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ABSTRACT. Given an $n \times n$ matrix M, a vector q in $\mathbb{R}^n$, and a polyhedral convex set $X = \{x | Ax \leq b, Bx = d\}$, where A is an $m \times n$ matrix and B is an $p \times n$ matrix, the affine variational inequality problem is to find $x \in X$ such that

$$(Mx + q)^T (y - x) \geq 0$$

for all $y \in X$. If M is positive semi-definite, the affine variational inequality can be transformed to a generalized complementarity problem, which can be solved in polynomial time using the path following method of Kojima et.al. The main contribution of this paper is that the particular structure of the problem is exploited, rather than artificial variables being introduced to construct a standard form problem.

1. INTRODUCTION

In this paper, we investigate a path following interior point algorithm for monotone affine variational inequalities. Given an $n \times n$ matrix M, a vector q in $\mathbb{R}^n$, and a polyhedral convex set

$$X = \{x | Ax \leq b, Bx = d\}$$

where A is an $m \times n$ matrix and B is an $p \times n$ matrix, the affine variational inequality problem, abbreviated as AVI(q, M, X), is to find $x \in X$ such that

$$\text{(AVI)} \qquad (Mx + q)^T (y - x) \geq 0, \text{ for all } y \in X.$$

In this paper, we assume that M is positive semi-definite, and we say that AVI(q, M, X) is monotone.

It is well known (see [3]) that AVI(q, M, X) is equivalent to the following complementarity problem

$$\text{(GLCP)} \qquad \begin{aligned} &(s, x, u) \in \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^m_+ \\ &H(s, x, u) = \begin{pmatrix} 0 & -B & 0 \\ B^T & M & A^T \\ 0 & -A & 0 \end{pmatrix} \begin{pmatrix} s \\ x \\ u \end{pmatrix} + \begin{pmatrix} d \\ q \\ b \end{pmatrix} \in \{0\} \times \{0\} \times \mathbb{R}^m_+ \\ &(s, x, u)^T H(s, x, u) = 0 \end{aligned}$$

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Complementarity problems of this form correspond to maximal monotone multifunction (see [6]) under the following condition

$$\operatorname{rank}\begin{pmatrix} 0 & -B \\ B^T & M \\ 0 & -A \end{pmatrix} = n + p \tag{1.1}$$

In this case, we call (GLCP) a generalized complementarity problem following the notation of Güler (see [2]). We show that any given AVI(q, M, X) can be reduced to an equivalent problem AVI$(\bar{q}, \bar{M}, \bar{X})$ such that (1.1) is satisfied. The path following method used by Kojima et.al. for solving linear complementarity problem can be adapted to solve this problem. Furthermore, this adaptation provides a polynomial algorithm for AVI(q, M, X) since the construction of AVI$(\bar{q}, \bar{M}, \bar{X})$ from AVI(q, M, X) can be achieved in polynomial time.

For convenience of analysis, we assume that $n \geq 2$, and $m \geq 1$. The case of $n \leq 1$ is trivial. In the case of $m = 0$, AVI(q, M, X) can be reduced to a system of linear equations.

The general scheme for solving AVI(q, M, X) therefore consists of following steps.

Step 1: Reduce AVI(q, M, X) to an equivalent problem AVI$(\bar{q}, \bar{M}, \bar{X})$ such that (1.1) is satisfied (see Section 2).

Step 2: Construct an artificial problem AVI(q', M', X') which has an easily generated starting point as outlined in Appendix A.

Step 3: Apply the path following algorithm given in Section 4 to find an approximate solution of AVI(q', M', X').

Step 4: Construct an exact solution of AVI(q', M', X') by using the technique described in Appendix B, and obtain a solution of AVI$(\bar{q}, \bar{M}, \bar{X})$ by dropping the artificial variable (or conclude that AVI$(\bar{q}, \bar{M}, \bar{X})$ is unsolvable).

Step 5: Construct a solution of AVI(q, M, X) by filling in zero components (see Section 2).

The rest of this paper is organized as follows. Section 2 deals with the issue of reducing AVI(q, M, X) to an equivalent problem satisfying (1.1). In Section 3 we show the existence of a central path by using existing results regarding complementarity problems as maximal monotone multifunctions. In Section 4 we describe our path following algorithm and justify its validity. There are three appendices; Appendix A shows how to construct an artificial problem with a easily generated starting point. Appendix B shows how to construct an exact solution of (GLCP) from an approximate solution generated by the path following algorithm. These are generalization of work found in [4]. Appendix C proves an algebraic property of positive semi-definite matrices which we use throughout the paper.

The following is a summary of our notation and the basic concepts employed. Given any matrix C and index sets α and β, $C_{\alpha\cdot}$ denotes the submatrix formed by those rows of C with indices in α, $C_{\cdot\beta}$ denotes the submatrix formed by those columns of C with indices in β, and $C_{\alpha\beta}$ denotes the submatrix formed by those elements of C with row indices in α and column indices in β. For any vector or matrix, a superscript T indicates the transpose and $\|\cdot\|_p$ denotes their p-norm, see [9]. For any vector v, $\operatorname{diag}(v)$ is the diagonal matrix whose diagonal elements are the components of v, $\operatorname{supp}(v)$ is the set of indices that correspond to non-zero components of v. Finally, for any closed convex set $S \subset \mathbb{R}^n$

$$\operatorname{rec} S := \{d \in \mathbb{R}^n \mid s + \lambda d \in S, \forall s \in S, \forall \lambda \geq 0\}$$

is the recession cone of S, and the set

$$L(S) := \{d \in \mathbb{R}^n \mid s + \mu d \in S, \forall s \in S, \forall \mu \in \mathbb{R}\}$$

is the lineality space of S (see [10]).

