Inf-Sup Conditions for Finite Difference Approximations of the Stokes Equations

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Dongho Shin† and John C. Strikwerda‡

Abstract. Inf-sup conditions are proven for three finite difference approximations of the Stokes equations by using the average values over certain oblique lines. This approximation technique preserves the divergence-free property which is important to the study of the Stokes equations. Moreover the approximation gives bounds on the norms of the divided differences of functions in the approximated space by the norms of the derivatives of functions in the continuous space. In a subsequent work we use the inf-sup conditions to prove estimates on the order of accuracy of the finite difference schemes.

Key words. inf-sup condition, finite difference schemes

AMS(MOS) subject classifications. 65N06, 65N22

1. Introduction. Inf-sup conditions, which have been introduced independently by Babŭska [3] and Brezzi [5], are important to study the linear boundary value problems with a constraint such as

Find $(u,p) \in X \times M$ *satisfying*

$$\begin{aligned} Au + B'p = f \quad & in\ X', \\ Bu = g \quad & in\ M' \end{aligned} \tag{1.1}$$

where X and M are two Hilbert spaces, X' and M' are their corresponding dual spaces, and $A \in L(X;X')$ and $B \in L(M;X')$ are two linear operators with $B' \in L(X;M')$ as the dual operator of B.

The linear operators A and B are associated with the bilinear forms

$$a(.,.) : X \times X \to \mathbf{R}, \qquad b(.,.) : X \times M \to \mathbf{R}.$$

Let $< .,. >$ denote the duality pairing between the spaces X and X' or M and M', then (1.1) is equivalent to the following variational problem :

Given $f \in X'$ *and* $g \in M'$, *find a pair* $(u,p) \in X \times M$ *such that*

$$\begin{aligned} a(u,v) + b(v,p) = < f, v > \qquad & \forall v \in X, \\ b(u,q) = < g, q > \qquad & \forall q \in M. \end{aligned} \tag{1.2}$$

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The inf-sup condition related to (1.2) is

$$\exists C > 0 \quad \textit{such that} \quad \inf_{p\in M} \sup_{u\in X} \frac{b(u,p)}{\|u\|_X \|p\|_M} \geq C. \tag{1.3}$$

The bilinear form $a(.,.)$ in (1.2) is related to the norm $\|.\|_X$ in (1.3) for most problems.

The inf-sup conditions in the continuous problems and the finite element problems are studied extensively in many places, for example Aziz and Babuška [2], Babuška [3], Brezzi [5], and Girault and Raviart [7], while the conditions are not studied yet in finite difference spaces. The inf-sup conditions for three finite difference approximations of the Stokes problem are proven in this paper for the first time.

Temam [13] and [14] used some finite difference approximations of the Stokes problem by using point evaluation and average value over an area or a line segment. Using the average values over horizontal and vertical line segments, Temam [14] mentions that the divergence-free property is kept in the resulting finite difference space. The divergence-free property holds in almost all applications. Hence it is important to study this finite difference space. However, it is not possible to bound the norm of the divided differences of functions in this finite difference space, or other finite difference spaces resulting from other approximations which we mentioned above, to the norm of the derivatives of functions in the function space from which the finite difference space is approximated.

It is necessary to evaluate the norm of divided differences or derivatives of a function to get the inf-sup conditions. In this paper, we introduce the finite difference spaces which come from the approximation using oblique line segments for this purpose. This approximation also preserves the divergence-free property by the same reason that the approximation using horizontal and vertical line segments does.

The finite difference schemes that we are interested in this paper are a staggered mesh scheme and the schemes that come from the backward and the forward differencings. These schemes are rather simple and hence serve well to the theoretical point of view. The proofs are done by setting a relation between a continuous space and its finite difference approximation space and uses the inf-sup condition of the continuous space.

2. Definitions. Let Ω be a domain in $\mathbf{R}^d$ and let Γ be its boundary. For simplicity, we focus on the case when $d = 2$, but the results in this paper will hold for any $d \geq 2$. We denote by $L^2(\Omega)$ the space of real functions defined on Ω which are integrable in the L^2 sense with the following usual inner product and norm

$$(u,v)_\Omega := \iint_\Omega uv \quad dA, \quad \|u\|_\Omega^2 := (u,u)_\Omega.$$

Let

$$H_0^1(\Omega) := \{\, u \in L^2(\Omega) \mid u_x, u_y \in L^2(\Omega) \text{ and } u|_\Gamma = 0 \,\}$$

have the following inner product and norm

$$(u,v)_{1,\Omega} := \iint_\Omega \vec{\nabla} u \cdot \vec{\nabla} v \quad dA, \quad \|u\|_{1,\Omega}^2 := (u,u)_{1,\Omega}$$

and

$$L_0^2(\Omega) := \{\, p \in L^2(\Omega) \,|\, (p,1)_\Omega = 0 \}.$$

We use the notation $\vec{u} = (u_i)$ for a vector. We shall often be concerned with two-dimensional vector functions with components in $L^2(\Omega)$ or $H_0^1(\Omega)$. The notation $L^2(\Omega)^2$, $H_0^1(\Omega)^2$ will be used for the product spaces. Define, for $\vec{u}$ and $\vec{v} \in L^2(\Omega)^2$,

$$(\vec{u},\vec{v})_\Omega := \sum_{i=1}^{2}(u_i,v_i)_\Omega \quad , \quad \|\vec{u}\|_\Omega^2 := (\vec{u},\vec{u})_\Omega$$

and, for $\vec{u}$ and $\vec{v} \in H_0^1(\Omega)^2$,

$$(\vec{u},\vec{v})_{1,\Omega} := \sum_{i=1}^{2}(u_i,v_i)_{1,\Omega} \quad , \quad \|\vec{u}\|_{1,\Omega}^2 := (\vec{u},\vec{u})_{1,\Omega}.$$

We also make some definitions analogous to the above on discrete subsets of the unit square S in $\mathbf{R}^2$. Let

$$S := \{\ (x,y) \in \mathbf{R}^2 \ |\ 0 < x,y < 1\ \}$$

and T its boundary. Let

$$h := \frac{1}{N}, \quad \text{for some } N \in \mathbf{N},$$
$$\mathbf{R}_h^2 := \{\ (lh,mh) \in \mathbf{R}^2 \ \ |\ \ l,m \in \mathbf{N}\ \},$$
$$S_h := \bar{S} \cap \mathbf{R}_h^2$$

where $\bar{S}$ is the closure of S.

