A Survey of the Kreiss Matrix Theorem for Power Bounded Families of Matrices and Its Extensions

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Abstract. We survey results related to the Kreiss Matrix Theorem, especially examining extensions of this theorem to Banach space and Hilbert space. The survey includes recent and established results together with proofs of many of the interesting facts concerning the Kreiss Matrix Theorem.

1. Introduction.

We present in this review many results related to the Kreiss Matrix Theorem for power-bounded families of matrices. Most of the results are known results, but we have organized the material in what is hoped to be a convenient order for the reader.

The Kreiss Matrix Theorem is of central important in the theory of finite difference methods for partial differential equations because it deals with necessary and sufficient conditions for stability of general systems. We present an example of a finite difference system to motivate the discussion. Because the Kreiss Matrix Theorem deals with the boundedness of iterates of families of linear operators, there are many related topics. We have therefore had to limit our discussion somewhat arbitrarily. We have tried to focus on the topics closest to the Kreiss Matrix Theorem, but also mention references to the more extensive literature.

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The organization of this review is as follows. In the remainder of this section we present some definitions and set forth our notation. In section 2 we present an example and discussion to motivate the theorem. In section 3 we state the original Kreiss Matrix Theorem and also some additional equivalent conditions, followed by a discussion of the relationships between these conditions for finite dimensional problems.

In sections 4 and 5 we discuss extensions of the Kreiss Matrix Theorem to Banach space and Hilbert space. We present several examples showing the sharpness of several estimates regarding the possible extension or non-extensions of the finite dimensional results to infinite dimensions. For these sections most of the proofs are straight-forward, although some deeper results are also presented. In section 6 we present some results on estimates for Cesàro sums of powers and relate these to the Kreiss Matrix Theorem. Finally, in section 7 we present the proof of the original Kreiss Matrix Theorem and the extensions as given in section 3.

In the remainder of this section we set our notation and giving some definitions. For $x, y \in \mathbb{C}^m$, $m \geq 1$, let $\langle x, y\rangle := y^*x$, where $*$ denotes the conjugate transpose, and let $\|x\| := \langle x, x\rangle^{\frac{1}{2}}$. We denote by $\mathcal{M}$ the collection of $m \times m$, complex-valued matrices, and $\mathcal{M}_h$ denotes the subclass of hermitian matrices, i.e., those with $H^* = H$. If $A \in \mathcal{M}$, $\|A\|$ denotes the operator norm induced by $\|\cdot\|$ on $\mathbb{C}^m$, Re $A := (A + A^*)/2$ the real part of A, Im $A := (A - A^*)/(2i)$ the imaginary part, and $R_\lambda(A) := (\lambda I - A)^{-1}$ is the resolvent for $\lambda \in \mathbb{C}$.

If $H, N \in \mathcal{M}_h$, then we say that $H \leq N$ if $\langle Hx, x\rangle \leq \langle Nx, x\rangle$ for all $x \in \mathbb{C}^m$. Given $\Omega \in \mathcal{M}_h$, the generalized numerical radius of $A \in \mathcal{M}$ is

$$r_\Omega(A) := \sup_{x \neq 0} \frac{|\langle \Omega Ax, x\rangle|}{\langle \Omega x, x\rangle},$$

and the spectral radius of A is $|\sigma(A)|$, defined to be the absolute value of the largest eigenvalue, where $\sigma(A)$ is the spectrum. Clearly, $|\sigma(A)| \leq r_\Omega(A)$ for any $\Omega \in \mathcal{M}_h$, but strict inequality can occur. If $\Omega = T^*T$, then it is easy to see that $r_\Omega(A) = r_I(TAT^{-1})$, where I is the identity matrix. Also, we have

$$r_I(A) \leq \|A\| \leq 2r_I(A).$$

The first inequality is easy to verify, but the second merits a brief description of its proof. We can decompose A as Re $A + i$Im A, each of which is normal and so has numerical radius equal to its norm, cf. [19]. Also, $r_I(A) = r_I(A^*)$. Therefore,

$$\begin{aligned} \|A\| \le \|\text{Re } A\| + \|\text{Im } A\| &= r_I(\text{Re } A) + r_I(\text{Im } A), \\ \le r_I(A) + r_I(A^*) &= 2r_I(A). \end{aligned}$$

A collection of complex numbers, $\{z_\nu\}_{\nu=1}^m$, is said to be nested with constant c if

$$|z_\mu - z_\nu| \le c|z_i - z_j|, \qquad 1 \le i \le \mu \le \nu \le j \le m.$$

Any finite set of complex numbers can be put into nested form simply by choosing one and then, successively, choosing the nearest of the remaining numbers until there are none remaining. The triangle inequality implies that c can be taken to be 2^m whenever there are exactly m numbers to be nested, see [51].

2. Motivation.

To motivate the discussion of the Kreiss Matrix Theorem we consider a simple system of equations for two functions $u(t, x)$ and $v(t, x)$.

$$\begin{aligned} u_t &= au_{xx} + bv_x \\ v_t &= cu_x \quad + dv_x \end{aligned} \tag{2.1}$$

where the subscripts denote differentiation, i.e., $u_t = \partial u/\partial t$, and the coefficients a and d are positive. We consider the pure initial value problem with data specified for u and v at $t = 0$.

To solve this system with the finite difference method, consider a grid defined by two positive constants k and h. The grid points are (t_n, x_m) with $t := nk$ for $n \in \boldsymbol{N}$ and $x_m := mh$ for $m \in \mathbb{Z}$. One possible consistent scheme for this system is

$$\begin{aligned} \frac{u_m^{n+1} - u_m^n}{k} &= a\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} + b\frac{v_{m+1}^n - v_{m-1}^n}{2h} \\ \frac{v_m^{n+1} - v_m^n}{k} &= c\frac{u_{m+1}^n - u_{m-1}^n}{2h} \qquad\qquad + d\frac{v_{m+1}^n - v_m^n}{h}\ . \end{aligned} \tag{2.2}$$

where u_m^n and v_m^n are the values of the discrete approximation at (t_n, x_m). For the special case with $b = c = 0$ this scheme is stable if

$$k \le h^2/(2a) \quad \text{and} \quad k \le h/d, \tag{2.3}$$

see [55] or [51]. This system is stable if for every positive value of T there is a constant C_T such that

$$\sum_{m=-\infty}^{\infty} |u_m^n|^2 + |v_m^n|^2 \le C_T \sum_{m=-\infty}^{\infty} |u_m^0|^2 + |v_m^0|^2$$

for $0 \le nk \le T$, with $(k,h) \in \Lambda$

The stability of such systems is analyzed using Fourier analysis. The quantities u_m^n and v_m^n are replaced with $\hat{u}^n e^{imh\xi}$ and $\hat{v}^n e^{imh\xi}$, respectively. This leads to the following system for $\hat{u}^n$ and $\hat{v}^n$.

$$\begin{pmatrix} \hat{u}^{n+1} \\ \hat{v}^{n+1} \end{pmatrix} = \begin{pmatrix} 1 - 4\frac{ak}{h^2}\sin^2 \frac{1}{2}h\xi & ib\frac{k}{h}\sin h\xi \\ ic\frac{k}{h}\sin h\xi & 1 + d\frac{k}{h}(e^{ih\xi} - 1) \end{pmatrix} \begin{pmatrix} \hat{u}^n \\ \hat{v}^n \end{pmatrix}$$

for each $\xi \in \mathbb{R}$. We write this system as

$$\hat{U}^{n+1} = G(k,h,\xi)\,\hat{U}^n ,$$

and stability is equivalent, via Parseval's relation, to

$$\|G(k,h,\xi)^N\| \le C_T \tag{2.4}$$

for $0 \le kN \le T$, all ξ, and $k,h \to 0$ appropriately, i.e., under some restrictions such as (2.3). It can be shown, from consistency of the scheme, that the constant C_T can be taken in the form $Ke^{\alpha T}$, for some constants α and K. Thus, the estimate (2.4) can be expressed as

$$\|\left(e^{-\alpha k}G(k,h,\xi)\right)^N\| \le K$$

for all N and ξ, and $k,h \to 0$ appropriately.

