Convergence Estimates for Finite Difference Approximations of the Stokes Equations

Dongho Shin
John C. Strikwerda

Technical Report #1106

August 1992

Convergence Estimates for Finite Difference Approximations of the Stokes Equations

Dongho Shin† and John C. Strikwerda‡

Abstract. For three finite difference approximations of the Stokes equations, the Schur complement Q_h of the linear system generated by each of these approximations is shown to have its condition number $\kappa(Q_h)$ independent of mesh size. This result is used to prove convergence estimates of the solutions generated by Q_h for these approximations. One of the convergence estimates is for a staggered mesh scheme and the estimate for this scheme shows that both the pressure and the velocity are second-order accurate.

Key words. pressure equation method, Stokes equations, incompressible Navier-Stokes equations, inf-sup conditions, finite difference schemes, iterative methods

AMS(MOS) subject classifications. 65F10, 65N06, 65N22

1. Introduction. The pressure equation (PE) method, a new fast iterative method to solve finite difference approximations of the Stokes and the incompressible Navier-Stokes equations has been introduced by Shin and Strikwerda [10]. The PE method and many other iterative methods to solve the Stokes equations are heavily dependent on the properties the Schur complements of the linear systems resulting from discretizations of these equations, which we investigate in this paper.

We first review the PE method. The steady-state Stokes equations in $\mathbf{R}^d$ are

$$\begin{aligned} \nabla^2\vec{u} - \vec{\nabla}p &= \vec{f} \\ \vec{\nabla}\cdot\vec{u} &= g \end{aligned} \qquad \text{in} \qquad \Omega \subset \mathbf{R}^d. \tag{1.1}$$

The velocity $\vec{u}$ is a vector of dimension d and the pressure p is a scalar. Consider the the Dirichlet boundary condition

$$\vec{u} = \vec{b} \qquad \text{on} \qquad \partial\Omega. \tag{1.2}$$

Let A_h, G_h and D_h be the operators generated by discretizations of the differential operators

$$\begin{pmatrix} \nabla^2 & 0 \\ 0 & \nabla^2 \end{pmatrix}, \qquad -\vec{\nabla}\,, \qquad \text{and} \quad (\vec{\nabla}\cdot),$$

respectively. The discretization of (1.1) and (1.2) may then be written as

$$\begin{aligned} A_h u_h + G_h p_h &= f_h \\ D_h u_h &= g_h \end{aligned} \tag{1.3}$$

---

† Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706.

‡ Department of Computer Sciences and Center for the Mathematical Sciences, University of Wisconsin-Madison, Madison, WI 49506. The work of this author was supported by the U.S.Army Research Office under grants DAAL03-87-K-0028 and DAAL03-91-G-0094.

and

$$u_h = b_h \qquad \partial\Omega,$$

respectively.

Note that

$$A_h u_h = f_h - G_h p_h, \qquad u_h = b_h \quad \text{on} \quad \partial\Omega \tag{1.4}$$

by the first row in (1.3) and (1.2). Hence

$$u_h = v_h - w_h \tag{1.5}$$

where

$$A_h v_h = f_h, \qquad v_h = b_h \quad \text{on} \quad \partial\Omega$$

and

$$A_h w_h = G_h p_h, \qquad w_h = 0 \quad \text{on} \quad \partial\Omega.$$

If A_0^{-1} and A_b^{-1} are the inverse operators of A_h using the zero boundary condition and the boundary condition for u_h, respectively, then

$$v_h = A_b^{-1} f_h \qquad \text{and} \qquad w_h = A_0^{-1} G_h p_h. \tag{1.6}$$

Using the second row in (1.3), (1.5), and (1.6), one obtains

$$D_h(v_h - w_h) = g_h$$

and

$$D_h A_0^{-1} G_h p_h = D_h A_b^{-1} f_h - g_h.$$

Thus

$$Q_h p_h = D_h A_b^{-1} f_h - g_h, \tag{1.7}$$

if we let

$$Q_h := D_h A_0^{-1} G_h.$$

The operator Q_h is called the Schur complement of the linear system (1.3).

In this paper, we show that, for three finite difference schemes, the operator Q_h is self-adjoint, positive definite with eigenvalues bounded independently of mesh size. In each of these cases, one can use the conjugate gradient (CG) method to solve (1.7), and the number of the CG iterations required to solve (1.7) should be independent of the grid parameters. The iterative method based on solving (1.7) by the CG method is called the PE method.

A popular iterative method to solve the Stokes and Navier-Stokes equations is the pressure poisson equation (PPE) method, see [4]. Applying the divergence operator ($\vec{\nabla}\cdot$) to the first row in (1.1), we have

$$\nabla^2(\vec{\nabla} \cdot \vec{u}) - \vec{\nabla} \cdot \vec{\nabla}\, p = \vec{\nabla} \cdot \vec{f}$$

and, by the second row in (1.1),

$$\nabla^2 p = \nabla^2 g - \vec{\nabla} \cdot \vec{f}.$$

The PPE method is based on solving the above Poisson equaton for pressure. A boundary condition for pressure is needed to solve this equation and it is not clear which boundary condition is appropriate, see [4]. The PE method is similar to the pressure poisson equation (PPE) method in the sense that they use equations for pressure only, but the PE method doesn't require a boundary condition for pressure.

2. Definitions and inf-sup conditions. Let Ω be a domain in $\mathbf{R}^d$ and let Γ be its boundary. For simplicity, we focus on the case when $d = 2$, but the results in this paper will hold for any $d \geq 2$. We denote by $L^2(\Omega)$ the space of real functions defined on Ω which are integrable in the L^2 sense with the following usual inner product and norm

$$(u,v)_\Omega := \iint_\Omega uv \ \ dA, \quad \|u\|_\Omega^2 := (u,u)_\Omega.$$

Let

$$H_0^1(\Omega) := \{ \, u \in L^2(\Omega) \mid u_x, u_y \in L^2(\Omega) \text{ and } u|_\Gamma = 0 \, \}$$

have the following inner product and norm

$$(u,v)_{1,\Omega} := \iint_\Omega \vec{\nabla} u \cdot \vec{\nabla} v \ \ dA, \quad \|u\|_{1,\Omega}^2 := (u,u)_{1,\Omega}$$

and

$$L_0^2(\Omega) := \{ \, p \in L^2(\Omega) \, | \, (p,1)_\Omega = 0 \}.$$

We use the notation $\vec{u} = (u_i)$ for a vector. We shall often be concerned with two-dimensional vector functions with components in $L^2(\Omega)$ or $H_0^1(\Omega)$. The notation $L^2(\Omega)^2$, $H_0^1(\Omega)^2$ will be used for the product spaces. Define, for $\vec{u}$ and $\vec{v} \in L^2(\Omega)^2$,

$$(\vec{u},\vec{v})_\Omega := \sum_{i=1}^{2} (u_i, v_i)_\Omega \quad , \quad \|\vec{u}\|_\Omega^2 := (\vec{u},\vec{u})_\Omega$$

and, for $\vec{u}$ and $\vec{v} \in H_0^1(\Omega)^2$,

$$(\vec{u},\vec{v})_{1,\Omega} := \sum_{i=1}^{2} (u_i, v_i)_{1,\Omega} \quad , \quad \|\vec{u}\|_{1,\Omega}^2 := (\vec{u},\vec{u})_{1,\Omega}.$$

We also make some definitions analogous to the above on discrete subsets of the unit square S in $\mathbf{R}^2$. Let

$$S := \{ \ (x,y) \in \mathbf{R}^2 \mid 0 < x, y < 1 \ \}$$

and T its boundary. Let

$$\begin{aligned} h &:= \frac{1}{N}, \quad \text{for some } N \in \mathbf{N}, \\ \mathbf{R}_h^2 &:= \{ \ (lh, mh) \in \mathbf{R}^2 \mid l, m \in \mathbf{N} \ \}, \\ S_h &:= \bar{S} \cap \mathbf{R}_h^2 \end{aligned}$$

where $\bar{S}$ is the closure of S.

