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THE APPROXIMATION ORDER OF BOX SPLINE SPACES

by

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To Professor I.J. Schoenberg, in memoriam.

ABSTRACT

Let M be a box spline associated with an arbitrary set of directions and suppose that $S(M)$ is the space spanned by the integer translates of M. In this note, the subspace of all polynomials in $S(M)$ is shown to be the joint kernel of a certain collection of homogeneous differential operators with constant coefficients. The approximation order from the dilates of $S(M)$ to smooth functions is thereby characterized. This extends a well-known result of de Boor and Höllig [BH], on box splines with integral direction sets.

The argument used is based on a new relation, valid for any compactly supported distribution ϕ, between the semi-discrete convolution $\phi *'$ and the distributional convolution $\phi *$.

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The approximation order of box spline spaces

A. Ron & N. Sivakumar

1. Introduction

Let Ξ be a real $s \times n$ matrix with non-zero columns. At times we think of Ξ as the collection of its column vectors, so that $\xi \in \Xi$ means that ξ is a column of Ξ and $Y \subset \Xi$ means that Y is an $s \times k$ $(k \le n)$ submatrix of Ξ. The **box spline** M_Ξ associated with Ξ is defined to be the Dirac distribution in case Ξ is empty (i.e., $n = 0$), and otherwise by the distributional rule

$$M_\Xi(\psi) := \int_{[0,1]^n} \psi(\Xi t)\, dt, \quad \forall \psi \in C(\mathbb{R}^s). \tag{1.1}$$

Its Fourier transform is given by

$$\widehat{M_\Xi}(w) = \prod_{\xi \in \Xi} \int_0^1 e^{-iw\cdot\xi\, t}\, dt. \tag{1.2}$$

In general M_Ξ can be identified with a positive measure supported on a compact polyhedral subset of the column span of Ξ, and, in case Ξ is of rank s, $M_\Xi : \mathbb{R}^s \to \mathbb{R}_+$ is a compactly supported piecewise polynomial function. Various specific relevant references on box splines are given in the sequel. For expository material on box splines, we refer the reader to [C] (and references therein) as well as to the forthcoming book of de Boor, Höllig and Riemenschneider [BHR].

The main purpose of this note is to characterize the *approximation order* of *box spline spaces*. For any compactly supported distribution ϕ, we define $S(\phi)$ to be the (infinite) span of the integer translates of ϕ:

$$S(\phi) := \operatorname{span}\{E^\alpha \phi : \ \alpha \in \mathbb{Z}^s\}, \tag{1.3}$$

with E^x, $x \in \mathbb{R}^s$, the translation operator:

$$E^x : f \mapsto f(\cdot + x). \tag{1.4}$$

A space of the form $S(M)$, for a box spline M, is referred to as a **box spline space**. To define the approximation order of $S(M)$, we need a way to *refine* this space. A refinement $S_h(\phi)$ (with h positive and small) of $S(\phi)$ can be obtained by scaling $S(\phi)$:

$$S_h(\phi) := \{f(\cdot/h) : \ f \in S(\phi)\}. \tag{1.5}$$

The **approximation order** of $S(\phi)$ (in the ∞-norm) is then the maximal integer d that satisfies

$$\operatorname{dist}_\infty(f, S_h(\phi)) = O(h^d), \quad \forall f \in W^d_\infty, \tag{1.6}$$

where W_∞^d is the usual Sobolev space. For a function ϕ, the study of approximation orders for the space $S(\phi)$ is significantly facilitated by the **Strang-Fix Conditions.** These conditions focus on the space $\Pi(\phi)$ of all polynomials in $S(\phi)$, and assert (cf. [R]) that the approximation order of $S(\phi)$ is the maximal integer d for which

$$\Pi_{d-1} \subset \Pi(\phi), \tag{1.7}$$

provided that $\widehat{\phi}(0) \neq 0$. (Here and elsewhere $\Pi := \Pi(\mathbb{R}^s)$ is the space of all complex valued s-variate polynomials and Π_k is the subspace of all polynomials of total degree at most k.) Consequently, the question of approximation orders is reduced to the identification of the space $\Pi(\phi)$.

