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ABSTRACT. A modified auction algorithm for solving the shortest path problem is presented and convergence is established. The proposed method differs from the standard auction algorithm in the way dual variables are updated. By relaxing the dual feasibility requirement we were able to substantially reduce the total number of iterations required by the auction algorithm to compute the shortest path. Computational results show the advantage of this new approach, especially when the number of intermediate nodes in the shortest path from the origin to the destination is large.

1. INTRODUCTION

In this paper we present a modified auction algorithm for solving the shortest path problem. Auction algorithms were first proposed by Bertsekas [1] (see also [2, 3]) for the assignment problem and later extended to general transportation problems [6, 8, 7]. A survey of the auction algorithms for network optimization problems is contained in [5, Chapter 4].

In [4], Bertsekas proposed an auction algorithm for the shortest path problem. For the single origin and single destination case, the algorithm can be viewed as an instance of the "naive" auction algorithm applied to a special type of assignment problem. Furthermore, it can be interpreted as a finitely terminating dual coordinate ascent method. Under the assumptions that there exists a path from the origin to the destination, that each cycle of the graph has positive length, that the forward star of each node is not empty and that all input data are integer, the algorithm terminates in pseudopolynomial time.

Recently, Pallottino and Scutellà [10] proposed two new versions of the auction algorithm for the shortest path problem. By opportunely pruning the original graph until it "collapses" in a shortest path tree, they developed strongly polynomial versions of the auction algorithm.

The standard auction algorithm for shortest path problems consists of 3 basic operations: path *extension*, path *contraction* and dual *price raise*. For the single origin and single destination case, the algorithm maintains a path starting at the origin and a set of *feasible* dual prices. At each iteration, the candidate path is either extended by adding a new node at the end of the path, or contracted by deleting the terminal node. When no extensions or contractions are possible, the value of dual variable corresponding to the terminal node in the candidate path is raised. The algorithm terminates when the destination node is reached.

The algorithm we propose differs from the standard auction algorithm in the way the dual variables are updated. We do not require that dual feasibility be maintained throughout

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the algorithm. This allows us to raise the dual prices higher than in the standard auction algorithm and, as a consequence, the number of path contractions is substantially reduced. In Section 3 we will show that (as in the standard auction algorithm) if a node i in the graph is reached, then a shortest path form the origin to node i has also been found. As a consequence, the pruning operation proposed in [10] can be applied. The convergence proof for our modified auction algorithm uses this pruning strategy to construct a feasible dual solution for the shortest path on the reduced graph.

The paper is organized as follows. In Section 2 we present in detail the Modified Auction Algorithm. Convergence of the algorithm is proved in Section 3. Finally, preliminary experiments are reported in Section 4.

2. The Modified Auction Algorithm

In this section we will describe in detail the Modified Auction Algorithm for the shortest path problem. For sake of simplicity we will discuss only the single origin and single destination case.

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ be a directed graph, where $\mathcal{N}$ is the set of nodes and $\mathcal{A}$ is the set of arcs. Let s be the source node and t the destination node. The single origin and single destination shortest path problem can be formulated as

$$\begin{array}{lll} & \text{minimize} & \displaystyle\sum_{(l,p)\in\mathcal{A}} c_{lp}x_{lp} \\ (1) & \text{subject to} & \displaystyle\sum_{(l,p)\in\mathcal{A}} x_{lp} - \sum_{(p,l)\in\mathcal{A}} x_{pl} = b_p, \quad \forall\, p \in \mathcal{N}, \\ & & x_{lp} \geq 0, \quad \forall\, (l,p) \in \mathcal{A}, \end{array}$$

where $b_p = 0$ for all $p \neq s, t$, $b_s = 1$, and $b_t = -1$. The dual of (1) is

$$\begin{array}{lll} (2) & \text{maximize} & \pi_s - \pi_t \\ & \text{subject to} & c_{lp} - \pi_l + \pi_p \geq 0, \quad \forall\, (l,p) \in \mathcal{A} \end{array}$$

where π_p is the dual variable associated with node p (sometimes referred to as the price of node p). At each iteration of the standard auction algorithm a dual feasible solution and a primal (infeasible) solution are available for which complementarity slackness holds. While maintaining complementarity slackness, the algorithm either constructs a new primal (not necessarily feasible) solution or a new dual feasible solution until a primal feasible (and hence optimal) solution is obtained.

By contrast, in our Modified Auction Algorithm we do not require that the dual variables remain feasible. The dual variable corresponding to the terminal node of the candidate path (let's call it node i) is raised to the second minimum value in the set

$$(3) \qquad \left\{-(c_{ki} - \pi_k + \pi_i),\ \ c_{ip} - \pi_i + \pi_p,\ (i,p) \in \mathcal{A}, \right\},$$

where node k is the immediate predecessor of node i in the candidate path.

