discussed in this section are defined with respect to $\mathfrak{I}$ and $\boldsymbol{S}$. In particular, an original dataset is of the following form: $\{(u_1, S_1), \ldots, (u_n, S_n)\}$, where $\{u_1, \ldots, u_n\} = \mathfrak{I}$ and S_i is a (possibly empty) subset of $\boldsymbol{S}$. For any ground expression E, all the individuals and the sensitive values involved in E are from $\mathfrak{I}$ and $\boldsymbol{S}$, respectively. Because $\mathscr{L}_{t,\sigma}^{\text{SVPI}}(\ell, k, m)$ and $\mathscr{L}_{basic}(k)$ are defined for the SVPI case, in this section, we assume each individual has exactly one sensitive value; i.e., $|S_i| = 1$ for all i. However, the definitions also apply to the MVPI case.

Definition: Knowledge language. *A (knowledge) language is a set of ground expressions.*

Note that, we define $\mathscr{L}_{t,\sigma}^{\text{SVPI}}(\ell, k, m)$ as an expression with variables, rather than languages. However, it is easily to derive a language from an expression K with variables, which is the set of all the ground expressions that can be derived from K. For ease of exposition, we slightly abuse the notation by using $\mathscr{L}_{t,\sigma}^{\text{SVPI}}(\ell, k, m)$ to also denote the language derived from expression $\mathscr{L}_{t,\sigma}^{\text{SVPI}}(\ell, k, m)$.

Definition: Language $\boldsymbol{\mathscr{L}_{t,\sigma}^{\text{SVPI}}(\ell, k, m)}$. *Language $\mathscr{L}_{t,\sigma}^{\text{SVPI}}(\ell, k, m)$ is the set of all the ground expressions that can be derived from expression $\mathscr{L}_{t,\sigma}^{\text{SVPI}}(\ell, k, m)$.*

In the rest of this section, $\mathscr{L}_{t,\sigma}^{\text{SVPI}}(\ell, k, m)$ is treated as a language.

Recall that $\mathscr{L}_{basic}(1) = \{((\wedge_{i\in[1,m]}\, x_i \in u_i[S]) \rightarrow (\vee_{j\in[1,n]}\, y_j \in v_j[S])) : m > 0, n > 0, u_i \in \mathfrak{I}, v_j \in \mathfrak{I}, x_i \in \boldsymbol{S}, y_j \in \boldsymbol{S}\}$,

$$\mathscr{L}_{basic}(k) = \{(\wedge_{i\in[1,k]}\, E_i) : E_i \in \mathscr{L}_{basic}(1)\}.$$

Also recall that

$$\mathscr{L}_{t,\sigma}^{\text{SVPI}}(\ell, k, m) = \{((\wedge_{i\in[1,\ell]}\, x_i \notin t[S]) \wedge (\wedge_{i\in[1,k]}\, y_i \in u_i[S]) \wedge (\wedge_{i\in[1,m]}\, (\sigma \in v_i[S] \rightarrow \sigma \in t[S]))) : \\ u_i \in \mathfrak{I}, u_i \neq t, v_i \in \mathfrak{I}, v_i \neq t, v_i \neq u_j, x_i \in \boldsymbol{S}, y_i \in \boldsymbol{S}\},$$

where t is a particular individual in $\mathfrak{I}$ and σ is a particular sensitive value in $\boldsymbol{S}$.

Definition: Expressibility. *A ground expression E is expressible in language $\mathscr{L}$ iff there exist an expression $K \in \mathscr{L}$ such that, for any possible original dataset* $\mathbf{W}$, *E and K are either both true on* $\mathbf{W}$ *or both false on* $\mathbf{W}$.

Recall that an expression is defined as a constraint that can be evaluated on a possible original dataset and returns either true or false. The syntax of an expression is application dependent. Thus, the above E and K may have different syntaxes. However, since both of them can be evaluated on an original dataset, the above expressibility is well-defined. For example, $E = \{\text{Ann, Bob}\}$, which is true on $\mathbf{W}$ if and only if Ann and Bob are in a same-value family (i.e., Ann and Bob have the same set of sensitive values) in **W**, and $K = (\wedge_{\sigma \in S} (\sigma \in \text{Ann}[S] \leftrightarrow \sigma \in \text{Bob}[S]))$, which is true on **W** if and only if the logic sentence is true on **W**. In this example, E is expressible in language $\{K\}$, which contains only one expression.

