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GLOBAL ERROR BOUNDS FOR MONOTONE AFFINE
VARIATIONAL INEQUALITY PROBLEMS

by

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Global Error Bounds for Monotone Affine Variational Inequality Problems[1]

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Abstract. A global "gradient projection" type error bound is given for monotone affine variational inequality problems. This new bound is based on a global error bound for monotone linear complementarity problems [6] and requires no assumptions on the monotone affine variational inequality other than solvability. The error bound can be used in conjunction with recent results of Tseng and Luo [9] to establish *global* linear convergence of any regular splitting method for solvable symmetric monotone affine variational inequality problems.

Key Words: Error bound, variational inequalities, linear complementarity problem

Abbreviated Title: Error bounds for variational inequalities

1. Introduction

We consider here the monotone affine complementarity problem [1, 3] of finding an $\bar{x}$ in X such that

$$(1.1) \qquad (x - \bar{x})(M\bar{x} + q) \geq 0, \quad \forall x \in X := \{x | Ax \geq b, \; x \geq 0\}$$

where M is an $n \times n$ real positive semidefinite matrix (not necessarily symmetric) and A is an $m \times n$ real matrix. When X is the nonnegative orthant, the problem becomes the classical monotone linear complementarity problem (LCP) [2, 7]. Our principal concern here is: Given an arbitrary point x in X, how far is it from the closed convex solution set $\bar{X}$ of (1.1), assuming that $\bar{X}$ is nonempty? Pang [8, Lemma 2] was the first to obtain global bound on this distance in terms of the following "gradient projection" map for the *strongly* monotone linear complementarity problem

$$(1.2) \qquad r(x) = x - (x - Mx - q)^+$$

where $(\cdot)^+$ denotes the orthogonal projection on X. Note that when M is symmetric, which need not be the case here, $r(x)$ is precisely the gradient projection map for $\min_{x \in X} \frac{1}{2} xMx + qx$. Since the solution set $\bar{X}$ of (1.1) is the set of zeros of r (see Proposition 2.3 below), the quantity $\|r(x)\|$ can be thought of as a *natural residual*. Unfortunately [6, Example 2.9] this natural residual fails to bound *globally* the error for the monotone linear complementarity problem. Recently, however, Tseng and Luo have established a constant multiple of $\|r(x)\|$ as a *local* error bound for the *general* affine variational inequality problem [9, Theorem 2.1], under a solvability assumption on (1.1). More recently, Luo and Tseng established the same quantity as a *global* error bound for the *monotone* affine variational inequality problem [4, Proposition 1] under the following assumption:

$$(1.3) \qquad Mu = 0, \; Au \geq 0, \; u \geq 0 \quad \text{imply that} \quad A_I u = 0, \; x_J = 0,$$

where I, J are the index sets of active constraints on the solution set $\bar{X}$, that is: $A_I x = b_I$, $x_J = 0$ for *all* x in $\bar{X}$. However, condition (1.3) is unknown *a priori*, and also restricts the validity of the error bound to problems satisfying this condition. Our purpose here is to give a bound in terms of $r(x)$ that involves no such restrictions.

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This however comes at the apparently inevitable expense of slightly more complicated expressions such as (3.1), (3.8) and (3.12) below. Thus it seems that in globalizing the error bound involving $\|r(x)\|$ one must either impose restrictions on the monotone affine linear complementarity problem as was done in [4], or one must use slightly more involved expressions. The global bounds of both approaches however, when specialized to local bounds, do not reproduce the known simple local bound [9, Theorem 2.1] without the additional assumption (1.3) of [4, Proposition 1], and in our case without a boundedness assumption on X. It would therefore be interesting to obtain a global error bound involving $\|r(x)\|$ which reduces to the simple local bound of [9, Theorem 2.1], without any additional assumptions other than solvability.

Our principal tool in deriving our bounds here is the error bound we obtained previously [6, Theorem 2.7] for the monotone linear complementarity problem. In Section 2 of this paper we give a simplified version of this error bound, and in Section 3 we derive our error bounds for (1.1), both for a possibly degenerate problem (Theorem 3.1) and a nondegenerate problem (Theorem 3.2). We then simplify our global error bound (3.1) for the possibly degenerate case by removing the term $\|r(x)\|^{\frac{1}{2}}$. We do this by combining (3.1) with the local error bound of Tseng and Luo [9, Theorem 2.1] and thus obtaining the simple residual (3.12) (Theorem 3.3). This simplified bound can be used in conjunction with results of Tseng and Luo [9] to establish *global* linear convergence of iterates of any regular splitting method for nondegenerate monotone affine variational inequality problems (Theorem 3.4). Section 4 contains some concluding remarks.

