Deformation of a Membrane Under Uniform Static Pressure

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Abstract

We analyze the deformation of an isotropic, homogeneous circular membrane due to a uniform static pressure applied to one side. The solution is obtained as a perturbation expansion in terms of the pressure. In our analysis the stress is approximated by taking only terms linear in the Euler strain, and thus the analysis applies only to deformations that have small strain. It does include nonlinear geometric effects since the strain is a nonlinear function of the deformation gradient. This theory does not include bending effects, which is reasonable in most paper and film applications.

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1. Introduction.

In this paper we present a derivation of equations describing the shape of a membrane disk subjected to a uniform static pressure on one side. We use a perturbation expansion in terms of the pressure and determine the deformation as a function of the pressure. The equations are nonlinear since the normal and radial displacements are not of the same order in the perturbation parameter.

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Our interest in this problem arises from a desire to better understand the burst test or Mullen test which is used throughout the paper, paperboard, and corrugated paperboard industries as a measure of product integrity. In the burst test, a paper specimen is clamped by circular, rigid clamps and loaded by a rubber diaphragm, which in turn is displaced by a fluid. The pressure causing the specimen to rupture is called the burst factor. This test was designed to simulate a paper maker's old test of paper, where paper makers would hold a sheet of paper and try to puncture the paper with their thumb.

Other researchers have analyzed the deformation of paper by one of two methods. Early work assumed the shape of the deformed surface was a spherical cap [1], [9], and [12]. This shape was used because it simplified the analysis and seemed reasonable based on previous work [3]. Recently, Suhling [10] has found that the shape of the deformation associated with this geometry is not spherical, but more nearly parabolic. This result agrees with the results obtained in this paper. His analysis used a form of von Karman plate theory for nonlinear materials. The system of partial differential equations was solved using the finite element numerical analysis technique. Suhling found that the contribution of bending was negligible; the dominant effect was due to membrane forces. His finite element results agree with experimental measurements of the out-of-plane deformations.

The previous theories have proved inadequate for predicting the deformation associated with this geometry because the shape is not spherical and the deflections are very large compared to the thickness. This paper provides an alternative analysis which imposes no limitation on the shape of deformation or the material constitutive behavior. The sole limitations of this analysis are that bending effects are negligible and the strains are small.

The equations for rotationally symmetric deformations were derived by Föppl [7] and generalized by Bromberg and Stoker [2]. These are the equations for the first significant terms we obtain in section 4. Analytical methods for obtaining the solution as a power series were presented by Henky [8] and analyzed by Dickey [4]. Dickey presented a numerical method, based on an integral equation, for obtaining the solution.

In this paper we present a more modern derivation of the equations, first obtaining a general form and then, by a perturbation analysis, obtaining the equations of Föppl. The advantage of our method is that it can be easily extended to cover the case of anisotropic media, as we intend to do in a subsequent paper. Secondly, we present an efficient numerical method for obtaining the solution of the system of equations. Finally we compare or results to experimental measurements.

The outline of the paper is as follows. In section 2 the basic system of equations is derived for a rather general membrane. In section 3 we discuss the constitutive relation for the membrane and in section 4 we begin the perturbation analysis of the equations. The solution of the first significant terms of the perturbation analysis is the topic of section 5, an efficient numerical method of evaluating the first significant terms is given in section 6. In section 7 we compare the analytical results to several experimental results. The conclusions of the analysis, as well as topics for further work, are stated in section 8. There are two appendices, the first discussing the Lamé moduli for a membrane, and the second is devoted to proving a result from section 5.

2. Derivation of the General Membrane Equations.

We model the membrane as a two-dimensional surface and consider its behavior under deformation. Take an undeformed two-dimensional body, call it Ω, and consider the deformed body, $\Psi(\Omega)$, to be the result of a mapping Ψ from Ω into R^3. In our application Ω is a disk of radius D. The deformation gradient of Ψ is $F = \nabla\Psi$. Using polar coordinates (ρ, φ) on the disk Ω in R^2, and cylindrical coordinates (r, θ, z) in R^3, we have $\Psi(\rho,\varphi) = (r(\rho,\varphi),\ \theta(\rho,\varphi),\ z(\rho,\varphi))$ and

$$F = \begin{pmatrix} \dfrac{\partial r}{\partial \rho} & \dfrac{1}{\rho}\dfrac{\partial r}{\partial \varphi} \\ r\dfrac{\partial \theta}{\partial \rho} & \dfrac{r}{\rho}\dfrac{\partial \theta}{\partial \varphi} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{1}{\rho}\dfrac{\partial z}{\partial \varphi} \end{pmatrix}.$$

The Jacobian for the transformation is $J = \det(F^T F)^{1/2}$. The unit normal to the surface $\Psi(\Omega)$ at $\Psi(\rho,\varphi)$, call it $\vec{m}$, is the cross-product of the two columns of F, normalized to have unit length. We have

$$\vec{m} = \begin{pmatrix} \dfrac{\partial\theta}{\partial\rho}\dfrac{\partial z}{\partial\varphi} - \dfrac{\partial z}{\partial\rho}\dfrac{\partial\theta}{\partial\varphi} \\ \dfrac{1}{r}\left(\dfrac{\partial z}{\partial\rho}\dfrac{\partial r}{\partial\varphi} - \dfrac{\partial r}{\partial\rho}\dfrac{\partial z}{\partial\varphi}\right) \\ \dfrac{\partial r}{\partial\rho}\dfrac{\partial\theta}{\partial\varphi} - \dfrac{\partial\theta}{\partial\rho}\dfrac{\partial r}{\partial\varphi} \end{pmatrix} \frac{1}{M} \tag{2.1}$$

where

$$M^2 = \left(\frac{\partial\theta}{\partial\rho}\frac{\partial z}{\partial\varphi} - \frac{\partial z}{\partial\rho}\frac{\partial\theta}{\partial\varphi}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial\rho}\frac{\partial r}{\partial\varphi} - \frac{\partial r}{\partial\rho}\frac{\partial z}{\partial\varphi}\right)^2 + \left(\frac{\partial r}{\partial\rho}\frac{\partial\theta}{\partial\varphi} - \frac{\partial\theta}{\partial\rho}\frac{\partial r}{\partial\varphi}\right)^2 .$$