A characterization of $\Pi(M_\Xi)$ is well known in case Ξ is an *integral* matrix (i.e., all entries in Ξ are integers). To describe it, we associate with each column ξ of Ξ the polynomial

$$p_\xi : x \mapsto \xi \cdot x,$$

and define

$$p_Y := \prod_{\xi \in Y} p_\xi, \qquad Y \subset \Xi. \tag{1.8}$$

Thus, $p_Y(D)$ is the product of the directional derivatives $p_\xi(D)$ (in any order). With

$$\mathbb{K}(\Xi) := \{Y \subset \Xi : \ \mathrm{rank}(\Xi \backslash Y) < s\}, \tag{1.9}$$

it is known that

$$\Pi(M_\Xi) = D(\Xi) := \{q \in \Pi : \ p_Y(D)q = 0, \ \forall Y \in \mathbb{K}(\Xi)\}. \tag{1.10}$$

Since $\Pi(M_\Xi)$ is obtained in (1.10) as the intersection (in Π, but as a matter of fact even in the distribution space $\mathcal{D}'(\mathbb{R}^s)$), of kernels of homogeneous differential operators with constant coefficients, it follows that

$$\Pi_{d-1} \subset \Pi(M_\Xi) \iff (\#Y \geq d, \quad \forall\, Y \in \mathbb{K}(\Xi)), \tag{1.11}$$

where $\#Y$ is the number of columns in the matrix Y, i.e., the cardinality of the multiset Y. Consequently, the approximation order of $S(M_\Xi)$ is the number

$$d_\Xi := \min\{\#Y : \ Y \in \mathbb{K}(\Xi)\}, \tag{1.12}$$

i.e., the lowest degree of the differential operators involved in the definition of $D(\Xi)$. These results were first established by de Boor and Höllig in [BH], and were also proved (with the aid of different arguments and in a slightly more general setting) by Dahmen and Micchelli in [DM].

As emphasized earlier, this characterization of the approximation order for $S(M_\Xi)$ is valid only when the underlying matrix Ξ is integral. For a general Ξ, it is still true, [BH], that M_Ξ is piecewise in $D(\Xi)$, so that $\Pi(M_\Xi) \subset D(\Xi)$ and the number d_Ξ given in (1.12) provides *an upper bound* for the approximation order of $S(M_\Xi)$. Yet, simple examples show that in general this bound is not attained and may be far from the actual approximation order. In this note we show that, surprisingly, $\Pi(M_\Xi)$ is always realizable as the common null-space of certain differential operators of the form $p_Y(D)$, $Y \subset \Xi$. We thereby extend the aforementioned results of [BH] and [DM] to non-integral matrices.

Our argument is based on the interplay between the convolution operator $M_\Xi *$ and related differential and difference operators, [BH]. To make this interplay effective in the setting here, we invoke, in section 2, the Poisson summation formula in a way that reduces the characterization of $\Pi(\phi)$ (for a compactly supported distribution ϕ), to the study of the action of the convolution operator $\phi*$ on the exponential spaces $e_\alpha\Pi$, $\alpha \in 2\pi\mathbb{Z}^s$ (henceforth, $e_\alpha(\cdot) = e^{i\alpha\cdot}$). This avoids the standard conversion of the problem into the Fourier transform domain. The main result is stated and proved in section 3, and is followed by some discussion and examples.

2. Semi-discrete convolution

Throughout this paper ϕ is assumed to be a compactly supported distribution (in s dimensions). We reserve the notation $\phi*$ for the standard distributional convolution operator (defined on $\mathcal{D}'$), and, following [B], use the notation $\phi*'$ for the semi-discrete convolution operator which is defined as

$$\phi*' : f \mapsto \phi*'f := \sum_{\alpha\in\mathbb{Z}^s} f(\alpha)E^{-\alpha}\phi \in S(\phi), \tag{2.1}$$

with f being any function defined (at least) on $\mathbb{Z}^s$.

The following result is useful in the study of $\Pi(\phi)$. For a *function* ϕ it can be found in [B].