A word about out notation. For a vector x in the n-dimensional space R^n, x_+ will denote the orthogonal projection on the nonnegative orthant R^n_+, that is $(x_+)_i := \max\{x_i, 0\}$, $i = 1, \ldots, n$; whereas x^+ will denote the orthogonal projection on the set $X := \{x | Ax \geq b, x \geq 0\}$, where A is a given $m \times n$ real matrix. The norm $\| \cdot \|$ will denote the Euclidean or 2-norm $(xx)^{\frac{1}{2}}$, while other norms will be appropriately subscripted. The transpose of a matrix M will be denoted by M'. For an $m \times n$ real matrix A, A_i will denote the ith row, while A_J will denote the set of rows A_i for $i \in J \subset \{i, \ldots, m\}$. Similar notation, b_i and b_J is used for a vector b in R^m. The identity matrix of arbitrary dimension will be denoted by I.

2. Preliminary Background: Simplification of Monotone LCP Error Bound

We begin by giving a simplification of the error bound of [6, Theorem 2.7] for the monotone linear complementary problem

$$(2.1) \qquad Nz + p \geq 0, \ z \geq 0, \ z(Nz + p) = 0$$

where N is a $k \times k$ real positive semidefinite matrix, not necessarily symmetric. We shall assume that its solution set $\bar{Z}$ is nonempty. By slight modification of the proofs of Lemma 2.5 and Theorem 2.7 of [6] and making use of Theorem 2.6 of [5] we obtain the following simplified error bounds in terms of another "natural" residual $s(z)$ (see (2.4) below).

2.1 Theorem (Error bound for monotone linear complementarity problems) Let N be positive semidefinite, let the solution set $\bar{Z}$ of (2.1) contain $\bar{z}$ and let

$$(2.2) \qquad \hat{N} := \frac{1}{2}(N + N') \ \text{and} \ d := 2\hat{N}\bar{z} + p$$

For any point z in R^k there exists a $\bar{z}(z)$ in $\bar{Z}$ such that [6, Theorem 2.7]

$$(2.3) \qquad \|z - \bar{z}(z)\| \leq \sigma(N, p) \cdot \left(s(z) + s(z)^{\frac{1}{2}}\right)$$

where $s(z)$ is the residual

$$(2.4) \qquad s(z) := \left\|\left(-Nz - p, \ -z, \ z(Nz + p)\right)_+\right\|,$$

and

$$(2.5) \qquad \sigma(N, p) := k^{\frac{1}{2}} \nu(N, p)^{\frac{1}{2}} \tau(N, p)$$

$$(2.6) \qquad \nu(N, p) := \min_{\bar{z} \in \bar{Z}} \|1, \ \bar{z}, \ N\bar{z} + p\|$$

$$(2.7) \qquad \tau(N, p) := \sup_{(u,v,\xi,z)} \left\{ \|u, v, \xi, z\| \left| \begin{array}{l} \left\|N'u + v - d\xi + \hat{N}^{\frac{1}{2}}z\right\|_1 = 1 \\ (u, v, \xi) \geq 0 \\ \text{Columns of } [N' \ I \ d \ \hat{N}^{\frac{1}{2}}], \\ \text{corresponding to nonzero} \\ \text{elements of } (u, v, z, \xi), \\ \text{are lin. indep.} \end{array} \right. \right\}$$

If the LCP (2.1) has some nondegenerate solution $\hat{z}$, that is $\hat{z} + N\hat{z} + p > 0$, then the bound (2.3) simplifies to [5, Theorem 2.6]

$$(2.8) \qquad \|z - \bar{z}(z)\| \leq \sigma(N,p)s(z),$$

with the term $\nu(N,p)$ deleted from (2.5), and the terms $\hat{N}^{\frac{1}{2}}z$ and $\hat{N}^{\frac{1}{2}}$ deleted from (2.7).

It was noted in [6, Example 2.9] that the square root term in the bound (2.3) is essential and cannot be dispensed with even locally.

In order to apply the above bounds to the monotone affine variational inequality problem (1.1) we state an equivalent characterization of (1.1) as a linear complementarity problem in R^{n+m} as follows.

