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INTERIOR DUAL LEAST 2-NORM ALGORITHM FOR LINEAR PROGRAMS

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Abstract

An interior algorithm is proposed for solving the dual of the least 2-norm formulation of a linear program. This is a convex quadratic problem with nonnegativity constraints only. Sixty six test problems, including sixty three Netlib problems were solved very accurately. The total time speedup of the algorithm for all 66 problems over MINOS 5.3 is 1.67. Linear convergence of the algorithm is also established.

1 Introduction

It is well known [Mangasarian & Meyer, 1979, Mangasarian, 1984] that a linear program

$$\min_x cx \ s.t. \ Ax \geq b, \ x \geq 0 \tag{1}$$

is solvable if and only if the quadratic program

$$\min_x cx + \frac{\epsilon}{2}xx \ s.t. \ Ax \geq b, \ x \geq 0 \tag{2}$$

is solvable by the same $\bar{x}$ for all $\epsilon \in (0, \bar{\epsilon}]$ for some $\bar{\epsilon} > 0$. If $x(\epsilon)$ solves the quadratic problem (2), then it is the solution of the linear program (1)

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which is closest to the origin in the 2-norm. The dual of the above quadratic program [Mangasarian, 1969] is

$$\max_{x} \quad -\frac{\epsilon}{2}xx + bu \tag{3}$$

$$s.t. \quad \epsilon x - A^t u + v - c = 0 \tag{4}$$

$$x \geq 0 \tag{5}$$

Elimination of x from the dual problem by using the constraint relation

$$x = \frac{1}{\epsilon}\left(A^t u + v - c\right)$$

leads to the following exterior penalty function with penalty parameter ϵ associated with the dual of linear program (1)

$$\min_{u,v} \frac{1}{2}\left\|A^t u + v - c\right\|^2 - \epsilon bu \ \ s.t.\ (u,v) \geq 0 \tag{6}$$

The Karush-Kuhn-Tucker optimality conditions for the quadratic problem (6) can be expressed as a symmetric linear complementarity problem

$$Mz + q \geq 0\ , \quad z \geq 0\ , \quad z(Mz+q) = 0 \tag{7}$$

upon making the following identifications

$$M := \begin{pmatrix} AA^t & A \\ A^t & I \end{pmatrix}\ , \quad q := \begin{pmatrix} -Ac - \epsilon b \\ -c \end{pmatrix}\ , \quad z := \begin{pmatrix} u \\ v \end{pmatrix} \tag{8}$$

Iterative SOR (successive overrelaxation) methods have been proposed for solving the symmetric linear complementarity problems [Mangasarian, 1977]. An SOR method which preserves the sparsity structure of the problem has been implemented to solve very large linear programs [Mangasarian & De Leone, 1986]. These large linear programs with up to 125,000 constraints and 500,000 variables are impossible to solve using the direct method such as the simplex.

Our approach to find the least 2-norm solution of a linear program is to use an interior penalty function. Since the only constraints present in the dual problem (6) are nonnegativity constraints, an initial starting point for the algorithm can be obtained trivially. The interiority of the iterates are easy to maintain by taking an appropriate stepsize. These facts constitute the motivation behind our dual interior penalty method.

We now briefly outline the contents of the paper. In Section 2, we describe the algorithm and in Section 3 we establish its linear convergence. In Section 4 we give computational results. In Section 5 we describe a refinement procedure to improve the accuracy of the optimal solutions obtained by the algorithm and in Section 6 we summarize the paper.

2 Interior Dual Least 2-Norm (IDLN) Algorithm

We consider the linear program given in the following standard form

$$\min_{x} cx \ s.t. \ Ax = b, \ x \geq 0 \tag{9}$$

and its dual

$$\max_{u,v} bu \ s.t. \ A^t u + v = c, \ v \geq 0 \tag{10}$$

We make the following assumption throughout regarding these linear programs.

Assumption 1 *The dual feasible region is nonempty and bounded. That is, the set* $\mathcal{V} := \{(u,v) | A^t u + v = c, v \geq 0\}$ *is nonempty and bounded.*

We note immediately that the following is a trivial consequence of the above assumption.

$$\mathcal{S} := \{(u,v) | A^t u + v = 0, v \geq 0, (u,v) \neq 0\} = \emptyset \tag{11}$$

By using a theorem of the alternative [Mangasarian, 1981, Theorem 1], we have that the following is implied by (11) and hence is a consequence of Assumption 1.

Lemma 2 *Suppose Assumption 1 holds. Then*

1. *The matrix A has full row rank.*
2. *The set $\mathcal{X} := \{x | Ax = b, x > 0\} \neq \emptyset$.*

The primal and dual least 2-norm formulations for the linear program (9) are

$$\min_{x} cx + \frac{\epsilon}{2} xx \ s.t. \ Ax = b, \ x \geq 0 \tag{12}$$

and

$$\min_{u,v} \frac{1}{2} \left\| A^t u + v - c \right\|^2 - \epsilon bu \ \ s.t. \ v \geq 0 \tag{13}$$

respectively, for some $\epsilon > 0$.

If $x(\epsilon)$ solves the primal problem (12) and $(u(\epsilon), v(\epsilon))$ solves the dual problem (13), then the following relation holds

$$x(\epsilon) = \frac{1}{\epsilon} \left(A^t u(\epsilon) - v(\epsilon) - c \right)$$

To get the solution of problem (13) by an the interior penalty method, one minimizes a sequence of the unconstrained subproblems

$$\min_{u,v} \frac{1}{2} \left\| A^t u + v - c \right\|^2 - \epsilon bu - \gamma^i \sum_{j=1}^{n} \log v_j \tag{14}$$

where $\{\gamma^i\}$ is a sequence of decreasing positive parameters. However, for the algorithm that we are proposing here, subproblem (14) is not solved exactly. For each penalty parameter γ^i, one Newton step is taken.

Define the function $F(u, v)$ as follows

$$F(u, v) := \frac{1}{2} \left\| A^t u + v - c \right\|^2 - \epsilon bu - \gamma^i \sum_{j=1}^{n} \log v_j$$

then its gradient and Hessian are

$$\nabla F(u, v) = \begin{pmatrix} \nabla_u F(u, v) \\ \nabla_v F(u, v) \end{pmatrix} = \begin{pmatrix} A\left(A^t u + v - c\right) - \epsilon b \\ A^t u + v - c - \gamma^i V^{-1} e \end{pmatrix}$$

$$\nabla^2 F(u, v) = \begin{pmatrix} AA^t & A \\ A^t & I + \gamma^i V^{-2} \end{pmatrix}$$

where $V := diag(v)$.

The Newton direction can then be obtained by solving the linear system

$$\nabla^2 F(u^i, v^i) \begin{pmatrix} u - u^i \\ v - v^i \end{pmatrix} + \nabla F(u^i, v^i) = 0$$

for u and v.

Since it is not known a priori, how small ϵ needs be in order that a solution of (13) yield the least 2-norm solution of the linear program (9), we start the algorithm with an arbitrary $\epsilon^0 > 0$ and decrease its value as we iterate. We now state the complete algorithm.

Algorithm IDLN

- Initialization

  1. Choose any $u^0 \in \mathbb{R}^m$, $v^0 \in \mathbb{R}^n_+$. Set $i = 0$
  2. Choose $\gamma^0 > \gamma_{min} > 0$ and $\epsilon^0 > \epsilon_{min} > 0$ and $0 < \alpha, \rho < 1$. (α, ρ are attenuation factors for γ and ϵ)

- Iteration

  1. Solve the linear system

$$\nabla^2 F(u^i, v^i) \begin{pmatrix} u - u^i \\ v - v^i \end{pmatrix} + \nabla F(u^i, v^i) = 0 \tag{15}$$

  Let $(\overline{u}^i, \overline{v}^i)$ be the solution of the above linear system.