If the coefficients of the system (2.1) are functions of t and x, then the stability of the system (2.2) depends on the relation (2.4) holding for all (t,x) as well. The family of matrices consists of the matrices $G(k,h,\xi)$ for values of h, k, and ξ, and for variable coefficients, the parameters t and x, as well. For such systems, methods from pseudo-difference operators is required, see Kreiss [31], Lax and Nirenberg [34], Michelson [42], Shintani and Toemeda [53], Yamaguti and Nogi [68], and Wade [65].

In general, the stability of finite difference systems such as (2.2) is equivalent to a family of matrices having powers that are uniformly bounded, such as the estimate (2.4). The Kreiss Matrix Theorem presents several conditions equivalent to the family being uniformly power-bounded.

Good references for the relationship between the Kreiss Matrix Theorem and finite difference methods for partial differential equations are the book by Richtmyer and Morton [51] and the reviews by Thomée [63] and by van Dorsselaer et al. [16]. Other related references are Brenner and Thomée [6], [7], Brenner, Thomée, and Wahlbin [8], Dahlquist et al. [14], Hersh and Kato [26], Kato [27], Kreiss [32], [30], Strikwerda and Wade [56], and Wade [66].

3. Statement of the Kreiss Matrix Theorem.

In this section we state the Kreiss Matrix Theorem for power-bounded families of matrices. We are given a family of matrices $\mathcal{F} \subset \mathcal{M}$. The question at hand is whether this family is uniformly power-bounded, i.e., whether $\|A^n\|$ is bounded by a constant that is the same for all $A \in \mathcal{F}$ and $n \in \boldsymbol{N}$.

Theorem 3.1. *(Kreiss Matrix Theorem) The following conditions are equivalent:*

[A] *There exists $C > 0$ such that $\|A^n\| \le C$ for all $A \in \mathcal{F}$ and $n \in \boldsymbol{N}$.*

[R] *There exists $C > 0$ such that $\|R_\lambda(A)\| \le C(|\lambda| - 1)^{-1}$ for all $A \in \mathcal{F}$ and $\lambda \in \mathbb{C}$ with $|\lambda| > 1$.*

[S] *There exists $C > 0$ such that for each $A \in \mathcal{F}$ there is a nonsingular $S \in \mathcal{M}$, with $\|S\|, \|S^{-1}\| \le C$, such that $\hat{A} := SAS^{-1}$ is upper triangular and*

$$\text{i)} |\hat{A}_{ii}| \le 1, \quad \textit{for} \quad 1 \le i \le m,$$

$$\text{ii)} |\hat{A}_{ij}| \le C \min\{1 - |\hat{A}_{ii}|, 1 - |\hat{A}_{jj}|\}, \quad \textit{for} \quad i < j.$$

[H] *There exists $C > 0$ such that for each $A \in \mathcal{F}$ there is $H \in \mathcal{M}_h$ such that $C^{-1}I \le H \le CI$ and $A^*HA \le H$.*

There are several other conditions that are equivalent to the original conditions. Some of these are stated in the next theorem. Another set of equivalent condition is presented in section 6.

Theorem 3.2. *The following conditions are equivalent to the conditions of Theorem 3.1:*

[B] *There exists $C > 0$ such that for each $A \in \mathcal{F}$ there is a unitary matrix U, such that $\hat{A} := UAU^*$ is upper triangular with nested diagonal and*

$$\text{i)} |\widehat{A}_{ii}| \leq 1, \quad \text{for} \quad 1 \leq i \leq m,$$

$$\text{ii)} |\widehat{A}_{ij}| \leq C \max\{1 - |\widehat{A}_{ii}|, 1 - |\widehat{A}_{jj}|, |\widehat{A}_{ii} - \widehat{A}_{jj}|\}, \quad \text{for} \quad i < j.$$

[N] *There exist $c_0, c_1 > 0$ such that for each $A \in \mathcal{F}$ there is $N \in \mathcal{M}_h$ such that $c_1^{-1} I \leq N \leq c_1 I$ and*

$$\text{Re}\,\big(N(I - zA)\big) \geq c_0(1 - |z|)I, \qquad \forall \quad z \in \mathbb{C}, \quad |z| \leq 1.$$

[Ω] *There exists $c > 0$ such that for each $A \in \mathcal{F}$ there is $\Omega \in \mathcal{M}_h$ such that $c^{-1}I \leq \Omega \leq cI$ and $r_\Omega(A) \leq 1$.*

Kreiss' original paper [30] contains the conditions [A], [R], [S], and [H]. The condition [B] is due to Buchanon in [9]. Condition [N] comes from Strikwerda and Wade [56] [65], and condition [Ω] was introduced by Tadmor in [60]. A complete proof of Theorems 3.1 and 3.2 is given in section 7.

The first condition, [A], is the object of study— it deals with the uniform power-boundedness of the family of matrices $\mathcal{F}$. The Buchanon criterion [B] deals only with unitary upper triangularizations, which always exist by Schur's lemma. The requirement is simply to check a certain relationship between the elements of any unitarily upper triangularized version which is first put into a convenient nested form. For this reason, this condition is a practical one, although it is not always easy to find upper triangularizations. Still, [B] is closer to being a practical condition than [S] because it would normally be difficult to directly check [S]. Condition [S] is mainly useful as a bridge between [R] and [H].

In the book by Richtmyer and Morton [51 p.82] two examples are presented showing the need for the nesting property in the Buchanon criterion [B]. We note, too, that a careful examination of the proof that [B] implies [S] (in section 7) brings to light the reason for the nesting requirement; also see the comments in [46].

The remaining conditions [H], [N], and [Ω] are closely linked. [H] and [N] are useful in actually proving the power-boundedness, or stability, and these both can be utilized in the study of stability for variable coefficient finite difference operators, cf. [32], [56], [65] or [66]. Condition [Ω] is less useful in the variable coefficient case, but it sometimes can be directly verified with $\Omega = I$, cf. [19] or [60].

If condition [A] holds with a given constant C, then condition [R] holds with the same constant. The reverse is not true, the constant for condition [A] must grow with the dimension of the space. Tadmor [60] was the first to prove this result. Improvements on the constant of proportionality were made by LeVeque and Trefethen [38], Lubich and Nevanlinna [39], and Spijker [54].

Theorem 3.3. *If condition* [R] *holds with constant* C_R *for any family* $\mathcal{F} \subset \mathcal{M}$ *of* $m \times m$ *matrices, then the constant* C_A *for condition* [A] *can be bounded as*

$$C_A \leq emC_R.$$

Example: If we consider the specific example

$$(a_m)_{ij} = \begin{cases} \gamma_m, & j = i + 1; \\ 0, & \text{else}, \end{cases} \qquad 1 \leq i, j \leq m,$$

where $\gamma_m := m$, then it is not hard to show that

$$(em - c)C_R \leq C_A, \qquad (m \text{ large}),$$

where $c > 0$ does not depend on m. So the linear dependence of the ratio between the constants in [A] and [R] is sharp.

For proving the stability of variable coefficient finite difference equations it is essential to be able to construct 'smooth' matrix families $\{H\}$ or $\{N\}$, cf. [32], [31], [42], [63], [66] and [67]. A basic question in this line of reasoning is whether one can always construct such a matrix H for condition [H] from the resolvent condition so that H is a continuous function of the matrix A. In the works just cited, restrictions are placed on the difference schemes so that there is a continuous dependence of H on A.