The idea of raising the dual prices to the second minimum value is not a novel idea for the auction algorithms. In fact, for the assignment problem the dual prices are updated in a similar fashion. However, as far as we know, no one has proposed this updating scheme for shortest path or minimum cost flow problems. The main reason for this may be that the above updating formula (3) does not guarantee dual feasibility.

The following technical assumptions are needed in order to guarantee that an initial dual feasible solution exists and the contraction operation is well defined.

Assumption 1. All cycles in the graph have positive length.
Assumption 2. The forward star for all nodes (except eventually node t) is not empty. Moreover at least two arcs are in the forward star for the starting node s.

We are now ready to present in detail our Modified Auction Algorithm.

Modified Auction Algorithm

Step 0: Set $x_{lp} = 0$ for all arcs. Choose $\pi \in \mathbb{R}^{|\mathcal{N}|}$ such that $c_{lp} - \pi_l + \pi_p \geq 0$ for all $(l,p) \in \mathcal{A}$.
Step 1: If node t is the terminal node of the candidate path, stop. Otherwise, let i be the terminal node for the candidate path.
Step 2: Let

$$S^+ := \Big\{(i,p) \in \mathcal{A} : \quad x_{ip} = 0\Big\}$$

$$S^- := \Big\{(l,i) \in \mathcal{A} : \quad x_{li} = 1\Big\}$$

Define

$$\gamma_1 = \min\left\{ \min_{(i,p)\in S^+} \Big\{c_{ip} - \pi_i + \pi_p\Big\}, \ \min_{(l,i)\in S^-} \Big\{-\Big(c_{li} - \pi_l + \pi_i\Big)\Big\} \right\}.$$

If $\gamma_1 = c_{ip^*} - \pi_i + \pi_{p^*}$ for some $(i,p^*) \in S^+$, then set

$$\gamma = \min\left\{ \min_{\substack{(i,p)\in S^+ \\ (i,p)\neq(i,p^*)}} \Big\{c_{ip} - \pi_i + \pi_p\Big\}, \ \min_{(l,i)\in S^-} \Big\{-\Big(c_{li} - \pi_l + \pi_i\Big)\Big\} \right\}.$$

If, instead, $\gamma_1 = -(c_{l^*i} - \pi_{l^*} + \pi_i)$ with $(l^*,i) \in S^-$, then set

$$\gamma = \min\left\{ \min_{(i,p)\in S^+} \Big\{c_{ip} - \pi_i + \pi_p\Big\}, \ \min_{\substack{(l,i)\in S^- \\ (l^*,i)\neq(l,i)}} \Big\{-\Big(c_{li} - \pi_l + \pi_i\Big)\Big\} \right\}.$$

Set $\pi_i = \pi_i + \gamma$
Step 3: If there exists a node j such that:

(4a) $$(i,j) \in \mathcal{A}, \ x_{ij} = 0, \ \pi_i > \pi_j + c_{ij}$$

then set $x_{ij} = 1$, and go to Step 1
If there exists a node j such that:

(4b) $$(j,i) \in \mathcal{A}, \ x_{ji} = 1, \ \pi_i > \pi_j - c_{ji}$$

then set $x_{ji} = 0$, and go to Step 1
If there exists a node j such that:

(4c) $$(i,j) \in \mathcal{A}, \ x_{ij} = 0, \ \pi_i = \pi_j + c_{ij}$$

then set $x_{ij} = 1$, and go to Step 1

Remark 1. Note that Assumption 2 guarantee that at each iteration of the Modified Auction Algorithm, there are at least two elements in $S^+ \cup S^-$.

Remark 2. For each arc $(l,p) \in \mathcal{A}$, let $r_{lp} := c_{lp} - \pi_l + \pi_p$ be the reduced cost for this arc. Then, the quantity γ_1 is the minimum of the smallest reduced cost for the arcs leaving node i and the negative of the reduced cost for the only arc in the path entering node i. Later we will show that all reduced costs for the arcs leaving node i are nonnegative while the reduced cost for the only arc in path entering node i is nonpositive. Therefore the quantities γ_1 and γ defined in Step 2 of the Modified Auction Algorithm are both nonnegative.

Remark 3. The choice of γ in Step 2 of the Modified Auction Algorithm ensures that after Step 3 if $\pi_i > \pi_j + c_{ij}$ for some $(i,j) \in \mathcal{A}$ with $x_{ij} = 0$, then $\pi_i \leq \pi_k - c_{ki}$ where k is the node immediately preceding node i in the candidate path. Similarly, if $\pi_i > \pi_k - c_{ki}$ where k is again the node immediately preceding node i in the candidate path, then $\pi_i \leq \pi_p + c_{ip}$ for all arcs $(i,p) \in \mathcal{A}$

Remark 4. In Step 3 the order in which the conditions (4a)–(4c) are tested is crucial to the convergence of the algorithm. When condition (4c) is satisfied a "degenerate" path extension is performed. Such extensions are only allowed when no "standard" path extensions or contractions are possible. Since the set S^- contains (except for the case $i = s$) only a single element, either an extension (degenerate or not) or a contraction is always possible.