  2. Update

$$\begin{aligned} x^{i+1} &:= \frac{1}{\epsilon^i}\left(A^t \overline{u}^i + \overline{v}^i - c\right) \\ u^{i+1} &:= \overline{u}^i \end{aligned} \tag{16}$$

  3. Compute stepsize λ

$$\lambda := \begin{cases} 1 & \text{if } \ \overline{v}^i \geq 0 \\ \min_{j \in J}\left(\frac{v^i_j}{v^i_j - \overline{v}^i_j}\right) & \text{otherwise} \end{cases} \tag{17}$$

  where $J := \{j | v^i_j - \overline{v}^i_j > 0\}$

  4. Update

$$v^{i+1} := v^i + 0.98\lambda\left(\overline{v}^i - v^i\right)$$

- Termination
  If $(x^{i+1}, u^{i+1}, v^{i+1})$ is feasible to the primal programs (9) and its dual and $|cx^{i+1} - bu^{i+1}|$ is sufficiently small, then stop
  Else

  1. Set $i := i + 1$
  2. if $\gamma^i > \gamma_{min}$ then $\gamma^{i+1} = \alpha\gamma^i$
     if $\epsilon^i > \epsilon_{min}$ then $\epsilon^{i+1} = \rho\epsilon^i$
  3. Go to Iteration

Remark 3 *Choosing an interior point to start this algorithm is trivial, since the dual problem (13) has only nonnegativity constraints. This is the main advantage of this algorithm over the primal algorithm implemented by Gill et al [1986], the dual affine algorithm implemented by Monma and Morton [1987] or the primal-dual affine algorithm implemented by McShane et al [1988] and Lustig [1988] where a Phase I is needed to start the algorithms.*

Remark 4 *The solution of the $m+n$ linear system (15) in the $m+n$ variables (u, v) can be achieved by first solving the m linear equations in m unknowns*

$$A\left[I - \left(I + \gamma(V^i)^{-2}\right)^{-1}\right] A^t \left(u - u^i\right) =$$
$$A\left(I + \gamma(V^i)^{-2}\right)^{-1} \nabla_v F(u^i, v^i) - \nabla_u F(u^i, v^i) \qquad (18)$$

for u and then computing

$$v - v^i = -\left(I + \gamma(V^i)^{-2}\right)^{-1} \left(\nabla_v F(u^i, v^i) + A^t \left(u - u^i\right)\right)$$

The Yale Sparse Matrix Package [S. C. Eisenstat et al, 1977 & 1982] was used to solve the system of linear equations (18) for all the numerical results reported in this paper.

Remark 5 *By using $(\overline{u}^i, \overline{v}^i)$ as opposed to (u^{i+1}, v^{i+1}) in computing x^{i+1}, we guarantee that the sequence $\{x^i\}$ is such that $Ax^i = b$, except for possibly x^0.*

3 Convergence of IDLN

The logarithmic penalty minimization problem associated with the dual problem (13) with penalty parameters $\epsilon^i > 0$ and $\gamma^i > 0$ that we are considering is

$$\min_{u,v} F(u,v) := \frac{1}{2}\left\|A^t u + v - c\right\|^2 - \epsilon^i bu - \gamma^i \sum_{j=1}^{n} \log v_j \tag{19}$$

Note that ϵ^i is an exterior penalty parameter for the dual linear program (10) and γ^i is an interior penalty parameter for (10). However, we note that ϵ^i need not go to zero [Mangasarian & Meyer, 1979].

The optimality condition for the above unconstrained problem is

$$\begin{aligned} A(A^t u + v - c) - \epsilon^i b &= 0 \qquad (20) \\ \gamma^i e - V(A^t u + v - c) &= 0 \qquad (21) \end{aligned}$$

where $V := diag(v)$. The Newton direction can then be obtained by solving linear system

$$\begin{pmatrix} AA^t & A \\ A^t & I \end{pmatrix}\begin{pmatrix} u^i \\ v^i \end{pmatrix} + \begin{pmatrix} -\epsilon b - Ac \\ -c \end{pmatrix} - \begin{pmatrix} 0 \\ \gamma^i (V^i)^{-1} e \end{pmatrix} +$$

$$\begin{pmatrix} AA^t & A \\ A^t & I + \gamma^i (V^i)^{-2} \end{pmatrix}\begin{pmatrix} u - u^i \\ v - v^i \end{pmatrix} = 0 \tag{22}$$

or equivalently

$$\begin{aligned} A(A^t u + v - c) - \epsilon^i b &= 0 \qquad (23) \\ A^t u + v - c - \gamma^i (V^i)^{-1} e + \gamma^i (V^i)^{-2}(v - v^i) &= 0 \qquad (24) \end{aligned}$$

where $u^i \in \mathbb{R}^m$ and $v^i \in \mathbb{R}^n_{++}$.
We denote the solution of the above system of linear equation by (u^{i+1}, v^{i+1}).

Define the descent directions

$$\begin{aligned} y^i &= u^{i+1} - u^i \\ z^i &= v^{i+1} - v^i \end{aligned}$$

and let

$$d^i = (V^i)^{-1}(v^{i+1} - v^i)$$

Premultiplying equation (24) by V^i gives

$$\begin{aligned} V^i(A^t u^{i+1} + v^{i+1} - c) &= \gamma^i e - \gamma^i (V^i)^{-1}(v^{i+1} - v^i) \\ &= \gamma^i (e - d^i) \end{aligned} \tag{25}$$

Premultiplying the Newton equation (22) by the diagonal matrix

$$\begin{pmatrix} I & 0 \\ 0 & V^i \end{pmatrix}$$

gives the following equation

$$\begin{pmatrix} I & 0 \\ 0 & V^i \end{pmatrix} \begin{pmatrix} A(A^t u^i + v^i - c) - \epsilon^i b \\ A^t u^i + v^i - c - \gamma (V^i)^{-1} e \end{pmatrix} +$$

$$\begin{pmatrix} I & 0 \\ 0 & V^i \end{pmatrix} \begin{pmatrix} AA^t & A \\ A^t & I + \gamma (V^i)^{-2} \end{pmatrix} \begin{pmatrix} u^{i+1} - u^i \\ v^{i+1} - v^i \end{pmatrix} = 0$$

which is equivalent to

$$\begin{pmatrix} A(A^t u^i + v^i - c) - \epsilon^i b \\ V^i (A^t u^i + v^i - c) - \gamma^i e \end{pmatrix} + \begin{pmatrix} AA^t & AV^i \\ V^i A^t & \gamma^i I + (V^i)^2 \end{pmatrix} \begin{pmatrix} y^i \\ d^i \end{pmatrix} = 0 \tag{26}$$

Define the matrix M^i

$$M^i := \begin{pmatrix} AA^t & AV^i \\ V^i A^t & \gamma^i I + (V^i)^2 \end{pmatrix} \tag{27}$$

and the residual vector (p^i, r^i)

$$\begin{pmatrix} p^i \\ r^i \end{pmatrix} := \begin{pmatrix} \epsilon^i b - A(A^t u^i + v^i - c) \\ \gamma^i e - V^i (A^t u^i + v^i - c) \end{pmatrix} \tag{28}$$

Premultiplying equation (26) by (y^i, d^i) gives

$$\left\langle \begin{pmatrix} y^i \\ d^i \end{pmatrix}, M^i \begin{pmatrix} y^i \\ d^i \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} p^i \\ r^i \end{pmatrix}, (M^i)^{-1} \begin{pmatrix} p^i \\ r^i \end{pmatrix} \right\rangle \tag{29}$$

The basic idea for the proof is as follows. Suppose that the residual vectors p^i and r^i are bounded at iteration i, then the Newton solution (u^{i+1}, v^{i+1}) and the vector $x^{i+1} := \frac{1}{\epsilon^i}(A^t u^{i+1} + v^{i+1} - c)$ are shown to be primal-dual feasible. Moreover, by careful updating of the parameters ϵ and γ, the boundedness of the residual vectors p^{i+1} and r^{i+1} are guaranteed. This proof is based on the convergence proof given by [Tseng, 1989] for the solution of a convex quadratic problem using the logarithmic penalty method. The linear convergence of the algorithm is also established using results given in [Mangasarian & DeLeone, 1988]. We note that Tseng has also established the linear convergence for this algorithm [Tseng, 1990].

We begin by stating the following lemmas regarding matrix M^i.

Lemma 6 *Let M be a symmetric real $n \times n$ matrix such that for all $x \in R^n$, $\langle x, Mx \rangle \geq \gamma \|x\|^2$ for some $\gamma > 0$, then $\langle x, M^{-1}x \rangle \leq \frac{1}{\gamma}\|x\|^2$ for all $x \in R^n$.*

Lemma 7 *Let $N = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$ be a symmetric invertible matrix. If A^{-1} and $(C - B^t A^{-1} B)^{-1}$ exist, then*

$$N^{-1} := \begin{pmatrix} A^{-1} + A^{-1}B(C - B^tA^{-1}B)^{-1}B^tA^{-1} & -A^{-1}B(C - B^tA^{-1}B)^{-1} \\ -(C - B^tA^{-1}B)^{-1}B^tA^{-1} & (C - B^tA^{-1}B)^{-1} \end{pmatrix}$$

Lemma 8 *Let*

$$M := \begin{pmatrix} AA^t & AV \\ VA^t & \gamma I + V^2 \end{pmatrix}$$

where A is an $m \times n$ real matrix with independent rows, V is an $n \times n$ positive diagonal matrix and $\gamma > 0$, then for all $(u, v) \in R^{m+n}$

1.

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, M \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle \geq \gamma \|v\|^2 \tag{30}$$

2.

$$\left\langle \begin{pmatrix} 0 \\ v \end{pmatrix}, M^{-1} \begin{pmatrix} 0 \\ v \end{pmatrix} \right\rangle \leq \frac{1}{\gamma} \|v\|^2 \tag{31}$$

Proof

1.