2. Transformation to a Generalized Complementarity Problem

The problem (GLCP) is a generalized complementarity problem if (1.1) holds. In general, a problem in the form of (GLCP) can be reduced to a smaller problem satisfying (1.1), which is again equivalent to a monotone affine variational inequality. Define the feasible set of (GLCP) by

$$S := \left\{(u,v) \,\middle|\, u, v \geq 0, v = Ax - b, Bx - d = 0, Mx + A^T u + B^T s + q = 0\right\} \tag{2.1}$$

Then the lineality space (see [10]) of S is

$$L(S) = \left\{(s,0,0,x) \,\middle|\, B^T s + Mx = 0, -Ax = 0, -Bx = 0\right\}$$

So, $L(S) = \{0\}$ if and only if (1.1) holds.

For convenience of notation, define

$$Q = \begin{pmatrix} 0 & -B \\ B^T & M \end{pmatrix} \qquad C = \begin{pmatrix} 0 & A \end{pmatrix}$$

(GLCP) can be reformulated as

$$\text{(GLCP')} \qquad \begin{aligned} &(z,u) \in \mathbb{R}^{p+n} \times \mathbb{R}^m_+ \\ &H(z,u) = \begin{pmatrix} Q & C^T \\ -C & 0 \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix} + \begin{pmatrix} q' \\ b \end{pmatrix} \in \{0\} \times \mathbb{R}^m_+ \\ &(z,u)^T H(z,u) = 0 \end{aligned}$$

where $z = \binom{s}{x}$ and $q' = \binom{d}{q}$.

Suppose $L(S) \neq \{0\}$, then the columns of the matrix $\binom{Q}{-C}$ are linearly dependent. There exists index sets α and β such that

$$\begin{pmatrix} Q \\ -C \end{pmatrix} = \begin{pmatrix} Q_{\cdot\alpha} & Q_{\cdot\beta} \\ -C_{\cdot\alpha} & -C_{\cdot\beta} \end{pmatrix}$$

and $\binom{Q_{\cdot\alpha}}{-C_{\cdot\alpha}}$ is a maximum subset of linearly independent columns of the matrix $\binom{Q}{-C}$. Thus

$$\begin{pmatrix} Q_{\cdot\beta} \\ -C_{\cdot\beta} \end{pmatrix} = \begin{pmatrix} Q_{\cdot\alpha} \\ -C_{\cdot\alpha} \end{pmatrix} P \tag{2.2}$$

for some $|\alpha| \times |\beta|$ matrix P.

Lemma 2.1. *Let α, β and P be as in (2.2), $\beta \neq \emptyset$. If (GLCP') is solvable, then there exists a solution $(\bar{z}, \bar{u})$ such that $\bar{z}_\beta = 0$.*

Proof. Let $(\bar{z}, \bar{u}) = (\bar{z}_\alpha, \bar{z}_\beta, \bar{u})$ be a solution of (GLCP'), then it is clear that $(\bar{z}_\alpha + P\bar{z}_\beta, 0, \bar{u})$ is the desired solution. $\square$

Lemma 2.2. *Define (GLCP″) by*

$$(\text{GLCP}'') \qquad \begin{aligned} &(w,u) \in \mathbb{R}^{p+n-|\beta|} \times \mathbb{R}^m_+ \\ &\tilde{H}(w,u) = \begin{pmatrix} Q_{\alpha\alpha} & (C^T)_{\alpha\cdot} \\ -C_{\cdot\alpha} & 0 \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} + \begin{pmatrix} q'_\alpha \\ b \end{pmatrix} \in \{0\} \times \mathbb{R}^m_+ \\ &(w,u)^T \tilde{H}(w,u) = 0 \end{aligned}$$

Then (z,u) is a solution of (GLCP′) with $z_\beta = 0$ if and only if (z_α, u) is a solution of (GLCP″).

Proof. If (z,u) is a solution of (GLCP′) with $z_\beta = 0$, then it is easily verified that (z_α, u) is a solution of (GLCP″).

If (z_α, u) is a solution of (GLCP″), then

$$Q_{\alpha\alpha} z_\alpha + (C^T)_{\alpha\cdot} u + q'_\alpha = 0 \tag{2.3}$$

$$-C_{\cdot\alpha} z_\alpha + b \in \mathbb{R}^m_+ \tag{2.4}$$

and

$$u^T(-C_{\cdot\alpha} z_\alpha + b) = 0 \tag{2.5}$$

Moreover, since the matrix $\begin{pmatrix} Q & C^T \\ -C & 0 \end{pmatrix}$ is positive semi-definite, we can apply Lemma C.3 to (2.2) resulting in

$$\begin{pmatrix} Q_{\beta\alpha} & Q_{\beta\beta} & (C^T)_{\beta\cdot} \end{pmatrix} = P^T \begin{pmatrix} Q_{\alpha\alpha} & Q_{\alpha\beta} & (C^T)_{\alpha\cdot} \end{pmatrix}$$

Also, taking into account (2.3), we have

$$\begin{aligned} \begin{pmatrix} Q_{\beta\alpha} & Q_{\beta\beta} & (C^T)_{\beta\cdot} \end{pmatrix} \begin{pmatrix} z_\alpha \\ 0 \\ u \end{pmatrix} + q'_\beta &= P^T \begin{pmatrix} Q_{\alpha\alpha} & Q_{\alpha\beta} & (C^T)_{\alpha\cdot} \end{pmatrix} \begin{pmatrix} z_\alpha \\ 0 \\ u \end{pmatrix} + q'_\beta \\ &= q'_\beta - P^T q'_\alpha \end{aligned}$$

If $q'_\beta - P^T q'_\alpha \neq 0$, then the system

$$\begin{pmatrix} Q_{\alpha\alpha} & Q_{\alpha\beta} & (C^T)_{\alpha\cdot} \\ Q_{\beta\alpha} & Q_{\beta\beta} & (C^T)_{\beta\cdot} \end{pmatrix} \begin{pmatrix} z_\alpha \\ 0 \\ u \end{pmatrix} + \begin{pmatrix} q'_\alpha \\ q'_\beta \end{pmatrix} = 0$$

is inconsistent, a contradiction to the solvability of (GLCP′) and Lemma 2.1. Hence

$$q'_\beta - P^T q'_\alpha = 0$$

Let $z_0 = (z_\alpha, 0)$, then

$$H(z_0, u) = \begin{pmatrix} 0 \\ 0 \\ -C_{\cdot\alpha} z_\alpha + b \end{pmatrix} \in \{0\} \times \mathbb{R}^m_+$$

follows from (2.3), (2.4). We also have $(z_0, u^T) H(z_0, u) = 0$ by reference to (2.5). □