The reader should also consult [21] or [62 §2] where questions related to the Kreiss Matrix Theorem for families of matrices of unbounded order arise in the context of spectral methods for partial differential equations. A theorem similar to Theorem 3.1 by Kreiss [31] gives necessary and sufficient conditions for the well-posedness of constant coefficient partial differential equations.

4. Extension of results to Banach Space.

An obvious question concerning the Kreiss Matrix Theorem is the possibility of extending the results to a Hilbert space or Banach space setting. Condition [S] is clearly finite dimensional in its statement. Indeed, Theorem 3.1 as stated is not true unless the dimension is finite. However questions about how much of the theorem can be extended are of interest. These questions are considered in the next sections.

It is obvious from the proof that the condition [A] implies condition [R] in Banach space. The proof that condition [R] implies conditions [B] and [S] is inherently finite dimensional, as is the proof that condition [S] implies condition [H]. Conditions [H], [N], and [Ω] require Hilbert space for their statements, and indeed, these conditions are related in Hilbert space, as shown in section 5. The implication that these conditions imply condition [A] is valid in Hilbert space.

At first we consider any given sequence $\{a_n\}_{n=0}^{\infty}$ in a Banach space $\mathcal{B}$, but we will later specialize to the case $a_n = A^n$. In the more general case we will use the notation $r(z)$ to denote $\sum_{n=0}^{\infty} a_n z^n$. In the special case where the sequence consists of powers of the matrix A then $R_\lambda(A) = r(\lambda^{-1})\lambda^{-1}$.

In the context of sequences the resolvent condition is equivalent to the following condition.

[R] There is a constant C such that for all $z \in \mathbb{C}$ with $|z| < 1$

$$\left|\sum_{n=0}^{\infty} a_n z^n\right| \leq \frac{C}{1-|z|} .$$

We first prove that when $\mathcal{B}$ is a Banach space condition [R] for sequences implies uniform linear growth of the sequences in $\mathcal{F}$. That is, [R] implies

$$\|a_n\| \leq eC(n+1), \qquad \forall \quad \{a_n\}_{n=0}^{\infty} \in \mathcal{F},\ n \in \boldsymbol{N} , \tag{4.1}$$

where C is the resolvent constant.

Inequality (4.1) is proved using the residue calculus. We have that

$$a_n = \frac{1}{2\pi i} \int_\gamma z^{-(n+1)} r(z)\, dz,$$

where γ is any simple closed contour circling the origin in the positive direction. For $n > 0$ we take γ to be the contour given by $|z| = 1 - 1/(n+1)$. Therefore

$$\begin{aligned} \|a_n\| &\leq c\frac{1}{2\pi} \int_0^{2\pi} \left(1 - \frac{1}{n+1}\right)^{-(n+1)} (n+1)\, d\theta, \\ &\leq 2ce(n+1), \end{aligned}$$

where c depends only on the resolvent constant. The result for $n = 0$ can be proved similarly. This proves our claim.

The estimate (4.1) is sharp, in the sense that a_n may grow linearly with n, even when the sequence is a sequence of powers. An example of an operator that satisfies condition [R] and whose powers satisfy

$$\|A^n\| = n + 1$$

has been given by Shields [52]. Because of its simplicity, and since we use it in section 6, we present it here.

Example: The Banach space $\mathcal{B}$ is the set of functions analytic in the open unit disc with the norm

$$\|f\| = \|f\|_\infty + \frac{1}{2\pi} \int_0^{2\pi} |f'(e^{i\theta})|\, d\theta.$$

$\mathcal{B}$ is a commutative Banach algebra under ordinary multiplication. We consider $\mathcal{B}$ also as a set of operators on $\mathcal{B}$ and it is easy to see that the operator norm coincides with the norm on $\mathcal{B}$. We consider the operator A defined as multiplication by z. It is then easy to check that the norm of A^n is $n + 1$.

The resolvent of A is the function $r_\lambda(z) = (\lambda - z)^{-1}$. The norm of the resolvent is

$$\|R_\lambda(A)\| = \frac{1}{|\lambda| - 1} + \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|\lambda - e^{i\theta}|^2}\, d\theta = \frac{1}{|\lambda| - 1} + \frac{1}{|\lambda|^2 - 1} \leq \frac{2}{|\lambda| - 1}.$$

(The integrand in the integral is the Poisson kernel.) Thus this operator satisfies the resolvent condition [R] but is not power-bounded.

Shields in [52] considers conditions under which Möbius transformations of a bounded operator are uniformly bounded. In particular, a power-bounded operator is Möbius bounded [52].

Several authors have considered the following stronger condition, referred to as the strong resolvent condition.

[SR] There exists $C > 0$ such that $\|R_z(A)^n\| \le C\,(|z|-1)^{-n}$ for all $A \in \mathcal{F}$ and $z \in \mathbb{C}$ with $|z| > 1$ and $n \in \boldsymbol{N}$.

It is easy to see that if condition [A] holds then condition [SR] holds with the same constant. As shown first by McCarthy [40], condition [SR] does not imply power-boundedness, i.e., condition [A]. In fact, the following result has been obtained by several authors, see Bollobás [3], Brenner and Thomée [6], Crabb [11], Crouzeix et al. [13], Friedland [18], McCarthy [40], Hedstrom [25], Kraaijevanger [29], and Lubich and Nevanlinna [39].

Theorem 4.1. *If a family of operators on a Banach space satisfies condition* [SR], *then there is a constant C such that*

$$\|A^n\| \le C(n+1)^{1/2}$$

for all $n \in \boldsymbol{N}$.

The growth rate in Theorem 4.1 is sharp, that is, there are families of operators for which the growth rate of $O(n^{1/2})$ is the best possible. Examples have been given in most of the works just cited. An example, due to T. Kato, see [12], is the family given by

$$K_\varepsilon = (1-\varepsilon D)^{-1}(1+\varepsilon D)$$

where D is differentiation and $\varepsilon \in (0,1]$. When considered as operating on $L^1(\mathbb{R})$ it can be shown that

$$\|K_{1/n}^n\| \ge c n^{1/2}$$

for some constant c.

Lubich and Nevanlinna [39] show that appropriate Möbius transformations of the shift on bounded sequences satisfies condition [SR], but the powers grow as $n^{1/2}$.

O. Nevanlinna [47] has suggested that the resolvent condition [R] be replaced by a similar condition on the Yosida approximation

$$Y_\lambda(A) = \lambda A(\lambda I - A)^{-1} \,.$$

Nevanlinna has the following result, see [47].

Theorem 4.2. *For a family of operators on a Banach space the following are equivalent:*

[A] *There exists* $C > 0$ *such that* $\|A^n\| \le C$ *for all* $A \in \mathcal{F}$ *and* $n \in \boldsymbol{N}$.

[Y] *There exists* $C > 0$ *such that* $\|Y_\lambda(A)^k\| \le C \left(\frac{|\lambda|}{|\lambda| - 1)}\right)^k$ *for all* $A \in \mathcal{F}$, $\lambda \in \boldsymbol{C}$ *with* $|\lambda| > 1$ *and* $k \in \boldsymbol{N}$ *and all* z *in a neighborhood of the origin.*

Actually this can be easily generalized to the following.

Theorem 4.3. *Let* $g(z)$ *be any function analytic in a neighborhood of the origin with nonnegative Taylor series coefficients, and* $g(0) = 0$ *and* $g'(0) = 1$, *then condition* [A] *is equivalent to there being a positive constant c such that*

$$\|g(zA)^k\| \le c\, g(|z|)^k$$

for all $k \in \boldsymbol{N}$.

The proof follows easily from the observation that

$$A^n = \lim_{z \to 0} z^{-n} g(zA)^n.$$

The works [17], [20], [22], [38], [41], [44], [58], [61] and [60] each contain aspects of the question regarding power-bounded operators. We mention [24 ch.23], [23 §6], and [59 ch.1,§11], which discuss similar types of questions in functional analysis. Also, there is an extensive literature on power-bounded operators whose spectrum consists of the set $\{1\}$, see the review by Zemánek [70] for a review of related results.