For an arbitrary discrete set Ω_h of the form

$$\Omega_h := \{\ (lh,mh) \in S_h \ |\ l_0 \le l \le l_1 \text{ and } m_0 \le m \le m_1\ \},$$

we define

$$\Omega_h^o := \{\ (lh,mh) \in S_h \ |\ l_0+1 \le l \le l_1-1\ ,\ m_0+1 \le m \le m_1-1\ \},$$
$$e(\Omega_h) := \{\ (lh,mh) \in S_h \ |\ l_0+1 \le l \le l_1\ ,\ m_0 \le m \le m_1\ \},$$
$$w(\Omega_h) := \{\ (lh,mh) \in S_h \ |\ l_0 \le l \le l_1-1\ ,\ m_0 \le m \le m_1\ \},$$
$$s(\Omega_h) := \{\ (lh,mh) \in S_h \ |\ l_0 \le l \le l_1\ ,\ m_0 \le m \le m_1-1\ \},$$
$$n(\Omega_h) := \{\ (lh,mh) \in S_h \ |\ l_0 \le l \le l_1\ ,\ m_0+1 \le m \le m_1\ \}$$

as the interior, east, west, south and the north sides of Ω_h and define

$$se(\Omega_h) := s(\Omega_h) \cap e(\Omega_h), \qquad sw(\Omega_h) := s(\Omega_h) \cap w(\Omega_h)$$
$$ne(\Omega_h) := n(\Omega_h) \cap e(\Omega_h), \qquad nw(\Omega_h) := n(\Omega_h) \cap w(\Omega_h).$$

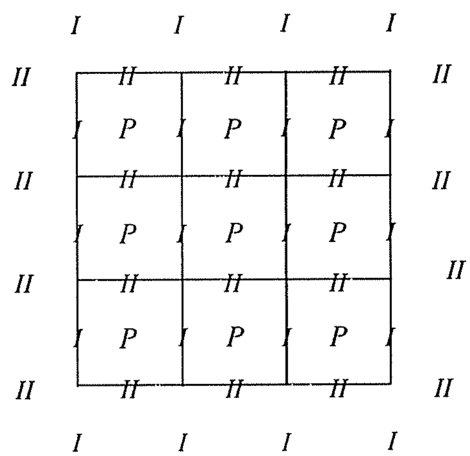
For the boundary Γ_h of Ω_h, we define

$$e(\Gamma_h), \qquad w(\Gamma_h), \qquad s(\Gamma_h), \qquad n(\Gamma_h)$$

as the east, west, south and north parts of Γ_h including the end points.

In this paper, we want to study both standard and staggered grids. The staggered mesh schemes use different grids that are staggered for the pressure and the velocity. A staggered grid is shown in Figure 1. The points marked by P, I, and II are where the pressure and the first and the second components of the velocity are defined, respectively.

Figure 1



Let

$$S_P := \{ \ (\ (l-\frac{1}{2})h \ , \ (m-\frac{1}{2})h \ ) \in S \mid l,m = 1,\ldots,N \ \},$$

$$S_I := \{ \ (\ lh \ , \ (m-\frac{1}{2})h \ ) \in S \mid l = 0,\ldots,N \ , \ m = 0,\ldots,N+1 \ \},$$

$$S_{II} := \{ \ (\ (l-\frac{1}{2})h \ , \ mh \ ) \in S \mid l = 0,\ldots,N+1 \ , \ m = 0,\ldots,N \ \},$$

then these are the sets for P, I, and II. Figure 1 shows S_P, S_I, and S_{II} when $N = 3$. Staggered mesh schemes have been used by Amsden and Harlow [1], Brandt and Dinar [4], Harlow and Welch [8], Patankar and Spalding [9], and Raithby and Schneider [10] and others.

Let $L^2(\Omega_h)$ be the space of all discrete functions defined on Ω_h with the following inner product and norm

$$(U,V)_{\Omega_h} := h^2 \sum_{(x,y)\in\Omega_h} U(x,y)V(x,y), \qquad \|U\|^2_{\Omega_h} := (U,U)_{\Omega_h}$$

and let

$$L_0^2(\Omega_h) := \{\ P \in L^2(\Omega_h)\ \mid\ (P,1)_{\Omega_h} = 0\ \},$$

then $L^2(\Omega_h)$ and $L_0^2(\Omega_h)$ are the discrete analogies of $L^2(\Omega)$ and $L_0^2(\Omega)$.

For notational convenience, we introduce

$$U_{l,m} := U(lh, mh),$$

and define the forward, backward and central differencings on the x axis and y axis, respectively, as

$$(\delta_{x+}U)_{l,m} := \frac{U_{l+1,m} - U_{l,m}}{h}, \qquad (\delta_{y+}U)_{l,m} := \frac{U_{l,m+1} - U_{l,m}}{h},$$

$$(\delta_{x-}U)_{l,m} := \frac{U_{l,m} - U_{l-1,m}}{h}, \qquad (\delta_{y-}U)_{l,m} := \frac{U_{l,m} - U_{l,m-1}}{h},$$

$$(\delta_{xo}U)_{l,m} := \frac{U_{l+\frac{1}{2},m} - U_{l-\frac{1}{2},m}}{h}, \qquad (\delta_{yo}U)_{l,m} := \frac{U_{l,m+\frac{1}{2}} - U_{l,m-\frac{1}{2}}}{h}.$$

Define the discrete gradients as

$$\vec{\nabla}_+ := (\delta_{x+}, \delta_{y+}), \qquad \vec{\nabla}_- := (\delta_{x-}, \delta_{y-}), \qquad \text{and} \quad \vec{\nabla}_o := (\delta_{xo}, \delta_{yo}),$$

and let ∇_h^2 be the five-point discrete Laplacian, then

$$\nabla_h^2 = \vec{\nabla}_- \cdot \vec{\nabla}_+ = \vec{\nabla}_+ \cdot \vec{\nabla}_-.$$

The inner product and the norm of

$$H_0^1(\Omega_h) := \{\ U \in L^2(\Omega_h)\ \mid\ U|_{\Gamma_h} = 0\ \}$$

are defined as

$$(U,V)_{1,\Omega_h} := (\vec{\nabla}_+ U, \vec{\nabla}_+ V)_{sw(\Omega_h)} = (\vec{\nabla}_- U, \vec{\nabla}_- V)_{ne(\Omega_h)},$$

$$\|U\|_{1,\Omega_h}^2 := (U,U)_{1,\Omega_h}$$

which are the sums over all points in Ω_h where difference quotients are defined. The inner product and the norm of the product spaces $L^2(\Omega_h)^2$ and $H_0^1(\Omega_h)^2$ are defined naturally from $L^2(\Omega_h)$ and $H_0^1(\Omega_h)$.