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} AA^t & AV \\ VA^t & \gamma I + V^2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle = \left\|A^t u + Vv\right\|^2 + \gamma \|v\|^2 \geq \gamma \|v\|^2$$

2. By Lemma 7 we have that

$$\left\langle \begin{pmatrix} 0 \\ v \end{pmatrix}, M^{-1} \begin{pmatrix} 0 \\ v \end{pmatrix} \right\rangle = \langle v, N^{-1}v \rangle$$

where

$$N^{-1} = \left(\gamma I + V^2 - VA^t(AA^t)^{-1}AV\right)^{-1}$$

or

$$N = \gamma I + V\left(I - A^t(AA^t)^{-1}A\right)V$$

Define $P := (I - A^t(AA^t)^{-1}A)$, then $P = P^2$ and we have the following

$$\langle v, Nv \rangle = \gamma \|v\|^2 + \|PVv\|^2 \geq \gamma \|v\|^2$$

Hence from Lemma 6 it follows that $\langle v, N^{-1}v \rangle \leq \frac{1}{\gamma} \|v\|^2$. □

In a similar fashion to [Kojima et al, 1989] we define the error function $E_\gamma : \mathbb{R}^m \times \mathbb{R}^n_{++} \to \mathbb{R}$ as

$$E_\gamma(u, v) := \left\| \gamma e - V(A^t u + v - c) \right\| \tag{32}$$

to measure the error in satisfying the optimality condition (21) by the solution of the Newton Equation (22). It is clear that $E_{\gamma^i}(u^{i+1}, v^{i+1}) = 0$ and $\epsilon^i b - A\left(A^t u^{i+1} + v^{i+1} - c\right) = 0$ if and only if (u^{i+1}, v^{i+1}) solves problem (19).

The next lemma gives bound to the error function E_{γ^i} at (u^{i+1}, v^{i+1}).

Lemma 9 *Define the residual vectors p^i and r^i*

$$\begin{aligned} p^i &:= \epsilon^i b - A(A^t u^i + v^i - c) \\ r^i &:= \gamma^i e - V^i(A^t u^i + v^i - c) \end{aligned}$$

and the matrices

$$\begin{aligned} P &= I - A^t(AA^t)^{-1}A \\ E^i &:= \left(I + (AA^t)^{-1}AV^i(\gamma^i I + V^i P V^i)^{-1} V^i A^t\right)(AA^t)^{-1} \\ F^i &:= -(AA^t)^{-1}AV^i(\gamma^i I + V^i P V^i)^{-1} \end{aligned}$$

and the variable η^i

$$\eta^i := \left\| E^i \right\| \left\| p^i \right\|^2 + 2 \left\| F^i \right\| \left\| p^i \right\| \left\| r^i \right\| \tag{33}$$

Let (u^{i+1}, v^{i+1}) *be the solution of the Newton Equation (22), then*

$$E_{\gamma^i}(u^{i+1}, v^{i+1}) \leq \eta^i + \left\langle r^i, \left(\gamma^i I + V^i P V^i\right)^{-1} r^i \right\rangle$$

Proof
Recall that

$$\begin{aligned} v^{i+1} &= v^i + z^i \\ &= v^i + V^i d^i \end{aligned}$$

Since $V^{i+1} = diag(v^{i+1})$, then $V^{i+1} = V^i + V^i D^i$.
We have the following

$$\begin{aligned}
& E_{\gamma^i}(u^{i+1}, v^{i+1}) \\
= & \left\| \gamma^i e - V^{i+1}(A^t u^{i+1} + v^{i+1} - c) \right\|_2 \\
= & \left\| \gamma^i e - (V^i + V^i D^i)(A^t u^{i+1} + v^{i+1} - c) \right\|_2 \\
= & \left\| \gamma^i e - V^i(A^t u^{i+1} + v^{i+1} - c) - D^i V^i (A^t u^{i+1} + v^{i+1} - c) \right\|_2 \\
= & \left\| \gamma^i e - \gamma^i e + \gamma^i d^i - D^i(\gamma^i e - \gamma^i d^i) \right\|_2 \quad (By Eqn. (25)) \\
= & \gamma^i \left\| D^i d^i \right\|_2 \\
\leq & \gamma^i \left\| D^i d^i \right\|_1 \\
= & \gamma^i \left\| d^i \right\|_2^2 & (34) \\
\leq & \left\langle \begin{pmatrix} y^i \\ d^i \end{pmatrix}, M^i \begin{pmatrix} y^i \\ d^i \end{pmatrix} \right\rangle \quad (by \;\; Lemma \; 8) \\
= & \left\langle \begin{pmatrix} p^i \\ r^i \end{pmatrix}, (M^i)^{-1} \begin{pmatrix} p^i \\ r^i \end{pmatrix} \right\rangle \quad (By \; Eqn. \; (29)) \\
= & \left\langle p^i, E^i p^i \right\rangle + 2 \left\langle p^i, F^i r^i \right\rangle + \left\langle r^i, (\gamma^i I + V^i P V^i)^{-1} r^i \right\rangle \quad (By \; Lemma \;\; 7) \\
\leq & \eta^i + \left\langle r^i, (\gamma^i I + V^i P V^i)^{-1} r^i \right\rangle & (35)
\end{aligned}$$

This completes the proof. □

For the next iteration, define the penalty parameter

$$\gamma^{i+1} = \alpha\gamma^i \tag{36}$$

where

$$\alpha := \frac{0.375 + \sqrt{n}}{0.5 + \sqrt{n}}$$

We are now ready to state the following important lemma.

Lemma 10 *Let γ^{i+1} be defined as in (36) and $V^i := diag(v^i)$ where $v^i \in \mathbb{R}^n_{++}$. Define the matrix M^{i+1} and the vector r^{i+1} as follows*

$$M^{i+1} := \begin{pmatrix} AA^t & AV^{i+1} \\ V^{i+1}A^t & \gamma^{i+1}I + (V^{i+1})^2 \end{pmatrix} \tag{37}$$

$$r^{i+1} := \gamma^{i+1}e - V^{i+1}(A^t u^{i+1} + v^{i+1} - c)$$

where (u^{i+1}, v^{i+1}) is the solution of the Newton equation (22). Suppose that

$$\left\langle r^i, \left(\gamma^i I + V^i P V^i\right)^{-1} r^i \right\rangle \leq 0.25\gamma^i \tag{38}$$

and that η^i as defined in (33) satisfies the following

$$\eta^i \leq 0.125\gamma^i \tag{39}$$

then

1. *The point (u^{i+1}, v^{i+1}) is feasible for the dual problem (13) and $x^{i+1} := \frac{1}{\epsilon^i}(A^t u^{i+1} + v^{i+1} - c)$ is feasible for the primal problem (12) with $\epsilon = \epsilon^i$ and the following holds*

$$\epsilon^i x_j^{i+1} v_j^{i+1} \leq \gamma^i \ \ \forall\, j = 1, 2, \ldots n \tag{40}$$

2. *The vector r^{i+1} is bounded as follows*

$$\left\langle r^{i+1}, \left(\gamma^{i+1} I + V^{i+1} P V^{i+1}\right)^{-1} r^{i+1} \right\rangle \leq 0.25\gamma^{i+1}$$

Proof

1. Let

$$\begin{aligned} y^i &= u^{i+1} - u^i \text{ and} \\ d^i &= (V^i)^{-1}(v^{i+1} - v^i) \end{aligned}$$

We will first show that under the above conditions $\|d^i\| < 1$. By lines (34) and (35) of proof of Lemma 9 we have

$$\begin{aligned} \left\|d^i\right\|^2 &\leq \frac{1}{\gamma^i}\left(\eta^i + \left\langle r^i, \left(\gamma^i I + V^i P V^i\right)^{-1} r^i\right\rangle\right) \\ &< 1 \end{aligned}$$

The fact that $\|d^i\| < 1$ and $v^i > 0$ imply that $v^{i+1} > 0$, hence the dual feasibility of (u^{i+1}, v^{i+1}).

From equation (25) and the definition of x^{i+1} we have

$$\begin{aligned} \epsilon^i x^{i+1} &= A^t u^{i+1} + v^{i+1} - c \\ &= \gamma^i (V^i)^{-1}(e - d^i) > 0 \end{aligned}$$

The equality constraint $Ax^{i+1} = b$ follows from the definition of x^{i+1} and the Newton equation (23). To establish relation (40), note that from equation (25), we have

$$\begin{aligned} \epsilon^i x^{i+1} &= \gamma^i (V^i)^{-1}(e - d^i) \\ &= \gamma^i (V^{i+1})^{-1}(D^i + I)(e - d^i) \\ &= \gamma^i (V^{i+1})^{-1}(e - D^i d^i) \\ &\leq \gamma^i (V^{i+1})^{-1} e \end{aligned}$$

Upon premultiplying the last relation by V^{i+1} we get $\epsilon^i x_j^{i+1} v_j^{i+1} \leq \gamma^i \ \forall j = 1, 2, \ldots n$.