To derive the equations describing the deformation of the membrane, we consider a small subdomain P of Ω. The deformed membrane contains the subdomain $\Psi(P)$ and we assume that the boundary transforms nicely, i.e., $\partial\Psi(P) = \Psi(\partial P)$. At a point q in $\partial\Psi(P)$ the outer unit normal to $\partial\Psi(P)$ in the tangent plane to $\Psi(P)$ will be denoted $\vec{\eta}$. If $\vec{\tau}$ is the unit tangent vector to $\partial\Psi(P)$, then $\vec{m} = \vec{\eta} \times \vec{\tau}$, assuming the usual orientations.

We now consider the static force balance on the membrane. This is simply the balance between the resultant pressure force and the force due to the deformation of the materials on each portion. Mathematically this is expressed as

$$-\int_{\Psi(P)} p_0 \vec{m}\, da = \int_{\partial\Psi(P)} T \cdot \vec{\eta}\, d\ell \tag{2.2}$$

where p_0 is the uniform static pressure per unit area, T is the Cauchy stress tensor, and ℓ is the arc length along $\partial\Psi(P)$.

Our first form of the force balance equation comes from applying the divergence theorem to (2.2). We obtain

$$-\int_{\Psi(P)} p_0 \vec{m}\, da = \int_{\Psi(P)} \operatorname{div}\,(T)\, da\,,$$

and since P is an arbitrary domain,

$$-p_0 \vec{m} = \operatorname{div}\, T\,. \tag{2.3}$$

Together with a constitutive relation expressing the stress T as a function of F, equation (2.3) describes the deformation of the membrane under the pressure force.

From a computational standpoint equation (2.3) is unsatisfactory since the divergence operator is defined in terms of the unknown surface. We now reformulate the problem to use derivatives on the undeformed region. This is essentially a change of variables in the integrals in (2.2).

By definition of the Jacobian we have that the left-hand side of (2.2) transforms to

$$-p_0 \int_{\Psi(P)} \vec{m}\, da = -p_0 \int_P \vec{m} J\, dA\,,$$

using dA as the measure on P, and where $J = \det(F^T F)^{1/2}$. Also, the right-hand side of (2.2) transforms to

$$\int_{\partial\Psi(P)} T\vec{\eta}\, d\ell = \int_{\partial P} TG\vec{n}\, dL$$

where $\vec{n}$ is the unit outer normal on ∂P,

$$G = F(F^T F)^{-1} J \tag{2.4}$$

and L is arc length on ∂P. The tensor G maps the outer normal on ∂P to the outer normal on $\partial\Psi(P)$. The tensor TG is the Piola-Kirchhoff tensor for the membrane. Applying the divergence theorem on P we have

$$-p_0 \int_P \vec{m} J\, dA = \int_P \mathrm{DIV}\ (TG)\, dA\,,$$

giving the equation

$$-p_0 \vec{m} J = \mathrm{DIV}\ (TG)\,. \tag{2.5}$$

We use the notation of Gurtin [6], in which DIV refers to the divergence operator in the undeformed coordinates, and div refers to the deformed coordinates.

Equation (2.5) is a nonlinear partial differential equation for the position vector of the deformed membrane as a function of its position in the undeformed membrane. Given the constitutive relations that relate T to the deformation gradient, the system (2.5) determines the displacement.

For the problem of the burst test, we consider paper to be a membrane and held fixed at the edge of the disk of radius D, giving the boundary conditions

$$r(D,\phi) = D, \quad \theta(D,\phi) = \phi, \quad \text{and} \quad z(D,\phi) = 0.$$

3. The Constitutive Relation.

In this section we address the determination of the stress tensor T as a function of the deformation gradient F, and material properties. By invariance to the observer, see [6], T has the form

$$T = F\tilde{T}F^T$$

where $\tilde{T}$ is some tensor function of F^TF. We also have that T vanishes when $F^TF = I$, i.e., there is no stress when there is no deformation.

Rather than consider the tensor $\tilde{T}$ in general, we consider the expansion of $\tilde{T}$ into linear and higher order terms, we have

$$\tilde{T} = \tilde{T}_1(E) + O(\|E\|^2)$$

where $\tilde{T}_1(\cdot)$ is a linear mapping of the material strain tensor E to the stress tensor $\tilde{T}$. Since $\tilde{T}_1$ is linear, by invariance of observer, see [6], $\tilde{T}_1$ must have the form

$$\tilde{T}_1 = 2\bar{\mu}E + \bar{\lambda}\,\mathrm{tr}(E)I \tag{3.1}$$

where $\bar{\mu}$ and $\bar{\lambda}$ are the Lamé constants for the two-dimensional surface. These constants are related to the three-dimensional Lamé constants μ and λ by

$$\bar{\mu} = \mu h \qquad \bar{\lambda} = \frac{2\mu\lambda}{2\mu + \lambda}h$$

where h is the thickness of the membrane, as shown in Appendix 1. We take for the material strain tensor the Euler strain $E = \frac{1}{2}(F^TF - I)$.

In terms of Young's modulus and Poisson's ratio, see Appendix 2. we can rewrite (3.1) as

$$\tilde{T}_1 = \frac{Yh}{1-\nu^2}\left((1-\nu)E + \nu\ \mathrm{tr}(E)I\right)$$
$$= \frac{Yh}{1-\nu^2}\left(\frac{1}{2}(1-\nu)(F^TF - I) + \frac{1}{2}\nu\ \mathrm{tr}(F^TF - I)I\right).$$

We approximate $\tilde{T}$ by the linear part $\tilde{T}_1$, restricting ourselves to the case of small strains. Thus the equation (2.5) can be written

$$-p_0\vec{m}J = \mathrm{DIV}\ (F\tilde{T}_1J)\,.$$

using the definition of G, see (2.4).