3. Inf-sup conditions for finite difference spaces. To show the inf-sup conditions for finite difference spaces which come from approximations of the Stokes problem, we begin with the related theory for partial differential equations. Refer to Aziz and Babŭska [2] for the proof of the next theorem.

Theorem 3.1. *Let Ω be a bounded domain with a Lipschitz-continuous boundary, then there exists a positive constant $C_{pde} = C_{pde}(\Omega)$ such that any $p \in L_0^2(\Omega)$ has a vector $\vec{u} \in H_0^1(\Omega)^2$ which satisfies*

$$\vec{\nabla} \cdot \vec{u} = p \quad in \quad \Omega \qquad and \qquad \|\vec{u}\|_{1,\Omega}^2 \le C_{pde}\, \|p\|_{\Omega}^2.$$

The above theorem implies the so-called inf-sup condition for the Stokes problem.

Theorem 3.2.

$$\inf_{p \in L_0^2(\Omega)\setminus\{0\}} \sup_{\vec{u} \in H_0^1(\Omega)^2} \frac{(\vec{\nabla} \cdot \vec{u}, p)_{\Omega}^2}{\|\vec{u}\|_{1,\Omega}^2 \|p\|_{\Omega}^2} \ge C_{pde}^{-1}.$$

By the next theorem, we will get the inf-sup conditions for finite difference spaces.

Theorem 3.3. *There exist positive constants C_1 and C_2, which are independent of h, such that*

(1) any $P \in L_0^2(S_P)$ has a vector $\vec{U} \in H_0^1(S_I) \times H_0^1(S_{II})$ which satisfies

$$(\vec{\nabla}_o \cdot \vec{U}, P)_{S_P} \ge C_1 \|P\|_{S_P}^2 \quad , \quad \|U_1\|_{1,S_I}^2 + \|U_2\|_{1,S_{II}}^2 \le C_2 \|P\|_{S_P}^2$$

(2) any $P \in L_0^2(S_h^o)$ has a vector $\vec{U} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))$ which satisfies

$$(\vec{\nabla}_- \cdot \vec{U}, P)_{S_h^o} \ge C_1 \|P\|_{S_h^o}^2 \quad , \quad \|U_1\|_{1,w(S_h)}^2 + \|U_2\|_{1,s(S_h)}^2 \le C_2 \|P\|_{S_h^o}^2$$

(3) any $P \in L_0^2(S_h^o)$ has a vector $\vec{U} \in H_0^1(e(S_h)) \times H_0^1(n(S_h))$ which satisfies

$$(\vec{\nabla}_+ \cdot \vec{U}, P)_{S_h^o} \ge C_1 \|P\|_{S_h^o}^2 \quad , \quad \|U_1\|_{1,e(S_h)}^2 + \|U_2\|_{1,n(S_h)}^2 \le C_2 \|P\|_{S_h^o}^2.$$

Setting $C := C_1^2 / C_2$, we get the following inf-sup conditions for some finite difference spaces.

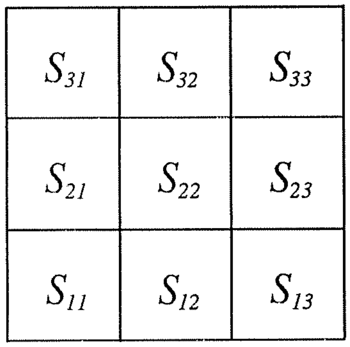
Theorem 3.4. *There exists a positive constant C, which is independent of h, such that*

$$(1) \qquad \sup_{\vec{U} \in H_0^1(S_I) \times H_0^1(S_{II})} \frac{(\vec{\nabla}_o \cdot \vec{U}, P)_{S_P}^2}{\|U_1\|_{1,S_I}^2 + \|U_2\|_{1,S_{II}}^2} \ge C \|P\|_{S_P}^2 \qquad \forall P \in L_0^2(S_P),$$

$$(2) \qquad \sup_{\vec{U} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))} \frac{(\vec{\nabla}_- \cdot \vec{U}, P)_{S_h^o}^2}{\|U_1\|_{1,w(S_h)}^2 + \|U_2\|_{1,s(S_h)}^2} \ge C \|P\|_{S_h^o}^2 \qquad \forall P \in L_0^2(S_h^o),$$

$$(3) \qquad \sup_{\vec{U} \in H_0^1(e(S_h)) \times H_0^1(n(S_h))} \frac{(\vec{\nabla}_+ \cdot \vec{U}, P)_{S_h^o}^2}{\|U_1\|_{1,e(S_h)}^2 + \|U_2\|_{1,n(S_h)}^2} \ge C \|P\|_{S_h^o}^2 \qquad \forall P \in L_0^2(S_h^o).$$

Figure 2



Proof of Theorem 3.3. Define

$$S_{l,m} := \{ (x,y) \in S \mid (l-1)h < x < lh \,,\, (m-1)h < y < mh \}$$

for $l, m = 1, \ldots, N$. Figure 2 shows $S_{l,m}$ when $N = 3$.

We first prove (1). Let $P \in L_0^2(S_P)$, then we define the piecewise constant function $p \in L^2(S)$ by

$$p|_{S_{l,m}} := P_{l-\frac{1}{2}, m-\frac{1}{2}}.$$

Note that

$$(p, 1)_S = (P, 1)_{S_P} = 0 \quad \text{and} \quad \|p\|_S = \|P\|_{S_P}.$$

Since $p \in L_0^2(S)$, by Theorem 3.2, there exists a vector $\vec{u} \in H_0^1(S)^2$ such that

$$\vec{\nabla} \cdot \vec{u} = p \quad \text{in} \quad S \qquad \text{and} \qquad \|\vec{u}\|_{1,S}^2 \le C_{pde} \|p\|_S^2. \tag{3.1}$$