For an arbitrary discrete set Ω_h of the form

$$\Omega_h := \{ \ (lh, mh) \in S_h \mid l_0 \leq l \leq l_1 \text{ and } m_0 \leq m \leq m_1 \ \},$$

we define

$$\Omega_h^o := \{\ (lh, mh) \in S_h \mid l_0 + 1 \le l \le l_1 - 1\ ,\ m_0 + 1 \le m \le m_1 - 1\ \},$$
$$e(\Omega_h) := \{\ (lh, mh) \in S_h \mid l_0 + 1 \le l \le l_1\ ,\ m_0 \le m \le m_1\ \},$$
$$w(\Omega_h) := \{\ (lh, mh) \in S_h \mid l_0 \le l \le l_1 - 1\ ,\ m_0 \le m \le m_1\ \},$$
$$s(\Omega_h) := \{\ (lh, mh) \in S_h \mid l_0 \le l \le l_1\ ,\ m_0 \le m \le m_1 - 1\ \},$$
$$n(\Omega_h) := \{\ (lh, mh) \in S_h \mid l_0 \le l \le l_1\ ,\ m_0 + 1 \le m \le m_1\ \}$$

as the interior, east, west, south and the north sides of Ω_h and define

$$se(\Omega_h) := s(\Omega_h) \cap e(\Omega_h), \qquad sw(\Omega_h) := s(\Omega_h) \cap w(\Omega_h)$$
$$ne(\Omega_h) := n(\Omega_h) \cap e(\Omega_h), \qquad nw(\Omega_h) := n(\Omega_h) \cap w(\Omega_h).$$

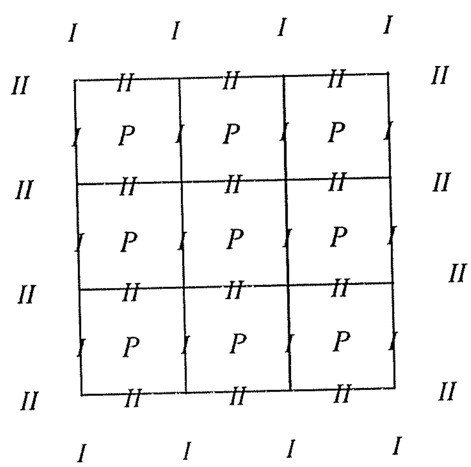
For the boundary Γ_h of Ω_h, we define

$$e(\Gamma_h), \qquad w(\Gamma_h), \qquad s(\Gamma_h), \qquad n(\Gamma_h)$$

as the east, west, south and north parts of Γ_h including the end points.

In this paper, we want to study both standard and staggered grids. The staggered mesh schemes use different grids that are staggered for the pressure and the velocity. A staggered grid is shown in Figure 1. The points marked by P, I, and II are where the pressure and the first and the second components of the velocity are defined, respectively.

Figure 1



Let

$$S_P := \{\ (\ (l - \frac{1}{2})h\ ,\ (m - \frac{1}{2})h\ ) \in S \mid l, m = 1, \ldots, N\ \},$$
$$S_I := \{\ (\ lh\ ,\ (m - \frac{1}{2})h\ ) \in S \mid l = 0, \ldots, N\ ,\ m = 0, \ldots, N + 1\ \},$$
$$S_{II} := \{\ (\ (l - \frac{1}{2})h\ ,\ mh\ ) \in S \mid l = 0, \ldots, N + 1\ ,\ m = 0, \ldots, N\ \},$$

then these are the sets for P, I, and II. Figure 1 shows S_P, S_I, and S_{II} when $N = 3$. Staggered mesh schemes have been used by Amsden and Harlow [1], Brandt and Dinar [2], Harlow and Welch [6], Patankar and Spalding [8], and Raithby and Schneider [9] and others.

Let

$$\begin{aligned}
S_w &:= \{ (x,y) \in \mathbf{R}^2 \,|\, 0 < x < 1-h \,,\, 0 < y < 1 \}, \\
S_e &:= \{ (x,y) \in \mathbf{R}^2 \,|\, h < x < 1 \,,\, 0 < y < 1 \}, \\
S_s &:= \{ (x,y) \in \mathbf{R}^2 \,|\, 0 < x < 1 \,,\, 0 < y < 1-h \}, \\
S_n &:= \{ (x,y) \in \mathbf{R}^2 \,|\, 0 < x < 1 \,,\, h < y < 1 \}, \\
S_1 &:= \{ (x,y) \in \mathbf{R}^2 \,|\, 0 < x < 1 \,,\, -\frac{h}{2} < y < 1 + \frac{h}{2} \}, \\
S_2 &:= \{ (x,y) \in \mathbf{R}^2 \,|\, -\frac{h}{2} < x < 1 + \frac{h}{2} \,,\, 0 < y < 1 \} \\
S_{sw} &:= S_s \cap S_w, \qquad \text{and} \qquad S_{ne} := S_n \cap S_e
\end{aligned}$$

be the continuous analogues of $w(S_h)$, $e(S_h)$, and so forth, respectively.

Let $L^2(\Omega_h)$ be the space of all discrete functions defined on Ω_h with the following inner product and norm

$$(U,V)_{\Omega_h} := h^2 \sum_{(x,y)\in\Omega_h} U(x,y)V(x,y), \qquad \|U\|^2_{\Omega_h} := (U,U)_{\Omega_h}$$

and let

$$L^2_0(\Omega_h) := \{ \; P \in L^2(\Omega_h) \;\; | \;\; (P,1)_{\Omega_h} = 0 \; \},$$

then $L^2(\Omega_h)$ and $L^2_0(\Omega_h)$ are the discrete analogies of $L^2(\Omega)$ and $L^2_0(\Omega)$.

For notational convenience, we introduce

$$U_{l,m} := U(lh, mh),$$

and define the forward, backward and central differencings on the x axis and y axis, respectively, as

$$\begin{aligned}
(\delta_{x+}U)_{l,m} &:= \frac{U_{l+1,m} - U_{l,m}}{h}, \qquad & (\delta_{y+}U)_{l,m} &:= \frac{U_{l,m+1} - U_{l,m}}{h}, \\
(\delta_{x-}U)_{l,m} &:= \frac{U_{l,m} - U_{l-1,m}}{h}, \qquad & (\delta_{y-}U)_{l,m} &:= \frac{U_{l,m} - U_{l,m-1}}{h}, \\
(\delta_{xo}U)_{l,m} &:= \frac{U_{l+\frac{1}{2},m} - U_{l-\frac{1}{2},m}}{h}, \qquad & (\delta_{yo}U)_{l,m} &:= \frac{U_{l,m+\frac{1}{2}} - U_{l,m-\frac{1}{2}}}{h}.
\end{aligned}$$

Define the discrete gradients as

$$\vec{\nabla}_+ := (\delta_{x+}, \delta_{y+}), \qquad \vec{\nabla}_- := (\delta_{x-}, \delta_{y-}), \qquad \text{and} \quad \vec{\nabla}_o := (\delta_{xo}, \delta_{yo}),$$

and let ∇_h^2 be the five-point discrete Laplacian, then

$$\nabla_h^2 = \vec{\nabla}_- \cdot \vec{\nabla}_+ = \vec{\nabla}_+ \cdot \vec{\nabla}_-.$$

The inner product and the norm of

$$H_0^1(\Omega_h) := \{ \ U \in L^2(\Omega_h) \ \mid \ U|_{\Gamma_h} = 0 \ \}$$

are defined as

$$(U,V)_{1,\Omega_h} := (\vec{\nabla}_+ U, \vec{\nabla}_+ V)_{sw(\Omega_h)} = (\vec{\nabla}_- U, \vec{\nabla}_- V)_{ne(\Omega_h)},$$
$$\|U\|_{1,\Omega_h}^2 := (U,U)_{1,\Omega_h}$$

which are the sums over all points in Ω_h where difference quotients are defined. The inner product and the norm of the product spaces $L^2(\Omega_h)^2$ and $H_0^1(\Omega_h)^2$ are defined naturally from $L^2(\Omega_h)$ and $H_0^1(\Omega_h)$.

The following inf-sup conditions are essential to study Q_h, the matrix in the pressure equation (1.7). Refer to Shin and Strikwerda [11] for the proofs.