We also now nondimensionalize the problem to prepare for the perturbation analysis. We replace the coordinates (ρ, φ) and (r, θ, z) by $(D\rho, \varphi)$ and (Dr, θ, Dz), where now ρ and r vary from 0 to 1, recall that D is the radius of the circular membrane. We also drop the subscript of 1 on $\tilde{T}_1$.

We then write the above equation as

$$-\varepsilon\vec{m}J = \mathrm{DIV}\ (F\tilde{T}J)\,, \tag{3.2}$$

where

$$\varepsilon = \frac{(1-\nu^2)Dp_0}{Yh}$$

and

$$\tilde{T} = (1-\nu)E + \nu\ \mathrm{tr}(E)I\,.$$

The boundary conditions for the burst test in nondimensional form are then

$$r(1,\phi) = 1, \quad \theta(1,\phi) = \phi, \quad \text{and} \quad z(1,\phi) = 0. \tag{3.3}$$

From now on all quantities are nondimensional.

4. Perturbation Expansion of the Solution.

We begin our analysis of equation (3.2) by performing a perturbation expansion of the solution using ε as the perturbation parameter. For $\varepsilon = 0$, it is easily checked that $\Psi(\rho,\varphi) = (\rho,\varphi,0)$ is a solution, we assume that this solution is unique.

We consider the solution of (3.2) for all values of ν with $0 \le \nu \le 1$, and small values of ε. Even though the interpretation of the results may not be physically applicable at $\nu = 1$, the mathematical problem is well-defined. The solutions we obtain at $\nu = 1$ may be regarded as an appropriate limiting case for ν near 1.

Because the deformation perpendicular to the plane of the membrane is a different order than the deformation within the plane, it is necessary to use fractional powers of ε in the expansion. We write

$$\begin{aligned} r &= \rho + \varepsilon^{1/3} r_1 + \varepsilon^{2/3} r_2 + O(\varepsilon)\,, \\ \theta &= \varphi + \varepsilon^{1/3}\theta_1 + \varepsilon^{2/3}\theta_2 + O(\varepsilon)\,, \\ z &= \varepsilon^{1/3} z_1 + \varepsilon^{2/3} z_2 + O(\varepsilon)\,. \end{aligned} \tag{4.1}$$

The choice of the fractions for the exponents on ε is a result of the analysis. In general, one should expand r, θ, and z in arbitrary powers of ε. It comes out of the analysis that those given in (4.1) are the only ones giving a solution. Notice that the subscripts on the terms of the expansions for r, θ, and z are equal to the power of $\varepsilon^{1/3}$.

The symbolic manipulation software MACSYMA [11] was used in the analysis given next. Its use reduced the possibility of errors in the manipulations and gave several valuable insights.

Our first result, the proof of which is in Appendix 2, is:

Theorem 4.1.

$$r_1(\rho,\varphi) = 0 \quad \text{and} \quad \theta_1(\rho,\varphi) = 0\,.$$

The displacement gradient F is expanded in powers of ε as

$$F = F_0 + \varepsilon^{1/3} F_1 + \varepsilon^{2/3} F_2 + O(\varepsilon)$$

and Theorem 4.1 gives us the forms

$$F_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \dfrac{\partial z_1}{\partial \rho} & \dfrac{1}{\rho}\dfrac{\partial z_1}{\partial \varphi} \end{pmatrix}.$$

$$F_2 = \begin{pmatrix} \dfrac{\partial r_2}{\partial \rho} & \dfrac{1}{\rho}\dfrac{\partial r_2}{\partial \varphi} \\ \rho\dfrac{\partial \theta_2}{\partial \rho} & \dfrac{r_2}{\rho} + \dfrac{\partial \theta_2}{\partial \varphi} \\ \dfrac{\partial z_2}{\partial \rho} & \dfrac{1}{\rho}\dfrac{\partial z_2}{\partial \varphi} \end{pmatrix}.$$

We then have

$$\begin{aligned} F^TF &= (F_0^T + \varepsilon^{1/3}F_1^T + \varepsilon^{2/3}F_2^T + O(\varepsilon))(F_0 + \varepsilon^{1/3}F_1 + \varepsilon^{2/3}F_2 + O(\varepsilon)) \\ &= F_0^TF_0 + \varepsilon^{1/3}(F_0^TF_1 + F_1^TF_0) + \varepsilon^{2/3}(F_0^TF_2 + F_2^TF_0 + F_1^TF_1) + O(\varepsilon^2) \\ &= I + \varepsilon^{2/3}\begin{pmatrix} 2\dfrac{\partial r_2}{\partial \rho} + \left(\dfrac{\partial z_1}{\partial \rho}\right)^2 & \dfrac{1}{\rho}\dfrac{\partial r_2}{\partial \varphi} + \rho\dfrac{\partial \theta_2}{\partial \rho} \\ \dfrac{1}{\rho}\dfrac{\partial r_2}{\partial \varphi} + \rho\dfrac{\partial \theta_2}{\partial \rho} & 2\dfrac{r_2}{\rho} + 2\dfrac{\partial \theta_2}{\partial \varphi} + \left(\dfrac{1}{\rho}\dfrac{\partial z_1}{\partial \varphi}\right)^2 \end{pmatrix} + O(\varepsilon). \end{aligned}$$