Let C_1 be any real number such that

$$0 < C_1 < 1$$

and let C_3 be the solution of

$$\frac{2 + C_{pde}}{\sqrt{2C_3}} = 1 - C_1$$

and

$$C_2 = C_3 C_{pde},$$

then our claim holds with these C_1 and C_2. Note that

$$C_3 = \frac{1}{2}\left(\frac{2 + C_{pde}}{1 - C_1}\right)^2 \ge 2$$

and

$$k := h/C_3 < h.$$

For $t \in [0,1]$, define the line segments

$$x_l^k(t) := \begin{cases} lh - kt, & \text{if } 1 \le l \le N-1; \\ lh, & \text{if } l = 0, N; \end{cases}$$

$$y_m^k(t) := \begin{cases} mh - kt, & \text{if } 1 \le m \le N-1; \\ mh, & \text{if } m = 0, N; \end{cases}$$

$$x_l^h(t) := \begin{cases} lh - ht, & \text{if } 1 \le l \le N; \\ lh, & \text{if } l = 0, N; \\ 1, & \text{if } l = N+1; \end{cases}$$

$$y_m^h(t) := \begin{cases} mh - ht, & \text{if } 1 \le m \le N; \\ mh, & \text{if } m = 0, N; \\ 1, & \text{if } m = N+1. \end{cases}$$

Let

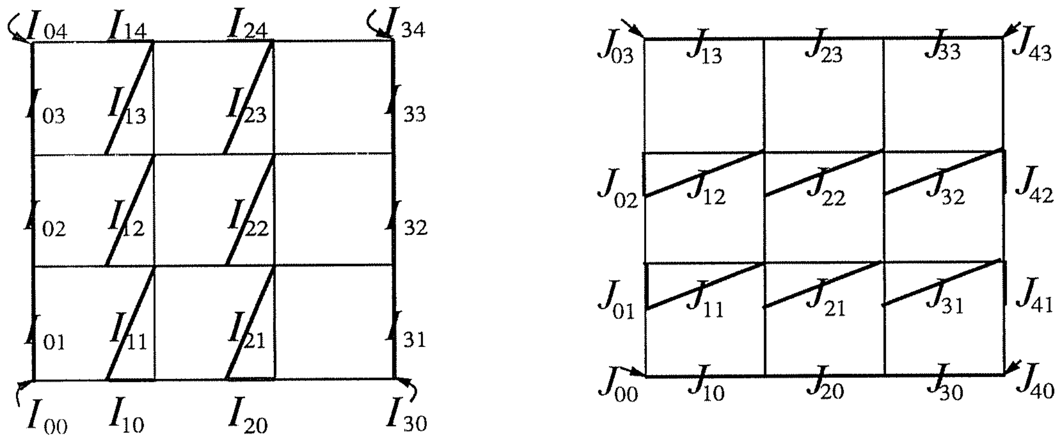
$$I_{l,m} := \{ \ (x_l^k(t), y_m^h(t)) \mid t \in [0,1] \ \},$$

for $l = 0, \ldots, N$ and $m = 0, \ldots, N+1$, and

$$J_{l,m} := \{ \ (x_l^h(t), y_m^k(t)) \mid t \in [0,1] \ \},$$

for $l = 0, \ldots, N+1$ and $m = 0, \ldots, N$, be line segments in $\bar{S}$. Figure 3 shows $I_{l,m}$ and $J_{l,m}$ when $N = 3$.

Figure 3



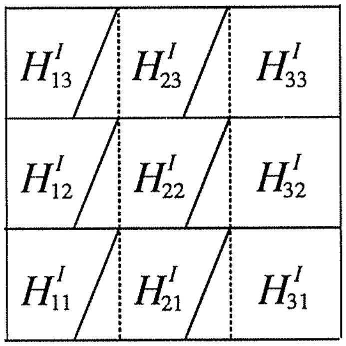
If

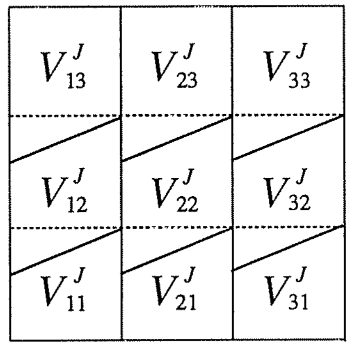
$$(U_1)_{l,m-\frac{1}{2}} := \int_0^1 u_1(x_l^k(t), y_m^h(t)) \, dt = \text{average value of } u_1 \text{ on } I_{l,m}$$

$$(U_2)_{l-\frac{1}{2},m} := \int_0^1 u_2(x_l^h(t), y_m^k(t)) \, dt = \text{average value of } u_2 \text{ on } J_{l,m},$$

then $\vec{U} \in H_0^1(S_I) \times H_0^1(S_{II})$ since $\vec{u} \in H_0^1(S)^2$.

Figure 4





Using H and V to denote "horizontal" and "vertical", respectively, we define

$$H^I_{l,m} := \text{the region between the line segments } I_{l,m} \text{ and } I_{l-1,m}$$
$$V^J_{l,m} := \text{the region between the line segments } J_{l,m} \text{ and } J_{l,m-1}$$

for $l, m = 1, \ldots, N$ and

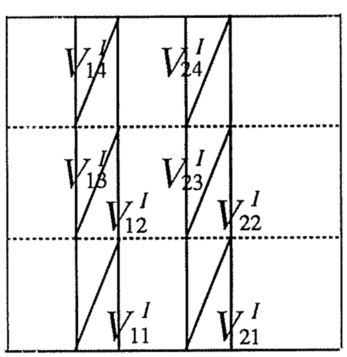
$$V^I_{l,m} := \text{the region between the line segments } I_{l,m} \text{ and } I_{l,m-1}$$

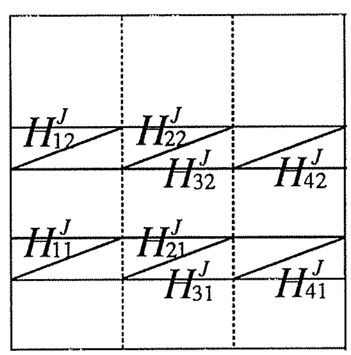
for $l = 1, \ldots, N-1$ and $m = 1, \ldots, N+1$ and

$$H^J_{l,m} := \text{the region between the line segments } J_{l,m} \text{ and } J_{l-1,m}$$

for $l = 1, \ldots, N+1$ and $m = 1, \ldots, N-1$. Figure 4 shows $H^I_{l,m}$ and $V^J_{l,m}$ and Figure 5 shows $V^I_{l,m}$ and $H^J_{l,m}$ when $N = 3$.