Theorem 2.1. *There exists a positive constant C, which is independent of h, such that*

$$(1) \qquad \sup_{\vec{U} \in H_0^1(S_I) \times H_0^1(S_{II})} \frac{(\vec{\nabla}_o \cdot \vec{U}, P)_{S_P}^2}{\|U_1\|_{1,S_I}^2 + \|U_2\|_{1,S_{II}}^2} \geq C\|P\|_{S_P}^2, \qquad \forall P \in L_0^2(S_P),$$

$$(2) \qquad \sup_{\vec{U} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))} \frac{(\vec{\nabla}_- \cdot \vec{U}, P)_{S_h^o}^2}{\|U_1\|_{1,w(S_h)}^2 + \|U_2\|_{1,s(S_h)}^2} \geq C\|P\|_{S_h^o}^2, \qquad \forall P \in L_0^2(S_h^o),$$

$$(3) \qquad \sup_{\vec{U} \in H_0^1(e(S_h)) \times H_0^1(n(S_h))} \frac{(\vec{\nabla}_+ \cdot \vec{U}, P)_{S_h^o}^2}{\|U_1\|_{1,e(S_h)}^2 + \|U_2\|_{1,n(S_h)}^2} \geq C\|P\|_{S_h^o}^2, \qquad \forall P \in L_0^2(S_h^o).$$

3. Approximations by Finite Differences. Three finite difference approximations Q_h are introduced in this section. Let $P \in L^2(S_h^o)$, then $\delta_{x-}P$ and $\delta_{y-}P$ are defined in $w(S_h^o)$ and $s(S_h^o)$, respectively. Note that

$$w(\Omega_h^o) = w(\Omega_h)^o$$

for any rectangular subset Ω_h of S_h. Hence if $\vec{U} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))$ is the solution of

$$\nabla_h^2\, U_1 = \delta_{x-}P \quad \text{in } w(S_h)^o \qquad \text{and} \qquad \nabla_h^2\, U_2 = \delta_{y-}P \quad \text{in } s(S_h)^o, \tag{3.1}$$

then

$$Q_- P := \vec{\nabla}_- \cdot \vec{U} \qquad \text{in} \quad S_h^o \tag{3.2}$$

is well-defined. The above finite difference problem is similar to the following partial differential problem : For $p \in L^2(S_w \cup S_s)$, define

$$Qp := \vec{\nabla} \cdot \vec{u} \qquad \text{in} \quad S_{sw} \tag{3.3}$$

where $\vec{u} \in H_0^1(S_w) \times H_0^1(S_s)$ is the solution of

$$\nabla^2 u_1 = p_x \quad \text{in } S_w \qquad \text{and} \qquad \nabla^2 u_2 = p_y \quad \text{in } S_s. \tag{3.4}$$

Similarly, for $P \in L^2(S_h^o)$, let $\vec{U} \in H_0^1(e(S_h)) \times H_0^1(n(S_h))$ be the solution of

$$\nabla_h^2\, U_1 = \delta_{x+} P \quad \text{in } e(S_h)^o \qquad \text{and} \qquad \nabla_h^2\, U_2 = \delta_{y+} P \quad \text{in } n(S_h)^o, \tag{3.5}$$

then

$$Q_+ P := \vec{\nabla}_+ \cdot \vec{U} \qquad \text{in} \quad S_h^o$$

is well-defined. The above finite difference problem is similar to the following partial differential problem : For $p \in L^2(S_e \cup S_n)$, define

$$Qp := \vec{\nabla} \cdot \vec{u} \qquad \text{in} \quad S_{ne}$$

where $\vec{u} \in H_0^1(S_e) \times H_0^1(S_n)$ is the solution of

$$\nabla^2 u_1 = p_x \quad \text{in } S_e \qquad \text{and} \qquad \nabla^2 u_2 = p_y \quad \text{in } S_n. \tag{3.6}$$

Another approximation comes from the staggered mesh schemes. For $P \in L^2(S_P)$, let $\vec{U} \in H_0^1(S_I) \times H_0^1(S_{II})$ be the solution of

$$\nabla_h^2\, U_1 = \delta_{xo} P \quad \text{in } S_I^o \qquad \text{and} \qquad \nabla_h^2\, U_2 = \delta_{yo} P \quad \text{in } S_{II}^o, \tag{3.7}$$

then

$$Q_s P := \vec{\nabla}_o \cdot \vec{U} \qquad \text{in} \quad S_P \tag{3.8}$$

is well-defined. The above finite difference problem is similar to the following partial differential problem : For $p \in L^2(S_1 \cup S_2)$, define

$$Qp := \vec{\nabla} \cdot \vec{u} \qquad \text{in} \quad S \tag{3.9}$$

where $\vec{u} \in H_0^1(S_1) \times H_0^1(S_2)$ is the solution of

$$\nabla^2 u_1 = p_x \quad \text{in } S_1 \qquad \text{and} \qquad \nabla^2 u_2 = p_y \quad \text{in } S_2. \tag{3.10}$$

4. Preliminaries. In this section, we get some basic results for the next sections and also show that $Q_\pm$ and Q_s are self-adjoint. The next lemma resembles integration by parts.

Lemma 4.1. *If $U, V \in L^2(\Omega_h)$ with $UV|_{e(\Gamma_h) \cup w(\Gamma_h)} = 0$, then*

$$(\delta_{x+} U, V)_{w(\Omega_h)} = -(U, \delta_{x-} V)_{e(\Omega_h)}.$$

Proof. Let

$$\Omega_h := \{\ (lh, mh) \in S_h \mid l_0 \le l \le l_1 \text{ and } m_0 \le m \le m_1\ \},$$

then we note that $U_{l,m}V_{l,m} = 0$ if $l = l_0$ or l_1. Hence

$$(\delta_{x+}U, V)_{w(\Omega_h)} = h^2 \sum_{m=m_0}^{m_1} \sum_{l=l_0}^{l_1-1} \left(\frac{U_{l+1,m} - U_{l,m}}{h}\right) V_{l,m}$$

$$= h\left(\sum_{m=m_0}^{m_1} \sum_{l=l_0+1}^{l_1} U_{l,m}V_{l-1,m} - \sum_{m=m_0}^{m_1} \sum_{l=l_0}^{l_1-1} U_{l,m}V_{l,m}\right)$$

$$= h\left(\sum_{m=m_0}^{m_1} \sum_{l=l_0+1}^{l_1} U_{l,m}V_{l-1,m} - \sum_{m=m_0}^{m_1} \sum_{l=l_0+1}^{l_1} U_{l,m}V_{l,m}\right)$$

$$= -h^2 \sum_{m=m_0}^{m_1} \sum_{l=l_0+1}^{l_1} U_{l,m} \left(\frac{V_{l,m} - V_{l-1,m}}{h}\right) = -(U, \delta_{x-}V)_{e(\Omega_h)}. \quad \square$$

A similar result for staggered grids is stated in the next lemma. Other similar results which arise from different discrete domains and different differencings will be used without proof.

Lemma 4.2. *Let $P \in L^2(S_P)$.*
(1) For $U \in H_0^1(S_I)$, $(U, \delta_{xo}P)_{S_I^\circ} = (-\delta_{xo}U, P)_{S_P}$.
(2) For $U \in H_0^1(S_{II})$, $(U, \delta_{yo}P)_{S_{II}^\circ} = (-\delta_{yo}U, P)_{S_P}$.

By Lemma 4.1, we get the next lemma.

Lemma 4.3. *For any $U \in H_0^1(\Omega_h)$,*
(1) $\|\delta_{x-}U\|^2_{e(\Omega_h)} = \|\delta_{x+}U\|^2_{w(\Omega_h)} = (U, -\delta_{x-}\delta_{x+}U)_{\Omega_h^\circ}$
(2) $\|\delta_{y-}U\|^2_{n(\Omega_h)} = \|\delta_{y+}U\|^2_{s(\Omega_h)} = (U, -\delta_{y-}\delta_{y+}U)_{\Omega_h^\circ}$
(3) $\|U\|^2_{1,\Omega_h} = (U, -\nabla_h^2\, U)_{\Omega_h^\circ}$.

Proof. Let

$$V = \begin{cases} \delta_{x+}U, & \text{in } w(\Omega_h); \\ \text{any finite number}, & \text{on } e(\Gamma_h), \end{cases}$$

then, by Lemma 4.1,

$$\|\delta_{x-}U\|^2_{e(\Omega_h)} = \|\delta_{x+}U\|^2_{w(\Omega_h)} = (\delta_{x+}U, \delta_{x+}U)_{w(\Omega_h)} = (\delta_{x+}U, V)_{w(\Omega_h)}$$
$$= (U, -\delta_{x-}V)_{e(\Omega_h)} = (U, -\delta_{x-}V)_{\Omega_h^\circ} = (U, -\delta_{x-}\delta_{x+}U)_{\Omega_h^\circ}.$$

The proof for (2) is similar. The statement (3) follows from (1) and (2). $\square$

Relation (3) in Lemma 4.3 extends to $H_0^1(\Omega_h)^2$ and implies that $-\nabla_h^2$ is positive definite, and hence the Schwarz inequality for $-\nabla_h^2$,

$$(\vec{V}, -\nabla_h^2 \vec{U})^2_{\Omega_h^o} \leq (\vec{V}, -\nabla_h^2 \vec{V})_{\Omega_h^o} (\vec{U}, -\nabla_h^2 \vec{U})_{\Omega_h^o}, \qquad (4.1)$$

holds for any $\vec{U}$ and $\vec{V}$ in $H_0^1(\Omega_h)^2$.

By Lemma 4.1 and Lemma 4.2, we can show that the approximations $Q_\pm$ and Q_s are self-adjoint.

Theorem 4.4.