This gives the expansion of the Jacobian $J = \det(F^TF)^{1/2}$ as

$$J^2 = 1 + \varepsilon^{2/3}\left\{2\left(\frac{\partial r_2}{\partial \rho} + \frac{r_2}{\rho} + \frac{\partial \theta_2}{\partial \varphi}\right) + \left(\frac{\partial z_1}{\partial \rho}\right)^2 + \left(\frac{1}{\rho}\frac{\partial z_1}{\partial \varphi}\right)^2\right\} + O(\varepsilon)$$

and

$$J = 1 + \varepsilon^{2/3}\left\{\left(\frac{\partial r_2}{\partial \rho} + \frac{r_2}{\rho} + \frac{\partial \theta_2}{\partial \varphi}\right) + \frac{1}{2}\left(\frac{\partial z_1}{\partial \rho}\right)^2 + \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial z_1}{\partial \varphi}\right)^2\right\} + O(\varepsilon)$$

We also have

$$E = \frac{1}{2}(F^TF - I) = \varepsilon^{2/3}E_2 + O(\varepsilon)$$

with

$$E_2 = \begin{pmatrix} \dfrac{\partial r_2}{\partial \rho} + \dfrac{1}{2}\left(\dfrac{\partial z_1}{\partial \rho}\right)^2 & \dfrac{1}{2}\left(\dfrac{1}{\rho}\dfrac{\partial r_2}{\partial \varphi} + \rho\dfrac{\partial \theta_2}{\partial \rho}\right) \\ \dfrac{1}{2}\left(\dfrac{1}{\rho}\dfrac{\partial r_2}{\partial \varphi} + \rho\dfrac{\partial \theta_2}{\partial \rho}\right) & \dfrac{r_2}{\rho} + \dfrac{\partial \theta_2}{\partial \varphi} + \dfrac{1}{2}\left(\dfrac{1}{\rho}\dfrac{\partial z_1}{\partial \varphi}\right)^2 \end{pmatrix}.$$

We obtain for equation (3.2)

$$F\tilde{T}J = TG = FEJ = \varepsilon^{2/3}F_0E_2 + \varepsilon F_1E_2 + O(\varepsilon^{4/3}).$$

Substitution of these expansions in (3.2) results in

$$-\varepsilon\vec{m}_0 = \varepsilon^{2/3}\mathrm{DIV}(F_0E_2) + \varepsilon\mathrm{DIV}(F_1E_2) + O(\varepsilon^{4/3})$$

where $\vec{m}_0$ is the unit vector in the positive z–direction. The system of equations for r_2, θ_2, and z_1 are obtained by taking the first two equations to $O(\varepsilon^{2/3})$ and the third equation to $O(\varepsilon)$.

The DIV operator acting on a tensor S_{ij} with $i = 1,2,3$ and $j = 1,2$ in polar coordinate representation has the form

$$(\mathrm{DIV}S)_1 = \frac{1}{\rho}\frac{\partial(\rho S_{11})}{\partial\rho} + \frac{1}{\rho}\frac{\partial S_{12}}{\partial\varphi} - \frac{S_{22}}{\rho}$$

$$(\mathrm{DIV}S)_2 = \frac{1}{\rho}\frac{\partial(\rho S_{21})}{\partial\rho} + \frac{1}{\rho}\frac{\partial S_{22}}{\partial\varphi} + \frac{S_{12}}{\rho}$$

$$(\mathrm{DIV}S)_3 = \frac{1}{\rho}\frac{\partial(\rho S_{31})}{\partial\rho} + \frac{1}{\rho}\frac{\partial S_{32}}{\partial\varphi}\,.$$

5. Solution for the first significant terms.

Because of the symmetry of the domain, and the symmetry of the boundary conditions, i.e., $r_2 = 0$, $\theta_2 = 0$ on the boundary of the disk, we can take θ_2 equal to zero everywhere and have r_2 and z_1 be independent of φ. This gives the system of ordinary differential equations

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\left(\frac{dr_2}{d\rho} + \nu\frac{r_2}{\rho} + \frac{1}{2}\left(\frac{dz_1}{d\rho}\right)^2\right)\right] - \frac{\nu}{\rho}\left(\frac{dr_2}{d\rho} + \frac{1}{2}\left(\frac{dz_1}{d\rho}\right)^2\right) - \frac{r_2}{\rho^2} = 0 \tag{5.1}$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\left(\frac{dz_1}{d\rho}\right)\left(\frac{dr_2}{d\rho} + \nu\frac{r_2}{\rho} + \frac{1}{2}\left(\frac{dz_1}{d\rho}\right)^2\right)\right] = -1\,. \tag{5.2}$$

The solution of the system (5.1) and (5.2) is given by the following theorem.

Theorem 5.1. *Let $L(\tau)$ be the solution of the differential equation*

$$\tau\frac{d^2L}{d\tau^2}+2\frac{dL}{d\tau}+\frac{2(1-\nu^2)}{L^2}=0 \tag{5.3}$$

with $L(0)=1+\nu$, and, for $\nu\neq 1$, let the functions $A(\tau)$ and $B(\tau)$ be given by

$$A(\tau)=\int_0^\tau \frac{1}{L(\sigma)}d\sigma \tag{5.4}$$

$$B(\tau)=(2\tau\frac{dL}{d\tau}+(1-\nu)L)/(1-\nu^2)\,, \tag{5.5}$$

and define the value of the positive parameter α by $B(\alpha)=0$, then solution of (5.1) and (5.2) is given by

$$\begin{aligned} r_2(\rho)&=\frac{1}{4}\rho B(\alpha\rho^2)\alpha^{-1/3}\\ z_1(\rho)&=(A(\alpha)-A(\alpha\rho^2))\alpha^{-2/3}\,. \end{aligned} \tag{5.6}$$

For $\nu=1$ the solution is

$$\begin{aligned} r_2(\rho)&=2^{-7/3}\rho(1-\rho^2)\\ z_1(\rho)&=2^{-2/3}(1-\rho^2). \end{aligned}$$

The solution of (5.3) is easily computed using a simple transformation and a Taylor series, from which the series expansions of $A(\tau)$ and $B(\tau)$ can be obtained. The result is an efficient numerical procedure for obtaining values for $r_2(\rho)$ and $z_1(\rho)$. The numerical procedure to evaluate the functions A and B is discussed in the section 6.