We can write

$$Q_{\alpha\alpha} = \begin{pmatrix} 0 & -\bar{B} \\ \bar{B}^T & \bar{M} \end{pmatrix} \qquad C_{\cdot\alpha} = \begin{pmatrix} 0 & \bar{A} \end{pmatrix}$$

for appropriately defined submatrices $\bar{A}$, $\bar{B}$, and $\bar{M}$ of A, B, and M respectively. Note that $\bar{M}$ is positive semi-definite and the matrix

$$\begin{pmatrix} 0 & -\bar{B} \\ \bar{B}^T & \bar{M} \\ 0 & -\bar{A} \end{pmatrix}$$

has full column rank. Therefore (GLCP$''$) is equivalent to AVI$(\bar{q}, \bar{M}, \bar{X})$ where

$$\bar{X} = \left\{ y \,\middle|\, \bar{A}y \leq \bar{b}, \bar{B}y = \bar{d} \right\}$$

and $\bar{q}$, $\bar{b}$, and $\bar{d}$ are vectors which consist of appropriate components of q, b and d respectively.

The procedure of reducing AVI(q, M, X) to AVI$(\bar{q}, \bar{M}, \bar{X})$ can be carried out as follows: Use Gaussian elimination to find a maximum subset of linearly independent columns for the matrix

$$\begin{pmatrix} 0 & -B \\ B^T & M \\ 0 & -A \end{pmatrix}$$

and hence construct the index sets α and β defined by (2.2). Drop the the rows and columns with indices in β from the matrix

$$\begin{pmatrix} 0 & -B & 0 & d \\ B^T & M & A^T & q \\ 0 & -A & 0 & b \end{pmatrix}$$

Define $\bar{M}$, $\bar{A}$, $\bar{B}$ $\bar{q}$, $\bar{b}$, and $\bar{d}$ as the remaining parts of M, A, B, q, b, and d. The problem AVI$(\bar{q}, \bar{M}, \bar{X})$ has thus been constructed. A solution of AVI$(\bar{q}, \bar{M}, \bar{X})$ is found by solving the equivalent problem (GLCP$''$). A solution of (GLCP), and hence a solution of AVI(q, M, X), can then be constructed from that of (GLCP$''$) by applying Lemma 2.2. The number of arithmetic operations required to construct AVI$(\bar{q}, \bar{M}, \bar{X})$ from AVI(q, M, X) and to reconstruct the solution of AVI(q, M, X) from that of AVI$(\bar{q}, \bar{M}, \bar{X})$ is bounded by $O((m+n+p)^3)$.

For the rest of this paper, we assume that we are given AVI(q, M, X) such that (1.1) holds.

3. Existence of the Central Path

We solve AVI(q, M, X) by solving the complementarity problem (GLCP) under assumption (1.1). We assume the set of interior points of S

$$S^0 := \left\{ (u, v) \,\middle|\, u, v > 0, v = Ax - b, Bx - d = 0, Mx + A^T u + B^T s + q = 0 \right\}$$

is nonempty. In Appendix A, we will show, for any given affine variational inequality, how to construct an equivalent problem such that $S^0 \neq \emptyset$.

We observe that the set

$$T = \{(u, v) \mid (s, u, v, x) \in S, \text{ for some } s, x\} \tag{3.1}$$

defines a multifunction (see [1]) on $\mathbb{R}^m$. For any $(u_i, v_i) \in T$, $i = 1, 2$,

$$\Delta u = u_2 - u_1, \text{ and } \Delta v = v_2 - v_1$$

and some appropriate Δs and Δx satisfy the following homogeneous equation

$$\begin{pmatrix} B^T & A^T & 0 & M \\ 0 & 0 & I & A \\ 0 & 0 & 0 & B \end{pmatrix} \begin{pmatrix} \Delta s \\ \Delta u \\ \Delta v \\ \Delta x \end{pmatrix} = 0 \tag{3.2}$$

Therefore

$$\Delta u^T \Delta v = \Delta x^T M^T \Delta x$$

It follows from the positive semi-definiteness if M that

$$\Delta u^T \Delta v \geq 0 \tag{3.3}$$

which implies that T is a monotone multifunction. The assumption (1.1) further implies that T is maximal. In order to see this, we first introduce a technical lemma, which is proven in [4].

Lemma 3.1. *Given p, r, $u \in \mathbb{R}^n$, $p + r = u$ and $p^T r \geq 0$ then*

$$\begin{aligned} \|p\|_2 &\leq \|u\|_2 \\ \|r\|_2 &\leq \|u\|_2 \\ \|p\|_2 \|r\|_2 &\leq \frac{1}{2} \|u\|_2^2 \end{aligned}$$

Using this lemma, we are able to prove the following result.

Lemma 3.2. *For any positive diagonal matrices D_1, D_2, and $r \in \mathbb{R}^m$, the equation*

$$\begin{pmatrix} 0 & D_1 & D_2 & 0 \\ B^T & A^T & 0 & M \\ 0 & 0 & I & A \\ 0 & 0 & 0 & B \end{pmatrix} \begin{pmatrix} \Delta s \\ \Delta u \\ \Delta v \\ \Delta x \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

has a unique solution.

Proof. It suffices to show that the homogeneous system

$$\begin{pmatrix} 0 & D_1 & D_2 & 0 \\ B^T & A^T & 0 & M \\ 0 & 0 & I & A \\ 0 & 0 & 0 & B \end{pmatrix} \begin{pmatrix} \Delta s \\ \Delta u \\ \Delta v \\ \Delta x \end{pmatrix} = 0 \tag{3.4}$$

has a unique solution.