(1) $(Q_\pm P_1, P_2)_{S_h^o} = (P_1, Q_\pm P_2)_{S_h^o}, \qquad \forall P_1, P_2 \in L^2(S_h^o)$

(2) $(Q_s P_1, P_2)_{S_P} = (P_1, Q_s P_2)_{S_P}, \qquad \forall P_1, P_2 \in L^2(S_P)$

Proof. Let $\vec{U}$, $\vec{V} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))$ be the solutions of

$$\nabla_h^2 U_1 = \delta_{x-} P_1 \quad \text{in } w(S_h)^o \qquad \text{and} \qquad \nabla_h^2 U_2 = \delta_{y-} P_1 \quad \text{in } s(S_h)^o$$

and

$$\nabla_h^2 V_1 = \delta_{x-} P_2 \quad \text{in } w(S_h)^o \qquad \text{and} \qquad \nabla_h^2 V_2 = \delta_{y-} P_2 \quad \text{in } s(S_h)^o,$$

respectively, then

$$Q_- P_1 = \vec{\nabla}_- \cdot \vec{U} \quad \text{and} \quad Q_- P_2 = \vec{\nabla}_- \cdot \vec{V} \qquad \text{in} \quad S_h^o.$$

Hence, by Lemma 4.1 and Lemma 4.3,

$$\begin{aligned}
(Q_- P_1, P_2)_{S_h^o} &= (\vec{\nabla}_- \cdot \vec{U}, P_2)_{S_h^o} = (\delta_{x-} U_1, P_2)_{e(w(S_h))} + (\delta_{y-} U_2, P_2)_{n(s(S_h))} \\
&= (U_1, -\delta_{x+} P_2)_{w(w(S_h))} + (U_2, -\delta_{y+} P_2)_{s(s(S_h))} \\
&= (U_1, -\nabla_h^2 V_1)_{w(S_h)^o} + (U_2, -\nabla_h^2 V_2)_{s(S_h)^o} \\
&= (\delta_{x+} U_1, \delta_{x+} V_1)_{w(w(S_h))} + (\delta_{y+} U_1, \delta_{y+} V_1)_{s(w(S_h))} \\
&\qquad + (\delta_{x+} U_2, \delta_{x+} V_2)_{w(s(S_h))} + (\delta_{y+} U_2, \delta_{y+} V_2)_{s(s(S_h))} \\
&= (\delta_{x+} V_1, \delta_{x+} U_1)_{w(w(S_h))} + (\delta_{y+} V_1, \delta_{y+} U_1)_{s(w(S_h))} \\
&\qquad + (\delta_{x+} V_2, \delta_{x+} U_2)_{w(s(S_h))} + (\delta_{y+} V_2, \delta_{y+} U_2)_{s(s(S_h))} \\
&\qquad = (Q_- P_2, P_1)_{S_h^o} = (P_1, Q_- P_2)_{S_h^o}.
\end{aligned}$$

The proof for Q_+ is similar. The proof for Q_s is similar, but we include the proof for completeness. Let $\vec{U}$, $\vec{V} \in H_0^1(S_I) \times H_0^1(S_{II})$ be the solutions of

$$\nabla_h^2 U_1 = \delta_{xo} P_1 \quad \text{in } S_I^o \qquad \text{and} \qquad \nabla_h^2 U_2 = \delta_{yo} P_1 \quad \text{in } S_{II}^o$$

and

$$\nabla_h^2 V_1 = \delta_{xo} P_2 \quad \text{in } S_I^o \qquad \text{and} \qquad \nabla_h^2 V_2 = \delta_{yo} P_2 \quad \text{in } S_{II}^o$$

respectively, then

$$Q_s P_1 = \vec{\nabla}_o \cdot \vec{U} \quad \text{and} \quad Q_s P_2 = \vec{\nabla}_o \cdot \vec{V} \qquad \text{in} \quad S_P.$$

Hence, by Lemma 4.2,

$$\begin{aligned}
(Q_s P_1, P_2)_{S_P} &= (\vec{\nabla}_o \cdot \vec{U}, P_2)_{S_P} = (\delta_{xo} U_1, P_2)_{S_P} + (\delta_{yo} U_2, P_2)_{n(S_{II})} \\
&= (U_1, -\delta_{xo} P_2)_{S_I^o} + (U_2, -\delta_{yo} P_2)_{S_{II}^o} = (U_1, -\nabla_h^2\, V_1)_{S_I^o} + (U_2, -\nabla_h^2\, V_2)_{S_{II}^o} \\
&= (\delta_{x+} U_1, \delta_{x+} V_1)_{w(S_I)} + (\delta_{y+} U_1, \delta_{y+} V_1)_{s(S_I)} + (\delta_{x+} U_2, \delta_{x+} V_2)_{w(S_{II})} + (\delta_{y+} U_2, \delta_{y+} V_2)_{s(S_{II})} \\
&= (\delta_{x+} V_1, \delta_{x+} U_1)_{w(S_I)} + (\delta_{y+} V_1, \delta_{y+} U_1)_{s(S_I)} + (\delta_{x+} V_2, \delta_{x+} U_2)_{w(S_{II})} + (\delta_{y+} V_2, \delta_{y+} U_2)_{s(S_{II})} \\
&= (Q_s P_2, P_1)_{S_P} = (P_1, Q_s P_2)_{S_P}. \quad \square
\end{aligned}$$

5. The Condition Number of Q_h. We first prove that $Q_\pm$ and Q_s are bounded above by 1.

Theorem 5.1.
(1) $(Q_\pm P, P)_{S_h^o} \leq \|P\|_{S_h^o}^2, \quad \forall P \in L^2(S_h^o)$
(2) $(Q_s P, P)_{S_P} \leq \|P\|_{S_P}^2, \quad \forall P \in L^2(S_P)$,
(3) $\|Q_\pm\|_{S_h^o}, \|Q_s\|_{S_P} \leq 1$.

Proof. Let $\vec{U} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))$ be the solution of (3.1), then

$$\begin{aligned}
\|\vec{\nabla}_- \cdot \vec{U}\|_{S_h^o}^2 &\leq \|U_1\|_{1,w(S_h)}^2 + \|U_2\|_{1,s(S_h)}^2 = (U_1, -\nabla_h^2\, U_1)_{w(S_h)^o} + (U_2, -\nabla_h^2\, U_2)_{s(S_h)^o} \\
&= (U_1, -\delta_{x+} P)_{w(w(S_h))} + (U_2, -\delta_{y+} P)_{s(s(S_h))} = (\delta_{x-} U_1, P)_{e(w(S_h))} + (\delta_{y-} U_2, P)_{n(s(S_h))} \\
&= (\vec{\nabla}_- \cdot \vec{U}, P)_{S_h^o} \leq \|\vec{\nabla}_- \cdot \vec{U}\|_{S_h^o} \|P\|_{S_h^o}
\end{aligned} \tag{5.1}$$

which implies

$$\|\vec{\nabla}_- \cdot \vec{U}\|_{S_h^o} \leq \|P\|_{S_h^o}.$$

By (3.2),

$$Q_- P = \vec{\nabla}_- \cdot \vec{U}$$

and hence

$$\|Q_- P\|_{S_h^o} = \|\vec{\nabla}_- \cdot \vec{U}\|_{S_h^o} \leq \|P\|_{S_h^o}. \tag{5.2}$$

Thus

$$(Q_- P, P)_{S_h^o} \leq \|Q_- P\|_{S_h^o} \|P\|_{S_h^o} \leq \|P\|_{S_h^o}^2.$$

The proof for Q_+ is similar. The proof for (2) is also similar, but we show it for completeness. Let $\vec{U} \in H_0^1(S_I) \times H_0^1(S_{II})$ be the solution of (3.7), then, by Lemma 4.2,

$$\begin{aligned}
\|\vec{\nabla}_o \cdot \vec{U}\|_{S_P}^2 &\leq \|U_1\|_{1,S_I}^2 + \|U_2\|_{1,S_{II}}^2 = (U_1, -\nabla_h^2\, U_1)_{S_I^o} + (U_2, -\nabla_h^2\, U_2)_{S_{II}^o} \\
&= (U_1, -\delta_{xo} P)_{S_I^o} + (U_2, -\delta_{yo} P)_{S_{II}^o} = (\vec{\nabla}_o \cdot \vec{U}, P)_{S_P} \leq \|\vec{\nabla}_o \cdot \vec{U}\|_{S_P} \|P\|_{S_P}
\end{aligned}$$

By this relation and (3.8), we obtain

$$\|Q_s P\|_{S_P} \le \|P\|_{S_P} \tag{5.3}$$

and hence

$$(Q_s P, P)_{S_P} \le \|Q_s P\|_{S_P} \|P\|_{S_P} \le \|P\|^2_{S_P}.$$

The statement (3) follows from (5.2) and (5.3). □

To prove that $Q_\pm$ and Q_s are bounded below, we need the next lemma.

Lemma 5.2.