We now prove Theorem 5.1. Equation (5.2) can be integrated once to give

$$\frac{dz_1}{d\rho}\left(\frac{dr_2}{d\rho}+\nu\frac{r_2}{\rho}+\frac{1}{2}\left(\frac{dz_1}{d\rho}\right)^2\right)=-\frac{\rho}{2} \tag{5.7}$$

using the condition that r_2, z_1, and their first derivatives are finite at $\rho=0$.

We now change variables with

$$\begin{aligned} \rho&=(\tau/\alpha)^{1/2}\\ r_2&=\frac{1}{4}\alpha^{-5/6}\tau^{1/2}B(\tau)\\ z_1&=Z_0-\alpha^{-2/3}A(\tau) \end{aligned} \tag{5.8}$$

with $B(0) = 1$ and $A(0) = 0$. The scaling with the parameter α and the boundary conditions are easily motivated by considering the Taylor series expansion of $r_2(\rho)$ and $z_1(\rho)$ around the origin. Since $r_2(1) = 0$, we obtain α as the solution to $B(\alpha) = 0$, and since $z_1(1) = 0$, we have that $Z_0 = \alpha^{-2/3}A(\alpha)$. Values of α and Z_0 as functions of ν are given in Table 1.

ν	α	Z_0
0.0	0.2350	0.7183
0.1	0.3000	0.7012
0.2	0.3778	0.6867
0.3	0.4708	0.6743
0.4	0.5823	0.6637
0.5	0.7162	0.6547
0.6	0.8778	0.6472
0.7	1.0744	0.6410
0.8	1.3161	0.6361
0.9	1.6173	0.6324
1.0	2.0000	0.6300

Table 1

With the transformation (5.8) the system of (5.1) and (5.7) becomes

$$\begin{aligned} &\tau B'' + 2B' + 8\tau AA'' + (6 - 2\nu)A'^2 = 0 \\ &2\tau B' + (1+\nu)B + 8\tau A'^2 = (A')^{-1} . \end{aligned} \tag{5.9}$$

Taking twice the first equation of this system and subtracting the derivative of the second equation gives the equation

$$(1-\nu)B' + 4(1-\nu)A'^2 = -((A')^{-1})' .$$

Setting $L(\tau) = (A'(\tau))^{-1}$, we obtain, for $\nu \neq 1$,

$$B' = -L'/(1-\nu) - 4/L^2 . \tag{5.10}$$

Note that for $\nu = 1$, we have that $A'(\tau)$ is constant. We consider the case $\nu = 1$ later.

Substituting the expression for B' from (5.10) in the second equation of (5.9) we obtain

$$(1+\nu)B = 2\tau L'/(1-\nu) + L\,. \tag{5.11}$$

This formula gives $L(0) = (1+\nu)B(0) = 1+\nu$.

Differentiating (5.11) and equating the derivative of B with the formula in (5.10), we obtain the equation for L

$$\tau L'' + 2L' + 2(1-\nu^2)/L^2 = 0\,,$$

which is (5.3), and with the boundary condition $L(0) = 1+\nu$ determines a unique solution. Once $L(\tau)$ is obtained, we then obtain $B(\tau)$ by (5.11) and obtain $A(\tau)$ by

$$A(\tau) = \int_0^\tau \frac{1}{L(\sigma)}d\sigma$$

which is equation (5.4).

The function L and the parameter α completely determine the solution of the system (5.1) and (5.2). This completes the proof of Theorem 5.1.

6. The Numerical Method to Evaluate z_1 and r_2.

In this section we discuss the numerical evaluation of the functions L, A, and B that, by Theorem 5.1, are used to determine the functions z_1 and r_2 by (5.6).

We first show how to compute $L(\tau)$. It is not hard to check that $L(\tau)$ for general ν is related to $L(\tau)$ for $\nu = 0$, which we call $L_0(\tau)$, by the relation

$$L(\tau) = (1+\nu)L_0\left(\frac{(1-\nu)}{(1+\nu)^2}\tau\right) \tag{6.1}$$

This relation was discovered with the aid of the symbolic manipulation language MACSYMA, [11].

We determine $L_0(\tau)$ as a power series in τ. Let

$$L_0(\tau) = \sum_{k=0}^{\infty} d_k \tau^k \tag{6.2}$$

with $d_0 = 1$. We rewrite equation (5.3) as

$$L_0^2(\tau L_0'' + 2L_0') = -2\,.$$

Substituting (6.2) in this expression we obtain

$$\sum_{m=1}^{\infty}\Big[\sum_{i+j+k=m} d_i d_j d_k k(k+1)\Big]\tau^{m-1} = -2\,.$$

For m equal to 1, we easily obtain $d_1 = -1$. For m greater than 1, we obtain

$$d_0^2 d_m m(m+1) = -\sum_{k=1}^{m-1}\Big(\sum_{j=0}^{m-k} d_{m-j-k} d_j\Big) d_k k(k+1)$$

to determine the coefficients d_m recursively.

The coefficients d_m grow geometrically. The radius of convergence was estimated using the ratio test on the first 100 terms, and was found to be between 0.47 and 0.48. Since we require the evaluation of L_0 for τ at most 0.25, the power series can be used with 20 or fewer terms. Some values of the coefficients are displayed in Table 2. The series is also used to evaluate L_0'.

The reciprocal of L_0 can be computed as

$$L_0(\tau)^{-1} = \sum_{k=0}^{\infty} c_k \tau^k$$

by the formulas $c_0 = 1$ and

$$c_m = -\sum_{k=0}^{m-1} c_k d_{m-k}\,.$$

Values of c_m are displayed in Table 2.