5. Extension of results to Hilbert Space.

Condition [H] of Theorem 3.1 is equivalent to the statement that the operator A is a contraction in the Hilbert space norm induced by H. Thus a conclusion from the

Kreiss Matrix Theorem is that in finite dimensional spaces every power-bounded operator is similar to a contraction. The question of whether every power-bounded operator in Hilbert space is similar a contraction has been answered in the negative by Foguel [17], see also the comments by Halmos [22]. Foguel presents an example of a power-bounded operator which is not equivalent to a contraction.

In a Hilbert space the conditions [H], [N], and [Ω] of section 3 are related as follows.

Theorem 5.1. *In a Hilbert space $\mathcal{H}$, conditions* [N] *and* [Ω] *are equivalent. Condition* [H] *implies conditions* [N] *and* [Ω].

We begin by proving that the condition [H] implies condition [N]. Assuming condition [H] holds, let $A \in \mathcal{F}$, let H be the corresponding positive definite hermitian matrix and let $z \in \mathbb{C}$ with $|z| \leq 1$. Expanding the relation

$$0 \leq (I - zA)^* \, H \, (I - zA) \, ,$$

we obtain

$$\begin{aligned} 0 &\leq H - 2\text{Re } (zHA) + |z|^2 A^* HA \\ &\leq H - 2\text{Re } (zHA) + |z|^2 H \\ &= 2\text{Re } (H(I - zA)) + \left(|z|^2 - 1\right) H. \end{aligned}$$

Thus

$$\frac{1}{2}\,(1 - |z|)\, H \leq \frac{1}{2}\left(1 - |z|^2\right) H \leq \text{Re } (H(I - zA)) \, ,$$

and condition [N] follows with N taken as H.

To see that condition [N] is equivalent to condition [Ω], note that if [N] holds then

$$\text{Re } \big(N(I - zA)\big) \geq c_0 \, (1 - |z|) \, I \geq 0.$$

This implies that

$$\text{Re } N \geq \text{Re } (NzA) \qquad \text{for all } |z| \leq 1.$$

Given $x \in \mathcal{H}$, choose z such that

$$\text{Re } (\langle NAx, x \rangle z) = |\langle NAx, x \rangle|.$$

Therefore,

$$|\langle NAx, x\rangle| \leq \langle Nx, x\rangle, \qquad x \in \mathcal{H},$$

which implies $r_N(A) \leq 1$. This proves that condition [Ω] holds with the matrix Ω taken to be N.

Conversely, assume that the condition [Ω] holds. We will show that this implies condition [N]. We have, for $x \in \mathcal{H}$ and $|z| \leq 1$,

$$\begin{aligned}
\text{Re } \langle \Omega\,(I - zA)\,x, x\rangle &= \text{Re } \langle \Omega x, x\rangle - \text{Re } \langle z\Omega Ax, x\rangle \\
&\geq \langle \Omega x, x\rangle - |z||\langle \Omega Ax, x\rangle| \\
&\geq \langle \Omega x, x\rangle - |z|\langle \Omega x, x\rangle \\
&= (1 - |z|)\,\langle \Omega x, x\rangle \\
&\geq c_\Omega^{-1}\,(1 - |z|)\,\langle x, x\rangle.
\end{aligned}$$

This implies condition [N] with $N := \Omega$, and $c_1 := c_\Omega$, and $c_0 := c_\Omega^{-1}$. ∎

Example: We show that the matrix N can be taken equal to H, but not the converse. Here is an example using a family of a single matrix.

$$\mathcal{F} := \left\{ \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix} \right\}.$$

Referring to [19], the numerical radius of A can be computed because it equals the spectral radius of its real part. This last claim is seen to be true because

$$\begin{aligned}
r_I(A) &= \max\{|\langle Ax, x\rangle| : x \in \mathbb{C}^2, \quad \|x\| = 1\}, \\
&= |\langle Ax_0, x_0\rangle|,
\end{aligned}$$

for some $\|x_0\| = 1$. Hence

$$r_I(A) = |\langle Ax_0, x_0\rangle| \leq \text{Re } \langle Ay_0, y_0\rangle \leq r_I(A)$$

where $(y_0)_i := |(x_0)_i|$, $i = 1, 2$, since the elements of A are nonnegative. Therefore

$$\begin{aligned}
r_I(A) &= \max\{|\text{Re } \langle Ax, x\rangle| : x \in \mathbb{C}^2 \quad \|x\| = 1\}. \\
&= \max\{|\langle \text{Re } Ax, x\rangle| : x \in \mathbb{C}^2 \quad \|x\| = 1\}, \\
&= r_I(\text{Re } A) \\
&= |\sigma(\text{Re } A)|
\end{aligned}$$

because Re A is normal. Since $|\sigma(\text{Re } A)| = 1$, N may be taken equal to $\Omega = I$ (using the fact that [N]$\Leftrightarrow$[Ω]), yet it is easy to check that $\|A\| > 1$ and so the matrix H cannot be the identity.

Condition [Ω] is equivalent to the resolvent condition with constant 1, see the text by Pazy [48]. For condition [Ω] we point out that an equivalent formulation is: there exists $c > 0$ such that for each $A \in \mathcal{F}$ there is $\Omega \in \mathcal{M}_h$ such that $c^{-1}I \leq \Omega \leq cI$ and $r_\Omega(A^n) \leq 1$ for all $n \in \boldsymbol{N}$. The reason this modified version of Tadmor's condition is equivalent to the original is simply that the (generalized) power inequality holds, cf. [60] or [24 #221], that is, $r_\Omega(A^n) \leq \left(r_\Omega(A)\right)^n$. Also, one can find an alternate proof that [Ω] implies [A] by adapting the solution of [24 #221]. Each proof amounts to the same basic idea— proving the power inequality for the generalized numerical radius.

Condition [N] is closely related to both the power inequality ([24 #221]) and also the sharp Gårding inequality from pseudodifference operator theory, cf. [56] or [66]. Additionally, condition [N] is a straight generalization of the condition on the symbol given in [34 1.12], in which the matrix N is the identity.

The interested reader may consult the works by Berger and Stampfli [2], Bonsall and Duncan [4] and [5], Crabb [10], Goldberg and Tadmor [19], Kato [28], Lenferink and Spijker [35], [36], and [37], and Reddy and Trefethen [50]. Many of the results for Hilbert space can be extended to Banach space by using the duality map, see Yosida [69], however, little appears to be gained in this regard.

6. Conditions on Cesàro Means.

In this section we discuss the relation of Abel and Cesàro means of the sequence of powers of an operator A on a Banach space $\mathcal{B}$. As in section 4 we start by considering general sequences in the Banach space and the function $r(z)$.

We define the Abel means of the sequence $\{a_n\}_{n=0}^{\infty}$ to be

$$\alpha(z) := \left(\sum_{n=0}^{\infty} a_n z^n\right) \Big/ \left(\sum_{n=0}^{\infty} |z|^n\right),$$

where $z \in \mathbb{C}$ with $|z| < 1$. If $z = \rho e^{i\phi}$ with $\rho < 1$, then

$$\alpha(z) = \left(\sum_{n=0}^{\infty} a_n z^n\right)(1-\rho) = (1-\rho)r(z).$$

Thus,

$$\|r(z)\| = (1-|z|)^{-1}\|\alpha(z)\|, \qquad \forall \quad |z|<1,$$

provided the above series converge for the given sequence.

Condition [R] is clearly equivalent to $\|\alpha(z)\| \le c$. It is well known that Abel means are related to Cesàro means, cf. [1], [64], or [71], and this leads naturally to the consideration of the kth Cesàro means of the sequence $\{a_n\}_{n=0}^{\infty}$. Cesàro means are important in ergodic theory, cf. [15 ch.5], however there appears to be no direct relation between that theory and the results discussed here. Most of the results in this section are in [57] by the authors, although some of the proofs here are simpler.