(1) $$(Q_- P, P)_{S_h^o} = \sup_{\vec{V} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))} \frac{(\vec{\nabla}_- \cdot \vec{V}, P)^2_{S_h^o}}{\|V_1\|^2_{1,w(S_h)} + \|V_2\|^2_{1,s(S_h)}}, \qquad \forall P \in L^2(S_h^o)$$

(2) $$(Q_+ P, P)_{S_h^o} = \sup_{\vec{V} \in H_0^1(e(S_h)) \times H_0^1(n(S_h))} \frac{(\vec{\nabla}_+ \cdot \vec{V}, P)^2_{S_h^o}}{\|V_1\|^2_{1,e(S_h)} + \|V_2\|^2_{1,n(S_h)}}, \qquad \forall P \in L^2(S_h^o)$$

(3) $$(Q_s P, P)_{S_P} = \sup_{\vec{V} \in H_0^1(S_I) \times H_0^1(S_{II})} \frac{(\vec{\nabla}_o \cdot \vec{V}, P)^2_{S_P}}{\|V_1\|^2_{1,S_I} + \|V_2\|^2_{1,S_{II}}}, \qquad \forall P \in L^2(S_P).$$

Proof. Let $\vec{U}$ be the solution of (3.1), then

$$(Q_- P, P)_{S_h^o} = (\vec{\nabla}_- \cdot \vec{U}, P)_{S_h^o} = (U_1, -\nabla_h^2\, U_1)_{w(S_h)^o} + (U_2, -\nabla_h^2\, U_2)_{s(S_h)^o}$$

by (5.1). Using (4.1), we have

$$(V_1, -\nabla_h^2\, U_1)^2_{w(S_h)^o} \le (V_1, -\nabla_h^2\, V_1)_{w(S_h)^o} (U_1, -\nabla_h^2\, U_1)_{w(S_h)^o}$$

$$(V_2, -\nabla_h^2\, U_2)^2_{s(S_h)^o} \le (V_2, -\nabla_h^2\, V_2)_{s(S_h)^o} (U_2, -\nabla_h^2\, U_2)_{s(S_h)^o}$$

for any nonzero $\vec{V} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))$. Hence

$$2(V_1, -\nabla_h^2\, U_1)_{w(S_h)^o} (V_2, -\nabla_h^2\, U_2)_{s(S_h)^o} \le 2\sqrt{(V_1, -\nabla_h^2\, V_1)_{w(S_h)^o}}$$
$$\sqrt{(U_1, -\nabla_h^2\, U_1)_{w(S_h)^o}} \sqrt{(V_2, -\nabla_h^2\, V_2)_{s(S_h)^o}} \sqrt{(U_2, -\nabla_h^2\, U_2)_{s(S_h)^o}}$$
$$\le (V_1, -\nabla_h^2\, V_1)_{w(S_h)^o} (U_2, -\nabla_h^2\, U_2)_{s(S_h)^o} + (U_1, -\nabla_h^2\, U_1)_{w(S_h)^o} (V_2, -\nabla_h^2\, V_2)_{s(S_h)^o}$$

and

$$\left((V_1, -\nabla_h^2\, U_1)_{w(S_h)^o} + (V_2, -\nabla_h^2\, U_2)_{s(S_h)^o} \right)^2 \le$$
$$\left((V_1, -\nabla_h^2\, V_1)_{w(S_h)^o} + (V_2, -\nabla_h^2\, V_2)_{s(S_h)^o} \right) \left((U_1, -\nabla_h^2\, U_1)_{w(S_h)^o} + (U_2, -\nabla_h^2\, U_2)_{s(S_h)^o} \right).$$

We have

$$(V_1, -\nabla_h^2\, V_1)_{w(S_h)^o} + (V_2, -\nabla_h^2\, V_2)_{s(S_h)^o} = \|V_1\|^2_{1,w(S_h)} + \|V_2\|^2_{1,s(S_h)}$$

and, by following the steps in (5.1),

$$(V_1, -\nabla_h^2\, U_1)^2_{w(S_h)^o} + (V_2, -\nabla_h^2\, U_2)^2_{s(S_h)^o} = (\vec{\nabla}_- \cdot \vec{V}, P)_{S_h^o}.$$

Thus (1) is proved. The proofs for (2) and (3) are similar. □

Theorem 2.1 and Lemma 5.2 imply that $Q_\pm$ and Q_s are bounded away from zero uniformly with respect to the mesh size h.

Theorem 5.3. *There exists a positive constant C_L which is independent of h such that*
(1) $\forall P \in L_0^2(S_h^o)$, $(Q_\pm P, P)_{S_h^o} \geq C_L \|P\|_{S_h^o}^2$
(2) $\forall P \in L_0^2(S_P)$, $(Q_s P, P)_{S_P} \geq C_L \|P\|_{S_P}^2$.

Theorem 5.1 and Theorem 5.3 imply that the condition numbers of $Q_\pm$ and Q_s are independent of h.

6. Preliminaries for Convergence Estimation. This section is to prepare to get the convergence rates of the solutions computed by $Q_\pm$ and Q_s. We define

$$|f|_{\Omega_h} := \sup_{(x,y)\in\Omega_h} |f(x,y)| \quad , \quad |f|_{r_o,\Omega_h} := \sum_{r\leq r_o} \sup_{(x,y)\in\Omega_h} |\partial^r f(x,y)|$$

where r_o is a positive integer and $\partial^r f$ denotes all possible r'th partial derivatives of f. For a vector $\vec{f} = (f_i)$, let

$$|\vec{f}|_{\Omega_h} := \sum_i |f_i|_{\Omega_h} \quad , \quad |\vec{f}|_{r_o,\Omega_h} := \sum_i |f_i|_{r_o,\Omega_h}.$$

The relation between $\|\ .\ \|$ and $|\ .\ |$ is stated in the next lemma.

Lemma 6.1. *Let Ω_h be a subset of S_h and $U \in L^2(\Omega_h)$, then*

$$\|U\|_{\Omega_h^o} \leq |U|_{\Omega_h^o}.$$

Proof. Let the number of points in Ω_h^o be M, then $M \leq (N-1)^2$. Hence

$$\|U\|_{\Omega_h^o}^2 \leq h^2 \sum_{(x,y)\in\Omega_h^o} |U(x,y)|^2 \leq h^2 M |U|_{\Omega_h}^2 = \frac{(N-1)^2}{N^2} |U|_{\Omega_h}^2 \leq |U|_{\Omega_h}^2. \quad \square$$

Refer to [3] for the next lemma which is a result of the maximum principle.

Lemma 6.2. *Let Ω_h be a subset of one of the sets S_h, S_I and S_{II} and let Γ_h be its boundary, then there exists a positive constant $C_M = C_M(\Omega_h)$ such that*

$$|U|_{\Omega_h^o} \leq C_M |\nabla_h^2 U|_{\Omega_h^o} + |U|_{\Gamma_h} \quad \text{for } U \in L^2(\Omega_h).$$

Moreover, there exists a positive constant $C_S = C_S(S)$ such that

$$|u|_S \leq C_S |\nabla^2 u|_S + |u|_T$$

for any u which is twice differentiable in S.

From the above lemma, we can estimate the norm of the difference quotients of the solution of the Poisson's equation.

Lemma 6.3. *Let $\Omega_h \subseteq S_h$ and $U \in H_0^1(\Omega_h)$ be such that*

$$\nabla_h^2 \, U = F \qquad in \quad \Omega_h^\circ,$$

then there exists a positive constant C_Q such that

$$\|U\|_{1,\Omega_h} \le C_Q |F|_{\Omega_h^\circ}.$$

Proof. By Lemma 4.3, Lemma 6.1 and Lemma 6.2,

$$\|U\|_{1,\Omega_h}^2 = (U, -\nabla_h^2 \, U)_{\Omega_h^\circ} = (U, -F)_{\Omega_h^\circ} \le \frac{1}{2}\left(\|U\|_{\Omega_h^\circ}^2 + \|F\|_{\Omega_h^\circ}^2 \right)$$

$$\le \frac{1}{2}\left(|U|_{\Omega_h^\circ}^2 + |F|_{\Omega_h^\circ}^2 \right) \le \frac{1}{2}\left(C_M |F|_{\Omega_h^\circ}^2 + |F|_{\Omega_h^\circ}^2 \right).$$

The claim follows by setting $C_Q = \sqrt{(C_M+1)/2}$. □

We prove the discrete Poincaré inequality.

Lemma 6.4. *Let $U \in H_0^1(S_h)$, then*
(1) $\|U\|_{S_h^\circ} \le \|\delta_{x\pm} U\|_{S_h^\circ}$ *and* $\|U\|_{S_h^\circ} \le \|\delta_{y\pm} U\|_{S_h^\circ}$
(2) $2\|U\|_{S_h^\circ}^2 \le \|U\|_{1,S_h}^2$.