Proposition 2.2. *Assume that $\phi*$ is 1-1 on Π (equivalently, $\widehat{\phi}(0) \neq 0$). Then*

$$\Pi(\phi) = \{p \in \Pi : \ \phi*'p \in \Pi\} = \phi*'\Pi(\phi). \tag{2.3}$$

Proof: Since $\Pi(\phi)$ is translation invariant, [B], it is an invariant space of $\phi*$; hence, the injectivity of $\phi*$ on Π implies that $\phi*\Pi(\phi) = \Pi(\phi)$. Set

$$Q := \{p \in \Pi : \ \phi*'p \in \Pi\}.$$

By [BR1], $\phi*'Q = \Pi(\phi)$, while, by [BR2],

$$\phi*'p \in \Pi \iff \phi*'p = \phi*p, \qquad p \in \Pi. \tag{2.4}$$

Thus $\phi*Q = \Pi(\phi)$. Since we have also shown that $\phi*\Pi(\phi) = \Pi(\phi)$, it follows that $Q = \Pi(\phi)$, and the proof is complete. ♠

The discussion presented above suggests the study of the map $\phi*'|_\Pi$ as a means of identifying the space $\Pi(\phi)$. It is therefore rather annoying to realize that, in contrast with the standard convolution operator $\phi*$, $\phi*'$ does not commute with non-integral translations, hence fails to commute with differentiation. This obstacle is being circumvented here with the aid of the identity

$$\phi*'f = \sum_{\alpha\in 2\pi\mathbb{Z}^s} \phi*(e_\alpha f), \tag{2.5}$$

which is valid under various conditions on the pair (ϕ, f). It is obtained by a (straightforward) application of Poisson's summation formula, [SW], as in [SF], [DM], [B] and [BR2], but, unlike these cited references, is not restricted to a polynomial or an exponential f. More importantly, it does not convert the problem into the Fourier transform domain, thus allowing us to exploit efficiently the favourable properties of the convolution operator $\phi*$.

Theorem 2.6. *Let ϕ be a distribution and f a continuous function such that (for some ordering of $\mathbb{Z}^s$) the series $\phi *' f$ and $\sum_{\alpha\in 2\pi\mathbb{Z}^s} \phi*(e_\alpha f)$ converge in $\mathcal{D}'$. Then (2.5) is valid if for some approximate identity $\{\sigma_h\}_{h>0}$, each function $[(\sigma_h*\phi)(x-\cdot)]f$, $x \in \mathbb{R}^s$, $h>0$, is in L_1 and satisfies Poisson's summation formula.*

Proof: Fix $x \in \mathbb{R}^s$ and $h > 0$, and set $\psi := [(\sigma_h*\phi)(x-\cdot)]f(\cdot)$. By assumption, $\sum_{\alpha\in\mathbb{Z}^s} \psi(\alpha) = \sum_{\alpha\in 2\pi\mathbb{Z}^s} \widehat{\psi}(\alpha)$. Now, on the one hand $\sum_{\alpha\in\mathbb{Z}^s}\psi(\alpha) = [(\sigma_h*\phi)*'f](x)$, while on the other hand, since $\psi \in L_1$,

$$\widehat{\psi}(\alpha) = \int_{\mathbb{R}^s} e^{-i\alpha\cdot t} f(t)(\sigma_h*\phi)(x-t)\,dt = (\sigma_h*\phi)*(e_{-\alpha}f)(x).$$

Consequently,

$$(\sigma_h*\phi)*'f = \sum_{\alpha\in 2\pi\mathbb{Z}^s} (\sigma_h*\phi)*(e_\alpha f),\ \forall h>0. \tag{2.7}$$

Since both $\phi*'f$ and $\sum_{\alpha\in 2\pi\mathbb{Z}^s}\phi*(e_\alpha f)$ converge, it follows from (2.7) that

$$\sigma_h*(\phi*'f) = \sigma_h*(\sum_{\alpha\in 2\pi\mathbb{Z}^s} \phi*(e_\alpha f)),\ \forall h>0,$$

and the desired result is obtained by letting $h \to 0$. ♠

In the case of interest here, ϕ is compactly supported, hence $\phi*'f$ always converges regardless of the growth rate of f at ∞. Furthermore, for a compactly supported ϕ, $\sigma_h*\phi \in \mathcal{D}$, so the function $[(\sigma_h*\phi)(x-\cdot)]f(\cdot)$ satisfies Poisson's summation formula if f is smooth enough. In particular, we obtain the following corollary.

Corollary 2.8. *If $\phi \in \mathcal{D}'$ is compactly supported and $f \in C^\infty$, then (2.5) holds provided that $\sum_{\alpha\in 2\pi\mathbb{Z}^s}\phi*(e_\alpha f)$ converges (in $\mathcal{D}'$).*

Our subsequent application concerns the very special case when ϕ is compactly supported and f is a polynomial. That (2.5) is always valid in this case is asserted by the following result.