For $k \in \boldsymbol{N}$ and any sequence $\{a_n\}_{n=0}^{\infty} \subset \mathcal{B}$, we define the kth Cesàro means by the following procedure:

$$\sigma_0(\{a_.\}, n) := a_n,$$

and

$$\sigma_{k+1}(\{a_.\}, n) := \left(\sum_{\nu=0}^{n} \binom{\nu+k}{k} \sigma_k(\{a_.\}, \nu)\right) \Big/ \binom{n+k+1}{k+1}.$$

We will write $\sigma_k(n)$ for $\sigma_k(\{a_.\}, n)$ when it is understood which sequence is being used. Explicit formulas for σ_1 and σ_2 are

$$\sigma_1(n) = \frac{1}{n+1}\sum_{\nu=0}^{n} a_\nu,$$

and

$$\sigma_2(n) = \frac{2}{(n+1)(n+2)}\sum_{\nu=0}^{n}(n+1-\nu)a_\nu.$$

For any $k \in \boldsymbol{N}$ it is easy to verify directly from the definition that

$$\|\sigma_{k+1}(n)\| \le \sup_{0\le\nu\le n}\{\|\sigma_k(\nu)\|\}.$$

In particular, if $\|a_n\| \le c$ for all $n \in \boldsymbol{N}$, then also $\|\sigma_k(n)\| \le c$ for all $n \in \boldsymbol{N}$ (and each $k \in \boldsymbol{N}$). The converse does not hold.

An important relationship exists between the function $r(z)$ and the higher Cesàro means. With $z = \rho e^{i\phi}$ and $\rho < 1$, we find for the sequence $\{a_n e^{in\phi}\}_{n=0}^{\infty}$ that

$$\begin{aligned} r(z) &= \sum_{n=0}^{\infty} a_n e^{in\phi} \rho^n, \\ &= \sum_{n=0}^{\infty} a_n e^{in\phi} \left((1-\rho) \sum_{\nu=n}^{\infty} \rho^\nu \right), \\ &= (1-\rho) \sum_{\nu=0}^{\infty} \rho^\nu \left(\sum_{n=0}^{\nu} a_n e^{in\phi} \right), \\ &= (1-\rho) \sum_{\nu=0}^{\infty} (\nu+1) \sigma_1(\nu) \rho^\nu, \end{aligned} \tag{6.1}$$

the interchange being justified if one side converges, cf. [71 p.78], or if there is absolute convergence of the double sum, cf. [64]. For instance, absolute convergence clearly holds if the a_n satisfy (4.1).

Applying this technique again gives

$$r(z) = (1-\rho)^2 e^{-i\phi} \sum_{\nu=0}^{\infty} \binom{\nu+2}{2} \sigma_2(\nu) \rho^\nu. \tag{6.2}$$

In general then, for any $k \in \boldsymbol{N}$, we obtain

$$r(z) = (1-\rho)^k \sum_{n=0}^{\infty} \binom{n+k}{k} \sigma_k(n) \rho^n, \tag{6.3}$$

provided only that one of the series converges, which will be the case in our applications. A similar result is contained in [33 K.3].

Theorem 6.1. *If $\{a_n\}_{n=0}^{\infty}$ is a sequence in a Banach space, then the following are equivalent.*

[R] *There is a constant C such that for all $z \in \mathbb{C}$ with $|z| < 1$*

$$\left| \sum_{n=0}^{\infty} a_n z^n \right| \leq \frac{C}{1-|z|}$$

[C2] *There exists a constant* $C > 0$ *such that*

$$\left\| \frac{2}{(n+1)(n+2)} \sum_{\nu=0}^{n} (n+1-\nu) e^{i\nu\phi} a_\nu \right\| \leq C$$

for all $\phi \in \mathbb{R}$ *and* $n \in \boldsymbol{N}$.

Proof:

Condition [C2] implies condition [R] by (6.3).

To prove that condition [R] implies condition [C2] we let $z = e^{-\zeta + i\phi}$ then condition [R] becomes

$$\|r(\zeta)\| = \left\| \sum_{n=0}^{\infty} a_n e^{in\phi} e^{-n\zeta} \right\| \leq \frac{C}{1 - e^{-\mathrm{Re}\,\zeta}}$$

for all ζ with Re $\zeta > 0$. By condition (4.1) the sum is absolutely convergent for Re $\zeta > 0$.

We now consider the integral

$$C_N = \frac{1}{2\pi i} \int_{\Gamma_N} \frac{e^{N\zeta}}{\zeta^2} r(\zeta)\, d\zeta$$

where the curve Γ_N is given by Re $\zeta = 1/N$ taken in the direction of increasing imaginary part.

Notice that for $k > N$ we have

$$\frac{1}{2\pi i} \int_{\Gamma_N} \frac{e^{N\zeta}}{\zeta^2} e^{-k\zeta}\, d\zeta = 0.$$

Also, that for $k \leq N$ we have

$$\frac{1}{2\pi i} \int_{\Gamma_N} \frac{e^{N\zeta}}{\zeta^2} e^{-k\zeta}\, d\zeta = \frac{1}{2\pi i} \int_{\Gamma_N} \frac{e^{(N-k)\zeta}}{\zeta^2}\, d\zeta = N - k$$

by deforming the contour to circle the origin.

Hence,

$$C_N = \sum_{n=0}^{N} (N-k) a_n e^{in\phi}.$$

To estimate the norm of C_N we have

$$\begin{aligned} \|C_N\| &\leq \frac{1}{2\pi} \int_{\Gamma_N} \frac{e^{N \operatorname{Re} \zeta}}{|\zeta|^2} \|r(\zeta)\| \, |d\zeta| \\ &\leq \frac{Ce}{2\pi \left(1 - e^{(-1/N)}\right)} \int_{\Gamma_N} \frac{1}{N^{-2} + t^2} \, dt \\ &\leq CeN^2 \end{aligned}$$

for N sufficiently large. This proves the theorem. ∎

We now consider the question of when the resolvent condition implies the first Cesàro mean condition. Let $\mathcal{B}$ be any Banach space and $\{a_n\}$ be a sequence in $\mathcal{B}$.

Theorem 6.2. *If a sequence in a Banach space satisfies condition* [R], *then*

$$\left\|\sigma_1(\{a.e^{i\cdot\phi}\}, n)\right\| \leq c \log(n+2), \qquad \forall \quad n \in \mathbb{N}\,, \ \phi \in \mathbb{R},$$

where c depends only on the resolvent constant.

The proof is similar in spirit to the result involving the second Cesàro means in Theorem 6.1. We consider the integral

$$C_N = \frac{1}{2\pi i} \int_{\Gamma_N} \frac{e^{N\zeta}}{\zeta} r(\zeta) \, d\zeta$$

where the curve Γ_N is given by $\operatorname{Re} z = 1/N$ taken in the direction of increasing imaginary part. Similar to the earlier result, we have

$$C_N = \sum_{n=0}^{N} a_n e^{in\phi}.$$

The estimate of C_N is

$$\begin{aligned} \|C_N\| &\leq \frac{1}{2\pi} \int_{\Gamma_N} \frac{e^{N \operatorname{Re} \zeta}}{|\zeta|} \|r(\zeta)\| \, |d\zeta| \\ &\leq \frac{Ce}{2\pi \left(1 - e^{(-1/N)}\right)} \int_{\Gamma_N} \frac{1}{|N + t|} \, dt \\ &\leq CeN \log N\,. \end{aligned}$$

for large N. The result follows immediately. ∎

The relationship between condition [R] and the Cesàro means of powers are essentially the same as for general sequences. However, there is one interesting result that does depend on the the sequence being powers. Even though the norm of the average σ_1 may be unbounded when condition [R] is satisfied, the squares of $\sigma_1(n)$ are bounded.