Suppose $(\Delta s, \Delta u, \Delta v, \Delta x)$ is a solution, then

$$D_1 \Delta u + D_2 \Delta v = 0$$

hence

$$D \Delta u + D^{-1} \Delta v = 0$$

where $D = (D_1 D_2^{-1})^{\frac{1}{2}}$. Notice that $(D\Delta u)^T(D^{-1}\Delta v) = \Delta u^T \Delta v \geq 0$ as a result of (3.3), so Lemma 3.1 applies, and we have

$$\|D\Delta u\|_2 \leq 0, \quad \|D^{-1}\Delta v\|_2 \leq 0$$

It follows that

$$\Delta u = 0, \quad \Delta v = 0$$

It then follows $\Delta s = 0$ and $\Delta x = 0$ since as rank $\begin{pmatrix} 0 & -B \\ B^T & M \\ 0 & -A \end{pmatrix} = n + p$. □

The next theorem follows as a direct consequence of the solvability of (3.4) and [2, Theorem 2.1].

Theorem 3.3. *Suppose T is defined by* (3.1). *Then T is maximal monotone.*

Our algorithm is based on the idea of tracing certain path in S, which is defined by

$$(u, v) \in T, \quad (u, v) \geq 0, \quad u_i v_i = \mu, \; 1 \leq i \leq m \tag{3.5}$$

To show that such a path exists, we need the following theorem which is a special case of [6, Theorem 2] provided that T is maximal monotone.

Theorem 3.4. *For maximal monotone multifunction T, the system* (3.5) *has unique solution for each $\mu > 0$.*

Such a path can be also characterized as the path of zeros of the following non–linear function

$$F(s, u, v, x, \mu) = (UV - \mu e, Mx + q + B^T s + A^T u, v + Ax - b, Bx - d)$$

under parameter μ, where $U = \text{diag}(u)$ and $V = \text{diag}(v)$. As a matter of fact, (s, u, v, x) solves (GLCP) if and only if

$$F(s, u, v, x, 0) = 0 \quad u, v \geq 0$$

The zeros of F, under parameter μ, form a continuous curve for $\mu > 0$ (see [6, Theorem 3]) referred to as the central path of S. The idea of tracing the central path is implemented by constructing Newton steps for F.

4. The Path Following Algorithm

We assume that all elements of the matrix

$$Q = \begin{pmatrix} M & q \\ A & b \\ B & d \end{pmatrix}$$

are integers. The size of the problem AVI(q, M, X) is defined by

$$L \;=\; 1 + \log(m + n + p)^2 + \lfloor \sum_{i=1}^{m+n+p} \sum_{j=1}^{n+1} \log(1 + |q_{ij}|) \rfloor$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x for any $x \in \mathbb{R}$, and q_{ij}'s are elements of the matrix Q. This quantity determines the accuracy required in solving affine variational inequalities and is used to devise a stopping criterion for our algorithm.

To solve (GLCP), we begin with an initial point (s^0, u^0, v^0, x^0) which is close to the central path, that is, a point in the set

$$S^\alpha := \left\{ (s,u,v,x) \in S^0 \,\middle|\, \|UVe - \zeta e\|_2 \le \alpha\zeta, \text{ where } \zeta = \frac{1}{m} u^T v \right\} \tag{4.1}$$

At each step, Newton's method is used to compute a new point in S^α such that ζ is reduced from the previous value by a constant factor. The algorithm terminates when ζ is sufficiently small. In Appendix A, we show how to construct such an initial point.

Given a point $(s^0, u^0, v^0, x^0) \in S^\alpha$, here is the algorithm:

Step 1: Choose $0 < \alpha \le \frac{1}{10}$, let $\delta = \frac{\alpha}{1-\alpha}$, and let $k = 0$.
Step 2: If $u^{kT} v^k < 2^{-4L}$, then stop.
Step 3: Let

$$\begin{aligned} \zeta &= u^{kT} v^k / m \\ \mu &= (1 - \delta / m^{\frac{1}{2}})\zeta \\ (s,u,v,x) &= (s^k, u^k, v^k, x^k) \end{aligned}$$

Step 4: Compute $(\Delta s, \Delta u, \Delta v, \Delta x)$ by constructing a Newton step for the function F, that is, solving

$$\begin{pmatrix} 0 & V & U & 0 \\ B^T & A^T & 0 & M \\ 0 & 0 & I & A \\ 0 & 0 & 0 & B \end{pmatrix} \begin{pmatrix} \Delta s \\ \Delta u \\ \Delta v \\ \Delta x \end{pmatrix} = \begin{pmatrix} UVe - \mu e \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.2}$$

and set

$$(s^{k+1}, u^{k+1}, v^{k+1}, x^{k+1}) = (s,u,v,x) - (\Delta s, \Delta u, \Delta v, \Delta x)$$

Step 5: Set $k = k+1$, and go to Step 1.

There are two crucial issues concerning the validity of the algorithm, one is the solvability of (4.2), and the other is the justification that each new iterate stays in S^α and that ζ is reduced. The solvability of (4.2) follows from is a direct consequence of Lemma 3.2. The following theorem proves that the sequence $\{(s^k, u^k, v^k, x^k)\}$ generated by the algorithm remains in S^α. Moreover $\zeta^k = \frac{1}{m} u^{kT} v^k$ decreases by a constant ratio $(1 - \frac{\delta}{6m^{\frac{1}{2}}})$ at each iteration. As a result, our algorithm stops in $O(m^{\frac{1}{2}} L)$ iterations, each of which requires $O((m+n+p)^3)$ operations to compute a new point. Therefore, it takes no more than $O(m^{\frac{1}{2}}(m+n+p)^3 L)$ arithmetic operations for the algorithm to find a point $\{(s^k, u^k, v^k, x^k)\}$ such that $u^{kT} v^k < 2^{-4L}$. In Appendix A, we show that an exact solution of AVI(q, M, X) can be constructed in no more than $O((m+n+p)^3)$ additional operations.