Proof. One can easily show that

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 \le n \sum_{i=1}^{n} a_i^2 \tag{6.1}$$

for any positive integer n. Since $U_{l,0} = 0$,

$$\sum_{m'=1}^{m} (\delta_{x-} U)_{l,m'} = \frac{U_{l,m}}{h} \qquad \text{for} \quad l, m = 1, \ldots, N-1. \tag{6.2}$$

Using (6.1) and (6.2), we get

$$U_{l,m}^2 \le h^2 \Big(\sum_{m'=1}^{m} (\delta_{x-} U)_{l,m'} \Big)^2 \le m h^2 \sum_{m'=1}^{m} (\delta_{x-} U)_{l,m'}^2 \le h \sum_{m'=1}^{N-1} (\delta_{x-} U)_{l,m'}^2$$

and

$$\sum_{m=1}^{N-1} U_{l,m}^2 \le (N-1) h \sum_{m'=1}^{N-1} (\delta_{x-} U)_{l,m'}^2 \le \sum_{m=1}^{N-1} (\delta_{x-} U)_{l,m}^2.$$

Hence

$$\|U\|_{S_h^\circ}^2 = h^2 \sum_{l=1}^{N-1} \sum_{m=1}^{N-1} U_{l,m}^2 \le h^2 \sum_{l=1}^{N-1} \sum_{m=1}^{N-1} (\delta_{x-} U)_{l,m}^2 = \|\delta_{x-} U\|_{l,m}^2.$$

The other inequalities in (1) are similar and (2) follows from (1). □

The general boundary value problems for second-order elliptic equations on a polygon are discussed by Grisvard [5]. The next theorem follows from the work for the zero Dirichlet boundary condition in [5].

Theorem 6.5. *Let k be a positive even integer and let Ω be a polygon with Γ its boundary. If $f \in H^k(\Omega)$, then the Poisson's equation*

$$\nabla^2 u = f \quad in\ \Omega \qquad with \quad u|_\Gamma = 0$$

has a unique solution $\vec{u} \in H_0^{k+2}(\Omega)^2$.

The proof for the following imbedding theorem is in [3].

Theorem 6.6. *For any positive integer k and a polygon Ω,*

$$H_0^k(\Omega) \subset C^{k-2}(\bar{\Omega}).$$

7. Convergence Estimation. The next lemma shows how smooth the solution of (3.4) is.

Lemma 7.1. *Let $\vec{u}$ be the solution of (3.4) with $p \in C^5(\overline{S_w} \cup \overline{S_s})$, then $\vec{u} \in C^4(\overline{S_w}) \times C^4(\overline{S_s})$.*

Proof. Since $p \in C^5(\overline{S_w} \cup \overline{S_s})$, we have

$$\vec{\nabla}\, p \in C^4(\overline{S_w}) \times C^4(\overline{S_s}) \subset H^4(S_w) \times H^4(S_s).$$

Applying Theorem 6.5 and Theorem 6.6 to the Poisson's equation (3.4), we have

$$\vec{u} \in H_0^6(S_w) \times H_0^6(S_s) \subset C^4(\overline{S_w}) \times C^4(\overline{S_s}). \quad \square$$

We show that Q_- gives a first-order accurate solution.

Theorem 7.2. *There exists a positive constant C such that the following is true : Let $p \in C^5(\overline{S_w} \cup \overline{S_s})$ be the solution of*

$$Qp = f \qquad in \quad \overline{S_{sw}} \tag{7.1}$$

and let $P \in L^2(S_h^o)$ be the solution of

$$Q_- P = f \qquad in \quad S_h^o \tag{7.2}$$

Let P be chosen up to a constant so that $p - P \in L_0^2(S_h^o)$. Let $\vec{u} \in C^4(\overline{S_w}) \times C^4(\overline{S_s})$ and $\vec{U} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))$ be the solutions of (3.4) and (3.1), respectively, then

$$\|u_1 - U_1\|_{w(S_h)} + \|u_2 - U_2\|_{s(S_h)} + \|p - P\|_{S_h^o} \le Ch\big(|u_1|_{4,S_w} + |u_2|_{4,S_s} + |p|_{2,S_w \cup S_s}\big).$$

Proof. Let $\vec{V} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))$ be the solution of

$$\nabla_h^2 V_1 = \delta_{x+}p \quad \text{in } w(S_h)^o \qquad \text{and} \qquad \nabla_h^2 V_2 = \delta_{y+}p \quad \text{in } s(S_h)^o,$$

then

$$Q_-p = \vec{\nabla}_- \cdot \vec{V} \qquad \text{in} \quad S_h^o \tag{7.3}$$

and

$$\begin{aligned} &\|u_1 - U_1\|_{w(S_h)^o} + \|u_2 - U_2\|_{s(S_h)^o} \leq \\ &\|u_1 - V_1\|_{w(S_h)^o} + \|u_2 - V_2\|_{s(S_h)^o} + \|V_1 - U_1\|_{w(S_h)^o} + \|V_2 - U_2\|_{s(S_h)^o}. \end{aligned} \tag{7.4}$$

Using Taylor expansions, for $(x, y) \in w(S_h)^o$, we have

$$\begin{aligned} \nabla_h^2 u_1(x,y) &= \nabla^2 u_1(x,y) + u_1^*(x,y) = p_x(x,y) + u_1^*(x,y) \\ \nabla_h^2 V_1(x,y) &= \delta_{x+}p(x,y) = p_x(x,y) + p_1^*(x,y) \end{aligned} \tag{7.5}$$

and, for $(x, y) \in s(S_h)^o$,

$$\begin{aligned} \nabla_h^2 u_2(x,y) &= \nabla^2 u_2(x,y) + u_2^*(x,y) = p_y(x,y) + u_2^*(x,y) \\ \nabla_h^2 V_2(x,y) &= \delta_{y+}p(x,y) = p_y(x,y) + p_2^*(x,y) \end{aligned} \tag{7.6}$$

where

$$\begin{aligned} u_1^*(x,y) &= \frac{h^2}{12}(u_1)_{xxxx}(x^1,y), \qquad p_1^*(x,y) = \frac{h}{2}p_{xx}(x^2,y) \\ u_2^*(x,y) &= \frac{h^2}{12}(u_2)_{yyyy}(x,y^3), \qquad p_2^*(x,y) = \frac{h}{2}p_{yy}(x,y^2) \end{aligned} \tag{7.7}$$

for some points (x^i, y) in S_w and (x, y^i) in S_s around (x, y).

By (7.5) and (7.6),

$$\begin{aligned} \nabla_h^2 (u_1 - V_1) &= E_1 := u_1^* - p_1^* \qquad \text{in } w(S_h)^o \\ \nabla_h^2 (u_2 - V_2) &= E_2 := u_2^* - p_2^* \qquad \text{in } s(S_h)^o \end{aligned} \tag{7.8}$$

and, by Lemma 6.1, Lemma 6.2, (7.7), and (7.8),

$$\begin{aligned} \|u_1 - V_1\|_{w(S_h)^o} + \|u_2 - V_2\|_{s(S_h)^o} &\leq C_M\big(|E_1|_{w(S_h)^o} + |E_2|_{s(S_h)^o}\big) \\ &\leq M_u h^2\big(|u_1|_{4,S_w} + |u_2|_{4,S_w}\big) + M_p h|p|_{2,S_w\cup S_s} \end{aligned} \tag{7.9}$$

for some positive constants M_u and M_p.

Using Lemma 4.1, Lemma 4.3, Theorem 5.1, Lemma 6.4, and the fact that $\vec{V}$, $\vec{U} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))$, we have

$$
\begin{aligned}
2\big(\,\|V_1-U_1\|^2_{w(S_h)^o}+\|V_2-U_2\|^2_{s(S_h)^o}\big) &\le \|V_1-U_1\|^2_{1,w(S_h)}+\|V_2-U_2\|^2_{1,s(S_h)}\\
&= \big(V_1-U_1,-\nabla_h^2\,(V_1-U_1)\big)_{w(S_h)^o}+\big(V_2-U_2,-\nabla_h^2\,(V_2-U_2)\big)_{s(S_h)^o}\\
&= \big(V_1-U_1,-\delta_{x+}(p-P)\big)_{w(w(S_h))}+\big(V_2-U_2,-\delta_{y+}(p-P)\big)_{s(s(S_h))}\\
&= \big(\delta_{x-}(V_1-U_1),p-P\big)_{e(w(S_h))}+\big(\delta_{y-}(V_2-U_2),p-P\big)_{n(s(S_h))}\\
&= \big(\vec{\nabla}_-\cdot(\vec{V}-\vec{U}),p-P\big)_{S_h^o} = \big(Q_-(p-P),p-P\big)_{S_h^o} \le \|p-P\|^2_{S_h^o}.
\end{aligned}
$$