Definition: Practical language. *A language $\mathscr{L}$ is impractical iff, for any release candidate $\mathbf{D}^*$ of any original dataset $\mathbf{D}$, for any $u \in \mathfrak{I}$ and $\sigma \in \boldsymbol{S}$, $\max_{K \in \mathscr{L}} \Pr(\sigma \in u[S] \mid K, \mathbf{D}^*) = b$, where b is a constant.*

An impractical language $\mathscr{L}$ is useless in defining a privacy criterion, because the breach probability (i.e., $\max_{K \in \mathscr{L}} \Pr(\sigma \in u[S] \mid K, \mathbf{D}^*)$) is independent of release candidate $\mathbf{D}^*$ under $\mathscr{L}$. In other words, if the data owner's original dataset **D** (which is also a particular release candidate) is unsafe under $\mathscr{L}$, then no release candidate of **D** can ever be safe under $\mathscr{L}$. Note that, in practice, almost no original dataset is safe.

Definition: Practical expressibility. *We say that a language $\mathscr{L}$ can practically express a ground expression E iff $\mathscr{L}$ is not impractical and E is expressible in $\mathscr{L}$.*

Proposition A8.1. *For any integer k and any expression E of form $\sigma \in u[S]$, where $\sigma \in \boldsymbol{S}$ and $u \in \mathfrak{I}$, $\mathscr{L}_{basic}(k)$ cannot practically express E.*

Proof: We will prove this proposition by contradiction. First, observe that, by the definition of $\mathscr{L}_{basic}(k)$, if $s \in t[S]$ is expressible in $\mathscr{L}_{basic}(k)$, for a particular $s \in \boldsymbol{S}$ and a particular $t \in \mathfrak{I}$, then for any $\sigma \in \boldsymbol{S}$ and $u \in \mathfrak{I}$, $\sigma \in u[S]$ is expressible in $\mathscr{L}_{basic}(k)$.

Now, we assume $s \in t[S]$ is expressible in $\mathscr{L}_{basic}(k)$, which is practical. Then, for any $\sigma \in \boldsymbol{S}$ and any $u \in \mathfrak{I}$, $\sigma \in u[S]$ is expressible in $\mathscr{L}_{basic}(k)$. Let $E_{\sigma \in u[S]}$ denote the expression in $\mathscr{L}_{basic}(k)$ equivalent to $\sigma \in u[S]$.

Thus, for any $\mathbf{D}^*$, any $\sigma \in \boldsymbol{S}$ and any $u \in \mathfrak{I}$, $\max_{K \in \mathscr{L}} \Pr(\sigma \in u[S] \mid K, \mathbf{D}^*) = \Pr(\sigma \in u[S] \mid E_{\sigma \in u[S]}, \mathbf{D}^*) = 1$.

We conclude that $\mathscr{L}_{basic}(k)$ is impractical, which results in a contradiction. ❑

If Bob $\in \mathfrak{I}$ and Flu $\in \boldsymbol{S}$, then a special case of Proposition A8.1 is that $\mathscr{L}_{basic}(k)$ cannot practically express Flu $\in$ Bob[S].

Comparison: For the comparison of $\mathscr{L}_{t,\sigma}^{\text{SVPI}}(\ell, k, m)$ and $\mathscr{L}_{basic}(k)$, no one is more expressive than the other. For example, $\mathscr{L}_{basic}(k)$ can practically express (Flu$\in$Bob[S] $\rightarrow$ AIDS$\in$Tom[S]), but $\mathscr{L}_{t,\sigma}^{\text{SVPI}}(\ell, k, m)$ cannot. $\mathscr{L}_{t,\sigma}^{\text{SVPI}}(\ell, k, m)$ can practically express Flu$\in$Bob[S], but $\mathscr{L}_{basic}(k)$ cannot, for any k. However, our $\mathscr{L}_{t,\sigma}^{\text{SVPI}}(\ell, k, m)$ is more intuitive and quantifies knowledge more precisely than $\mathscr{L}_{basic}(k)$.