2. Now the proof of the second part of the lemma

$$\left[\frac{1}{\gamma^{i+1}}\left\langle r^{i+1}, \left(\gamma^{i+1} I + V^{i+1} P V^{i+1}\right)^{-1} r^{i+1}\right\rangle\right]^{\frac{1}{2}}$$

$$\begin{aligned}
&= \left[\frac{1}{\gamma^{i+1}}\left\langle\begin{pmatrix}0\\ r^{i+1}\end{pmatrix}, (M^{i+1})^{-1}\begin{pmatrix}0\\ r^{i+1}\end{pmatrix}\right\rangle\right]^{\frac{1}{2}}\\
&\leq \frac{1}{\gamma^{i+1}}\left\|r^{i+1}\right\| \quad (By\ \ Lemma\ 8)\\
&= \frac{1}{\gamma^{i+1}}\left\|\gamma^{i+1}e - V^{i+1}(A^t u^{i+1} + v^{i+1} - c)\right\|\\
&= \frac{1}{\alpha\gamma^{i}}\left\|\alpha\gamma^{i} - V^{i+1}(A^t u^{i+1} + v^{i+1} - c)\right\| \quad (Definition\ of\ \gamma^{i+1})\\
&\leq \frac{1}{\alpha\gamma^{i}}\left(E_{\gamma^i}(u^{i+i}, v^{i+1}) + (1-\alpha)\gamma^i\left\|e\right\|\right)\\
&\leq \frac{1}{\alpha\gamma^{i}}\left(\eta^i + \left\langle r^i, \left(\gamma^i I + V^i P V^i\right)^{-1} r^i\right\rangle\right) + \frac{1-\alpha}{\alpha}\left\|e\right\| \quad (By Lemma\ 9)\\
&\leq \frac{1}{\alpha}\left(0.125 + 0.25 + \sqrt{n}\right) - \sqrt{n}\\
&\leq 0.5
\end{aligned}$$

and hence the proof of the lemma. □.

The next 2 lemmas establish the boundedness of u^{i+1}, v^{i+1} and x^{i+1} under Assumptions 1. We will show that if the conditions (38) and (39) of Lemma 10 are satisfied, then x^{i+1} is bounded. The proof is similar that of [Polyak, 1987] for the gradient projection algorithm. The boundedness of x^{i+1} together with the assumption that the dual feasible set is bounded establish the boundedness of (u^{i+1}, v^{i+1}).

Lemma 11 *Suppose that the conditions (38) and (39) of Lemma 10 are satisfied by* $(u^i, v^i) \in \mathbb{R}^m \times \mathbb{R}^n_{++}$. *Let* (u^{i+1}, v^{i+1}) *be the solution of the Newton equation (22) and let* x^* *be a solution of the linear program (9). If the parameters* γ^i *and* ϵ^i *are such that* $\gamma^i \leq \sqrt{\epsilon^i}$, *then*

$$\left\|x^{i+1} - x^*\right\|^2 \leq 2n + \left\|x^*\right\|^2 \tag{41}$$

where $x^{i+1} = \frac{1}{\epsilon^i}\left(A^t u^{i+1} + v^{i+1} - c\right)$.

Proof
From the Newton equation (23) we have

$$A(A^t u^{i+1} + v^{i+1} - c) = \epsilon^i b$$

which gives

$$u^{i+1} = (AA^t)^{-1}\left(\epsilon^i b - A(v^{i+1} - c)\right)$$

hence

$$\begin{aligned} x^{i+1} &= \left(I - A^t(AA^t)^{-1}A\right)\left(\frac{1}{\epsilon^i}(v^{i+1} - c)\right) + A^t(AA^t)^{-1}b \\ &= P_Q\left(\frac{1}{\epsilon^i}(v^{i+1} - c)\right) \end{aligned} \tag{42}$$

where $P_Q(x)$ is the projection of x onto the set $Q := \{z | Az = b\}$. By the Minimum Principle applied to the above projection problem (42),

$$0 \geq \left\langle \frac{1}{\epsilon^i}(v^{i+1} - c) - x^{i+1}, x^* - x^{i+1} \right\rangle \tag{43}$$

or equivalently

$$\begin{aligned} 0 &\geq \left\langle v^{i+1} - c - \epsilon^i x^{i+1}, x^* - x^{i+1} \right\rangle \\ &= -\left\langle c, x^* - x^{i+1} \right\rangle + \left\langle v^{i+1}, x^* \right\rangle - \left\langle v^{i+1}, x^{i+1} \right\rangle - \epsilon^i \left\langle x^{i+1}, x^* - x^{i+1} \right\rangle \\ &\geq \epsilon^i \left\langle -x^{i+1}, x^* - x^{i+1} \right\rangle - n\frac{\gamma^i}{\epsilon^i} \\ &= \frac{1}{2}\epsilon^i \left(\left\| x^{i+1} \right\|^2 + \left\| x^{i+1} - x^* \right\|^2 - \| x^* \|^2 \right) - n\frac{\gamma^i}{\epsilon^i} \end{aligned} \tag{44}$$

The second inequality follows from the fact that $cx^* \leq cx^{i+1}, \langle v^{i+1}, x^* \rangle \geq 0$ and $\epsilon^i x_j^{i+1} v_j^{i+1} \leq \gamma^i \ \forall j = 1, 2, \ldots n$. Rearranging the terms in (44) and multiplying by $2/\epsilon^i$ gives

$$\begin{aligned} \left\| x^{i+1} - x^* \right\|^2 &\leq 2n\frac{\gamma^i}{(\epsilon^i)^2} + \| x^* \|^2 - \left\| x^{i+1} \right\|^2 \\ &\leq 2n + \| x^* \|^2 \end{aligned}$$

Hence the proof is complete. □

In the next lemma, we establish the boundedness of (u^{i+1}, v^{i+1}).

Lemma 12 *Suppose that the point $(u^i, v^i) \in \mathbb{R}^m \times \mathbb{R}^n_{++}$ satisfies the conditions (38) and (39) of Lemma 10. Let (u^{i+1}, v^{i+1}) be the solution of the Newton equation (22). Furthermore, suppose that the set $\mathcal{V} := \{(u, v) | A^t u + v =$*

$c, v \geq 0\}$ is bounded. Then there exists a constant $\tau < \infty$ depending only on the matrix A and the vectors b and c of the linear program (9) such that

$$\left\| u^{i+1}, v^{i+1} \right\| \leq \tau \tag{45}$$

Proof

Define the set $\mathcal{V}^i$ as follows

$$\mathcal{V}^i := \{(u, v) | A^t u + v = c - \epsilon^i x^{i+1}\} \tag{46}$$

Note that $\mathcal{V}^i$ is nonempty by the construction of $x^{i+1} = \frac{1}{\epsilon^i}(A^t u^{i+1} + v^{i+1} - c)$. We claim that the set $\mathcal{V}^i$ is bounded. In the previous lemma it was shown that x^{i+1} is bounded, hence if the set $\mathcal{V}^i$ is unbounded, then there exists $(\overline{u}, \overline{v})$ such that

$$\begin{aligned} A^t \overline{u} + \overline{v} &= 0 \\ \overline{v} &\geq 0 \\ (\overline{u}, \overline{v}) &\neq 0 \end{aligned}$$

Then for any point $(w, z) \in \mathcal{V}$, we have that $(w + \lambda \overline{u}, z + \lambda \overline{v}) \in \mathcal{V}$ for any $\lambda \geq 0$, which contradicts the assumption that the set $\mathcal{V}$ is bounded. Hence $\mathcal{V}^i$ is bounded.

Consider now the following nonconvex problem

$$\max_{u,v} \|(u, v)\| \quad \text{s.t. } A^t u + v = c + \epsilon^i x^{i+1}, \ v \geq 0 \tag{47}$$

This problem has a solution, since we have just shown that its feasible set is bounded. By the generalized theorem of the existence of basic feasible solution [Mangasarian & T.H. Shiau, 1987], it follows that there must exist a basic solution. Let the basis matrix B^i denote the n by n nonsingular submatrix of $[A^t \quad I]$ corresponding to the basic solution $(\tilde{u}, \tilde{v})$ of problem (47). We have

$$\begin{aligned} \|\tilde{u}, \tilde{v}\| &= \left\| (B^i)^{-1} (c + \epsilon^i x^{i+1}) \right\| \\ &\leq \left\| (B^i)^{-1} \right\| \left\| c + \epsilon^i x^{i+1} \right\| \end{aligned}$$

Since there are only finite number of basis matrices in $[A^t \quad I]$, and since both ϵ^i and x^{i+1} are bounded, we conclude that there must exist $\tau < \infty$ such that

$$\left\| u^{i+1}, v^{i+1} \right\| \leq \|\tilde{u}, \tilde{v}\| \leq \tau$$

and this completes the proof. □

From the above lemma, we have that both $\max_V \|AV\|$ and $\max_V \|VA^t\|$ s.t. $v \in \mathcal{V}^i$ and $V := diag(v)$ are finite, where $\mathcal{V}^i$ is the set defined by (46).

The next lemma shows that if the attenuation factor $\rho \in (0,1)$ for decreasing ϵ^i is chosen carefully then the assumption (39) of Lemma 10 holds at iteration $i+1$.