Theorem 2.9. *Let ϕ be a compactly supported distribution, and p a polynomial. Then*

$$\phi*'p = \sum_{\alpha\in 2\pi\mathbb{Z}^s} \phi*(e_\alpha p). \tag{2.10}$$

Proof: In view of Corollary 2.8, we need only to show that the sum $\sum_{\alpha\in 2\pi\mathbb{Z}^s}\phi*(e_\alpha p)$ converges in $\mathcal{D}'$. To see this, note that $\widehat{\phi}$ (as well as any of its derivatives) is an entire function of exponential type with polynomial growth on $\mathbb{R}^s$, and therefore the distribution $\lambda_\alpha := (\phi*(e_\alpha p))\widehat{}$ is of the form

$$\lambda_\alpha : f \mapsto \sum_{|\beta|\le \deg p} c_\beta(\alpha)(D^\beta f)(\alpha),$$

with coefficients $\{c_\beta(\alpha)\}_\alpha$ of polynomial growth in α. This implies that $\sum_{\alpha\in 2\pi\mathbb{Z}^s}(\phi*(e_\alpha p))\widehat{}$ converges in $\mathcal{S}'$, hence so does $\sum_{\alpha\in 2\pi\mathbb{Z}^s}\phi*(e_\alpha p)$; *a fortiori* this latter sum converges in $\mathcal{D}'$. ♠

Theorem 2.9 leads to the following corollary, the first part of which was established in [BR2].

Corollary 2.11. *For a compactly supported ϕ and a polynomial p,*

$$\phi *' p \in \Pi \iff \phi *' p = \phi * p \iff \phi *(e_\alpha p) = 0,\ \forall \alpha \in 2\pi \mathbb{Z}^s \backslash 0.$$

Proof: By Theorem 2.9, $\phi *' p - \phi * p$ has its Fourier transform supported on $2\pi\mathbb{Z}^s\backslash 0$, hence can never be a non-trivial polynomial, while obviously, $\phi * p$ is always a polynomial, whence the first equivalence. As to the second, Theorem 2.9 clearly implies that $\phi *' p = \phi * p$ iff $\sum_{\alpha\in\mathbb{Z}^s\backslash 0} \phi *(e_\alpha p) = 0$, yet this latter sum can vanish only if each of its summands vanishes, e.g., since the supports of the Fourier transforms of these summands are pairwise disjoint. ♠

The following is a typical application of Corollary 2.11. Given a matrix $K_{s\times k}$, we employ the notation

$$\nabla^K := \prod_{\xi\in K} \nabla^\xi := \prod_{\xi \in K} (1 - E^{-\xi}), \tag{2.12}$$

and record the following straightforward fact for use here and later.

Lemma 2.13. *Let $\xi \in \mathbb{R}^s$, $\alpha \in \mathbf{C}^s$ and $q \in \Pi$. If $\nabla^\xi(e_\alpha) = 0$ (equivalently, if $e_\alpha(\xi) = 1$), then $\nabla^\xi(e_\alpha q) = e_\alpha \nabla^\xi q$. Otherwise, ∇^ξ is injective on $e_\alpha \Pi$. Consequently, for every translation invariant space $Q \subset \Pi$,*

$$e_\alpha \nabla^\xi Q \subset \nabla^\xi (e_\alpha Q).$$

Corollary 2.14. *Let ϕ be a compactly supported distribution. Then for any matrix $K_{s\times k}$,*

$$\Pi(\nabla^K \phi) \subset \Pi(\phi).$$

Proof: Set $\psi := \nabla^K \phi$. By [BR1], there exists a translation invariant $Q \subset \Pi$ such that $\psi *' Q = \Pi(\psi)$. Since $\psi *' Q \subset \Pi$, we may invoke Corollary 2.11 to obtain

$$\phi *(\nabla^K(e_\alpha Q)) = \psi *(e_\alpha Q) = 0, \qquad \forall \alpha \in 2\pi\mathbb{Z}^s\backslash 0. \tag{2.15}$$