3. Convergence of the Modified Auction Algorithm

In this section we will establish the convergence of the Modified Auction Algorithm. First we show in Lemma 3.1 that at each iteration the reduced costs are nonnegative for the arcs not in the candidate path and nonpositive for the arcs in the candidate path. Then, we will show that each time a node is reached the candidate path is a shortest path from the origin node s to its terminal node. Finally, we will prove that the Modified Auction Algorithm terminates in a finite number of steps if all cycles in the original graph have positive costs, if a path from the source node s to the terminal node t exists and if all input data are integer.

Lemma 3.1. *At each iteration of the Modified Auction Algorithm*

- *the arcs $(l,p) \in \mathcal{A}$ corresponding to $x_{lp} = 1$ determine a simple path from the origin to a node $i \in \mathcal{N}$,*
- *dual variables (not necessarily feasible) are available for which the following conditions are satisfied:*

$$(5a) \qquad (l,p) \in \mathcal{A}, \ \ x_{lp} = 0 \implies r_{lp} = c_{lp} - \pi_l + \pi_p \geq 0$$

$$(5b) \qquad (l,p) \in \mathcal{A}, \ \ x_{lp} = 1 \implies r_{lp} = c_{lp} - \pi_l + \pi_p \leq 0$$

Proof Clearly the initial choice of the primal and dual variables satisfy all the conditions above. The initial path only includes the starting node s. Let's now assume that at the beginning of the generic iteration of the Modified Auction Algorithm the arcs $(l,p) \in \mathcal{A}$ with $x_{lp} = 1$ determine a simple path from the source node to a node i and that dual variables π satisfying (5a) and (5b) are available. We will show that after performing Steps 2 and 3, a new simple path is constructed and new dual variables satisfying (5a) and (5b) are obtained.

Let i be the terminal node of the candidate path. Note that the quantity γ specified in Step 2 of the algorithm is nonnegative since (by assumption hypothesis) all the reduced costs for the arcs in S^+ are nonnegative and the reduced cost for the only entry in S^- is nonpositive. In Step 2 of the Modified Auction Algorithm, all the dual prices are unchanged except π_i whose value is increased by γ. Set $\pi_i^{new} = \pi_i + \gamma$.

Clearly, the reduced costs for all arcs $(l,p) \in \mathcal{A}$ with $x_{lp} = 0$ and not incident to node i remain nonnegative. Moreover, since the value of the dual variable π_i is increased, the reduced costs are nonnegative for all arcs entering node i.

We must now consider two distinct cases. Suppose that in Step 3 we extend our candidate path to include a new node j. This only happens if

$$\pi_i^{new} \geq \pi_j + c_{ij}, \text{ and } \pi_i^{new} \leq \pi_k - c_{kj}$$

where k is the immediate predecessor of node i in the candidate path. Therefore the reduced costs for the arcs (i,j) and (k,i) are both nonpositive. The choice of γ in Step 2 of the Modified Auction Algorithm also guarantees that for any other arc $(i,p) \in \mathcal{A}$, $p \neq j$, the reduced cost is nonnegative. Note that it is not possible that the newly reached node j was already in the path from s to i. In fact, suppose, by contradiction, that the candidate path was

$$\Big\{s, \ldots, j, l_1, l_2, \ldots, l_r = k, i\Big\}.$$

Then, by summing the reduced costs over the arcs of the cycle

$$(j,l_1),\ (l_1,l_2),\ \ldots,\ (l_r,i),\ (i,j)$$

we obtain

$$c_{jl_1} + c_{l_1 l_2} + \ldots, c_{l_r i} + c_{ij} \leq 0$$

which contradicts Assumption 1.

Suppose now that we contract our path. Note that this only happens when no extensions are possible, that is there are no arcs (i,p) leaving node i with $\pi_i^{new} > \pi_p + c_{ip}$. Therefore, the reduced cost for all the arcs leaving node i are nonnegative. Again, let k be the node immediately preceding node i in the candidate path. Condition (4b) guarantee that $\pi_i^{new} > \pi_k - c_{ki}$ and hence also the reduced cost for the arc (k,i) is nonnegative. □

As part of the above lemma, we also showed that the dual variables are only increased during the algorithm. The next lemma shows that the candidate paths constructed by the algorithm are shortest paths from the origin to their terminal nodes.