Theorem 4.1. *Let $(s,u,v,x) \in S^0$ satisfy*

$$\|UVe - \zeta e\|_2 \le \alpha\zeta \quad \textit{with} \quad \zeta = \frac{1}{m} u^T v$$

for $\alpha \in (0, \frac{1}{10})$. Let

$$\mu = (1 - \delta / m^{\frac{1}{2}})\zeta$$

Suppose $(\Delta s, \Delta u, \Delta v, \Delta x)$ *is a solution of* (4.2), *and*

$$(\bar{s}, \bar{u}, \bar{v}, \bar{x}) = (s, u, v, x) - (\Delta s, \Delta u, \Delta v, \Delta x)$$

Then, $(\bar{u}, \bar{v}) > 0$, *and*

$$\begin{aligned} \left\| \bar{U}\bar{V}e - \bar{\zeta}e \right\|_2 &\leq \alpha\bar{\zeta} \\ \bar{\zeta} = \frac{1}{m}\bar{u}^T\bar{v} &\leq (1 - \frac{\delta}{6m^{\frac{1}{2}}})\zeta \end{aligned}$$

Proof. Since

$$V\Delta u + U\Delta v = UVe - \mu e$$

and according to (3.3)

$$\Delta u^T \Delta v \geq 0$$

All the estimates for proving of Theorem 1 of [4] are valid (also see [2, Section 5]), and our theorem is therefore proven. □

Remark. In fact, the conclusion of Theorem 4.1 can be strengthened (see [4]) to allow $\alpha \leq \frac{2}{10}$ and consequently we have

$$\bar{\zeta} \leq (1 - \frac{\delta}{2m^{\frac{1}{2}}})\zeta$$

The rank-one update procedure described in [4] can also be incorporated to save $O((m + n + p)^{\frac{1}{2}})$ arithmetic operations for each iteration and hence provide an $O(m^{\frac{1}{2}}(m + n + p)^{\frac{5}{2}}L)$ algorithm.

5. Conclusion

The problem AVI(q, M, X) can be reduced to a generalized complementarity problem. Such a complementarity problem can be solved in polynomial time by using a path following method similar to the one proposed by Kojima et. al. in [4]. The complexity of solving AVI(q, M, X) is $O(m^{\frac{1}{2}}(m+n+p)^3 L)$, which can be further reduced to $O(m^{\frac{1}{2}}(m+n+p)^{\frac{5}{2}}L)$ if we incorporate a more sophisticated rank-one update scheme.

Appendix A. Constructing an Feasible Problem

Given AVI(q, M, X), we construct an artificial problem with an easily generated starting point. We can test whether S is empty by solving a single linear program. Suppose S is nonempty. Then there exist an extreme point $(s^0, u^0, v^0, x^0) \in S$ since $L(S) = \{0\}$. We will show in Appendix B that

$$\left\| (s^0, u^0, v^0, x^0) \right\|_\infty \leq \frac{2^{2L}}{(m + n + p)^2} \tag{A.1}$$

Consider AVI(q', M', X') with

$$\begin{aligned} q' &= \begin{pmatrix} q \\ (m+1)2^{3L} - e^T b \end{pmatrix} \\ M' &= \begin{pmatrix} M & -A^T e \\ e^T A & 0 \end{pmatrix} \\ X' &= \left\{ \begin{pmatrix} y & t \end{pmatrix} \mid y \in X, t \geq 0 \right\} \\ &= \left\{ \left(\begin{smallmatrix} y \\ t \end{smallmatrix}\right) \,\middle|\, \left(\begin{smallmatrix} A & -e \\ 0 & -1 \end{smallmatrix}\right) \left(\begin{smallmatrix} y \\ t \end{smallmatrix}\right) \leq \left(\begin{smallmatrix} b \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} B & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} y \\ t \end{smallmatrix}\right) = d \right\} \\ &= \{ y' \mid A'y' \leq b', B'y' = d' \} \end{aligned}$$

where $y' = \left(\begin{smallmatrix} y \\ t \end{smallmatrix}\right)$. Then $\left(\begin{smallmatrix} M & -A^T e \\ e^T A & 0 \end{smallmatrix}\right)$ is positive semi-definite, and rank $\left(\begin{smallmatrix} 0 & -B' \\ B'^T & M' \\ 0 & -A' \end{smallmatrix}\right) = n + 1 + p$. Let

$$S'_0 \;:=\; \left\{ (u', v') > 0 \,\middle|\, v' = A'x' - b', B'x' - d' = 0, M'x' + A'^T u' + B'^T s' + q' = 0 \right\}$$

then $S'_0 \neq \emptyset$. In fact, if we set

$$\begin{aligned} &s' = s^0, \;\; x' = (x^0, 2^{3L}) \\ &u' = \begin{pmatrix} 2^{3L} e + u^0 & e^T(Ax^0 - b - u^0) + 2^{3L} \end{pmatrix}, \\ &v' = \begin{pmatrix} 2^{3L} e + b - Ax^0 & 2^{3L} \end{pmatrix} \end{aligned}$$

then $u', v' > 0$ as a result of (A.1). It is also obvious from direct algebraic verification that

$$\begin{aligned} B'x' &= d' \\ v' + A'x' &= b' \\ B'^T s' + A'^T u' + M'x' + q' &= 0 \end{aligned}$$

So, $(s', u', v', x') \in S'_0$.