Theorem 6.3. *If a family of operators in a Banach space satisfies condition* [R] *of Theorem* 3.1 *and* 6.1, *then*

$$\|\sigma_1(\{A\cdot e^{i\cdot\phi}\}, n)\| \leq c\log(n+2)$$

and

$$\|\sigma_1(\{A\cdot e^{i\cdot\phi}\}, n)^2\| \leq c,$$

for all $n \in \mathbb{N}$ and $\phi \in \mathbb{R}$, where c depends only on the resolvent constant.

The logarithmic growth estimate follows from Theorem 6.2. By elementary manipulation one can easily check that

$$\begin{aligned}((n+1)\sigma_1(n))^2 &= \sum_{j=0}^{n}(j+1)A^j + \sum_{j=n+1}^{2n}(2n+1-j)A^j,\\ &= \binom{2n+2}{2}\sigma_2(2n) - 2\binom{(n-1)+2}{2}\sigma_2(n-1).\end{aligned}$$

From this relationship we can conclude that

$$\|\sigma_1(\{A\cdot e^{i\cdot\phi}, n)^2\| \leq 3 \sup_{0\leq\nu\leq 2n} \|\sigma_2(\{A\cdot e^{i\cdot\phi}\}, \nu)\|.$$

The theorem follows immediately from this relation. ∎

Example: To show that the logarithmic growth of Theorem 6.2 and 6.3 estimate is sharp, we consider the example of Shields [52] discussed in section 4. For $a_n = A^n$ the first Cesàro mean is

$$\sigma_1(N) = \frac{1}{N+1}\frac{1-z^{N+1}}{1-z}$$

and the derivative for $z = e^{i\theta}$ is

$$\sigma_1(N)' = \frac{1}{N+1}\left(\frac{-(N+1)z^N}{1-z} + \frac{1-z^{N+1}}{(1-z)^2}\right)$$
$$= \frac{2ie^{i(N-1)\theta/2}}{N+1}\left(\frac{(N+1)e^{iN\theta/2}\sin(\theta/2) - \sin(N+1)\theta/2}{-4\sin^2\theta/2}\right) .$$

Therefore,

$$\|\sigma_1(N)'\| \geq \frac{1}{2\pi}\int_0^{2\pi} |\sigma_1'(e^{i\theta})|\, d\theta$$
$$= \frac{1}{\pi(N+1)}\int_0^{2\pi}\left|\frac{(N+1)e^{iN\theta/2}\sin(\theta/2) - \sin(N+1)\theta/2}{\sin^2\theta/2}\right| d\theta$$
$$\geq \frac{1}{\pi}\int_0^{2\pi}\left|\frac{\sin(N\theta/2)}{\sin\theta/2}\right| d\theta$$
$$\geq C\log(N) .$$

7. Proof of the Kreiss Matrix Theorem.

We present the complete proof of the Kreiss Matrix Theorem in a rigorous manner, and we follow the order

$$[A]\Rightarrow[C1]\Rightarrow[C2]\Rightarrow[R]\Rightarrow[B]\Rightarrow[S]\Rightarrow[H]\Rightarrow[N]\Rightarrow[\Omega]\Rightarrow[A].$$

We give here what we believe to be the best proofs which allow this order of implications. The following methods of proof are partly derived from the existing literature, mainly [30], [44], [45], [46], [49], [51], [56], [60], and [65], except that we have simplified many aspects. We think that including condition [B] between conditions [R] and [S] simplifies the details of the proof and includes [B] in the sequence in a more understandable manner. To prove that [Ω] implies [A], we essentially follow [60], but for the reader's convenience we have spelled out all of the tricky details which are not contained in [60], in particular those from [49]. Our modifications of existing methods are so interspersed in the pages to follow that it would be too difficult to cite each individual source. Therefore, we omit most citations in this section, instead leaving to the reader to consult the works listed in our bibliography.

Several works contain interesting proofs of these results. Miller and Strang [44] give a novel way to construct the matrix families $\{S\}$ in condition [S]. Miller [43] has a method for constructing the matrix family $\{H\}$ for condition [H] directly from the resolvent condition [R]. See also Morton and Schecter [46].

We now begin the proof of Theorem 3.1. The proof that [A] $\Rightarrow$ [C1] and that [C1] $\Rightarrow$ [C2] are straightforward using the results of section 6.

[C2] $\Rightarrow$ [R]. Essentially this implication is the identity (6.2) except that some care must be exercised to be sure the series appearing in the derivation of identity (6.3) converge. Since we only know that the second Cesàro means are uniformly bounded, we need to justify the two step process of switching the summation indices used to derive that identity. However, condition [C2] immediately implies (take differences $\sigma_2(A,n) - \sigma_2(A,n-1)$) that

$$\|A^n\| \leq c(n+2),$$

and the interchange of the summations is justified to prove (6.1). We can then proceed to apply the same technique again to arrive at (6.2). This gives

$$\|R_\lambda(A)\| \leq (\rho-1)^2\rho^{-3}\sum_{n=0}^{\infty} C\binom{n+2}{2}\rho^{-n} = \frac{C}{\rho-1}$$

and the proof is completed.

[R] $\Rightarrow$ [B]. If $B \in \mathcal{M}$, then $|B_{ij}| \leq |\langle Be_i, e_j\rangle| \leq \|B\|$, where $\{e_i\}_{i=1}^m$ are the standard coordinate unit vectors. Choose any unitary matrix U (via Schur's lemma) which puts A into upper triangular form with the main diagonal nested. Note that

$$\|R_\lambda(UAU^*)\| = \|UR_\lambda(A)U^*\| = \|R_\lambda(A)\|,$$

so UAU^* also satisfies the resolvent condition [R] with the same constant. For notational convenience we denote UAU^* again by A. From the definition of the resolvent:

$$(\lambda I - A)R_\lambda(A) = I, \qquad |\lambda| > 1. \tag{7.1}$$

This equation implies that, if λ is not an eigenvalue of A and $1 \leq i \leq m$,

$$|R_\lambda(A)_{ii}| = |\lambda - A_{ii}|^{-1}.$$

Also, by considering (7.1) for $i < j$, we see that

$$|A_{ij}| \leq |\lambda - A_{jj}| \sum_{i \leq \mu < j} |(\lambda - A)_{i\mu}||R_\lambda(A)_{\mu j}|, \qquad i < j \leq m. \tag{7.2}$$

The resolvent condition clearly implies the von Neumann condition [B], i. The conclusion [B], ii is reached by an inductive argument on the number of upper diagonals.

To begin, assume that $1 \leq i < m$ and $j = i + 1$. The freedom of choice of λ will be exploited to obtain a bound for $|A_{ij}|$ as follows.

Equation (7.2) implies

$$|A_{ij}| \leq C|\lambda - A_{jj}||\lambda - A_{ii}| \left(|\lambda| - 1\right)^{-1},$$

where the resolvent condition and the comment at the beginning of the proof have been employed to bound the term $|R_\lambda(A)_{ij}|$ by $c(|\lambda| - 1)^{-1}$. If $|A_{jj}| \leq \frac{1}{2}$ then choose $\lambda := 2$ to conclude that

$$\begin{aligned} |A_{ij}| &\leq 8c, \\ &\leq 16c \max\{1 - |A_{ii}|, 1 - |A_{jj}|, |A_{ii} - A_{jj}|\}. \end{aligned}$$

Conversely, if $\frac{1}{2} \leq |A_{jj}| \leq 1$ then let $\lambda := t\bar{A}_{jj}^{-1}$ for $t > 1$. This implies

$$|A_{ij}| \leq c\frac{|t - |A_{jj}|^2||t - \bar{A}_{jj}A_{ii}|}{|A_{jj}|(t - |A_{jj}|)},$$

which gives, upon letting $t \to 1$,

$$\begin{aligned} |A_{ij}| &\leq c\frac{(1 + |A_{jj}|)|1 - \bar{A}_{jj}A_{ii}|}{|A_{jj}|}, \\ &\leq 4c|1 - |A_{jj}|^2 + \bar{A}_{jj}(A_{jj} - A_{ii})|, \\ &\leq 12c \max\{1 - |A_{ii}|, 1 - |A_{jj}|, |A_{ii} - A_{jj}|\}. \end{aligned}$$

Combining these two cases yields

$$|A_{ij}| \leq 16c \max\{1 - |A_{ii}|, 1 - |A_{jj}|, |A_{ii} - A_{jj}|\}, \tag{7.3}$$

which is [B], ii for the case $j = i + 1$.