Lemma 3.2. *At each iteration of the Modified Auction Algorithm, the simple path from the starting node s to node $i \in \mathcal{N}$ determined by the primal variables set to 1 is a shortest path from s to i.*

Proof We already showed that, at each step of the Modified Auction Algorithm, the arcs corresponding to primal variables equal to 1 determine a simple path from the origin s to a node $i \in \mathcal{N}$. Since our initial path includes only node s, the assertion is true at the beginning of the first iteration of the algorithm. Suppose now that at the beginning of the generic iteration the candidate path starting at node s and ending at node i is a shortest path from s to i. If in Step 3 we contract our path back to node k, the new path is again a shortest path from s to k [9, Proposition 2.1, page 56]. When the candidate path is extended to include node j, a new set of dual variables is also constructed for which conditions (5a) and (5b) are satisfied. Let P be the candidate path from s to j and P' an alternative path from the same two nodes. Also, let l be the last node that is in both paths. If $l = i$, then since the subpath of P from s to i is a shortest path from s to i, it follows that the sum of the arc costs for the path P is no greater than the sum of the arc costs for the path P'. Suppose now $l < i$ and let[1]

$$\begin{aligned} P &= \left\{s, \dots, l, l_1, l_2, \dots, l_r, i, j\right\} \\ P' &= \left\{s, \dots, l, l'_1, l'_2, \dots, l'_{r'} j\right\}. \end{aligned}$$

Since the arcs (l, l_1), (l_1, l_2), (l_r, i), (i, j) are in the candidate path while the arcs (l, l'_1), (l'_1, l'_2), $(l'_{r'}, i)$ are not, we have

$$c_{ll_1} + c_{l_1 l_2} + c_{l_r i} + c_{ij} - \pi_l + \pi_j \le 0 \le c_{ll'_1} + c_{l'_1 l'_2} + c_{l'_{r'} j} - \pi_l + \pi_j$$

and hence

$$c_{ll_1} + c_{l_1 l_2} + c_{l_r i} + c_{ij} \le c_{ll'_1} + c_{l'_1 l'_2} + c_{l'_{r'} j}.$$

Since ([9, Proposition 2.1, page 56]) the subpath of P from s to l, is a shortest path from s to l the total sum of the costs for the arcs in P is smaller than the sum of the costs for the arcs in P'. □

Corollary 3.3. *If the Modified Auction Algorithm terminates, then a shortest path from the origin node s to the destination node t has been found.*

The optimality of the candidate paths (for the standard auction algorithm) is the fundamental property used by Pallottino and Scutellà in [10] to construct their polynomial time versions of auction algorithm. The key observation is that the candidate path P is a shortest path from s to its terminal node i and hence the first time that node i is reached we can delete from the original graph all the arcs in the backward star of i except arc (k, i) where k is the immediate predecessor of i in the path. We will use a similar approach to show convergence of our modified auction algorithm.

Let

$$P(r) = \left\{i_0 = s, i_1, i_2, \dots, i_k, i_{k+1} = i\right\}$$

be the candidate path constructed at iteration r of the Modified Auction Algorithm and let $\pi^{(r)}$ be the (not necessarily feasible) dual variables available at iteration r. Define $\mathcal{A}(r)$ as the subset of the arcs in $\mathcal{A}$ obtained by deleting all the arcs in the backward star of the nodes in the candidate path $P(r)$ except their immediate predecessor in the path and let $\mathcal{G}(r) = (\mathcal{N}, \mathcal{A}(r))$ be the subgraph of $\mathcal{G}$. Corresponding to the variables $\pi^{(r)}$, construct a new set of dual prices $\hat{\pi}^{(r)}$ as follows:

$$\hat{\pi}^{(r)}_p = \pi^{(r)}_p, \quad \forall p \notin P(r) \tag{6a}$$

$$\hat{\pi}^{(r)}_i = \pi^{(r)}_i, \tag{6b}$$

$$\hat{\pi}^{(r)}_{i_j} = \pi^{(r)}_{i_{j+1}} + c_{i_j i_{j+1}} \quad \text{for } j = k, k-1, \dots, 0 \tag{6c}$$

The next two lemmas demonstrate that the dual variables $\hat{\pi}^{(r)}$ are feasible for the dual problem:

$$\begin{aligned} &\text{maximize} && \pi_s - \pi_t \\ &\text{subject to} && c_{lp} - \pi_l + \pi_p \ge 0, \quad \forall\, (l, p) \in \mathcal{A}(r). \end{aligned} \tag{7}$$

[1] Note that $l = s$ is possible.