Lemma 13 *Let (u^{i+1}, v^{i+1}) be the solution of the Newton equation (22) and $x^{i+1} = \frac{1}{\epsilon^{i+1}}(A^t u^{i+1} + v^{i+1} - c)$. Suppose that $(u^i, v^i) \in \mathrm{IR}^m \times \mathrm{IR}^n_{++}$ satisfy the conditions (38) and (39) of Lemma 10 and that the sequence $\{\gamma^k\}$ and $\{\epsilon^k\}$ are such that*

$$\begin{aligned} 0 &< \{\gamma^k\} \leq \gamma_{max} \quad (48) \\ \text{and } 0 &< \{\epsilon^k\} \leq \epsilon_{max} \quad (49) \end{aligned}$$

Define the constants

$$\begin{aligned} K_1 &= \left\|(AA^t)^{-1}\right\| \\ K_2^i &= \max\{\max_V \|AV\|, \max_V \left\|VA^t\right\|\} \ \ s.t.\ A^t u + v = c + \epsilon^i x^{i+1}, v \geq 0 \\ C_1^i &= (\gamma_{max} + K_1(K_2^i)^2)K_1\epsilon_{max}^2 \|b\|^2 / \gamma^{i+1} \\ C_2^i &= (4/\alpha)\epsilon_{max}\sqrt{n}\,\|b\|\,K_1 K_2^i \end{aligned}$$

where $\alpha = (0.375 + \sqrt{n})/(0.5 + \sqrt{n})$ and $\gamma^{i+1} = \alpha\gamma^i$. If

$$\epsilon^{i+1} = \rho^i \epsilon^i$$

where

$$1 > \rho^i \geq 1 - \delta^i \quad (50)$$

and

$$0 < \delta^i \leq \left(-C_2^i + \sqrt{(C_2^i)^2 + 0.5\gamma^{i+1}C_1^i}\right)/2C_1^i \quad (51)$$

then we have

$$\eta^{i+1} \leq 0.125\gamma^{i+1}$$

where

$$\eta^{i+1} := \left\|E^{i+1}\right\| \left\|p^{i+1}\right\|^2 + 2\left\|F^{i+1}\right\| \left\|p^{i+1}\right\| \left\|r^{i+1}\right\|$$

$$
\begin{aligned}
p^{i+1} &:= \epsilon^{i+1} b - A(A^t u^{i+1} + v^{i+1} - c) \\
r^{i+1} &:= \gamma^{i+1} e - V^{i+1}(A^{t+1} u^{i+1} + v^{i+1} - c) \\
E^{i+1} &:= \left(I + (AA^t)^{-1} A V^{i+1} (\gamma^{i+1} I + V^{i+1} P V^{i+1})^{-1} V^{i+1} A^t\right)(AA^t)^{-1} \\
F^{i+1} &:= -(AA^t)^{-1} A V^{i+1} (\gamma^{i+1} I + V^{i+1} P V^{i+1})^{-1}
\end{aligned}
$$

Proof

We will first compute the bounds on the norms of the residual vectors p^{i+1} and r^{i+1}.

1.

$$
\begin{aligned}
\left\|p^{i+1}\right\| &= \left\|\epsilon^{i+1} b - A(A^t u^{i+1} + v^{i+1} - c)\right\| \\
&= \left\|\epsilon^{i} b - A(A^t u^{i+1} + v^{i+1} - c) + \epsilon^{i+1} b - \epsilon^{i} b\right\| \\
&= (1 - \rho^i)\epsilon^i \|b\| \quad (By \;\; Eqn.(23))
\end{aligned}
$$

2.

$$
\begin{aligned}
\left\|r^{i+1}\right\| &= \left\|\gamma^{i+1} e - V^{i+1}(A^{t+1} u^{i+1} + v^{i+1} - c)\right\| \\
&\leq \gamma^{i+1}\sqrt{n} + \left\|\epsilon^i V^{i+1} x^{i+1}\right\| \\
&\leq 2\gamma^i \sqrt{n} \quad (By \;\; Eqn.(40))
\end{aligned}
$$

Next we compute the bounds on the norm of the matrices E^{i+1} and F^{i+1}.

1.

$$
\begin{aligned}
\left\|E^{i+1}\right\| &= \left\|\left(I + (AA^t)^{-1} A V^i (\gamma^{i+1} I + V^{i+1} P V^{i+1})^{-1} V^{i+1} A^t\right)(AA^t)^{-1}\right\| \\
&\leq (1 + \frac{1}{\gamma^{i+1}} K_1 (K_2^i)^2) K_1
\end{aligned}
$$

2.

$$
\begin{aligned}
\left\|F^{i+1}\right\| &= \left\|-(AA^t)^{-1} A V^{i+1} (\gamma^{i+1} I + V^{i+1} P V^{i+1})^{-1}\right\| \\
&\leq \frac{1}{\gamma^{i+1}} K_1 K_2^i
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\eta^{i+1} &= \left\|E^{i+1}\right\|\left\|p^{i+1}\right\|^2 + 2\left\|F^{i+1}\right\|\left\|p^{i+1}\right\|\left\|r^{i+1}\right\| \\
&\leq \frac{1}{\gamma^{i+1}}(\gamma^{i+1} + K_1(K_2^i)^2)K_1(1-\rho^i)^2(\epsilon^i)^2\|b\|^2 \\
&\quad + 2\frac{1}{\gamma^{i+1}}K_1K_2^i(1-\rho^i)\epsilon^i\|b\| 2\gamma^i\sqrt{n} \\
&\leq \epsilon_{max}^2\left(\gamma_{max} + K_1(K_2^i)^2\right)K_1\|b\|^2(1-\rho^i)^2/\gamma^{i+1} \\
&\quad + (4/\alpha)\epsilon_{max}K_1K_2^i\|b\|\sqrt{n}(1-\rho^i) \\
&\leq C_1^i(1-\rho^i)^2 + C_2^i(1-\rho^i) \\
&\leq C_1^i(\delta^i)^2 + C_2^i(\delta^i) \quad (By \;\; Eqn.(50)) \\
&\leq 0.125\gamma^{i+1} \quad (By \;\; Eqn.(51))
\end{aligned}
$$

This completes the proof of the lemma. □

By using the results from the Lemmas 10, 11, 12 and 13, we can now establish the following theorem regarding the IDLN algorithm.

Theorem 14 *Let $(u^i, v^i) \in \mathbb{R}^m \times \mathbb{R}^n_{++}$ be the $i-th$ iterate of the IDLN Algorithm with parameter $\epsilon = \epsilon^i$ and $\gamma = \gamma^i$, $\gamma^i \leq \sqrt{\epsilon^i}$ such that the following two conditions are satisfied*

$$\left\langle r^i, \left(\gamma^i I + V^i P V^i\right)^{-1} r^i \right\rangle \leq 0.25\gamma^i$$

and

$$\eta^i \leq 0.125\gamma^i$$

where r^i is the residual vector defined by Eqn. (28) and η^i is the real number defined by (33) in Lemma 9. Suppose that (u^{i+1}, v^{i+1}) is the solution of the Newton equation (22). If we let

$$
\begin{aligned}
x^{i+1} &= \frac{1}{\epsilon^i}(A^t u^{i+1} + v^{i+1} - c) \\
\gamma^{i+1} &= \alpha\gamma^i \\
\epsilon^{i+1} &= \rho^i \epsilon^i
\end{aligned}
$$

where $\alpha = (0.375 + \sqrt{n})/(0.5 + \sqrt{n})$ and $\rho^i \in (0, 1)$ satisfying the condition (50) in Lemma 13, then

1. *The triple* $(x^{i+1}, u^{i+1}, v^{i+1})$ *is bounded and is feasible for the primal-dual problems (12) and (13) with* $\epsilon = \epsilon^i$ *and the following holds*

$$\epsilon^i x_j^{i+1} v_j^{i+1} \leq \gamma^i \ \ \forall j = 1, 2, \ldots n$$

2. *The following bounds are satisfied for* (u^{i+1}, v^{i+1}).

$$\eta^{i+1} \leq 0.125\gamma^{i+1}$$

and

$$\left\langle r^{i+1}, \left(\gamma^{i+1} + V^{i+1} P V^{i+1}\right)^{-1} r^{i+1} \right\rangle \leq 0.25\gamma^{i+1}$$

□

The idea for the linear convergence proof of IDLN comes from a proof given by [Mangasarian & DeLeone, 1988] for the least 2-norm solution of linear programs, in which they give error bounds for a class of more general problems. The problem they consider is

$$min_x f(x) \ \ s.t. \ x \in S := \{x | x \geq 0, g(x) \leq 0\} \tag{52}$$

We begin by restating their main result.