Figure 5





Note that $H^I_{l,m}$, $V^I_{l,m}$, $H^J_{l,m}$ and $V^J_{l,m}$ are quadrilaterals or triangles such that

$$\begin{aligned} area(H^I_{l,m}) &\le area(H^I_{N,N}) = h^2 + hk/2 \\ area(V^J_{l,m}) &\le area(V^J_{N,N}) = h^2 + hk/2 \\ area(V^I_{l,m}) &\le area(V^I_{N-1,N}) = hk \\ area(H^J_{l,m}) &\le area(H^J_{N,N-1}) = hk. \end{aligned} \tag{3.2}$$

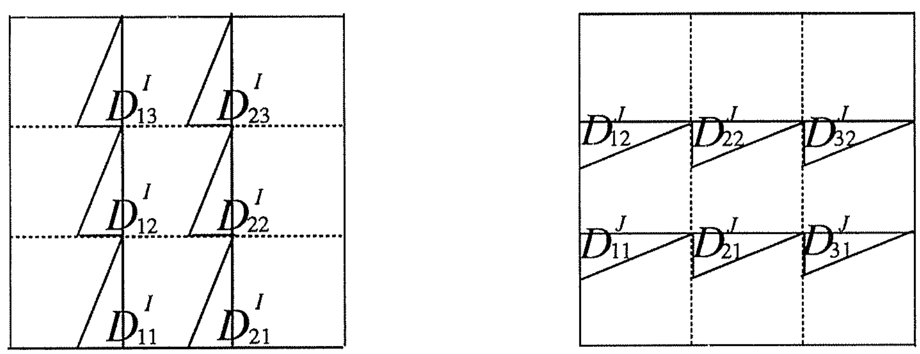
Define

$$D^I_{l,m} := \begin{cases} S_{l,m} \setminus H^I_{l,m}, & \text{if } l,m = 1,\dots,N; \\ \emptyset, & \text{if } l = 0 \text{ and } m = 1,\dots,N. \end{cases}$$

$$D^J_{l,m} := \begin{cases} S_{l,m} \setminus V^J_{l,m}, & \text{if } l,m = 1,\dots,N; \\ \emptyset, & \text{if } l = 1,\dots,N \text{ and } m = 0. \end{cases}$$

Figure 6 shows $D^I_{l,m}$ and $D^J_{l,m}$ when $N = 3$.

Figure 6



Note that $D^I_{l,m}$ and $D^J_{l,m}$ are triangles or empty sets and that

$$\max_{1 \le l,m \le N} area(D^I_{l,m}) = \max_{1 \le l,m \le N} area(D^J_{l,m}) = area(D^I_{1,1}) = kh/2 = h^2/2C_3. \tag{3.3}$$

Define

$$D^I := \bigcup_{l,m=1}^{N} D^I_{l,m} \quad \text{and} \quad D^J := \bigcup_{l,m=1}^{N} D^J_{l,m}.$$

Let's first evaluate $(\vec{\nabla}_o \cdot \vec{U}, P)_{S_P}$ in terms of $\|P\|^2_{S_P}$. By (3.1),

$$\|P\|^2_{S_P} = \|p\|^2_S = (\vec{\nabla} \cdot \vec{u}, p)_S.$$

For $l, m = 1, \ldots, N$,

$$
\begin{aligned}
(\delta_{xo} U_1)_{l-\frac{1}{2},m-\frac{1}{2}} &= \frac{(U_1)_{l,m-\frac{1}{2}} - (U_1)_{l-1,m-\frac{1}{2}}}{h} \\
&= \frac{1}{h}\left(\int_0^1 u_1(x_l^k(t), y_m^h(t))\ dt - \int_0^1 u_1(x_{l-1}^k(t), y_m^h(t))\ dt \right) \\
&= \frac{1}{h} \int_0^1 \{ u_1(x_l^k(t), y_m^h(t)) - u_1(x_{l-1}^k(t), y_m^h(t)) \}\ dt \\
&= \frac{1}{h} \int_0^1 \int_{x_{l-1}^k(t)}^{x_l^k(t)} (u_1)_x(x, y_m^h(t))\ dxdt.
\end{aligned}
$$

For any $y = y_m^h(t)$, define

$$
x_{l-1}^I(y) := x_{l-1}^k(t) \quad \text{and} \quad x_l^I(y) := x_l^k(t).
$$

By a change of variable, we have

$$
(\delta_{xo} U_1)_{l-\frac{1}{2},m-\frac{1}{2}} = \frac{1}{h^2} \int_{(m-1)h}^{mh} \int_{x_{l-1}^I(y)}^{x_l^I(y)} (u_1)_x(x,y)\ dxdy = \frac{1}{h^2} \iint_{H_{l,m}^I} (u_1)_x(x,y)\ dA. \quad (3.4)
$$

Similarly one gets

$$
(\delta_{yo} U_2)_{l-\frac{1}{2},m-\frac{1}{2}} = \iint_{V_{l,m}^J} (u_2)_y(x,y)\ dA \quad \text{for} \quad l, m = 1, \ldots, N.
$$

By the way that P was defined,

$$
(\vec{\nabla} \cdot \vec{u}, p)_S = \sum_{l,m=1}^{N} \iint_{S_{l,m}} \big((u_1)_x + (u_2)_y \big) P_{l-\frac{1}{2},m-\frac{1}{2}}\ dA.
$$

Thus we have

$$
\begin{aligned}
(\vec{\nabla} \cdot \vec{u}, p)_S - (\vec{\nabla}_o \cdot \vec{U}, P)_{S_P} &= \sum_{l,m=1}^{N} \Big(\iint_{S_{l,m}} \big((u_1)_x + (u_2)_y \big)\, dA - \iint_{H_{l,m}^I} (u_1)_x\, dA \\
- \iint_{V_{l,m}^J} (u_2)_y\, dA \Big) P_{l-\frac{1}{2},m-\frac{1}{2}} &= \sum_{l,m=1}^{N} \Big(\iint_{D_{l,m}^I} (u_1)_x\, dA - \iint_{D_{l-1,m}^I} (u_1)_x\, dA \qquad (3.5) \\
+ \iint_{D_{l,m}^J} (u_2)_y\, dA - \iint_{D_{l,m-1}^J} (u_2)_y\, dA \Big) P_{l-\frac{1}{2},m-\frac{1}{2}}.
\end{aligned}
$$

By the Schwarz inequality, we have

$$\Big| \iint_{D^I_{l,m}} (u_1)_x \; dA \Big| = \sqrt{area(D^I_{l,m})} \sqrt{\iint_{D^I_{l,m}} (u_1)_x^2 \; dA}.$$