Lemma 3.4. *Let $\pi^{(r)}$ be the dual variables available at iteration r of the Modified Auction Algorithm. Then*

$$\hat{\pi}^{(r)} \leq \pi^{(r)} \tag{8}$$

where $\hat{\pi}^{(r)}$ is given by (6)

Proof From (6a) and (6b) we have that $\hat{\pi}_p^{(r)} = \pi_p^{(r)}$ for all node $p \notin P$ and $\hat{\pi}_i^{(r)} = \pi_i^{(r)}$. Moreover, for all $l = 0, \dots, k$

$$\hat{\pi}_{i_l}^{(r)} = \sum_{j=l}^{k} c_{i_j i_{j+1}} + \pi_i^{(r)} \tag{9}$$

But from Lemma 3.1, the reduced costs $c_{i_j,i_{j+1}} - \pi_{i_j}^{(r)} + \pi_{i_{j+1}}^{(r)}$ are nonnegative for all $j = 0, \dots, k$ and hence

$$0 \geq \sum_{j=l}^{k} \left(c_{i_j,i_{j+1}} - \pi_{i_j}^{(r)} + \pi_{i_{j+1}}^{(r)} \right) = -\pi_{i_l}^{(r)} + \pi_i^{(r)} + \sum_{j=l}^{k} c_{i_j,i_{j+1}} = -\pi_{i_l}^{(r)} + \hat{\pi}_{i_l}^{(r)}$$

□

Lemma 3.5. *For all $(l,p) \in \mathcal{A}(r)$ we have*

$$\hat{r}_{lp} := c_{lp} - \hat{\pi}_l^{(r)} + \hat{\pi}_p^{(r)} \geq 0 \tag{10}$$

where $\hat{\pi}^{(r)}$ is the vector given by (6)

Proof From Lemma 3.1 and (6a) we have that the reduced costs are nonnegative for all the arcs (l,p) with both l and p not in $P(r)$. Condition (6b) ensures that the reduced costs are 0 for the arcs in the candidate path. Finally, since $\hat{\pi}^{(r)} \leq \pi^{(r)}$, dual feasibility holds for all arcs leaving nodes in the path and reaching nodes not in $P(r)$. □

The way the graph $\mathcal{G}(r)$ is constructed, ensures that the set of nodes that can be reached from the source node s in $\mathcal{G}(r)$ is the same as the set of nodes reachable in $\mathcal{G}$. Therefore, if a shortest path from s to t exists in the graph $\mathcal{G}$, it must also exist in the graph $\mathcal{G}(r)$ and hence the maximization problem (7) (which is feasible since (2) is feasible) cannot be unbounded above.

The following assumptions, in addition to Assumptions 1 and Assumptions 2, are needed in order to guarantee finite termination of the Modified Auction Algorithm.

Assumption 3. There exists a path from s to t.
Assumption 4. All input data are integer.

We already showed that during the Modified Auction Algorithm, the dual prices associated with the nodes are never decreased. Moreover, each time a contraction is done, the dual price for the end node of the candidate path is raised by a positive quantity.

Lemma 3.6. *Suppose that Assumptions* 1, 2 *and* 3 *hold. Then, each time we perform a path contraction from a node, the price of this node is raised by a positive quantity.*

Lemma 3.7. *A path contraction is only performed if in Step 3 of the Modified Auction Algorithm*

$$\pi_i > \pi_k - c_{ki}$$

where k is the node immediately proceeding i in the candidate path. But the reduced cost for the arc (k, i) was previously nonpositive, and hence in Step 2 the dual price associated with node i was raised by a positive quantity.

Lemma 3.8. *Suppose that the algorithm does not terminate. Then, under the Assumptions* 1, 2 *and* 4, *at least one of the dual prices goes to* $+\infty$.

Proof If the algorithm does not terminate, there is at least one node visited an infinite number of times. Since the graph has no cycles of length less than or equal to 0, an infinite number of extensions (and, more important, contractions) involve this node. At each contraction, the dual price associated with the node is raised by at least 1 (since we assumed that all input data are integer) and hence the dual price for this node goes to $+\infty$. □

The previous lemma shows that, when the algorithm does not terminate, there exists at least one node (let's call it i) such that

$$\lim_r \pi_i^{(r)} = +\infty.$$

From the sequence $\pi^{(r)}$, we can extract a subsequence $\pi^{(r_j)}$ with the following properties:

(1) the paths $P(r_j)$ are identical for all j,
(2) Node i is the end node for these identical paths.

Such subsequence exists because (under the assumption that the algorithm does not terminate) node i is the terminal node for an infinite number of candidate paths and the number of distinct (shortest) paths from s to i is finite. Since the candidate paths $P(r_j)$ are identical, also identical are the set of arcs $\mathcal{A}(r_j)$ for all j. Let $\mathcal{G}' = (\mathcal{N}, \mathcal{A}')$ with $\mathcal{A}' = \mathcal{A}(r_j)$. The variables $\hat{\pi}^{(r_j)}$ are feasible for the problem

$$\text{(11)} \qquad \begin{array}{ll} \text{maximize} & \pi_s - \pi_t \\ \text{subject to} & c_{lp} - \pi_l + \pi_p \geq 0, \quad \forall\, (l,p) \in \mathcal{A}'. \end{array}$$