Theorem 15 *(Mangasarian & DeLeone, 1988, Theorem 2.2) Let* $f : \mathbb{R}^n \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}^m$ *be differentiable on* $\mathbb{R}^n$, *let* f *be strongly convex on* $\mathbb{R}^n$ *with positive constant* k, *and let* g *be convex on* $\mathbb{R}^n$. *Let either* g *be linear and* $S \neq \emptyset$, *or let* g *satisfy the Slater constraint qualification that is,*

$$g(\hat{x}) < 0, \quad \hat{x} > 0$$

for some $\hat{x} \in \mathbb{R}^n$. *Then for any* $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m_+$ *the distance* $\|x - \overline{x}\|$ *to the unique solution of (52) is bounded by*

$$k^{\frac{1}{2}} \|x - \overline{x}\| \leq$$
$$[x\nabla_x L(x,u) - ug(x) + \alpha \|(-\nabla_x L(x,u))_+\|_1 + \beta \|g(x)_+\|_\infty + \gamma \|(-x)_+\|_\infty]^{\frac{1}{2}}$$

where

$$\begin{aligned} L(x,u) &:= f(x) + ug(x) \\ \alpha &:= min_{x \in S} (\|x\|_\infty + \|\nabla f(x)\|_1 / k) \\ \beta &:= min_{(u,v) \in W} \|u\|_1 \\ \gamma &:= min_{(u,v) \in W} \|v\|_1 \end{aligned}$$

where $W \subset \mathbb{R}_+^{m+n}$ is the nonempty closed convex polyhedral set of optimal multipliers (u, v) of the convex program (52) associated with the constraints $g(x) \leq 0, x \geq 0$. □

By using the above theorem, we will show that if we impose a stronger condition on the parameter ϵ^{i+1} then the algorithm IDLN is linearly convergent.

Theorem 16 *Let $(u^{i+1}, v^{i+1}, x^{i+1}), \alpha$ and ρ be as in Theorem 14. Suppose that all the conditions in Theorem 14 are satisfied and suppose that the parameter ϵ is decreased as follows*

$$\epsilon^{i+1} = \overline{\rho}^i \epsilon^i \tag{53}$$

where $1 > \overline{\rho}^i := max\{\alpha^{\frac{1}{4}}, \rho^i\}$, ρ^i as defined by (50) and γ is decreased as follows

$$\gamma^{i+1} = \alpha\gamma^i \tag{54}$$

where $\alpha = (0.375 + \sqrt{n})/(0.5 + \sqrt{n})$, then the sequence $\{x^i\}$ converges to $\overline{x}$, the unique least 2-norm solution of (9) with the linear root rate [Ortega, 1970]

$$\left\|x^{i+1} - \overline{x}\right\| \leq \delta(\alpha^{\frac{1}{4}})^{i+1} \quad for \quad i \geq \overline{i} \tag{55}$$

for some constant δ and some integer $\overline{i}$.

Proof
Let $L(x, u) = cx + \frac{\epsilon^i}{2}xx - u^t(Ax - b)$, then

$$\nabla_x L(x^{i+1}, u^{i+1}) = c + \epsilon^i x^{i+1} - A^t u^{i+1} = v^{i+1} > 0$$

By Theorem 14, we have that

$$\begin{aligned} v^{i+1} &\geq 0 \\ x^{i+1} &\geq 0 \\ \epsilon^i x_j^{i+1} v_j^{i+1} &\leq \gamma^i \ \ \forall j = 1, 2, \ldots n \ \ and \\ Ax^{i+1} &= b \end{aligned}$$

Let $\overline{x}(\epsilon^i)$ be the solution of the quadratic problem (12) with $\epsilon = \epsilon^i$. It follows from Theorem 15 that

$$\begin{aligned}
\left\|x^{i+1} - \overline{x}(\epsilon^i)\right\| &= \frac{1}{\sqrt{\epsilon^i}}\left\langle x^{i+1}, v^{i+1}\right\rangle^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{\epsilon^i}}\left(n\gamma^i/\epsilon^i\right)^{\frac{1}{2}} \\
&= \left(n\gamma^i\right)^{\frac{1}{2}}/\epsilon^i \\
&\leq \left(n(\alpha)^i\gamma^0\right)^{\frac{1}{2}}/(\alpha^{\frac{1}{4}})^i\epsilon^0 \\
&= \delta(\alpha^{\frac{1}{4}})^{i+1}
\end{aligned}$$

where $\delta = \sqrt{n\gamma^0}/(\epsilon^0\alpha^{\frac{1}{4}})$. Now let $\overline{i}$ be the smallest integer such that $\epsilon^{\overline{i}} \leq \overline{\epsilon}$ where $\overline{\epsilon}$ is that defined below (2). Combining the last result and the fact that $\overline{x} = \overline{x}(\epsilon^i)$ for $i \geq \overline{i}$, we have

$$\begin{aligned}
\left\|x^{i+1} - \overline{x}\right\| &\leq \left\|x^{i+1} - \overline{x}(\epsilon^i)\right\| + \left\|\overline{x}(\epsilon^i) - \overline{x}\right\| \\
&= \left\|x^{i+1} - \overline{x}(\epsilon^i)\right\| \\
&\leq \delta(\alpha^{\frac{1}{4}})^{i+1}
\end{aligned}$$

This establishes the linear convergence of the iterates. □

Remark 17 *The condition $\gamma^i \leq \sqrt{\epsilon^i}$ required in Theorem 14 will be satisfied for all i if we let $\epsilon^0 = (\gamma^0)^2$ and if $\{\epsilon^i\}$ and $\{\gamma^i\}$ are decreased according to Eqns. (53) and (54).*

Remark 18 *The parameter ϵ^k in (49) need not go to zero. Let $\overline{i}$ be the smallest integer such that $\epsilon^{\overline{i}} \leq \overline{\epsilon}$ where $\overline{\epsilon}$ is defined below expression (2). If for all $k > \overline{i}$ we fix $\epsilon^k = \overline{\epsilon}$, then the linear convergence of the algorithm still holds.*

4 Numerical results

The algorithm IDLN was implemented in FORTRAN and run on a DECstation 3100 under the Ultrix 2.1 Operating System. The source code was compiled using the "-O" option. All floating point operations are done in double precision. All times reported here were obtained by calling the system subroutine *etime()*.

The initial values of ϵ and γ are

$$\epsilon^0 = \gamma^0 = 1.0d0$$

The initial value of the dual variable u is

$$u^0 = 0.0d0$$

and the initial value of the dual variable v is

$$v^0 = 6.0d0$$

(for problems *Pilot.we, Scagr25, Sc205 and Truss3*, $v^0 = 6.0d1$).

If $\epsilon^i > 1.d-13$, then

$$\epsilon^{i+1} = \frac{\epsilon^i}{4.0d0}$$

and if $\gamma^i > 1.d-18$, then

$$\gamma^{i+1} = \begin{cases} \gamma^i/1.2d0 & \text{if } \|x^{i+1} - x^i\|^2 > 1.d05 \\ \gamma^i/2.0d0 & \text{if } \|x^{i+1} - x^i\|^2 > 1.d03 \\ \gamma^i/3.0d0 & \text{if } \|x^{i+1} - x^i\|^2 > 1.d01 \\ \gamma^i/3.5d0 & \text{if } \|x^{i+1} - x^i\|^2 > 1.d-1 \\ \gamma^i/4.0d0 & \text{otherwise} \end{cases}$$

Finally, the program is terminated if one of the following conditions is satisfied

- $$\left| \frac{cx^i - cx^{i+1}}{cx^i} \right| \leq 5.d-08$$

  and

  $$\left| \frac{cx^{i+1} - bu^{i+1}}{cx^{i+1} + bu^{i+1}} \right| \leq 5.d-08$$

- $$\left|\frac{bu^i - bu^{i+1}}{bu^{i+1}}\right| \leq 5.d-08$$

$$and$$

$$\left|\frac{cx^{i+1} - bu^{i+1}}{cx^{i+1} + bu^{i+1}}\right| \leq 5.d-08$$

At the termination of the IDLN algorithm, the following iterations are executed to improve the feasibility of the primal and dual solutions. This refinement technique is essentially due to Gay [Gay, 1989].

- $it = 0$
- while $it \leq itmax$ do
-   - if $x_i \geq 1.d-08$ then $D_{ii} = x_i^2$
      else $D_{ii} = 1.d-12$.
    - update $x = DA^t(ADA^t)^{-1}b$
    - update $u = (ADA^t)^{-1}ADc$
    - if $\|(-x)_+\| \leq \sigma_1$ and $\|Ax - b\| \,/\, \|b\| \leq \sigma_2$ and $\|(A^tu - c)_+\| \leq \sigma_3$
      then stop
      else $it = it + 1$

We tested the algorithm on 66 linear test problems, 63 of which are from the Netlib collection. The dimension of these 66 problems are given in Tables 1 and 2. In columns 3, 4 and 5 of these tables we list the number of rows (including the objective row), columns and nonzeros of matrix A of the linear program in its original MPS format. The next 3 columns show the size of the linear programs after the data is preprocessed so that these linear programs can be written in standard format (9).

The algorithm was implemented using FORTRAN 77 and run on a DECstation 3100. For comparison purpose, we solved these problems using MINOS 5.3 [B.A. Murtagh & M.A. Saunders, 1983] which is a linear programming package based on the simplex method. MINOS was run using the default parameter setting. The results that we obtained on the 66 test problems are listed in Tables 3-6.