We further show that

$$\|U'V'e - \zeta' e\|_2 \leq \alpha \zeta'$$

for $\alpha = \frac{1}{10}$, where

$$U' = \operatorname{diag}(u'), \quad V' = \operatorname{diag}(v'), \text{ and } \zeta' = \tfrac{1}{m+1} u'^T v'$$

Let $K = m + n + p$, then by the definition of L

$$2^L \geq 2^{m+p} \cdot K^2 \tag{A.2}$$

since

$$2^L = (m+n+p)^2 \prod_{i=1}^{m+n+p} \prod_{j=1}^{n+1} (1 + |q_{ij}|)$$

and there is at least one non-zero element in each row of the matrix $\left(\begin{smallmatrix} A & b \\ B & d \end{smallmatrix}\right)$. Hence

$$2^{-L} \leq \frac{1}{2^{m+p} \cdot K^2} \tag{A.3}$$

By definition u', v', also taking into account (A.1) and (A.3)

$$\begin{aligned}\left(1-\frac{1}{K^4}\right)2^{3L} &\leq u_i' \leq \left(1+\frac{1}{K^4}\right)2^{3L}\\ \left(1-\frac{1}{2K^4}\right)2^{3L} &\leq v_i' \leq \left(1+\frac{1}{2K^4}\right)2^{3L}\end{aligned}$$

for $1 \leq i \leq m+1$. Multiply u_i's by v_i's and notice that $K = m+n+p \geq 3$, then

$$\left(1-\frac{2}{K^4}\right)2^{6L} \leq u_i'v_i \leq \left(1+\frac{2}{K^4}\right)2^{6L},\ 1 \leq i \leq m+1$$

Therefore

$$\left(1-\frac{2}{K^4}\right)2^{6L} \leq \zeta' \leq \left(1+\frac{2}{K^4}\right)2^{6L} \tag{A.4}$$

Hence

$$|u_i'v_i' - \zeta'| \leq \frac{4}{K^4}2^{6L},\ 1 \leq i \leq m+1$$

Taking into account (A.4)

$$\|U'V'e - \zeta'e\|_2 = \left\{\sum_{i=1}^{m+1} |u_i'v_i' - \zeta'|^2\right\}^{\frac{1}{2}} \leq \frac{4(m+1)^{\frac{1}{2}}}{K^4-2}\zeta'$$

Knowing that $K = m+n+p \geq m+2$ and $m \geq 1$

$$\frac{4(m+1)^{\frac{1}{2}}}{K^4-2} \leq \frac{4(m+1)^{\frac{1}{2}}}{(m+2)^4-2} \leq \frac{1}{10}$$

We conclude that

$$\|U'V'e - \zeta'e\|_2 \leq \alpha\zeta'$$

for $\alpha = \frac{1}{10}$.

Now, considering that the size of the new problem

$$L' = 1 + \lfloor \log(m+1+n+1+p)^2 + \sum_{i=1}^{m+1+n+1+p} \sum_{j=1}^{n+1+1} \log(1+|q_{ij}'|)\rfloor$$

where q_{ij}''s are elements of $\begin{pmatrix} M' & q' \\ A' & b' \\ B' & d' \end{pmatrix}$ and noticing that

$$\begin{aligned}\log(1+|A_{i\cdot}^Te|) &\leq \sum_{j=1}^{m} \log(1+|A_{ij}^T|)\\ \log(1+|2^{4L} - e^Tb|) &\leq 4L + \sum_{j=1}^{m} \log(1+|b_j|)\\ m+1+n+1+p &\leq 2(m+n+p)\end{aligned}$$

we have the following estimate

$$\begin{aligned} L' &\le 1 + \lfloor \log(m+1+n+1+p)^2 + 3 \sum_{i=1}^{m+n+p} \sum_{j=1}^{n+1} \log(1+|q_{ij}|) + 1 + m + 4L \rfloor \\ &\le 1 + \lfloor 4\log(m+n+p)^2 + 4 \sum_{i=1}^{m+n+p} \sum_{j=1}^{n+1} \log(1+|q_{ij}|) \rfloor + 4L \\ &\le 8L \end{aligned}$$

This guarantees that we can solve the artificial problem in $O(m^{\frac{1}{2}}(m+n+p)^3 L)$ arithmetic operations.

The following theorem shows that given any solution $\bar{x}' = (\bar{x}, \bar{x}_{m+1})$ of AVI(q', M', X'), if $\bar{x}_{m+1} = 0$ then $\bar{x}$ is a solution of AVI(q, M, X); otherwise AVI(q, M, X) is unsolvable.

Theorem A.1. *Suppose the* AVI(q, M, X) *is solvable, and* $\bar{x}' = (\bar{x}, \bar{x}_{m+1})$ *is a solution of* AVI(q', M', X'). *Then* $\bar{x}_{m+1} = 0$ *and* $\bar{x}$ *solves* AVI(q, M, X).

Proof. Since AVI(q, M, X) is solvable, there exists $(\tilde{s}, \tilde{u}, \tilde{v}, \tilde{x})$ satisfying (GLCP) and that $\|\tilde{u}\|_\infty$, $\|\tilde{v}\|_\infty$ are hence bounded by $2^{2L}/(m+n+p)^2$. As a result, $(m+1)2^{3L} - e^T(\tilde{u}+\tilde{v}) > 0$. Let

$$\tilde{s}' = \tilde{s},\ \tilde{u}' = \begin{pmatrix} \tilde{u} & \tilde{u}_{m+1} \end{pmatrix},\ \tilde{v}' = \begin{pmatrix} \tilde{v} & 0 \end{pmatrix},\ \text{and } \tilde{x}' = \begin{pmatrix} \tilde{x} & 0 \end{pmatrix}$$

where

$$\begin{aligned} \tilde{u}_{m+1} &= (m+1)2^{3L} - e^T(\tilde{u}+\tilde{v}) \\ &= (m+1)2^{3L} + e^T(A\tilde{x} - b - \tilde{u}) \end{aligned}$$

then, it can be directly verified that $(\tilde{s}', \tilde{u}', \tilde{v}', \tilde{x}')$ solves AVI(q', M', X'). For any other solution $(\bar{s}', \bar{u}', \bar{v}', \bar{x}')$ such that

$$\bar{u}' = (\bar{u}, \bar{u}_{m+1}),\ \bar{v}' = (\bar{v}, \bar{v}_{m+1}),\ \text{and } \bar{x}' = (\bar{x}, \bar{x}_{m+1})$$