The series for $A(\tau)$ is then

$$A(\tau) = \frac{\tau}{1+\nu}\sum_{m=0}^{\infty}\frac{c_m}{m+1}\left(\frac{1-\nu}{(1+\nu)^2}\tau\right)^m.$$

m	d_m	c_m
0	1.00000	1.00000
1	−1.00000	1.00000
2	−0.66667	1.66667
3	−0.72222	3.05556
4	−0.94444	5.83333
5	−1.37037	11.38889
6	−2.12522	22.55423
7	−3.45240	45.10716
8	−5.80372	90.86327
9	−10.01663	184.04195
10	−17.65217	374.39085
11	−31.63996	764.28079
12	−57.51381	1564.70984
13	−105.78964	3211.18846
14	−196.56074	6603.77030
15	−368.41284	13604.66947
16	−695.78218	28070.73418
17	−1322.86550	57997.25720
18	−2530.07012	119972.39115
19	−4864.60373	248438.36363
20	−9397.79906	514957.71646

Table 2

The value of α can be determined by Newton's method with $B(\tau)$ evaluated by (5.11) and $B'(\tau)$ evaluated by (5.10). Relatively few Newton iterations are required to determine the value of α.

For $\nu = 1$, we have that $A'(\tau)$ is constant, as we observed from equation (5.10). The system (5.9) is easily seen to have the solution

$$A(\tau) = \frac{1}{2}\tau$$

$$B(\tau) = 1 - \frac{1}{2}\tau\,.$$

These formulas can also be obtained from definitions (5.4), (5.5), and (6.1) by taking the limit as ν tends to 1. For $\nu = 1$ the value of α is 2, and from (5.8) we have

$$r_2(\rho) = 2^{-7/3}\rho(1-\rho^2)$$
$$z_1(\rho) = 2^{-2/3}(1-\rho^2).$$

In Figure 1 the displacement is displayed for the case with $\nu = 0.3$ and $\varepsilon = 0.2$. The dots mark the deformed positions of the points whose undeformed positions are $\rho = 0$, 0.25, 0.5, 0.75, and 1.

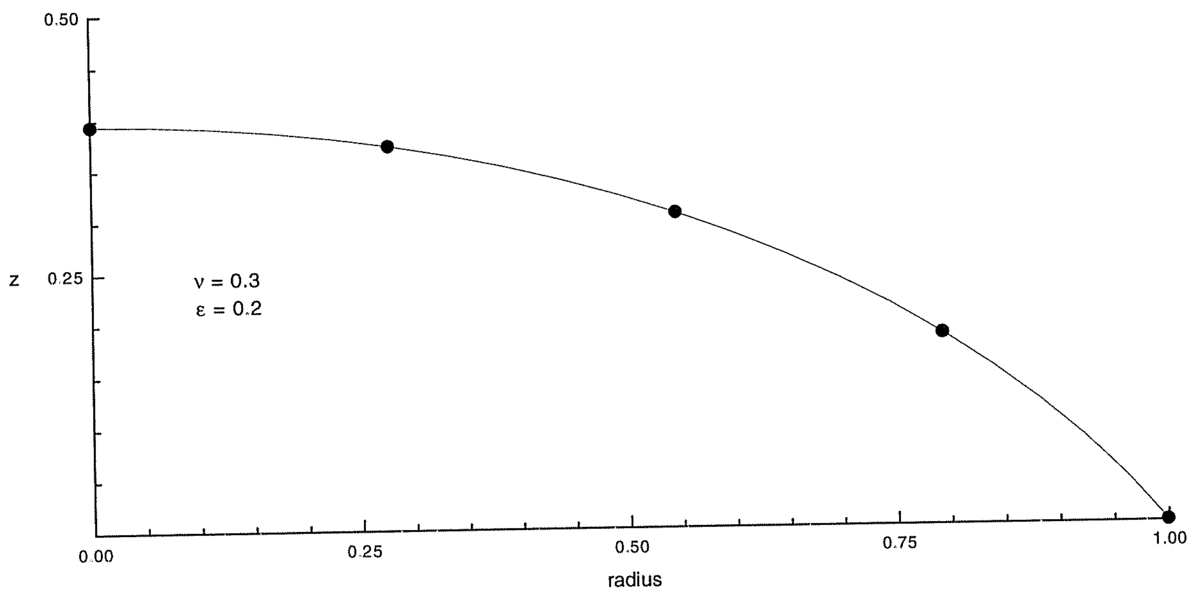

Figure 1

7. Comparison with Experiments.

In this section we apply the results of the analysis to compare with experimental results. For the first comparison we use the data obtained in tests conducted by the U.S. Navy on the deflection of steel plates [5]. We compare our results with experiments using steel disks of radius 10 inches and either one-eighth or one-sixteenth of an inch thick. Although the primary interest in these experiments was with the plastic deformation at large strains, there are sufficient data at low strains in the elastic deformation range with

thickness	pressure (lbs. / in^2)	deflection experimental	deflection analysis
1/16	25	0.31	0.32
1/16	50	0.42	0.42
1/16	100	0.50	0.51
1/8	50	0.35	0.33
1/8	100	0.46	0.42
1/8	200	0.62	0.53

Comparison of experimental and analytical results for STS
Table 3

which to compare our analytical results. Also, the tests show that the steel was essentially isotropic for small deformations, but was orthotropic for plastic deformations.

There are several qualitative agreements that can be made immediately. Several of the figures in [5] show that the pressure is proportional to the cube of the center deflection. Selected values that show good agreement are given in Table 3, showing the results for special treatment steel (STS). From the stress-strain curves, Young's modulus is computed to be $Y = 3.0\ 10^7$ lbs. per ft^3 and we take Poisson's ratio to be 0.29, which is typical of most steels. The value of ε is

$$\varepsilon = \frac{(1-\nu^2)Dp}{Yh}$$

where h is the thickness of the disk. The formula for the approximate center displacement is $\varepsilon^{1/3}D\ Z_0$ where Z_0 is the center deflection computed from (5.6), see Table 1. For higher pressures, the center deflection was larger than that predicted by this analysis, reflecting either the limited validity of the approximation from the perturbation analysis, the plastic behavior of the material, or both.