2.2 Proposition A point $\bar{x}$ in R^n solves the affine variational inequality (1.1) if and only if $\bar{x}$ and some $\bar{u}$ in R^m solve the following linear complementarity problem

$$(2.9) \qquad w = \begin{pmatrix} M & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} q \\ -b \end{pmatrix} \geq 0, \quad (x \quad u) \geq 0, \quad (x \quad u)w = 0$$

Proof An $\bar{x}$ is a solution of (1.1) if and only if $\bar{x} \in \arg\min_{x \in X} x(M\bar{x} + q)$. This is so if and only if $\bar{x}$ and some $\bar{u}$ satisfy the Karush-Kuhn-Tucker conditions (2.9) for the linear program $\min_{x \in X} x(M\bar{x} + q)$. ∎

The LCP (2.9) can also be interpreted as the following gradient projection optimality condition for (1.1).

2.3 Proposition A point $\bar{x}$ is a solution of (1.1) if and only if $r(x) = 0$, where $r(x)$ is defined by (1.2).

Proof The LCP (2.9) constitutes the Karush-Kuhn-Tucker conditions for $\bar{x} = \arg\min_{x \in X} \frac{1}{2}\|x - (\bar{x} - M\bar{x} - q)\|^2$, which is equivalent to $r(\bar{x}) = 0$. ∎

With the above preliminaries, we are ready to derive our error bounds for the monotone affine variational inequality (1.1).

3. Error Bounds for the Monotone Affine Variational Inequality

We first derive the global error bounds (3.1) and (3.8) below, for (1.1), by using the error bounds (2.3) and (2.8) on the equivalent LCP formulation (2.9) of the monotone variational inequality (1.1). We then simplify our error bound (3.1) to (3.12) for the possibly degenerate case, by making use of Tseng and Luo's local error bound [9, Theorem 2.1].

3.1 Theorem (Global error bound for monotone affine variational inequality)

Let M be positive semidefinite and let (1.1) be solvable. For each $x \in X$ there exists an $\bar{x}(x) \in \bar{X}$ such that

$$(3.1) \qquad \|x - \bar{x}(x)\| \leq \tau(M, A, q, b)\big(1 + \|y(x)\|\big)\big(\|r(x)\| + \|r(x)\|^{\frac{1}{2}}\big)$$

where

$$(3.2) \qquad r(x) := x - y(x) := x - (x - Mx - q)^+,$$

$$(3.3) \qquad \tau(M, A, q, b) := 1 + \sigma(N, p) \cdot \max\{\|I - M\|, \|I - M\|^{\frac{1}{2}}\},$$

and σ is defined in (2.5) with

$$(3.4) \qquad N = \begin{pmatrix} M & -A^T \\ A & 0 \end{pmatrix}, \quad p = \begin{pmatrix} q \\ -b \end{pmatrix}, \quad k = n + m$$

Proof Let $x \in X$ and let $y := y(x) := (x - Mx - q)^+$, that is y is the orthogonal projection of $x - Mx - q$ on X. Then

$$y = \arg\min_{z \in X} \frac{1}{2}\|z - (x - Mx - q)\|^2$$

Hence y and some $u \in R^m$ satisfy the Karush-Kuhn-Tucker conditions

$$(3.5) \qquad \begin{aligned} s := y - x + Mx + q - A^T u \geq 0, \quad ys = 0, \quad y \geq 0 \\ Ay - b \geq 0, \quad u(Ay - b) = 0, \quad u \geq 0 \end{aligned}$$

Consequently

$$(3.6) \qquad \begin{aligned} My - A^T u + q = s + (I - M)(x - y), \\ y(My - A^T u + q) = y(I - M)(x - y), \quad y \geq 0 \\ Ay - b \geq 0, \quad u(Ay - b) = 0, \quad u \geq 0 \end{aligned}$$

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The right-hand sides of the expressions in (3.6) can be considered as residuals generated by the point (y, u) when substituted into the LCP (2.9). Hence the residual $s(z)$ of (2.4), with N and p as defined in (3.4) and $z = \begin{pmatrix} y \\ u \end{pmatrix}$, can be used to bound the error in $\begin{pmatrix} y \\ u \end{pmatrix}$, in solving (2.9), and thus the error in y in solving (1.1). By using the error bound (2.3) with $z = \begin{pmatrix} y \\ u \end{pmatrix}$, we have that for any given $x \in X$ and the corresponding $\begin{pmatrix} y \\ u \end{pmatrix}$ satisfying (3.5)-(3.6), there exist $(\bar{x}(x), \bar{u}(x))$ that solve the LCP (2.9) (and hence $\bar{x}(x)$ solves the affine variational inequality (1.1)) and such that

$$\|y - \bar{x}(x)\| \leq \left\| \begin{matrix} y - \bar{x}(x) \\ u - \bar{u}(x) \end{matrix} \right\| \leq \sigma(N, p) \Big[\big\| (-s - (I - M)(x - y), y(I - M)(x - y))_+ \big\|$$
$$+ \big\| \quad \prime\prime \quad \big\|^{\frac{1}{2}} \Big]$$

$$\leq \sigma(N, p) \Big[\big\| (-(I - M)(x - y), y(I - M)(x - y))_+ \big\|$$
$$+ \big\| \quad \prime\prime \quad \big\|^{\frac{1}{2}} \Big]$$