In Tables 3 and 4 we list

$$\begin{aligned}
Relative\ \ Error &:= \left|\frac{bu - cx^*}{cx^*}\right| \\
Primal\ Infeasibility &= \frac{\|Ax - b\|}{\|b\|} \\
Dual\ \ Infeasibility &= \frac{\|(A^t u - c)_+\|}{\|(-c)_+\| + 1.0} \\
Duality\ \ Gap &= \left|\frac{cx - bu}{cx + bu}\right| \\
Complementarity &= \frac{\|X(c - A^t u)\|}{\|x\|\,\|u\|}
\end{aligned}$$

where cx^* is the optimal objective value reported by MINOS and $X := diag(x)$.

A relative error in the objective value that is less than 1.d-14 is listed as 0.00E+00.

We note that for most problems IDLN solutions have better primal feasibility than the solutions obtained by IPP Algorithm described in [Setiono, 1990]. In the primal algorithm, a Newton direction is computed in the primal space, i.e. the descent direction p is such that $Ap = 0$ and the primal variable is updated $x^{i+1} = x^i + \alpha p$. As i increases, the error $\|Ax^i - b\|$ accumulates and this will lead to a deterioration in the feasibility of the primal solution. In contrast, by taking the Newton step in the dual space, the primal feasibility $Ax^i = b$ depends only on the accuracy of the current Newton step.

On these 66 linear programs, we obtained the following results. On 10 problems the relative error of the objective value is greater than 5.d-10, on 4 problems the relative primal feasibility is greater that 5.d-10. On 19 problems the relative dual feasibility is greater that 5.d-10 and on 6 problems the the duality gap is greater than 5.d-10. On all problems the complementarity is less than 5.d-10. IDLN solved 28 of the 66 problems faster than MINOS 5.3. The violation in the nonnegativity constraint of the primal variable, $\|(-x)_+\|_\infty$ is less than 5.d-8 for all problems, except for one (Bnl2). The total time taken by IDLN to solve all the problems is 7391 seconds, while the total time for MINOS 5.3 to solve these problems is 12324 seconds. This gives a total time speedup of 1.67 in favor of IDLN.

5 Summary

We have described and implemented a linearly convergent algorithm for finding the least 2-norm solution of a linear program. A logarithmic penalty approach is applied to the dual reformulation of the problem to find this solution. This dual reformulation of the problem allows us to start the algorithm without a Phase I and generates primal solutions with better primal feasibility than primal interior methods.

Pr. No.	Problem Name	Original			Adjusted		
		rows	columns	nonzeros	rows	columns	nonzeros
1	25fv47	822	1571	11127	820	1876	10705
2	Adlittle	57	97	465	56	138	424
3	Afiro	28	32	88	27	51	102
4	Agg	489	163	2541	488	615	2862
5	Agg2	517	302	4515	516	758	4750
6	Agg3	517	302	4531	516	758	4756
7	Bandm	306	472	2659	305	472	2494
8	Beaconfd	174	262	3476	173	295	3408
9	Blend	75	83	521	74	114	522
10	Bnl1	644	1175	6129	642	1586	5532
11	Bnl2	2325	3489	16124	2324	4486	14996
12	Bore3d	234	315	1525	246	346	1473
13	Brandy	221	249	2150	193	303	2202
14	Capri	272	353	1786	446	641	2230
15	Cre-a	3517	4067	19054	3428	7248	18168
16	Cre-c	3069	3678	16922	2986	6411	15977
17	Czprob	930	3523	14173	1158	3562	10937
18	D2q06c	2172	5167	35674	2171	5831	33081
19	Degen2	445	534	4449	444	757	4201
20	Degen3	1504	1818	26230	1503	2604	25432
21	E226	224	282	2767	223	472	2768
22	Fffff800	525	854	6235	524	1028	6401
23	Finnis	498	614	2714	619	1141	2959
24	Gfrd-pnc	617	1092	3467	876	1420	2965
25	Grow15	301	645	5665	900	1245	6820
26	Grow22	441	946	8318	1320	1826	10012
27	Grow7	141	301	2633	420	581	3172
28	Israel	175	142	2358	174	316	2443
29	Kb2	44	41	291	52	77	331
30	Lotfi	154	308	1086	153	366	1136
31	Pilot.we	723	2789	9218	1256	3384	10255
32	Rabo	391	576	5510	317	560	5201
33	Recipe	92	180	752	211	300	903

Table 1: **LP dimensions**

Pr. No.	Problem Name	Original			Adjusted		
		rows	columns	nonzeros	rows	columns	nonzeros
34	Sc105	106	103	281	105	163	340
35	Sc205	206	203	552	205	317	665
36	Sc50a	51	48	131	50	78	160
37	Sc50b	51	48	119	50	78	148
38	Scagr25	472	500	2029	471	671	1725
39	Scagr7	130	140	553	129	185	465
40	Scfxm1	331	457	2612	330	600	2732
41	Scfxm2	661	914	5229	660	1200	5469
42	Scfxm3	991	1371	7846	990	1800	8206
43	Scorpion	389	358	1708	388	466	1534
44	Scrs8	491	1169	4029	490	1275	3288
45	Scsd1	78	760	3148	77	760	2388
46	Scsd6	148	1350	5666	147	1350	4316
47	Scsd8	398	2750	11334	397	2750	8584
48	Sctap1	301	480	2052	300	660	1872
49	Sctap2	1091	1880	8124	1090	2500	7334
50	Sctap3	1481	2480	10734	1480	3340	9734
51	Share1b	118	225	1182	117	253	1179
52	Share2b	97	79	730	96	162	777
53	Ship04l	403	2118	8450	360	2166	6380
54	Ship04s	403	1458	5810	360	1506	4400
55	Ship08l	779	4283	17085	712	4363	12882
56	Ship08s	779	2387	9501	712	2467	7194
57	Ship12l	1152	5427	21597	1042	5533	16276
58	Ship12s	1152	2763	10941	1042	2869	8284
59	Stocfor1	118	111	474	117	165	501
60	Stocfor2	2158	2031	9492	2157	3045	9357
61	Truss1	201	1602	6586	200	1602	4984
62	Truss2	501	4312	17896	500	4312	13584
63	Truss3	1001	8806	36642	1000	8806	27836
64	Vtp.base	199	203	914	347	477	1331
65	Wood1p	245	2594	70216	244	2595	70216
66	Woodw	1099	8405	37478	1098	8418	37487

Table 2: **LP dimensions (continued)**

Pr. No.	Problem Name	Rel. Error	Primal Infeasibility	Dual Infeasibility	Duality Gap	Comple-mentarity
1	25fv47	1.83E-14	1.32E-10	7.39E-11	2.69E-10	1.46E-13
2	Adlittle	2.39E-12	1.57E-16	3.73E-12	1.22E-09	5.46E-11
3	Afiro	0.00E+00	5.09E-17	6.12E-17	6.12E-17	1.52E-17
4	Agg	2.77E-14	4.68E-17	2.78E-14	1.49E-14	5.35E-19
5	Agg2	0.00E+00	1.25E-16	1.15E-15	2.25E-14	5.21E-17
6	Agg3	1.71E-11	1.65E-14	3.73E-09	2.31E-14	2.98E-14
7	BandM	0.00E+00	3.50E-15	6.64E-16	1.39E-12	2.83E-18
8	Beaconfd	0.00E+00	1.36E-14	2.13E-15	1.65E-14	5.29E-21
9	Blend	1.46E-12	1.69E-12	1.25E-11	5.49E-14	2.69E-14
10	Bnl1	1.40E-07	3.33E-12	7.48E-07	1.02E-09	4.68E-15
11	Bnl2	5.12E-10	4.15E-11	6.23E-07	5.65E-16	2.34E-18
12	Bore3d	0.00E+00	5.97E-14	3.50E-14	2.05E-14	1.06E-22
13	BrandY	0.00E+00	2.04E-14	4.48E-15	7.49E-17	1.16E-24
14	Capri	0.00E+00	1.74E-16	8.02E-14	7.25E-13	2.61E-22
15	Cre-a	1.80E-07	4.24E-12	4.16E-09	3.34E-13	4.71E-18
16	Cre-c	0.00E+00	1.43E-13	2.09E-10	1.35E-14	2.24E-18
17	CzProb	0.00E+00	1.50E-14	8.17E-11	1.41E-14	3.19E-19
18	D2q06c	2.10E-07	1.39E-10	3.60E-10	5.60E-08	9.25E-18
19	Degen2	3.99E-11	3.09E-14	8.85E-12	2.95E-13	1.03E-17
20	Degen3	9.78E-08	1.25E-09	1.28E-11	1.65E-11	5.94E-18
21	E226	0.00E+00	3.77E-13	5.84E-16	3.97E-13	5.31E-23
22	Fffff800	7.76E-09	2.04E-16	2.45E-13	2.10E-16	3.60E-25
23	Finnis	5.79E-07	6.40E-14	4.61E-06	2.17E-11	1.83E-13
24	Gfrd-Pnc	0.00E+00	1.72E-14	1.33E-10	4.72E-15	9.73E-22
25	Grow15	0.00E+00	1.08E-16	1.89E-15	0.00E+00	1.25E-18
26	Grow22	0.00E+00	1.20E-16	1.89E-15	9.26E-17	1.03E-18
27	Grow7	2.10E-14	1.12E-16	1.37E-15	2.34E-16	2.80E-18
28	Israel	4.69E-09	2.27E-16	3.12E-07	3.34E-09	1.37E-10
29	Kb2	0.00E+00	1.17E-10	1.20E-13	1.44E-13	1.37E-17
30	Lotfi	0.00E+00	3.99E-14	4.03E-15	2.45E-14	7.38E-20
31	Pilot.we	1.41E-06	8.37E-10	4.43E-14	9.48E-08	1.92E-16
32	Rabo	4.14E-09	3.89E-15	2.77E-06	8.39E-13	1.31E-14
33	Recipe	0.00E+00	8.85E-18	1.45E-16	5.13E-11	3.36E-20