It is also reported in [5] that the strain at several points on the disk is proportional to the square of the center deflection (Fig. 31 of [5]). This agrees with our analysis that shows that strain is proportional to $\varepsilon^{2/3}$ and the center deflection is proportional to $\varepsilon^{1/3}$.

A second set of comparisons can be made with the finite element calculations of Suhling [10] to model the deformation of paper board in the burst test. Although his model is for an anisotropic material, there is good agreement with several data sets. For his data giving the center deflection as a function of pressure (actually pressure divided by thickness) there is excellent agreement with the conclusion that the center deflection is proportional to the pressure to the 1/3 power, see Table 7.1 of [10]. Over the range of p_0/h from 0.125 to 30.0 the value of $Z_0(p_0/h)^{1/3}$ varies between 0.02493 and 0.02499. Also the normal stresses at the center were proportional to p_0/h to the two-thirds power over the same range of p_0/h.

Appendix 1. The Lamé moduli for the two-dimensional membrane.

In this appendix we determine the Lamé moduli and constitutive equations for two-dimensional membranes by analysis of three-dimensional bodies using the theory of elasticity.

Consider the standard linear relation between stress T and strain E for a three-dimensional isotropic elastic solid,

$$T = 2\mu E + \lambda \operatorname{tr}(E) I \; . \tag{7.1}$$

The constants μ and λ are the Lamé moduli of the material. We consider the case where the material is assumed to be in a plane stress condition, i.e., $T_{3,3} = T_{1,3} = T_{2,3} = 0$. This situation arises when considering a membrane, for which the third dimension of the body is negligible in thickness, and we are concerned only with the stresses and strains in the plane of the membrane. We actually define the membrane by asserting that all stresses lie in the plane of the membrane.

We wish to reformulate (7.1) as

$$t = 2\bar{\mu}\; e + \bar{\lambda} \operatorname{tr}(e)\; I \; . \tag{7.2}$$

where t and e are the two-dimensional stress and strain, respectively, in the plane of the membrane. The form of the stress in terms of the strain given in (7.1) is the most general form for a linear function of the strain that is invariant under change of observer when the material is isotropic. The same is also true for (7.2), however for two dimensions there is the additional requirement that there be a line of symmetry, or equivalently, that the two sides of the two-dimensional surface be indistinguishable.

Note that the symbol I is used for both the three-dimensional identity matrix, or tensor, in (7.1) and the two-dimensional identity matrix in (7.2). This should cause no difficulty, especially if one remembers that $\mathrm{tr}(I)$ is 2 or 3, depending on whether it is the two-dimensional or three-dimensional case.

Consider equation (7.1) for the situation in which T has the form

$$T = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$$

and we easily determine that E has the form

$$E = \begin{pmatrix} e & 0 \\ 0 & E_{3,3} \end{pmatrix} .$$

From (7.1) and $T_{3,3} = 0$ we obtain

$$E_{3,3} = -\frac{\lambda}{2\mu}\ \mathrm{tr}(E) .$$

and since $\mathrm{tr}(E) = \mathrm{tr}(e) + E_{3,3}$ we obtain

$$\mathrm{tr}(E) = \frac{2\mu}{2\mu + \lambda}\ \mathrm{tr}(e) .$$

Therefore, from (7.1), we may write the upper left two-by-two block as

$$t = 2\mu\, e + \frac{2\mu\lambda}{2\mu + \lambda}\ \mathrm{tr}(e)\ I$$

from which we conclude that

$$\bar{\mu} = \mu \quad \text{and} \quad \bar{\lambda} = \frac{2\mu\lambda}{2\mu + \lambda} .$$

For linear three-dimensional elasticity, Young's modulus Y and Poisson's ratio ν are defined by

$$Y = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda} \quad \text{and} \quad \nu = \frac{\lambda}{2(\mu + \lambda)}$$

or, in terms of μ and $\bar{\lambda}$,

$$Y = \frac{4\mu(\mu+\bar{\lambda})}{2\mu+\bar{\lambda}} \quad \text{and} \quad \nu = \frac{\bar{\lambda}}{2\mu+\bar{\lambda}}.$$

With these definitions we can rewrite (7.1) as

$$T = Y\frac{1-\nu}{(1+\nu)(1-2\nu)}\left((1-\nu)\,E + \nu\,\mathrm{tr}(E)\,I\right)\ .$$

and we can rewrite (7.2) as

$$t = \frac{Y}{1-\nu^2}\left((1-\nu)\,e + \nu\,\mathrm{tr}(e)\,I\right)\ .$$

If, as in section 3, the stress in the membrane is measured per unit length, then the stress t and strain e are averaged over the thickness of the membrane, giving

$$\bar{t} = \frac{Yh}{1-\nu^2}\left((1-\nu)\,\bar{e} + \nu\,\mathrm{tr}(\bar{e})\,I\right)\ ,$$

where the bars over the stress and strain reflect the averages through the membrane, and also that $\bar{t}$ is measured as force per unit length. This gives the Lamé moduli as used in section 3.

Appendix 2. The proof of Theorem 4.1.