Hence

$$\big(\,\|V_1-U_1\|_{w(S_h)^o}+\|V_2-U_2\|_{s(S_h)^o}\big)^2 \le 2\big(\,\|V_1-U_1\|^2_{w(S_h)^o}+\|V_2-U_2\|^2_{s(S_h)^o}\big) \le \|p-P\|^2_{S_h^o}$$

and

$$\|V_1-U_1\|_{w(S_h)^o}+\|V_2-U_2\|_{s(S_h)^o} \le \|p-P\|_{S_h^o}. \tag{7.10}$$

Combining (7.4), (7.9), and (7.10), we have

$$
\begin{aligned}
&\|u_1-U_1\|_{w(S_h)^o}+\|u_2-U_2\|_{s(S_h)^o} \le \\
&M_u h^2\big(\,|u_1|_{4,S_w}+|u_2|_{4,S_s}\big)+M_p h|p|_{2,S_w\cup S_s}+\|p-P\|_{S_h^o}.
\end{aligned} \tag{7.11}
$$

Now let's estimate $\|p-P\|_{S_h^o}$. By (3.3), Theorem 5.3, (7.1), (7.2), and (7.3), we have

$$
\begin{aligned}
C_L\|p-P\|_{S_h^o} &\le \|Q_-(p-P)\|_{S_h^o} = \|Q_-p-Qp\|_{S_h^o}\\
&= \|\vec{\nabla}_-\cdot\vec{V}-\vec{\nabla}\cdot\vec{u}\|_{S_h^o} \le \|\vec{\nabla}_-\cdot(\vec{V}-\vec{u})\|_{S_h^o}+\|\vec{\nabla}_-\cdot\vec{u}-\vec{\nabla}\cdot\vec{u}\|_{S_h^o}
\end{aligned} \tag{7.12}
$$

and, by Lemma 6.3, (7.7), and (7.8),

$$
\begin{aligned}
&\|\vec{\nabla}_-\cdot(\vec{V}-\vec{u})\|_{S_h^o} \le \|u_1-V_1\|_{1,w(S_h)}+\|u_2-V_2\|_{1,s(S_h)}\\
&\le C_Q\big(\,|E_1|_{w(S_h)^o}+|E_2|_{s(S_h)^o}\big) \le Q_u h^2\big(\,|u_1|_{4,S_w}+|u_2|_{4,S_s}\big)+Q_p h|p|_{2,S_w\cup S_s}
\end{aligned} \tag{7.13}
$$

for some positive constants Q_u and Q_p.

Using Taylor expansions, we get

$$\vec{\nabla}_-\cdot\vec{u}(x,y)=\vec{\nabla}\cdot\vec{u}(x,y)-\frac{h}{2}\big(\,\vec{u}_{xx}(\tilde{x},y)+\vec{u}_{yy}(x,\tilde{y})\big) \quad \text{in } S_h^o$$

where $(\tilde{x},y)$ and $(x,\tilde{y})$ are points in S around (x,y). Thus

$$\|\vec{\nabla}_-\cdot\vec{u}-\vec{\nabla}\cdot\vec{u}\|_{S_h^o} \le \frac{h}{2}|\vec{u}|_{2,S}. \tag{7.14}$$

By (7.12), (7.13), and (7.14), we have

$$\|p-P\|_{S_h^o} \le C_p h\big(\,|u_1|_{4,S_w}+|u_2|_{4,S_s}+|p|_{2,S_w\cup S_s}\big) \tag{7.15}$$

for some positive constant C_p. The claim follows from (7.11) and (7.15). □

The proof of the next lemma is similar to that of Lemma 7.1.

Lemma 7.3. *Let $\vec{u}$ be the solution of* (3.6) *with $p \in C^5(\overline{S_e} \cup \overline{S_n})$, then $\vec{u} \in C^4(\overline{S_e}) \times C^4(\overline{S_n})$.*

It is proved similarly that Q_+ gives a first-order accurate solution.

Theorem 7.4. *There exists a positive constant C such that the following is true : Let $p \in C^5(\overline{S_e} \cup \overline{S_n})$ be the solution of*

$$Qp = f \qquad in \quad \overline{S_{ne}}$$

and let $P \in L^2(S_h^o)$ be the solution of

$$Q_+ P = f \qquad in \quad S_h^o$$

Let P be chosen up to a constant so that $p - P \in L_0^2(S_h^o)$. Let $\vec{u} \in C^4(\overline{S_e}) \times C^4(\overline{S_n})$ and $\vec{U} \in H_0^1(e(S_h)) \times H_0^1(n(S_h))$ be the solutions of (3.6) *and* (3.5), *respectively, then*

$$\|u_1 - U_1\|_{e(S_h)} + \|u_2 - U_2\|_{n(S_h)} + \|p - P\|_{S_h^o} \le Ch\big(|u_1|_{4,S_e} + |u_2|_{4,S_n} + |p|_{2,S_e \cup S_n}\big).$$

The proof of the next lemma is also similar to that of Lemma 7.1.

Lemma 7.5. *Let $\vec{u}$ be the solution of* (3.10) *with $p \in C^5(\overline{S_1} \cup \overline{S_2})$, then $\vec{u} \in C^4(\overline{S_1}) \times C^4(\overline{S_2})$.*

We show the second-order accuracy of the solution computed by staggered mesh schemes.

Theorem 7.6. *There exists a positive constant C such that the following is true : Let $p \in C^5(\overline{S_1} \cup \overline{S_2})$ be the solution of*

$$Qp = f \qquad in \quad S \tag{7.16}$$

and let $P \in L^2(S_P)$ be the solution of

$$Q_s P = f \quad in \quad S_P \tag{7.17}$$

Let P be chosen up to a constant so that $p - P \in L_0^2(S_P)$. Let $\vec{u} \in C^4(\overline{S_1}) \times C^4(\overline{S_2})$ and $\vec{U} \in H_0^1(S_I) \times H_0^1(S_{II})$ be the solutions of (3.10) *and* (3.7), *respectively, then*

$$\|u_1 - U_1\|_{S_I^o} + \|u_2 - U_2\|_{S_{II}^o} + \|p - P\|_{S_P} \le Ch^2\big(|u_1|_{4,S_1} + |u_2|_{4,S_2} + |p|_{3,S}\big).$$

Proof. Let $\vec{V} \in H_0^1(S_I) \times H_0^1(S_{II})$ be the solution of

$$\nabla_h^2 \; V_1 = \delta_{xo} p \qquad \text{in} \quad S_I^o \qquad \text{and} \qquad \nabla_h^2 \; V_2 = \delta_{yo} p \qquad \text{in} \quad S_{II}^o,$$

then

$$Q_s p = \vec{\nabla}_o \cdot \vec{V} \qquad \text{in } S_P \tag{7.18}$$

and

$$\|u_1-U_1\|_{S_I^o}+\|u_2-U_2\|_{S_{II}^o} \;\le\; \|u_1-V_1\|_{S_I^o}+\|u_2-V_2\|_{S_{II}^o}+\|V_1-U_1\|_{S_I^o}+\|V_2-U_2\|_{S_{II}^o}. \tag{7.19}$$

Using Taylor expansions, for $(x,y) \in S_I^o$, we have

$$\begin{aligned} \nabla_h^2 \; u_1(x,y) &= \nabla^2 u_1(x,y) + u_1^*(x,y) = p_x(x,y) + u_1^*(x,y) \\ \nabla_h^2 \; V_1(x,y) &= \delta_{xo} p(x,y) + p_1^*(x,y) \end{aligned} \tag{7.20}$$

and, for $(x,y) \in S_{II}^o$,

$$\begin{aligned} \nabla_h^2 \; u_2(x,y) &= \nabla^2 u_2(x,y) + u_2^*(x,y) = p_y(x,y) + u_2^*(x,y) \\ \nabla_h^2 \; V_2(x,y) &= \delta_{yo} p(x,y) + p_2^*(x,y) \end{aligned} \tag{7.21}$$

where

$$\begin{aligned} u_1^*(x,y) &= \frac{h^2}{12}(u_1)_{xxxx}(x^1,y), \qquad & p_1^*(x,y) &= \frac{h}{2} p_{xxx}(x^2,y) \\ u_2^*(x,y) &= \frac{h^2}{12}(u_2)_{yyyy}(x,y^3), \qquad & p_2^*(x,y) &= \frac{h}{2} p_{yyy}(x,y^4) \end{aligned} \tag{7.22}$$

for some points (x^i,y) and (x,y^i) in S around (x,y).