But Q is translation invariant, and therefore, by Lemma 2.13, $\phi *(e_\alpha \nabla^K Q) \subset \phi *(\nabla^K(e_\alpha Q))$. Thus,

$$\phi *(e_\alpha \nabla^K Q) = 0, \qquad \forall \alpha \in 2\pi\mathbb{Z}^s\backslash 0,$$

which, together with Corollary 2.11, implies that

$$\phi *' \nabla^K Q = \phi * \nabla^K Q = (\nabla^K \phi) * Q = \psi * Q = \psi *' Q = \Pi(\psi).$$

Hence $\Pi(\psi) \subset \Pi(\phi)$ as claimed. ♠

Remark 2.16. In case the matrix K is integral, the preceding corollary becomes trivial, since then $S(\nabla^K \phi) = S(\phi)$, hence also $\Pi(\nabla^K\phi) = \Pi(\phi)$. However, the situation for a general K is subtle, since the space $S(\nabla^K\phi)$ might be very different from $S(\phi)$. In particular, taking ϕ to be the characteristic function of the interval $[0, 1)$, we check that $\Pi(\phi) = \Pi_0(\mathbb{R})$ while $\Pi((1 - E^{.5})\phi) = \{0\}$, so that the inclusion in the corollary might be proper. Further, choosing ϕ to be the characteristic function of the interval $[0, .5)$, we get $\Pi(\phi) = \{0\}$, while $\Pi((1 + E^{-.5})\phi) = \Pi_0(\mathbb{R})$, showing thereby that the corollary above does not extend to arbitrary difference operators (Lemma 2.13 does not carry over). Finally, note *that no regularity assumption on ϕ has been made here*, namely, the possibility $\widehat{\phi}(0) = 0$ has not been excluded.

3. Box splines

We say that ϕ **provides a partition of unity** if $1 \in S(\phi)$, i.e., if $\Pi(\phi)$ contains Π_0. Since $\Pi_0 \subset \Pi(\phi) \iff 1 \in \Pi(\phi)$, it follows from Proposition 2.2 and (2.4) that whenever $\widehat{\phi}(0) \neq 0$, ϕ provides a partition of unity if and only if

$$\phi *' 1 = \text{const.} \tag{3.1}$$

Suppose now that Ξ is an $s \times n$ matrix, and define

$$\mathbb{K}_U(\Xi) := \left\{ \, Y \subset \Xi \ : \ M_{\Xi\backslash Y} \text{ does not provide a partition of unity} \, \right\}.$$

We shall see later that the set $\mathbb{K}_U(\Xi)$ can be determined directly from Ξ without any direct recourse to box splines.

Theorem 3.2. *For a matrix* $\Xi_{s\times n}$,

$$\Pi(M_\Xi) = \cap_{Y\in\mathbb{K}_U(\Xi)} \ker p_Y(D) =: D_U(\Xi). \tag{3.3}$$

We note that the theorem is trivial if M_Ξ does not provide a partition of unity (since then $\Pi(M_\Xi) = \{0\} = D_U(\Xi)$). We may therefore assume that $\Pi_0 \subset \Pi(M_\Xi)$; in particular, Ξ is of rank s. Also, if Ξ is an *integral* matrix and $V \subset \Xi$, then M_V provides a partition of unity if and only if rank $V = s$, [BH]. Thus Theorem 3.2 extends the result quoted in (1.10).

Roughly speaking, there are two different approaches towards the proof of the *integral* case of Theorem 3.2. One method, [DM], is based on a clever calculation of the derivatives of the Fourier transform $\widehat{M_\Xi}$ on the lattice $2\pi\mathbb{Z}^s$ (cf. Lemma 3.1, Theorem 3.1, and Proposition 3.2 of [DM]). The other method [BH], [BAR], is based on the identity, [BH],

$$p_K(D)M_\Xi = \nabla^K M_{\Xi\backslash K}, \tag{3.4}$$

which is valid for every $K \subset \Xi$. Both approaches extend to the non-integral case. Here we exploit the latter method as a demonstration of the utility of Theorem 2.9 and Corollary 2.14. In the proof, we make use of the following simple lemmata.