For induction, assume that [B], ii holds with constant c_1 for all $1 \le i \le m - \mu + 1$ and $j - i < \mu$, where μ is some integer $2 \le \mu < m$. Equation (7.2) for $j = i + \mu$ and $1 \le i \le m - \mu$, plus the resolvent condition, implies

$$|A_{ij}| \le c\,(|\lambda| - 1)^{-1}\,|\lambda - A_{jj}| \sum_{i \le \nu < j} |(\lambda I - A)_{i\nu}|.$$

Note that $1 - |A_{\nu\nu}| \le 1 - |A_{jj}| + |A_{\nu\nu} - A_{jj}|$, which is bounded above by $2\max\{1 - |A_{jj}|, |A_{\nu\nu} - A_{jj}|\}$. The inductive hypothesis and the nesting property can now be used to conclude that

$$|A_{ij}| \le c|\lambda - A_{jj}|(|\lambda| - 1)^{-1}\Bigg(|\lambda - A_{ii}| + c_1 2^{m+1} \sum_{i<\nu<j} \max\{1 - |A_{ii}|, 1 - |A_{jj}|, |A_{ii} - A_{jj}|\}\Bigg).$$

This implies

$$|A_{ij}| \le \frac{c|\lambda - A_{ii}||\lambda - A_{jj}|}{(|\lambda| - 1)} \\ \frac{+\, cc_1 m 2^{m+1}|\lambda - A_{jj}|\max\{1 - |A_{ii}|, 1 - |A_{jj}|, |A_{ii} - A_{jj}|\}}{(|\lambda| - 1).} \tag{7.4}$$

If $|A_{jj}| \le \frac{1}{2}$, then choosing $\lambda := 2$ gives

$$|A_{ij}| \le (16c + 5cc_1 m 2^m)\max\{1 - |A_{ii}|, 1 - |A_{jj}|, |A_{ii} - A_{jj}|\}. \tag{7.5}$$

Conversely, if $\frac{1}{2} \le |A_{jj}| \le 1$ then choose $\lambda := t\bar{A}_{jj}^{-1}$, $t > 1$. By the same reasoning as that used to conclude inequality (7.3), as $t \to 1$, (7.4) yields

$$|A_{ij}| \le \left(12c + cc_1 m 2^{m+2}\right)\max\{1 - |A_{ii}|, 1 - |A_{jj}|, |A_{ii} - A_{jj}|\}. \tag{7.6}$$

Combining the two bounds (7.5) and (7.6) finishes the induction and therefore yields condition [B], ii.

[B] $\Rightarrow$ [S]. Assume, without loss of generality, that A is in upper triangular form with diagonal elements nested and satisfies [S], i. To prove [S], we construct the matrix S as a product of special matrices. For $1 \le i < j \le m$ define a matrix $P^{(i,j)}$ by

$$P^{(i,j)}_{\mu\nu} := \begin{cases} \frac{I_{\mu\nu} + \delta_{i\mu}\delta_{j\nu}A_{ij}}{A_{ii} - A_{jj}} & \text{if } \min\{1 - |A_{ii}|, 1 - |A_{jj}|\} < |A_{ii} - A_{jj}|; \\ I_{\mu\nu} & \text{else,} \end{cases}$$

where $\delta_{..}$ is the Kronecker delta and $1 \le \mu, \nu \le m$. (The essence of this matrix is really 2×2 in nature.)

Now it is possible to define the matrix S; it is defined in two stages. For $1 \le \nu \le m-1$ we let

$$S_\nu := \prod_{\substack{1 \le i \le m-\nu \\ j-i=\nu}} P^{(i,j)},$$

and define

$$S := S_{m-1} S_{m-2} \cdots S_1.$$

The inverse of $P^{(i,j)}$ is easily computed to be

$$P^{(i,j)-1}_{\mu\nu} = \begin{cases} \frac{I_{\mu\nu} - \delta_{i\mu}\delta_{j\nu}A_{ij}}{A_{ii}-A_{jj}} & \text{if } \min\{1-|A_{ii}|, 1-|A_{jj}|\}|A_{ii}-A_{jj}|; \\ I_{\mu\nu} & \text{else.} \end{cases}$$

Since

$$\max\{1-|A_{ii}|, 1-|A_{jj}|, |A_{ii}-A_{jj}|\} \le 2\max\{\min\{1-|A_{ii}|, 1-|A_{jj}|\}, |A_{ii}-A_{jj}|\},$$

it is easy to see that

$$\|S\|, \|S^{-1}\| \le (1+2c)^{m^2}.$$

Condition [S], i is a consequence of the fact that SAS^{-1} has the same main diagonal as A. It only remains to show that if $\widehat{A} := SAS^{-1}$ then

$$|\widehat{A}_{ij}| \le c_1 \min\{1-|\widehat{A}_{ii}|, 1-|\widehat{A}_{jj}|\}, \qquad \text{if} \quad i<j. \tag{7.7}$$

Consider the action of $P^{(i,j)} \cdot P^{(i,j)-1}$ for $1 \le i < m$ and $i < j \le m$. By explicit calculation using the upper triangularity of the three matrices involved, if $1 \le \mu \le \nu \le m$ then

$$\left(P^{(i,j)} A P^{(i,j)-1}\right)_{\mu\nu} = \sum_{\mu \le k \le \nu} \sum_{k \le l \le \nu} P^{(i,j)}_{\mu k} A_{kl} P^{(i,j)-1}_{l\nu}. \tag{7.8}$$

After using the definition of the $P^{(i,j)}$ to analyze (7.8) the following statements become clear. If either $\nu - \mu = j - i$ and $\mu \ne i$ or $\nu - \mu < j - i$ then equation (7.8) equals $A_{\mu\nu}$. If $\mu = i$ and $\nu = j$ and $\min\{1-|A_{ii}|, 1-|A_{jj}|\} \ge |A_{ii} - A_{jj}|$, then (7.8) equals $A_{\mu\nu}$ because $P^{(i,j)} = I$. If $\mu = i$ and $\nu = j$ and $\min\{1-|A_{ii}|, 1-|A_{jj}|\} < |A_{ii} - A_{jj}|$ then a simple calculation using the definition of $P^{(i,j)}$ shows that (7.8) equals zero. The elements of $P^{(i,j)}AP^{(i,j)-1}$ which are on diagonals above the $(j-i)^{th}$ are changed, but in a bounded way; equation (7.8) shows why this is true. By the nested property of the main diagonal it is easy to verify that the following holds for S_1:

$$|\left(S_1AS_1^{-1}\right)_{ij}| \leq \begin{cases} c\min\{1-|A_{ii}|, 1-|A_{jj}|\} & j=i+1; \\ c_m \max\{1-|A_{ii}|, 1-|A_{jj}|, |A_{ii}-A_{jj}|\} & j>i+1; \\ |A_{ii}| & j=i; \\ 0 & j<i, \end{cases}$$

where $c_m := 2^{m+1}m^2(1+2c)^{m^2}$.