In this appendix we prove Theorem 4.1. To order $\varepsilon^{1/3}$, by (2.1) the vector $\vec{m}$ is

$$\vec{m} = \begin{pmatrix} -\varepsilon^{1/3}\dfrac{\partial z_1}{\partial\rho} \\ -\varepsilon^{1/3}\dfrac{\partial z_1}{\rho\partial\varphi} \\ 1+O(\varepsilon^{1/3}) \end{pmatrix}\frac{1}{M} + O(\varepsilon^{2/3}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + O(\varepsilon^{1/3})\,.$$

The displacement gradient F is

$$F = F_0 + \varepsilon^{1/3}F_1 + O(\varepsilon^{2/3})$$

where

$$F_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} \dfrac{\partial r_1}{\partial \rho} & \dfrac{\partial r_1}{\rho \partial \varphi} \\ \rho \dfrac{\partial \theta_1}{\partial \rho} & \dfrac{\partial \theta_1}{\partial \varphi} + \dfrac{r_1}{\rho} \\ \dfrac{\partial z_1}{\partial \rho} & \dfrac{1}{\rho} \dfrac{\partial z_1}{\partial \varphi} \end{pmatrix}.$$

To evaluate J and $F^T F - I$, we first evaluate $F^T F$.

$$\begin{aligned} F^T F &= (F_0^T + \varepsilon^{1/3} F_1^T + O(\varepsilon^{2/3}))(F_0 + \varepsilon^{1/3} F_1 + O(\varepsilon^{2/3})) \\ &= F_0^T F_0 + \varepsilon^{1/3}(F_0^T F_1 + F_1^T F_0) + O(\varepsilon^{2/3}) \\ &= I + 2\varepsilon^{1/3} \begin{pmatrix} \dfrac{\partial r_1}{\partial \rho} & \dfrac{\partial r_1}{\rho \partial \varphi} \\ \rho \dfrac{\partial \theta_1}{\partial \rho} & \dfrac{\partial \theta_1}{\partial \varphi} + \dfrac{r_1}{\rho} \end{pmatrix} + O(\varepsilon^{2/3}). \end{aligned}$$

So

$$E = \frac{1}{2}(F^T F - I) = \varepsilon^{1/3} E_1 + O(\varepsilon^{2/3}),$$

with

$$E_1 = \begin{pmatrix} \dfrac{\partial r_1}{\partial \rho} & \dfrac{\partial r_1}{\rho \partial \varphi} \\ \rho \dfrac{\partial \theta_1}{\partial \rho} & \dfrac{\partial \theta_1}{\partial \varphi} + \dfrac{r_1}{\rho}. \end{pmatrix}$$

Also $J = \det(F^T F)^{1/2}$, hence

$$J = 1 + O(\varepsilon^{1/3})$$

and so,

$$FEJ = \varepsilon^{1/3} F_0 E_1 + O(\varepsilon^{2/3}),$$

or,

$$FEJ = \varepsilon^{1/3} \begin{pmatrix} \dfrac{\partial r_1}{\partial \rho} & \dfrac{\partial r_1}{\rho \partial \varphi} \\ \rho \dfrac{\partial \theta_1}{\partial \rho} & \dfrac{\partial \theta_1}{\partial \varphi} + \dfrac{r_1}{\rho} \\ 0 & 0 \end{pmatrix} + O(\varepsilon^{2/3}).$$

Using the first two equations of the system (3.2), we have two equations for r_1 and θ_1.

$$
\begin{aligned}
0 &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial r_1}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 r_1}{\partial \varphi^2} - \frac{1}{\rho}\left(\frac{\partial \theta_1}{\partial \varphi} + \frac{r_1}{\rho}\right) \\
0 &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial \theta_1}{\partial \rho}\right) + \frac{1}{\rho}\frac{\partial^2 \theta_1}{\partial \varphi^2} + \frac{2}{\rho^2}\frac{\partial r_1}{\partial \varphi} .
\end{aligned}
\tag{7.3}
$$

Note that the boundary conditions for r_1 and θ_1 on the boundary of the disk are that they both vanish, see (3.3) and (4.1).

By multiplying the first equation of (7.3) by ρr_1 and integrating over the disk, we obtain, after some integration by parts,

$$
0 = \iint_D \left(\left(\frac{\partial r_1}{\partial \rho}\right)^2 + \left(\frac{1}{\rho}\frac{\partial r_1}{\partial \varphi}\right)^2 + \left(\frac{r_1}{\rho}\right)^2 + \left(\frac{r_1}{\rho}\frac{\partial \theta_1}{\partial \varphi}\right) \right) \rho \, d\rho d\varphi
$$

Similarly, by multiplying the second equation in (7.3) by $\rho^2\theta_1$ and integrating over the disk, we obtain

$$
0 = \iint_D \left(\left(\frac{\partial \rho\theta_1}{\partial \rho}\right)^2 + \left(\frac{\partial \theta_1}{\partial \varphi}\right)^2 - 2\left(\frac{r_1}{\rho}\frac{\partial \theta_1}{\partial \varphi}\right) \right) \rho \, d\rho d\varphi .
$$

Adding one-half of this second equation to the preceding one, we obtain

$$
0 = \iint_D \left(\left(\frac{\partial r_1}{\partial \rho}\right)^2 + \left(\frac{1}{\rho}\frac{\partial r_1}{\partial \varphi}\right)^2 + \left(\frac{r_1}{\rho}\right)^2 + \frac{1}{2}\left(\frac{\partial \rho\theta_1}{\partial \rho}\right)^2 + \frac{1}{2}\left(\frac{\partial \theta_1}{\partial \varphi}\right)^2 \right) \rho \, d\rho d\varphi .
$$

Since the integrand is a sum of squares and must be nonnegative, we conclude that r_1 and θ_1 vanish on the disk. This completes the proof of Theorem 4.1.

8. Conclusions.

We have given an analysis of the deformation of a circular membrane when subjected to a uniform static pressure. We have shown that the analytical results agree well with the experimental data using steel plates subjected to high pressures and finite element calculations for membranes.

The analysis leads to a efficient procedure for the calculation of the membrane shape as a function of Poisson's ratio, Young's modulus, thickness, membrane radius, and the static pressure for small strains.

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