(By 2-norm monotonicity and $s \geq 0$)

$$\leq \sigma(N, p) \Big[((1 + \|y\|)\|I - M\| \|x - y\|)$$
$$+ (\quad \prime\prime \quad)^{\frac{1}{2}} \Big]$$

(By 2-norm monotonicity)

$$\leq (\tau(M, A, q, b) - 1)(1 + \|y\|)(\|x - y\| + \|x - y\|^{\frac{1}{2}})$$

(By Definition (3.3))

$$\leq (\tau(M, A, q, b) - 1)(1 + \|y(x)\|)(\|r(x)\| + \|r(x)\|^{\frac{1}{2}})$$

(By Definition (3.2))

Since $\|x - \bar{x}(x)\| \leq \|r(x)\| + \|y - \bar{x}(x)\|$, the desired bound (3.1) follows from the inequality between the first and last terms in the string of inequalities above. ∎

If the monotone affine variational inequality (1.1) is nondegenerate in the sense that the corresponding LCP (2.9) has some nondegenerate solution, then by using the

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corresponding error bound (2.8) of Theorem 2.1, the error bound (3.1) above simplifies as follows.

3.2 Theorem (Error bound for nondegenerate monotone affine variational inequality) Let M be positive semidefinite and let (1.1) be solvable with $\hat{x}$ being a nondegenerate solution of (1.1), that is $\hat{x}$ and a corresponding $\hat{u}$ solve (2.9) such that

$$(3.7) \qquad \hat{x} + M\hat{x} - A^T\hat{u} + q > 0, \quad \hat{u} + A\hat{x} - b > 0$$

For each $x \in X$ there exists $\bar{x}(x) \in \bar{X}$ such that

$$(3.8) \qquad \|x - \bar{x}(x)\| \leq \tau(M, A, q, b)(1 + \|y(x)\|)\|r(x)\|$$

where $r(x)$ and $y(x)$ are defined in (3.2), $\tau(M, A, q, b)$ is defined in (3.3)-(3.4) with $\|I - M\|^{\frac{1}{2}}$ deleted, and the $\sigma(N, p)$ appearing in (3.3) is defined by (2.5) with the term $\nu(N, p)^{\frac{1}{2}}$ deleted from it, and the terms $\hat{N}^{\frac{1}{2}}z$ and $\hat{N}^{\frac{1}{2}}$ deleted from (2.7).

It turns out that the simpler residual $(1 + \|y\|)\|r(x)\|$ of (3.8) can also be used to give an error bound for the possibly degenerate monotone affine variational inequality problem also, provided we are willing to modify the constant $\tau(M, A, q, b)$ employed in (3.8). This can be achieved (as pointed out to me by my student Jun Ren) by combining the global error bound given by (3.1) with the local error bound of Tseng-Luo [9, Theorem 2.1] which is the following for the solvable monotone affine variational inequality problem (1.1): There exist $\varepsilon > 0$ and $\delta > 0$ such that for each $x \in X$ there exist an $\bar{x}(x) \in \bar{X}$ such that

$$(3.9) \qquad \|x - \bar{x}(x)\| \leq \delta\|r(x)\| \quad \forall x \in X \colon \|r(x)\| \leq \varepsilon$$

Hence applying our bound (3.1) to $x \in X$ such that $\|r(x)\| \geq \varepsilon$, or equivalently $\|r(x)\|^{\frac{1}{2}} \leq \varepsilon^{-\frac{1}{2}}\|r(x)\|$, we obtain

$$(3.10) \quad \|x - \bar{x}(x)\| \leq (1 + \varepsilon^{-\frac{1}{2}})\tau(M, A, q, b)(1 + \|y(x)\|)\|r(x)\| \quad \forall x \in X \colon \|r(x)\| \geq \epsilon$$

The error bounds (3.9) and (3.10) can be combined by defining

$$(3.11) \qquad \alpha := \max\{\delta, (1 + \epsilon^{-\frac{1}{2}})\tau(M, A, q, b)\}$$

to give the following simplified global bound for (1.1).