Applying the inequality to the other integrals in the last row in (3.5) and using (3.3), we have

$$\begin{aligned} \big| (\vec{\nabla} \cdot \vec{u}, p)_S - (\vec{\nabla}_o \cdot \vec{U}, P)_{S_P} \big| \le \frac{h}{\sqrt{2C_3}} \sum_{l,m=1}^{N} \Bigg(\sqrt{\iint_{D^I_{l,m}} (u_1)_x^2 \; dA} \; + \\ \sqrt{\iint_{D^I_{l-1,m}} (u_1)_x^2 \; dA} + \sqrt{\iint_{D^J_{l,m}} (u_2)_y^2 \; dA} + \sqrt{\iint_{D^J_{l,m-1}} (u_2)_y^2 \; dA} \Bigg) \big| P_{l,m} \big|. \end{aligned} \tag{3.6}$$

Note that

$$\big| P_{l-\frac{1}{2},m-\frac{1}{2}} \big| \sqrt{\iint_{D^I_{l,m}} (u_1)_x^2 \; dA} \le \frac{1}{2} \Big(h \, \big| P_{l-\frac{1}{2},m-\frac{1}{2}} \big|^2 + \frac{1}{h} \iint_{D^I_{l,m}} (u_1)_x^2 \; dA \Big).$$

Using the same inequality to each term in the right side of (3.6), we get

$$\begin{aligned} & \big| (\vec{\nabla} \cdot \vec{u}, p)_S - (\vec{\nabla}_o \cdot \vec{U}, P)_{S_P} \big| \\ & \le \frac{1}{2\sqrt{2C_3}} \sum_{l,m=1}^{N} \Big(4h^2 \, \big| P_{l-\frac{1}{2},m-\frac{1}{2}} \big|^2 + \iint_{D^I_{l,m}} (u_1)_x^2 \; dA + \\ & \qquad \iint_{D^I_{l-1,m}} (u_1)_x^2 \; dA + \iint_{D^J_{l,m}} (u_2)_y^2 \; dA + \iint_{D^J_{l,m-1}} (u_2)_y^2 \; dA \Big) \\ & \le \frac{1}{2\sqrt{2C_3}} \Big(4\|P\|^2_{S_P} + 2\big(\iint_{D^I} (u_1)_x^2 \; dA + \iint_{D^J} (u_2)_y^2 \; dA \big) \Big) \\ & = \frac{1}{\sqrt{2C_3}} \big(2\|p\|^2_S + \|\vec{u}\|^2_{1,S} \big) \le \frac{(2 + C_{pde})}{\sqrt{2C_3}} \|p\|^2_S \end{aligned}$$

by (3.1). Hence

$$(\vec{\nabla}_o \cdot \vec{U}, P)_{S_P} \ge (\vec{\nabla} \cdot \vec{u}, p)_S - \frac{(2 + C_{pde})}{\sqrt{2C_3}} \|p\|^2_S = C_1 \|p\|^2_S = C_1 \|P\|^2_{S_P}. \tag{3.7}$$

Next we estimate $\|U_1\|_{1,S_I}$ in terms of $\|u_1\|_{1,S}$. By (3.2) and (3.4), we have

$$\begin{aligned} \|\delta_{x-} U_1\|^2_{1,e(S_I)} = \|\delta_{xo} U_1\|^2_{S_P} = h^2 \sum_{l,m=1}^{N} \Big(\frac{1}{h^2} \iint_{H^I_{l,m}} (u_1)_x(x,y) \; dA \Big)^2 \\ \le \frac{1}{h^2} \sum_{l,m=1}^{N} area(H^I_{l,m}) \iint_{H^I_{l,m}} (u_1)_x^2 \; dA = 2 \iint_S (u_1)_x^2 \; dA. \end{aligned} \tag{3.8}$$

Note that, for $l = 1, \ldots, N-1$ and $m = 1, \ldots, N+1$,

$$(\delta_{y-} U_1)_{l-\frac{1}{2}, m-\frac{1}{2}} = \frac{1}{h} \Big(\int_0^1 u_1(x_l^k(t), y_m^h(t)) \, dt -$$

$$\int_0^1 u_1(x_l^k(t), y_{m-1}^h(t)) \, dt \Big) = \frac{1}{h} \int_0^1 \int_{y_{m-1}^h(t)}^{y_m^h(t)} (u_1)_y(x_l^k(t), y) \, dy dt.$$

For any $x = x_l^k(t)$, if we let

$$y_{m-1}^I(x) := y_{m-1}^h(t), \quad y_m^I(x) := y_m^h(t),$$

then by a change of variable

$$(\delta_{y-} U_1)_{l-\frac{1}{2}, m-\frac{1}{2}} = \frac{1}{hk} \int_{lh-k}^{lh} \int_{y_{m-1}^I(x)}^{y_m^I(x)} (u_1)_y(x,y) \, dy dx = \frac{1}{hk} \iint_{V_{l,m}^I} (u_1)_y(x,y) \, dA.$$

Hence

$$\begin{aligned} \|\delta_{y-} U_1\|_{n(S_I)}^2 &= h^2 \sum_{l=1}^{N-1} \sum_{m=1}^{N+1} \Big(\frac{1}{hk} \iint_{V_{l,m}^I} (u_1)_y \, dA \Big)^2 \\ &\le \frac{1}{k^2} \sum_{l=1}^{N-1} \sum_{m=1}^{N+1} area(V_{l,m}^I) \iint_{V_{l,m}^I} (u_1)_y^2 \, dA \le C_3 \iint_S (u_1)_y^2 \, dA. \end{aligned} \tag{3.9}$$

By (3.8) and (3.9), we get

$$\begin{aligned} \|U_1\|_{1,S_I}^2 &= \|\delta_{x-} U_1\|_{e(S_I)}^2 + \|\delta_{y-} U_1\|_{n(S_I)}^2 \\ &\le 2 \iint_S (u_1)_x^2 \, dA + C_3 \iint_S (u_1)_y^2 \, dA \le C_3 \, \|u_1\|_{1,S}^2 \end{aligned}$$

since $C_3 \ge 2$. Similarly it follows that

$$\|U_2\|_{1,S_{II}}^2 \le C_3 \, \|u_2\|_{1,S}^2.$$

Hence

$$\|U_1\|_{1,S_I}^2 + \|U_2\|_{1,S_{II}}^2 \le C_3 \|\vec{u}\|_{1,S}^2 \le C_2 \|p\|_S^2. \tag{3.10}$$

The proof of statement (1) in Theorem 3.4 follows from (3.7) and (3.10).