The matrix S is constructed out of an ordered product of the $m-1$ matrices $\{S_\nu\}_{\nu=1}^{m-1}$. These give similar estimates at each successive step in the matrix transformation $S_{m-1}\cdots S_1AS_1^{-1}\cdots S_{m-1}^{-1}$. Now it is clear that after continuing up to SAS^{-1} this process implies that equation (7.7) holds for some constant c_1 depending only on m and c. This completes the proof that condition [B] implies [S].

[S] $\Rightarrow$ [H]. Define the diagonal matrix D_ε for $0<\varepsilon<1$ by $D_{\varepsilon ij} := \delta_{ij}\varepsilon^{m-i}$. Given $A \in \mathcal{F}$, and $S \in \mathcal{M}$ satisfying condition [S], our goal is to prove that

$$\left(SAS^{-1}\right)^* D_{\varepsilon_0}^2 \left(SAS^{-1}\right) \leq D_{\varepsilon_0}^2 \tag{7.9}$$

for some ε_0 depending only on c_1 and m. This would imply

$$A^*\left(S^*D_{\varepsilon_0}^2 S\right)A \leq S^*D_{\varepsilon_0}^2 S.$$

Condition [H] would then follow from this inequality if $H := S^*D_{\varepsilon_0}^2 S$ since in this case H is hermitian and for $x \in \mathbb{C}^m$,

$$\varepsilon_0^{2(m-1)}\langle Sx, Sx\rangle \leq \langle S^*D_{\varepsilon_0}^2 Sx, x\rangle \leq \langle Sx, Sx\rangle,$$

$$\varepsilon_0^{2(m-1)}c_0^{-2}\langle x, x\rangle \leq \langle S^*D_{\varepsilon_0}^2 Sx, x\rangle \leq c_0^2\langle x, x\rangle.$$

It is therefore only necessary to show that (7.9) holds. Equation (7.9) is equivalent to

$$\left(D_{\varepsilon_0}\widehat{A}D_{\varepsilon_0}^{-1}\right)^*\left(D_{\varepsilon_0}\widehat{A}D_{\varepsilon_0}^{-1}\right) \leq I, \tag{7.10}$$

where $\widehat{A} := SAS^{-1}$ is upper triangular. Defining B_{ε_0} to be $D_{\varepsilon_0}\widehat{A}D_{\varepsilon_0}^{-1}$, it is easy to check that $(B_{\varepsilon_0})_{ij} = \varepsilon_0^{j-i}\widehat{A}_{ij}$ for $i \leq j$. Equation (7.10) is now equivalent to

$$B_{\varepsilon_0}^* B_{\varepsilon_0} \leq I. \tag{7.11}$$

Given $x \in \mathbb{C}^m$, $\|B_{\varepsilon_0}x\|^2 = \sum_{1\leq i\leq m} |\sum_{i\leq j\leq m} \varepsilon_0^{j-i}\widehat{A}_{ij}x_j|^2$. The Cauchy-Schwarz inequality implies that

$$\|B_{\varepsilon_0}x\|^2 \leq \sum_{1\leq i\leq m} \left((\sum_{i\leq j\leq m} \varepsilon_0^{j-i}|\widehat{A}_{ij}|)(\sum_{i\leq j\leq m} \varepsilon_0^{j-i}|\widehat{A}_{ij}||x_j|^2) \right).$$

The condition [S], ii implies, if $\varepsilon_0 \leq (cm)^{-1}$,

$$\begin{aligned}
\|B_{\varepsilon_0}x\|^2 &\leq \sum_{1\leq i\leq m} \left((|\widehat{A}_{ii}| + \sum_{i<j\leq m} \varepsilon_0^{j-i}c(1-|\widehat{A}_{ii}|))(\sum_{i\leq j\leq m} \varepsilon_0^{j-i}|\widehat{A}_{ij}||x_j|^2) \right), \\
&\leq \sum_{1\leq i\leq m} \left((|\widehat{A}_{ii}| + 1 - |\widehat{A}_{ii}|)(\sum_{i\leq j\leq m} \varepsilon_0^{j-i}|\widehat{A}_{ij}||x_j|^2) \right), \\
&= \sum_{1\leq j\leq m} \sum_{1\leq i\leq j} \varepsilon_0^{j-i}|\widehat{A}_{ij}||x_j|^2, \\
&\leq \sum_{1\leq j\leq m} \left(|\widehat{A}_{jj}|^2|x_j|^2 + |x_j|^2 \sum_{1\leq i<j} \varepsilon_0^{j-i}c(1-|\widehat{A}_{jj}|) \right), \\
&\leq \sum_{1\leq j\leq m} (|\widehat{A}_{jj}| + 1 - |\widehat{A}_{jj}|)|x_j|^2, \\
&\leq \|x\|^2.
\end{aligned}$$

This proves inequality (7.11), which implies condition [H] via inequalities (7.9) and (7.10).

[H] $\Rightarrow$ [N] and [N] $\Rightarrow$ [Ω] have been proved in section 5.

[Ω] $\Rightarrow$ [A]. To begin this implication, we define $\{\zeta_k\}_{k=1}^n$ to be the n^{th} roots of unity, where

$n \in \boldsymbol{N}$. The following relationships hold for all complex numbers z (see [49]):

$$a)1 - z^n = \prod_{1 \le k \le n} (1 - \zeta_k z),$$

$$b)1 = \frac{1}{n} \sum_{1 \le j \le n} \prod_{\substack{k \ne j \\ 1 \le k \le n}} (1 - \zeta_k z).$$

It is clear that the same identities hold with z replaced by λA, where $\lambda \in \mathbb{C}$, $|\lambda| = 1$ and $A \in \mathcal{F}$. This is true because (a) and (b) are simply algebraic relationships between the various coefficients involving the n^{th} roots of unity. The same coefficients are obtained in the matrix equation.

Assume now that the condition $[\Omega]$ holds. For $x \in \mathbb{C}^m$, define

$$x_j := \prod_{\substack{k \ne j \\ 1 \le k \le n}} (I - \zeta_k \lambda A) x.$$

The condition [N] implies that, since $|\zeta_j \lambda| = 1$,

$$\begin{aligned}
0 &\le \frac{1}{n} \sum_{1 \le j \le n} \text{Re } \langle \Omega(I - \zeta_j \lambda A) x_j, x_j \rangle, \\
&= \frac{1}{n} \sum_{1 \le j \le n} \text{Re } \langle \Omega(I - \zeta_j \lambda A) \prod_{\substack{k \ne j \\ 1 \le k \le n}} (I - \zeta_k \lambda A) x, x_j \rangle, \\
&= \text{Re } \frac{1}{n} \sum_{1 \le j \le n} \langle \Omega(I - \lambda^n A^n) x, x_j \rangle, \\
&= \text{Re } \langle \Omega(I - \lambda^n A^n) x, \frac{1}{n} \sum_{1 \le j \le n} x_j \rangle, \\
&= \text{Re } \langle \Omega(I - \lambda^n A^n) x, x \rangle.
\end{aligned}$$

Hence, upon choosing λ such that Re $\bar{\lambda}^n \langle \Omega A^n x, x \rangle = |\langle \Omega A^n x, x \rangle|$, there results

$$|\langle \Omega A^n x, x \rangle| \le \langle \Omega x, x \rangle.$$

Thus

$$r_\Omega(A^n) \le 1.$$

Since the matrix Ω is positive definite and hermitian, it has a nonsingular square root, T. This means that, for $n \in \mathbb{N}$,

$$\|A^n\| \leq \|T\| \, \|T^{-1}\| \, \|TA^nT^{-1}\|,$$
$$\leq 2\|T\| \, \|T^{-1}\| \, r_I(TA^nT^{-1}).$$

Since $\|T\|$ and $\|T^{-1}\|$ are bounded by $\|\Omega\|^{\frac{1}{2}}$ and $\|\Omega\|^{-\frac{1}{2}}$, respectively, this yields

$$\|A^n\| \leq 2cr_\Omega(A^n) \leq 2c,$$

which is condition [A]. ∎

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