we have

$$\tilde{u'}^T \tilde{v}' = \tilde{u'}^T \bar{v}' + \bar{u'}^T \tilde{v}' + (\tilde{u}' - \bar{u}')^T (\tilde{v}' - \bar{v}')$$

by Lemma 2.1 of [5]. Since $(\tilde{u}' - \bar{u}')^T(\tilde{v}' - \bar{v}') \ge 0$ as a result of (3.3), and $\tilde{u'}^T \tilde{v}' = 0$

$$\tilde{u'}^T \bar{v}' + \bar{u'}^T \tilde{v}' \le 0$$

But, $(\tilde{u}', \tilde{v}') \ge 0$, $(\bar{u}', \bar{v}') \ge 0$, $\tilde{u}_{m+1} = (m+1)2^{3L} - e^T(\tilde{u}+\tilde{v}) > 0$, therefore $\bar{v}_{m+1} = \bar{x}_{m+1} = 0$. Now, it follows from

$$\begin{aligned} B'\bar{x}' - d' &= 0 \\ -A'\bar{x}' + b' &= \bar{v}' \\ B'^T \bar{s}' + A'^T \bar{u}' + M'\bar{x}' + q' &= 0 \end{aligned}$$

that

$$\begin{aligned} B\bar{x} - d &= 0 \\ -A\bar{x} + b &= \bar{v} \\ B^T \bar{s}' + A^T \bar{u} + M\bar{x} + q &= 0 \end{aligned}$$

Also, $\bar{u}, \bar{v} \geq 0$, and

$$0 \leq \bar{u}^T \bar{v} \leq \bar{u}'^T \bar{v}' = 0$$

Therefore $(\bar{s}', \bar{u}, \bar{v}, \bar{x})$ solves (GLCP). In another words, $\bar{x}$ solves AVI(q, M, X). □

Appendix B. Computing an Exact Solution

The path following method finds a point $(\bar{s}, \bar{u}, \bar{v}, \bar{x}) \in S$ such that

$$\bar{u}^T \bar{v} < 2^{-4L}$$

However, a solution (s, u, v, x) of (GLCP) satisfies

$$(s, u, v, x) \in S \tag{B.1}$$

$$u^T v = 0 \tag{B.2}$$

Again, S is defined by

$$S = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \geq 0 \;\middle|\; \begin{pmatrix} B^T & A^T & 0 & M \\ 0 & 0 & I & A \\ 0 & 0 & 0 & B \end{pmatrix} \begin{pmatrix} s \\ u \\ v \\ x \end{pmatrix} = \begin{pmatrix} -q \\ b \\ d \end{pmatrix} \right\}$$

Notice that lineality space $L(S)$ of S is zero, due to our assumptions that the rank $\begin{pmatrix} 0 & -B \\ B^T & M \\ 0 & -A \end{pmatrix} = n + p$. So, if (GLCP) has a solution, it has a solution which is an extreme point of S. In fact, such an extreme point can be characterized by the following Lemma.

Lemma B.1. *Given $(s, u, v, x) \in S$, suppose*

$$\alpha = \mathrm{supp}(u)$$
$$\beta = \mathrm{supp}(v)$$

Then (s, u, v, x) is an extreme point of S if and only if

$$Q(\alpha, \beta) = \begin{pmatrix} B^T & A^T_{\cdot\alpha} & 0 & M \\ 0 & 0 & I_{\cdot\beta} & A \\ 0 & 0 & 0 & B \end{pmatrix} \tag{B.3}$$

has full column rank. In this case, we also say that (s, u, v, x) is basic.

See [7, Section 3.4.4] for a proof.

Our objective is to construct a basic solution, in polynomial time, from an approximate solution given by the path following method. The following lemma points to the existence of certain basic solution that is closely related to a given approximate solution. For convenience of notation, we use

$$w = (s, u, v, x)$$

and

$$K(w) = \left\{ p \leq k \leq p + 2m \;\middle|\; w_k < 2^{-2L} \right\}$$
$$K(w)^c = \{ k \mid k \notin K(w) \}$$

for any point $(s, u, v, x) \in S$.

Lemma B.2. *For any point*

$$\bar{w} = (\bar{s}, \bar{u}, \bar{v}, \bar{x}) \in S$$

there exists basic solution

$$w^* = (s^*, u^*, v^*, x^*)$$

of (B.1) *such that*

$$w_k^* = 0$$

for all $k \in K(\bar{w})$.

Proof. For any basic solution (s^*, u^*, v^*, x^*) of S, each of its non-zero component is expressible as Δ_1/Δ_2 by Cramer's rule, in which Δ_i's are determinants of square submatrices of

$$Q = \begin{pmatrix} B^T & A^T & 0 & M \\ 0 & 0 & I & A \\ 0 & 0 & 0 & B \end{pmatrix} \tag{B.4}$$

as a result of the previous Lemma.

By definition of L, these Δ_i's are bounded by $2^{2L}/(m+n+p)^2$. Hence for any extreme point $w = (s, u, v, x)$

$$w_k = 0 \quad \text{if} \quad w_k < (m+n+p)^2 2^{-2L} \tag{B.5}$$

$$w_k \geq (m+n+p)^2 2^{-2L} \quad \text{if} \quad w_k > 0 \tag{B.6}$$

Given any point $\bar{w} = (\bar{s}, \bar{u}, \bar{v}, \bar{x}) \in S$, it can be written as

$$\bar{w} = \sum_{i=1}^{N} c_i w^i + r$$

where (s^i, u^i, v^i, x^i)'s are vertices, $c_i \geq 0$, $\sum_{i=1}^{N} c_i = 1$, $r \in \text{rec}S$ with $r_k \geq 0$ for $p+1 \leq k \leq p+2m$. In particular

$$\bar{w}_k = \sum_{i=1}^{N} c_i z_k^i + r_k, \qquad p+1 \leq k \leq p+2m$$

By Caratheodory (see [10, Theorem 17.1]), we can assume $N \leq 2m+n+p+1$, and we can hence find a j such that