Now let's prove the statement (2) in Theorem 3.4. Let $P \in L_0^2(S_h^o)$ and define

$$p|_{S_{l,m}} := P_{l,m}, \qquad l, m = 1, \ldots, N-1,$$

then $p \in L_0^2(S_{sw})$. Hence there exists a vector $\vec{u} \in H_0^1(S_{sw})^2$ such that

$$\vec{\nabla} \cdot \vec{u} = p \text{ in } S_{sw} \qquad \text{and} \qquad \|\vec{u}\|_{1,S_{sw}}^2 \le C_{pde} \|p\|_{S_{sw}}^2.$$

Define

$$x_l^k(t) := \begin{cases} lh - kt, & \text{if } 1 \le l \le N-2; \\ lh, & \text{if } l = 0, N-1; \end{cases}$$

$$y_m^k(t) := \begin{cases} mh - kt, & \text{if } 1 \le m \le N-2; \\ mh, & \text{if } m = 0, N-1; \end{cases}$$

$$x_l^h(t) := \begin{cases} lh - ht, & \text{if } 1 \le l \le N-1; \\ lh, & \text{if } l = 0, N-1; \\ (N-1)h, & \text{if } l = N; \end{cases}$$

$$y_m^h(t) := \begin{cases} mh - ht, & \text{if } 1 \le m \le N-1; \\ mh, & \text{if } m = 0, N-1; \\ (N-1)h, & \text{if } m = N. \end{cases}$$

Let

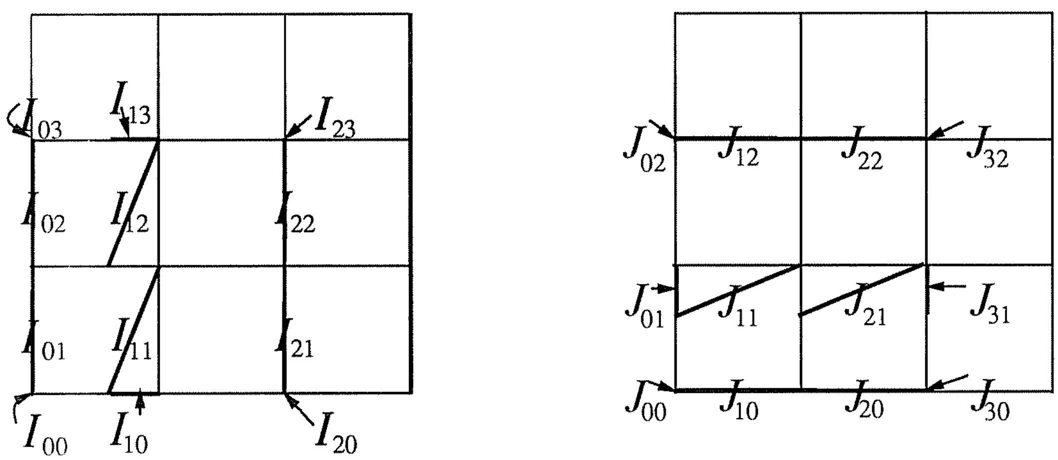
$$I_{l,m} := \{ \ (x_l^k(t), y_m^h(t)) \mid t \in [0,1] \ \},$$

for $l = 0, \dots, N-1$ and $m = 0, \dots, N$, and

$$J_{l,m} := \{ \ (x_l^h(t), y_m^k(t)) \mid t \in [0,1] \ \},$$

for $l = 0, \dots, N$ and $m = 0, \dots, N-1$, be line segments in $\bar{S}$. Figure 7 shows $I_{l,m}$ and $J_{l,m}$ when $N = 3$.

Figure 7



If

$$(U_1)_{l,m} := \int_0^1 u_1(x_l^k(t), y_m^h(t)) \ dt = \text{average value of } u_1 \text{ on } I_{l,m}$$

$$(U_2)_{l,m} := \int_0^1 u_2(x_l^h(t), y_m^k(t)) \ dt = \text{average value of } u_2 \text{ on } J_{l,m},$$

then $\vec{U} \in H_0^1(w(S_h)) \times H_0^1(s(S_h))$ since $\vec{u} \in H_0^1(S_{sw})^2$. The proof that $\vec{U}$ satisfies the required properties is similar to the proof for statement (1). The proof for (3) is similar to the proof of (2). □

4. Relation with finite element spaces. One may ask whether one could use some inf-sup conditions in simple well-known finite element spaces to get the inf-sup conditions for the finite difference spaces which are proven in this paper. In other words, to prove (1) in Theorem 3.3, can we use some functions $\vec{u}_e$ and p_e defined on some simple finite element spaces such that

$$\frac{\left(\vec{\nabla}_o \cdot \vec{U}, P\right)_{S_P}^2}{\|U_1\|_{1,S_I}^2 + \|U_2\|_{1,S_{II}}^2} \simeq \frac{\left(\vec{\nabla} \cdot \vec{u}_e, p_e\right)_S^2}{\|\vec{u}_e\|_{1,S}^2} \geq C\|p_e\|_S^2 \simeq C\|P\|_{S_P} \tag{4.1}$$

for some constant C? Hence we want to construct $\vec{U}$ from a given $P \in L_0^2(S_P)$ satisfying some conditions, using the functions p_e and $\vec{u}_e$. The actual process of this is to define p_e from P, $\vec{u}_e$ from p_e and finally $\vec{U}$ from $\vec{u}_e$. Since P is defined on S_h^o, p_e and $\vec{u}_e$ need to be defined on some triangles using $\{S_{l,m}\}$ basically.

The first relation in (4.1) may hold if

$$(\vec{\nabla}_o \cdot \vec{U}, P)_{S_P} \simeq (\vec{\nabla} \cdot \vec{u}_e, p_e)_S \quad \text{and} \quad \|U_1\|_{1,S_I}^2 + \|U_2\|_{1,S_{II}}^2 \simeq \|\vec{u}_e\|_{1,S}^2.$$

The first condition is equivalent to

$$h^2 \sum_{l,m=1}^{N} \left(\delta_{xo} U_1 + \delta_{yo} U_2\right)_{l-\frac{1}{2},m-\frac{1}{2}} P_{l-\frac{1}{2},m-\frac{1}{2}} \simeq$$

$$\sum_{l,m=1}^{N} \iint_{S_{l,m}} \left((u_{e1})_x + (u_{e2})_y\right)(x,y) p_e(x,y)\, dx dy.$$