Table 3: **IDLN:Interior Dual Least 2-Norm Results**

Pr. No.	Problem Name	Rel. Error	Primal Infeasibility	Dual Infeasibility	Duality Gap	Comple-mentarity
34	Sc105	0.00E+00	3.27E-14	1.61E-17	9.53E-16	1.11E-22
35	Sc205	0.00E+00	8.72E-14	4.07E-17	2.21E-14	5.89E-24
36	Sc50a	1.54E-14	4.55E-15	3.44E-17	4.62E-15	9.53E-23
37	Sc50b	0.00E+00	1.43E-15	7.34E-17	4.06E-16	2.96E-23
38	Scagr25	6.77E-14	5.52E-13	1.14E-13	8.11E-13	1.63E-14
39	Scagr7	3.09E-08	1.66E-13	3.66E-13	9.98E-12	2.71E-13
40	Scfxm1	0.00E+00	8.83E-12	1.15E-09	1.89E-12	2.32E-18
41	Scfxm2	2.74E-14	3.95E-12	6.99E-09	3.36E-11	1.69E-16
42	Scfxm3	1.82E-14	8.56E-12	2.29E-06	1.18E-11	3.69E-17
43	Scorpion	0.00E+00	2.90E-12	1.23E-11	6.46E-13	1.78E-20
44	Scrs8	0.00E+00	1.50E-12	4.03E-10	1.00E-10	2.10E-19
45	ScSd1	6.61E-12	5.85E-12	5.76E-10	2.17E-12	2.84E-13
46	ScSd6	2.12E-12	1.24E-10	2.27E-08	1.20E-11	7.94E-14
47	ScSd8	2.76E-13	3.76E-13	2.04E-09	2.44E-14	2.70E-15
48	ScTap1	0.00E+00	1.17E-13	1.01E-13	6.40E-14	1.07E-18
49	ScTap2	0.00E+00	1.74E-13	5.41E-14	9.89E-15	3.55E-18
50	ScTap3	0.00E+00	2.42E-13	4.84E-14	9.74E-15	3.04E-18
51	Share1b	0.00E+00	9.17E-14	6.48E-16	8.93E-14	3.74E-17
52	Share2b	2.39E-14	1.27E-11	1.35E-13	1.33E-12	7.56E-17
53	Ship04l	0.00E+00	3.54E-12	5.33E-12	5.24E-13	8.73E-18
54	Ship04s	0.00E+00	1.17E-12	9.10E-10	2.02E-13	1.87E-18
55	Ship08l	0.00E+00	1.76E-13	3.77E-09	2.71E-14	2.04E-17
56	Ship08s	5.20E-14	1.03E-13	9.72E-08	4.85E-15	2.41E-15
57	Ship12l	0.00E+00	2.11E-10	1.61E-09	2.47E-13	3.05E-17
58	Ship12s	0.00E+00	2.62E-13	6.13E-09	1.68E-14	3.37E-18
59	Stocfor1	0.00E+00	2.95E-12	6.37E-14	1.50E-15	2.39E-19
60	Stocfor2	1.28E-13	2.30E-10	1.28E-10	1.03E-09	1.33E-13
61	Truss1	0.00E+00	2.60E-14	8.85E-14	1.59E-16	3.67E-18
62	Truss2	1.38E-14	4.21E-14	3.79E-13	4.70E-15	1.15E-18
63	Truss3	0.00E+00	2.55E-10	2.18E-12	4.85E-11	1.48E-18
64	Vtp.base	4.62E-12	5.78E-15	2.98E-08	2.28E-12	1.74E-22
65	Wood1p	0.00E+00	9.04E-09	5.26E-13	5.66E-11	1.80E-17
66	Woodw	7.66E-14	2.50E-07	3.19E-10	3.61E-10	2.64E-18

Table 4: **IDLN:Interior Dual Least 2-Norm Results (continued)**

Pr. No.	Problem Name	IDLN Iter.	MINOS 5.3 (seconds)	IDLN (seconds)	Minos/IDLN Time Ratio
1	25fv47	80	339.47	206.06	1.65
2	Adlittle	27	0.97	1.26	0.77
3	Afiro	33	0.31	0.93	0.33
4	Agg	47	4.43	33.78	0.13
5	Agg2	39	7.71	42.80	0.18
6	Agg3	36	7.76	39.73	0.20
7	BandM	42	9.64	8.72	1.11
8	Beaconfd	40	3.45	9.18	0.38
9	Blend	35	1.20	1.96	0.61
10	Bnl1	55	42.18	31.95	1.32
11	Bnl2	86	609.54	1256.82	0.48
12	Bore3d	35	3.01	5.21	0.58
13	BrandY	65	6.43	11.35	0.57
14	Capri	57	4.46	17.26	0.26
15	Cre-a	89	592.03	179.97	3.29
16	Cre-c	63	665.41	123.92	5.37
17	CzProb	64	75.20	38.11	1.97
18	D2q06c	79	6299.60	2018.32	3.12
19	Degen2	32	29.19	32.25	0.91
20	Degen3	55	720.18	996.71	0.72
21	E226	57	7.59	10.46	0.73
22	Fffff800	61	27.37	65.29	0.42
23	Finnis	82	10.75	23.12	0.46
24	Gfrd-Pnc	40	18.17	8.34	2.18
25	Grow15	41	18.37	23.90	0.77
26	Grow22	43	34.51	38.06	0.91
27	Grow7	40	4.98	10.44	0.48
28	Israel	64	4.09	55.34	0.07
29	Kb2	31	0.67	1.33	0.50
30	Lotfi	56	3.83	4.54	0.84
31	Pilot.we	103	229.08	120.84	1.90
32	Rabo	47	16.52	105.24	0.16
33	Recipe	40	1.04	2.64	0.39

Table 5: **Comparison between Minos 5.3 and IDLN (DECstation 3100)**

Pr. No.	Problem Name	IDLN Iter.	MINOS 5.3 (seconds)	IDLN (seconds)	Minos/IDLN Time Ratio
34	Sc105	35	0.88	1.42	0.62
35	Sc205	45	1.99	2.41	0.83
36	Sc50a	33	0.45	1.03	0.44
37	Sc50b	33	0.43	1.02	0.42
38	Scagr25	57	8.49	6.96	1.22
39	Scagr7	46	1.27	1.91	0.66
40	Scfxm1	51	7.75	10.42	0.74
41	Scfxm2	58	22.99	24.83	0.93
42	Scfxm3	61	45.64	40.70	1.12
43	Scorpion	31	4.57	3.91	1.17
44	Scrs8	62	18.71	15.75	1.19
45	ScSd1	28	4.62	3.73	1.24
46	ScSd6	33	17.51	7.22	2.43
47	ScSd8	29	97.11	13.38	7.26
48	ScTap1	38	4.55	4.97	0.92
49	ScTap2	40	32.03	33.46	0.96
50	ScTap3	42	61.23	45.33	1.35
51	Share1b	55	1.64	3.70	0.44
52	Share2b	30	2.90	1.99	1.46
53	Ship04l	33	12.00	11.70	1.03
54	Ship04s	32	7.49	8.10	0.92
55	Ship08l	34	31.08	26.02	1.19
56	Ship08s	33	16.45	14.42	1.14
57	Ship12l	34	67.34	34.45	1.95
58	Ship12s	36	31.66	19.04	1.66
59	Stocfor1	30	1.07	1.67	0.64
60	Stocfor2	41	182.63	51.37	3.56
61	Truss1	36	23.43	13.80	1.70
62	Truss2	37	176.07	80.37	2.19
63	Truss3	47	930.90	253.38	3.67
64	Vtp.base	42	2.25	5.39	0.38
65	Wood1p	62	165.14	842.80	0.20
66	Woodw	59	542.64	277.83	1.95
-	**TOTAL**	-	12324.05	7390.83	1.67

Table 6: **Comparison between Minos 5.3 and IDLN (DECstation 3100) (continued)**

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