We already noticed that the dual price for node t is never modified and hence $\pi_t^{(r_j)} = \hat{\pi}_t^{(r_j)}$. Moreover, the quantity $\hat{\pi}_s^{(r_j)} - \hat{\pi}_t^{(r_j)}$ must remain bounded since the variables $\hat{\pi}^{(r)}$ are feasible for (11) and hence $\hat{\pi}_s^{(r_j)}$ must remain bounded. But, since $\hat{\pi}_i^{(r_j)} \to +\infty$, we have

$$\lim_j \hat{\pi}_s^{(r_j)} - \hat{\pi}_i^{(r_j)} = -\infty$$

which is impossible since $\hat{\pi}_s^{(r_j)} - \hat{\pi}_i^{(r_j)}$ is constant and equal to the sum of the costs for all the arcs in $P(r_j)$.

The following theorem summarizes the convergence results for the Modified Auction Algorithm.

Theorem 3.9. *Under the Assumptions* 1–4, *the Modified Auction Algorithm terminates after a finite number of extensions and contractions. The final path is a shortest path from the origin node s to the target node t.*

4. Computational Results

We compare the results obtained with our Modified Auction Algorithm with the results for a standard auction algorithm implementing the graph reduction operation as specified in [10]. To generate our problems we used the "gridgen" generator written by Yusin Lee and

Jim Orlin for grid-like networks. This graph generator, in addition to the arcs connecting the nodes in the grid (that can be two-way arcs or one-way arcs), introduces supplemental arcs by uniformly picking random node pairs. The number of these supplemental arcs is controlled by a parameter specifying the average node degree.

We tested the single origin and single destination version of the Modified Auction Algorithm (Figures 1, 2 and 5) as well the single origin and multiple destinations version (Figures 3 and 4). In the second case the stopping criterion in step 1 of the Modified Auction Algorithm is modified and termination occurs only when all nodes in the graph have been terminal nodes for at least one candidate path.

All the results are graphically displayed in Figures 1 to 5. For a specific number of nodes or grid size we report the range of variation for

$$\frac{T_{old} - T_{new}}{T_{old}}$$

where T_{old} is the time for the standard auction algorithm and T_{new} is the time for our Modified Auction Algorithm. The average value is indicated by an asterisk.

In Figures 1 and 2 we display the results for the single origin and single destination case for graph with number of nodes varying from 1,000 to 10,000 for almost square grids (Figure 1) and very skew, long grids (Figure 2).

Figures 3 and 4, instead, deal with the single origin multiple destination case for the same almost square or long grid graphs considered before.

In all our 4 test sets our Modified Auction Algorithm outperformed the standard auction algorithm, and in some case the time reduction was almost 50%. The reduction time observed by using the Modified Auction Algorithm is more pronounced (an average of over 28%) for the single origin and single destination case than for the single origin and multiple destination case for which the average time reduction was about 18.5%.

In general we observed a greater time reduction when very long candidate paths are constructed. In this case the Modified Auction Algorithm, by raising the dual prices to higher values, avoids useless contractions.

This observation motivated us to experiment on some pathological cases. We considered again grid graphs but now only the arcs belonging to the grid are generated. Moreover, nodes at opposite ends of the grid were chosen as origin and destination nodes. Figure 5 reports comparative results with the standard auction algorithm for this class of problems. We observed a substantial time reduction. In some instances our Modified Auction Algorithm was over 10 times faster than the standard auction algorithm and for skewed grid we observed an average time reduction of over 68%. The reduction is due to the fact that for these graphs the intermediate paths contains a large number of nodes and the Modified Auction Algorithm requires a smaller number of contractions before reaching the optimal solution.

5. Conclusions

We presented a modified auction algorithm for solving the shortest path problem. The algorithm differs from the standard auction algorithm because the dual feasibility requirement is relaxed. As a consequence, the total number of contractions is substantially reduced. Computational results show the advantage of this new approach in particular when the number of intermediate nodes in the shortest path from the origin to the destination is large.