$$c_j \geq \frac{1}{2m+n+p+1}$$

Now, we claim that w^j satisfies (B.5) and (B.6). Otherwise

$$w_k^j > 0$$

for some $k \in K(\bar{w})$, so

$$w_k^j \geq (m+n+p)^2 2^{-2L}$$

hence

$$\bar{w}_k = \sum_{i=1}^{N} c_i w_k^i + r_k \geq c_j w_k^j \geq \frac{(m+n+p)^2}{2m+n+1} 2^{-2L} > 2^{-2L}$$

a contradiction. $\square$

Suppose $\hat{w} = (\hat{s}, \hat{u}, \hat{v}, \hat{x})$ is a point in S satisfying $\hat{u}^T \hat{v} < 2^{-4L}$. By using a method given in [4, Appendix B], we can move from $\hat{w}$ to a point $\bar{w} \in S$ in $O((m+n+p)^3)$ arithmetic operations, with $\bar{w}$ satisfying $K(\hat{w}) \subset K(\bar{w})$ and that the set of columns of the matrix Q with indices in $K(\bar{w})^c$ is linearly independent. Consider the system of equations

$$\begin{array}{ll} Qw = q' & w \geq 0 \\ w_k = 0 & for \;\; k \in K(\bar{w}) \end{array}$$

where $q' = \begin{pmatrix} -q \\ b \\ d \end{pmatrix}$. According to the previous lemma, this system is satisfied by a solution $w^* = (s^*, u^*, v^*, x^*)$ of (B.1) such that $w_k^* = 0$ for $k \in K(\bar{w})$. But since $\hat{u}^T \hat{v} < 2^{-4L}$ we have

$$\hat{u}_k < 2^{-2L} \quad or \quad \hat{v}_k < 2^{-2L}$$

for each $1 \leq k \leq m$. Hence

$$\bar{u}_k < 2^{-2L} \quad or \quad \bar{v}_k < 2^{-2L}$$

for each $1 \leq k \leq m$. Therefore

$$u_k^* = 0 \quad or \quad v_k^* = 0$$

So, we see that w^* is a solution of (GLCP). Considering that $K(\bar{w}) \subset K(w^*)$ and that the set of columns of Q with indices in $K(\bar{w})^c$ are linearly independent, we know that w^* can be solved from the equation

$$Qw = q'$$

in $O((m+n+p)^3)$ arithmetic operations.

Appendix C. An Algebraic Property of PSD Matrices

We begin with the following simple fact.

Lemma C.1 ([8, Result 1.6]). *Let M be a positive semi-definite matrix, and assume*

$$M = \begin{pmatrix} 0 & u^T \\ 0 & M' \end{pmatrix}$$

then $u = 0$.

Consequently, we have the following corollaries.

Corollary C.2. *Let M be an $n \times n$ positive semi-definite matrix, and let*

$$\gamma \subset \{1, 2, \cdots, n\}$$

Assume $M_{\cdot\gamma} = 0$, then $M_{\gamma\cdot} = 0$.

Proof. Apply the previous Lemma to each index of γ. □

Corollary C.3. *Let M be an $n \times n$ positive semi-definite matrix, and γ, α, β be a partition of $\{1, 2, \cdots, n\}$, so that*

$$M = \begin{pmatrix} M_{\cdot\gamma} & M_{\cdot\alpha} & M_{\cdot\beta} \end{pmatrix}$$

Assume that

$$M_{\cdot\gamma} = M_{\cdot\alpha} P$$

for some $|\alpha| \times |\gamma|$ matrix P, then

$$M_{\gamma\cdot} = P^T M_{\alpha\cdot}$$

Proof.

$$\begin{aligned}
\begin{pmatrix} I & -P^T & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} M \begin{pmatrix} I & 0 & 0 \\ -P & I & 0 \\ 0 & 0 & I \end{pmatrix} &= \begin{pmatrix} I & -P^T & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} 0 & M_{\cdot\alpha} & M_{\cdot\beta} \end{pmatrix} \\
&= \begin{pmatrix} I & -P^T & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} 0 & M_{\gamma\alpha} & M_{\gamma\beta} \\ 0 & M_{\alpha\alpha} & M_{\alpha\beta} \\ 0 & M_{\beta\alpha} & M_{\beta\beta} \end{pmatrix} \\
&= \begin{pmatrix} 0 & * & * \\ 0 & M_{\alpha\alpha} & M_{\alpha\beta} \\ 0 & M_{\beta\alpha} & M_{\beta\beta} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & M_{\alpha\alpha} & M_{\alpha\beta} \\ 0 & M_{\beta\alpha} & M_{\beta\beta} \end{pmatrix}
\end{aligned}$$

where the last equality follows from Lemma C.1. It now follows that

$$\begin{aligned}
M &= \begin{pmatrix} I & P^T & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & M_{\alpha\alpha} & M_{\alpha\beta} \\ 0 & M_{\beta\alpha} & M_{\beta\beta} \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ P & I & 0 \\ 0 & 0 & I \end{pmatrix} \\
&= \begin{pmatrix} P^T M_{\alpha\alpha} P & P^T M_{\alpha\alpha} & P^T M_{\alpha\beta} \\ M_{\alpha\alpha} P & M_{\alpha\alpha} & M_{\alpha\beta} \\ M_{\alpha\beta} P & M_{\alpha\beta} & M_{\beta\beta} \end{pmatrix}
\end{aligned}$$

therefore

$$\begin{aligned}
M_{\gamma\cdot} &= \begin{pmatrix} P^T M_{\alpha\alpha} P & P^T M_{\alpha\alpha} & P^T M_{\alpha\beta} \end{pmatrix} \\
&= P^T \begin{pmatrix} M_{\alpha\alpha} P & M_{\alpha\alpha} & M_{\alpha\beta} \end{pmatrix} \\
&= P^T M_{\alpha\cdot}
\end{aligned}$$

□

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