By (7.20) and (7.21),

$$\begin{aligned} \nabla_h^2 \; (u_1 - V_1)(x,y) &= E_1(x,y) := u_1^* - p_1^* \qquad &\text{in} \quad S_I^o \\ \nabla_h^2 \; (u_2 - V_2)(x,y) &= E_2(x,y) := u_2^* - p_2^* \qquad &\text{in} \quad S_{II}^o. \end{aligned} \tag{7.23}$$

and, by Lemma 6.1, Lemma 6.2, (7.22), and (7.23),

$$\begin{aligned} \|u_1 - V_1\|_{S_I^o} + \|u_2 - V_2\|_{S_{II}^o} &\le C_M\big(|E_1|_{S_I^o} + |E_2|_{S_{II}^o} \big) \\ &\le C_m h^2 \big(|u_1|_{4,S_1} + |u_2|_{4,S_2} + |p|_{3,S} \big) \end{aligned} \tag{7.24}$$

for some positive constant C_m.

Using Lemma 4.2, Lemma 4.3, Theorem 5.1, Lemma 6.4, and the fact that $\vec{V}$, $\vec{U} \in H_0^1(S_I) \times H_0^1(S_{II})$,

$$\begin{aligned} 2\big(\|V_1 - U_1\|_{S_I^o}^2 + \|V_2 - U_2\|_{S_{II}^o}^2 \big) &\le \|V_1 - U_1\|_{1,S_I}^2 + \|V_2 - U_2\|_{1,S_{II}}^2 \\ &= \big(V_1 - U_1, -\nabla_h^2 \; (V_1 - U_1) \big)_{S_I^o} + \big(V_2 - U_2, -\nabla_h^2 \; (V_2 - U_2) \big)_{S_{II}^o} \\ &= \big(V_1 - U_1, -\delta_{xo}(p - P) \big)_{S_I^o} + \big(V_2 - U_2, -\delta_{yo}(p - P) \big)_{S_{II}^o} \\ &= \big(\vec{\nabla}_o \cdot (\vec{V} - \vec{U}), p - P \big)_{S_P} = \big(Q_s(p - P), p - P \big)_{S_P} \le \|p - P\|_{S_P}^2. \end{aligned}$$

Hence

$$\left(\|V_1 - U_1\|_{S_I^\circ} + \|V_2 - U_2\|_{S_{II}^\circ} \right)^2 \le 2\left(\|V_1 - U_1\|_{S_I^\circ}^2 + \|V_2 - U_2\|_{S_{II}^\circ}^2 \right) \le \|p - P\|_{S_P}^2$$

and

$$\|V_1 - U_1\|_{S_I^\circ} + \|V_2 - U_2\|_{S_{II}^\circ} \le \|p - P\|_{S_P}. \tag{7.25}$$

Combining (7.19), (7.24), and (7.25), we have

$$\|u_1 - U_1\|_{S_I^\circ} + \|u_2 - U_2\|_{S_{II}^\circ} \le C_m h^2 \left(|u_1|_{4,S_1} + |u_2|_{4,S_2} + |p|_{3,S} \right) + \|p - P\|_{S_P}. \tag{7.26}$$

Now let's estimate $\|p - P\|_{S_h^\circ}$. By (3.9), (7.16), (7.17), (7.18), and Theorem 5.3,

$$\begin{aligned} C_L\|p - P\|_{S_P} &\le \|Q_s(p - P)\|_{S_P} = \|Q_s p - Qp\|_{S_P} \\ &= \|\vec{\nabla}_o \cdot \vec{V} - \vec{\nabla} \cdot \vec{u}\|_{S_P} \le \|\vec{\nabla}_o \cdot (\vec{V} - \vec{u})\|_{S_P} + \|\vec{\nabla}_o \cdot \vec{u} - \vec{\nabla} \cdot \vec{u}\|_{S_P} \end{aligned} \tag{7.27}$$

and, by Lemma 6.4, (7.22), and (7.23),

$$\begin{aligned} \|\vec{\nabla}_o \cdot (\vec{V} - \vec{u})\|_{S_P} &\le \|u_1 - V_1\|_{1,S_I} + \|u_2 - V_2\|_{1,S_{II}} \\ &\le C_Q\left(|E_1|_{S_I^\circ} + |E_2|_{S_{II}^\circ} \right) \le C_q h^2 \left(|u_1|_{4,S_1} + |u_2|_{4,S_2} + |p|_{3,S} \right) \end{aligned} \tag{7.28}$$

for some positive constant C_q.

Using Taylor expansions around $(x, y) \in S_P$, we have

$$\begin{aligned} \vec{\nabla}_o \cdot \vec{u}(x,y) &= \vec{\nabla} \cdot \vec{u}(x,y) \\ &+ \frac{1}{6h}\left(\frac{h}{2}\right)^3 \left((u_1)_{xxx}(x^1, y) - (u_1)_{xxx}(x^2, y) + (u_2)_{yyy}(x, y^1) - (u_2)_{yyy}(x, y^2) \right) \end{aligned}$$

where (x^i, y) are points in S_1 and (x, y^i) are points in S_2 around (x, y). Hence

$$\|\vec{\nabla}_o \cdot \vec{u} - \vec{\nabla} \cdot \vec{u}\|_{S_P} \le C_o h^2 \left(|u_1|_{3,S_1} + |u_2|_{3,S_2} \right) \tag{7.29}$$

for some positive constant C_o. By equations (7.27), (7.28), and (7.29),

$$\|p - P\|_{S_P} \le C_p h^2 \left(|u_1|_{4,S_1} + |u_2|_{4,S_2} + |p|_{3,S} \right) \tag{7.30}$$

for some positive constant C_p. By (7.26) and (7.30), the claim follows. □

One may ask whether

$$Q_o P := \frac{Q_- P + Q_+ P}{2}$$

would give a second-order accurate solution. Note that the domains of the velocity parts in the solutions of (3.1) and (3.5) are different. If one changes (3.1) and (3.5) so that the domains of the velocity parts are same, then the domains of the pressure parts become different. This difference in domains makes problems for getting a second-order accurate solution.

8. Conclusions. The condition numbers of the matrices generated by three finite difference approximations of the Stokes problem in the pressure equation (PE) method are shown to be independent of mesh size. Moreover the convergence estimations of the solutions generated by these matrices are shown to be first or second-order accurate. These results were basically by the inf-sup conditions that are proved by Shin and Strikwerda [11]. Current research is on getting the inf-sup conditions for other finite difference approximations.

The PE method has been extended to the Navier-Stokes equations for low Reynolds numbers by Shin and Strikwerda [10]. Many algorithms that use differenct linearizing techniques could be applied for the extension of the PE method to the Navier-Stokes equations. Research is continuing on getting a better algorithm that works for higher Reynolds numbers.

Work is also being done on applying the PE method to time-dependent problems on more general domains. The method works better with polar domains since the Poisson equation can be solved directly with the help of the line SOR method. A lot of domains in applications are decomposed into some rectangular and some polar domains. Using the Schwarz alternating procedure [7] with a parallel algorithm, the PE method should work efficiently.

References

[1] A. A. Amsden and F. H. Harlow, *The SMAC method: a numerical technique for calculating incompressible fluid flows*, *Los Alamos Scientific Laboratory Rept. LA-4370*, Los Alamos, N.M., 1970.

[2] A. Brandt and N. Dinar, *Multigrid solutions to elliptic flow problems*, in *Numerical Methods for Partial Differential Equations*, S. V. Parter, ed., Academic Press, Inc., New York, 1979, pp. 53-148.

[3] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, New York, NY, 1983.

[4] P. M Gresho, *Some interesting issues in incompressible fluid dynamics, both in the continuum and in numerical simulation*, Advances in Applied Mechanics, 28 (1991), pp. 45-140.

[5] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, Marshfields, Mass., 1985.

[6] F. H. Harlow and J. E. Welch, *A domain decomposition method for incompressible viscous flow of fluid with free surface*, Physics Fluids, 8 (1965), pp. 2182-2189.

[7] P. L. Lions, *On the Schwarz alternating method. I*, in *Proc. First Intern. Symp. on Domain Decomposition Methods for Partial Differential Equations*, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, ed., SIAM, Phila., PA, 1988, pp. 1-42.

[8] S. V. Patankar and D. B. Spalding, *A calculation procedure for heat, mass, and momentum in three dimensional parabolic flows*, Internat. J. Heat Mass Trans., 15 (1972), pp. 1787-1806.

[9] G. E. Raithby and G. D. Schneider, *Numerical solution of problems in incompressible fluid flow: Treatment of the velocity-pressure coupling*, Num. Heat Trans., 2 (1979), pp. 417-440.

[10] D. H. Shin and J. C. Strikwerda, *Fast solvers for finite difference approximations for the Stokes and Navier-Stokes equations*, to appear.

[11] D. H. Shin and J. C. Strikwerda, *Inf-sup conditions for finite difference approximations of the Stokes equations*, to appear.