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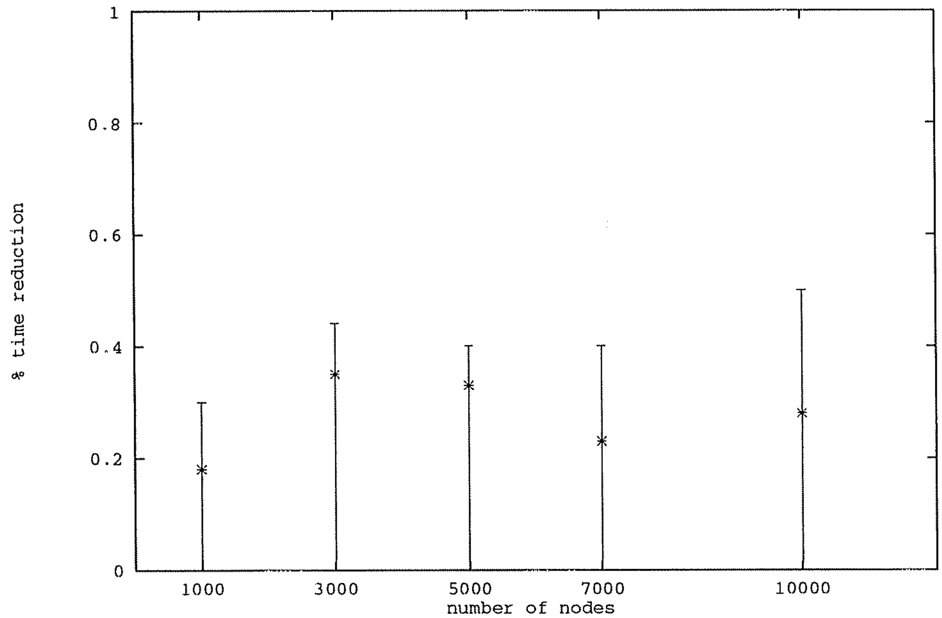

FIGURE 1: *Percent of time reduction for the single origin, single destination problem. Almost square grid. Average number of incident arcs per node = 4.*

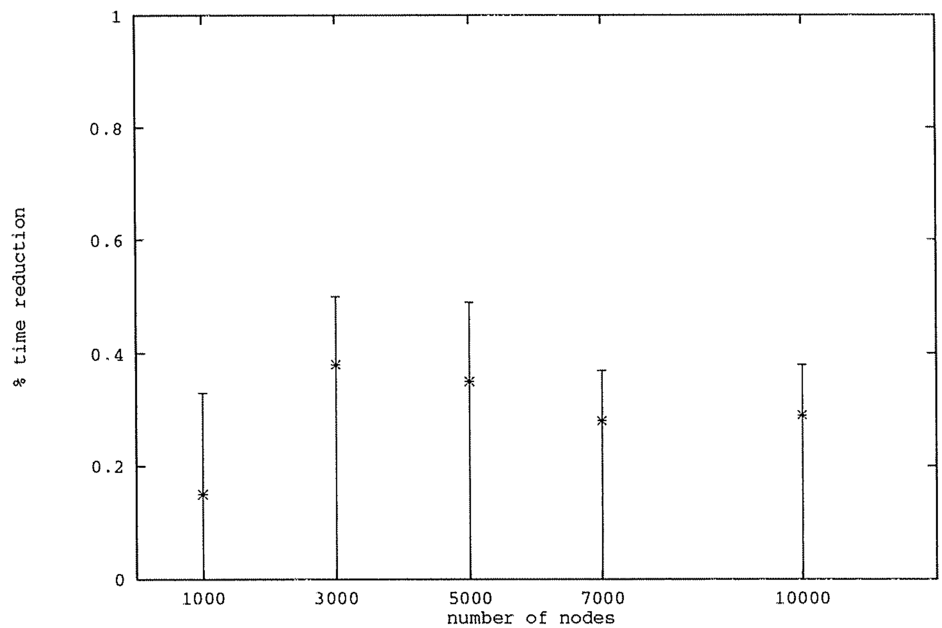

FIGURE 2: *Percent of time reduction for the single origin, single destination problem. Long grid. Average number of incident arcs per node = 4.*

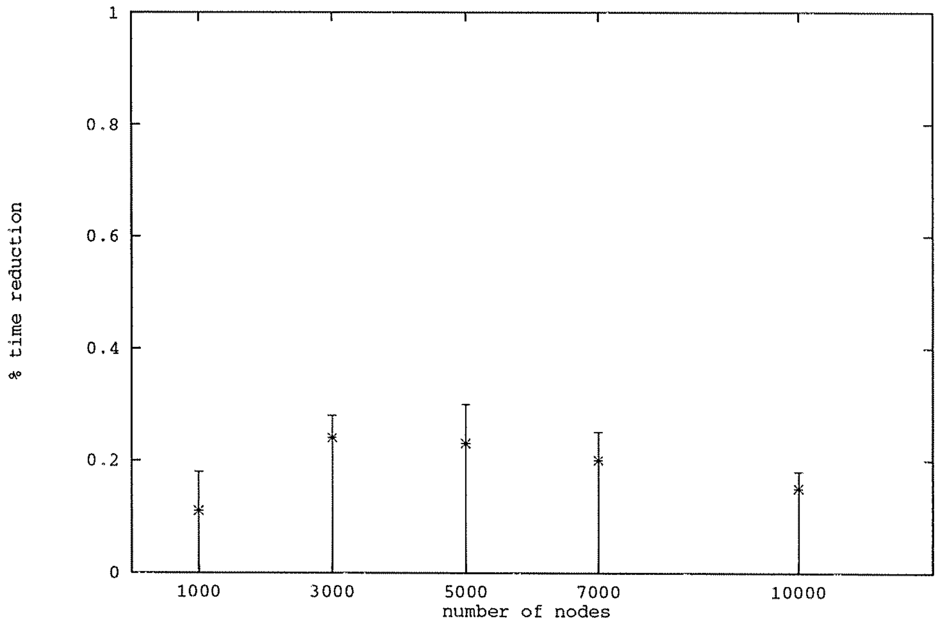

FIGURE 3: *Percent of time reduction for the single origin, multiple destination problem. Almost square grid. Average number of incident arcs per node = 4.*