This may hold when

$$p_e(x,y) \simeq P_{l-\frac{1}{2},m-\frac{1}{2}} \qquad \text{in} \quad S_{l,m} \tag{4.2}$$

$$h^2(\delta_{xo} U_1)_{l-\frac{1}{2},m-\frac{1}{2}} \simeq \iint_{S_{l,m}} (u_{e1})_x(x,y)\, dx dy \tag{4.3}$$

$$h^2(\delta_{yo} U_2)_{l-\frac{1}{2},m-\frac{1}{2}} \simeq \iint_{S_{l,m}} (u_{e2})_y(x,y)\, dx dy. \tag{4.4}$$

Let $T_{l,m}$ be the boundary of $S_{l,m}$ and let $w(T_{l,m})$ and $e(T_{l,m})$ be the west and east parts of $T_{l,m}$, respectively, then (4.3) is equivalent to

$$h\left((U_1)_{l,m-\frac{1}{2}} - (U_1)_{l-1,m-\frac{1}{2}}\right) \simeq \int_{e(T_{l,m})} u_{e1}(x,y)\, dy - \int_{w(T_{l,m})} u_{e1}(x,y)\, dy$$

$$= \int_{e(T_{l,m})} u_{e1}(x,y)\, dy - \int_{e(T_{l-1,m})} u_{e1}(x,y)\, dy$$

and hence

$$(U_1)_{l,m-\frac{1}{2}} \simeq \frac{1}{h}\int_{e(T_{l,m})} u_{e1}(x,y)dy. \tag{4.5}$$

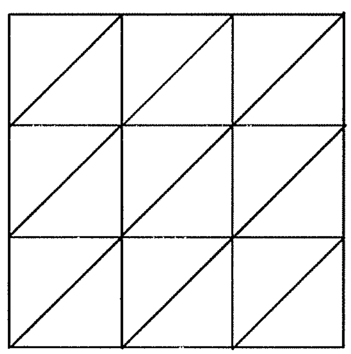
Similarly (4.4) is equivalent to

$$(U_2)_{l-\frac{1}{2},m} \simeq \frac{1}{h}\int_{n(T_{l,m})} u_{e2}(x,y)dx. \tag{4.6}$$

If $\vec{u}_e$ is a piecewise linear function defined using some triangles, then the value of $\vec{u}_e$ at the middle of each side of any triangle is the average value of $\vec{u}_e$ on the side. Hence (4.2), (4.5) and (4.6) may hold when p_e is a piecewise constant function and $\vec{u}_e$ is a piecewise linear constant function. The inf-sup conditions of this case for both the conforming and non-conforming finite element spaces for Stokes equations are discussed in [6].

Consider the case when one uses the triangles generated by dividing $S_{l,m}$ into two equal triangles as we show in Figure 8. If one forces the exact equality in (4.5) and (4.6) for this case, then one may not be able to relate $\delta_{yo}U_1$ and $\delta_{xo}U_2$ with $(u_{e1})_y$ and $(u_{e2})_x$, respectively, since $e(T_{l,m})$ and $n(T_{l,m})$ are straight line segments. Hence U_1 and U_2 need to be defined using the average values on some oblique line segments which are slightly deviated from those straight line segments. This is the reason that we used $I_{l,m}$ and $J_{l,m}$ in the proof of the inf-sup conditions for finite difference spaces. Hence we encounter with the same difficulty in proving the inf-sup conditions for finite difference spaces by using either the conditions in finite element spaces with the triangles given in Figure 8 or the the conditions in partial differential spaces.

Figure 8



Consider using the triangles, which are shown in Figure 9, generated by line segments $I_{l,m}$ and $J_{l,m}$ instead of those triangles given in Figure 8. Since the intersection of two adjacent triangles need to have a side of each of these two triangles, we need to introduce more line segments in $S_{l,m}$. Figure 10 shows the triangles with these extra line segments. Unfortunately, it is not possible to relate $\|U_1\|_{1,S_I}$ and $\|U_2\|_{1,S_{II}}$ to $\|u_{e1}\|_{1,S}$ and $\|u_{e2}\|_{1,S}$, respectively, if we use these triangles.

Figure 9

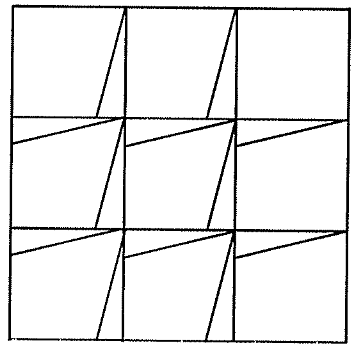


Figure 10

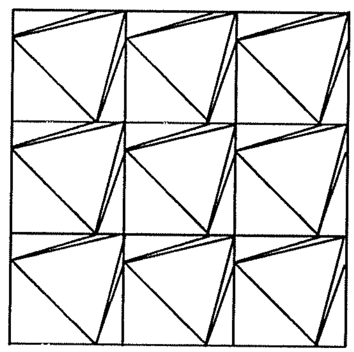
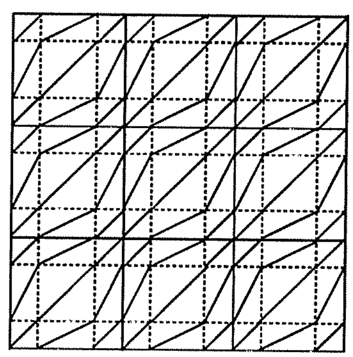


Figure 11



One could use triangles using line segments which are little modified from $I_{l,m}$ and $J_{l,m}$ which we show in Figure 11, but it is harder to get inf-sup conditions for finite difference spaces using the inf-sup conditions for the finite element spaces with these triangles than using the conditions for partial differential spaces. Hence the use of inf-sup conditions in finite element spaces doesn't offer much help for proving the inf-sup conditions in finite difference spaces.

5. Conclusion. The inf-sup conditions are proved for three finite difference approximations of the Stokes problem by using the average value over some oblique lines. The finite difference approximations use a staggered mesh scheme and the schemes resulting from the the backward and the forward differencings.

If Q_h is the Schur complement of the linear system generated by one of the finite difference approximations that we discussed in this paper, the inf-sup conditions that we proved in this paper can be used to prove that the condition number $\kappa(Q_h)$ is independent of mesh size h and to prove the convergence estimation of the solution generated by Q_h, which we will report in the forthcoming paper [12]. These results of Q_h support the use of the pressure equation method, a new fast iterative method introduced by Shin and Strikwerda [11], and other iterative methods to solve the finite difference approximations of the Stokes and the incompressible Navier-Stokes equations, since the Schur complement Q_h plays an important role in studying those equations.

Future research on the inf-sup conditions for other finite difference approximations and for other linear boundary value problems with a constraint needs to be done.

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