3.3 Theorem (Simplified global error bound for monotone affine variational inequality) Let M be positive semidefinite and let (1.1) be solvable. There exists an $\alpha > 0$ such that for each $x \in X$ there exists an $\bar{x}(x) \in \bar{X}$ such that

$$(3.12) \qquad \|x - \bar{x}(x)\| \leq \alpha\big(1 + \|y(x)\|\big)\|r(x)\|$$

As an interesting application of our error bound (3.12) we can establish a *global* linear convergence rate for the *symmetric* monotone affine variational inequality problem (1.1). We first note that by Tseng and Luo's Theorem 3.1 [9], the sequence $\{x^i\}$ generated by a regular splitting (B, C) of M, that is $M = B + C$, $B - C$ positive definite, $x^0 \in X$ and

$$(3.13) \qquad x^{i+1} = \big(x^{i+1} - (Bx^{i+1} + Cx^i + q)\big)^+ \quad i = 0, 1, \ldots$$

is convergent to a solution of (1.1), with a linear rate of convergence for $i \geq \bar{i}$ for some $\bar{i}$. Hence the corresponding sequence $\{y(x^i)\}$, $y(x^i) := (x^i - Mx^i - q)^+$ is bounded, say by γ. Consequently, our error bound (3.12) becomes the following for the sequence $\{x^i\}$

$$(3.14) \qquad \|x^i - \bar{x}(x^i)\| \leq \alpha(1 + \gamma)\|r(x^i)\|$$

Combining this "global" error bound with analysis similar to that of Tseng and Luo [9, Theorem 3.1], which uses a local error bound to obtain a "local" linear rate, we obtain the following global linear convergence rate result.

3.4 Theorem (Global linear convergence of regular splitting methods) Let M be symmetric positive semidefinite and let (1.1) be solvable. The regular splitting algorithm (3.13) iterates $\{x^i\}$ converge globally linearly to a solution of (1.1).

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4. Concluding Remarks

We have obtained global error bounds for the monotone variational inequality problem (1.1), the simplest of which is given by (3.12). For bounded x or $y(x)$, the error bound (3.12) behaves as the local Tseng-Luo bound [9, Theorem 2.1]. Essentially, this fact used in Theorem 3.4 to obtain a global linear convergence rate result for a regular splitting method for the symmetric monotone affine linear complementarity problem (1.1). It would be interesting to obtain global error bounds that reduce locally to the local error bound of [9], that is (3.12) with $\|y(x)\|$ replaced by a quantity which is bounded locally. However, we note that the term $\|y(x)\|$ is needed in the global error bound (3.12) in order to apply to the Example 2.10 of [6]

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} , \quad q = \begin{bmatrix} 1 \\ -2 \end{bmatrix} , \quad \bar{X} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} , \quad x(t) = \begin{bmatrix} t \\ 1 \end{bmatrix} , \quad t \to \infty$$

for which a multiple of the natural residual $\|r(x)\|$ fails to be an error bound.

We remark finally, that by a slight change in the proof of Theorem 3.1 above we can obtain the following alternate bound to (3.8) for the nondegenerate case

$$(3.8a) \qquad \|x - \bar{x}(x)\| \le \beta(M, A, q, b) t(x)$$

for some constant $\beta(M, A, q, b)$ and residual $t(x)$ defined by

$$(3.2a) \qquad t(x) := \|r(x)\| + \big(y(x)(I - M)r(x)\big)_+$$

where $r(x)$ and $y(x)$ are defined in (3.2). The significance of this bound is that when applied to the nondegenerate monotone linear complementarity problem we have from (3.6), by deleting A, bound u, that

$$(4.1) \qquad y(x)(I - M)r(x) = y(x)\big(My(x) + q\big)$$

and thus $t(x)$ of (3.2a) becomes the following modified residual for a nondegenerate monotone linear complementarity problem

$$(3.2b) \qquad t(x) := \|r(x)\| + \Big(y(x)\big(My(x) + q\big)\Big)_+$$

This residual is the sum of a gradient projection condition and a complementarity condition, both of which are zero at a solution point. We have also gotten rid of the term $(1 + \|y(x)\|)$ multiplying the residual in (3.8).

We note finally that all bounds given in this paper involving the term $\left(1+\|y(x)\|\right)$, such as (3.1), (3.8) and (3.12), can be replaced, by virtue of the definition (3.2) of $y(x)$, by another upper bound as follows

$$(4.1) \qquad \left(1 + \|y(x)\|\right) \leq \nu(M, q)(1 + \|x\|)$$

where

$$(4.2) \qquad \nu(M, q) := \max\left\{1 + \|q\|, \|I - M\|\right\}$$

I am indebted to Dr. Z.-Q. Luo for this observation.

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