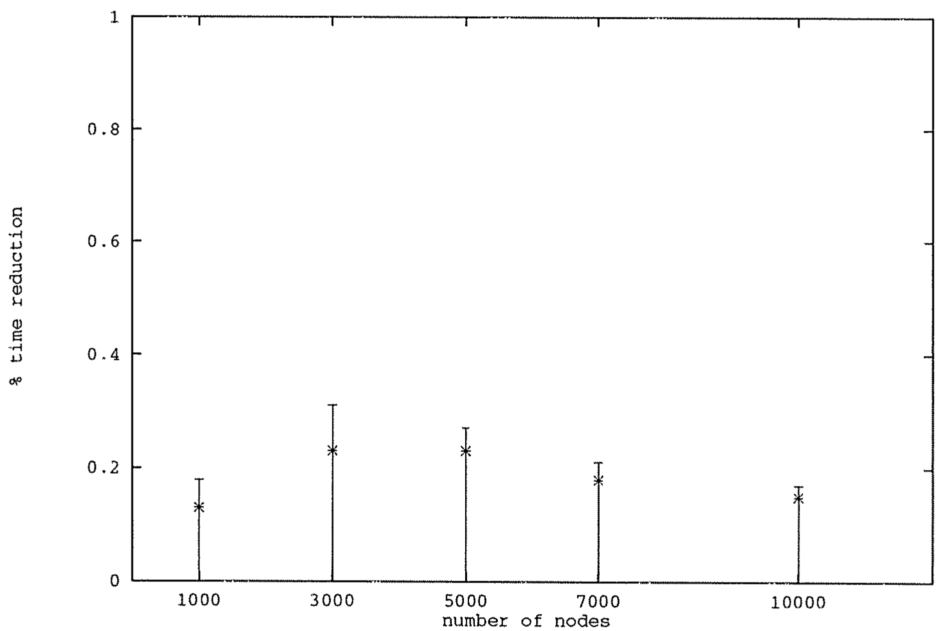

FIGURE 4: *Percent of time reduction for the single origin, multiple destination problem. Long grid. Average number of incident arcs per node = 4.*

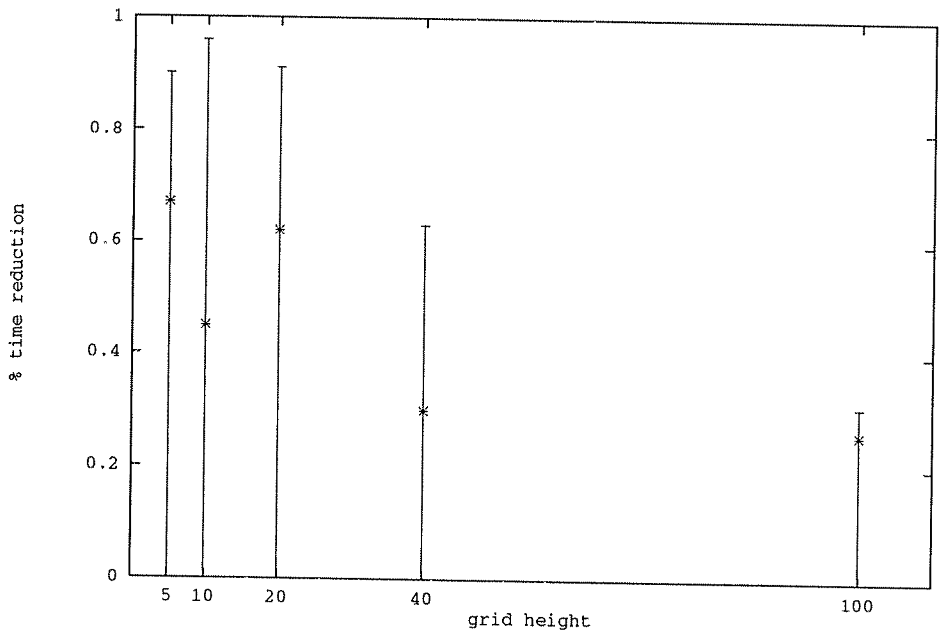

FIGURE 5: *Percent of time reduction for the single origin, single destination problem. Number of nodes = 10,000. Varying grid size. Grid arcs only.*