Lemma 3.5. *For every* $K \in \mathbb{K}_U(\Xi)$,

$$\Pi(M_{\Xi\backslash K}) = \{0\}.$$

Proof: This follows directly from the definition of $\mathbb{K}_U(\Xi)$, the fact that $\Pi(\phi)$ is always translation invariant, [B], and the fact that every non-trivial translation invariant polynomial space contains Π_0. ♠

Lemma 3.6. *Let $\xi \in \Xi$ and $\alpha \in \mathbf{C}^s$. Then*

$$M_\xi * e_\alpha = 0 \iff (\nabla^\xi e_\alpha = 0 \text{ and } p_\xi(D) e_\alpha \neq 0).$$

Proof: The proof follows from the facts that

$$\nabla^\xi e_\alpha = 0 \iff e_\alpha(-\xi) = 1,$$

$$p_\xi(D) e_\alpha \neq 0 \iff p_\xi(\alpha) \neq 0,$$

$$M_\xi * e_\alpha = 0 \iff \widehat{M_\xi}(\alpha) = \frac{e_\alpha(-\xi) - 1}{-ip_\xi(\alpha)} = 0,$$

and the fact that the univariate function $e^z - 1$ has only simple zeros. ♠

Proof of Theorem 3.2. We first prove that

$$\Pi(M_\Xi) \subset D_U(\Xi). \tag{3.7}$$

Since $\widehat{M_\Xi}(0) = 1$, it suffices to show, in view of Proposition 2.2, that

$$M_\Xi *' \Pi(M_\Xi) \subset D_U(\Xi).$$

Let $p \in \Pi(M_\Xi)$ and $K \in \mathbb{K}_U(\Xi)$. By Proposition 2.2, $M_\Xi *' p \in \Pi$ and by (3.4),

$$\Pi \ni p_K(D)(M_\Xi *' p) = (\nabla^K M_{\Xi \backslash K}) *' p \in S(\nabla^K M_{\Xi \backslash K}).$$

Hence, by Corollary 2.14 and Lemma 3.5,

$$p_K(D)(M_\Xi *' p) \in \Pi(M_{\Xi \backslash K}) = \{0\}.$$

Thus, $M_\Xi *' p \in \cap_{K \in \mathbb{K}_U(\Xi)} \ker p_K(D) = D_U(\Xi)$, as desired.

Next we prove that

$$\Pi(M_\Xi) \supset D_U(\Xi). \tag{3.8}$$

Let $q \in D_U(\Xi)$. In view of Proposition 2.2 and Corollary 2.11, it suffices to show that $M_\Xi * (e_\alpha q) = 0$ for all $\alpha \in 2\pi \mathbb{Z}^s \backslash 0$. To that end, fix $\alpha \in 2\pi\mathbb{Z}^s \backslash 0$, and set

$$K := \{\xi \in \Xi : \ M_\xi * e_\alpha = 0\} = \{\xi \in \Xi : \ \widehat{M_\xi}(\alpha) = 0\}.$$

We shall show that $M_K * (e_\alpha q) = 0$, which would imply that $M_\Xi * (e_\alpha q) = 0$, since $M_\Xi = M_{\Xi \backslash K} * M_K$. We first observe that (3.4) (with Ξ there replaced by K) implies that for any $f \in \mathcal{D}'(\mathbb{R}^s)$,

$$\nabla^K f = M_K * p_K(D) f. \tag{3.9}$$

Next we observe that $\widehat{M_\xi}(\alpha) \neq 0$ for $\xi \in \Xi \backslash K$, so $M_{\Xi \backslash K} * e_\alpha = e_\alpha \widehat{M}_{\Xi \backslash K}(\alpha) \neq 0$. By Corollary 2.11, $M_{\Xi \backslash K} *' 1 \notin \Pi$, i.e., $K \in \mathbb{K}_U(\Xi)$. Consequently, $p_K(D) q = 0$, whence by (3.9),

$$\nabla^K q = M_K * p_K(D) q = 0. \tag{3.10}$$

As $M_\xi * e_\alpha = 0$ for every $\xi \in K$, Lemma 3.6 allows us to deduce that $\nabla^\xi e_\alpha = 0$. Hence, by Lemma 2.13 and (3.10), $\nabla^K (e_\alpha q) = e_\alpha \nabla^K q = 0$. Appealing to (3.9) once again, we obtain

$$p_K(D)(M_K * (e_\alpha q)) = \nabla^K (e_\alpha q) = 0. \tag{3.11}$$

Finally, for every $\xi \in K$, $p_\xi(D) e_\alpha \neq 0$ by Lemma 2.13. This means that each $p_\xi(D)$, $\xi \in K$, is 1-1 on $e_\alpha \Pi$, hence so is $p_K(D)$. Thus, since $M_K * (e_\alpha q) \in e_\alpha \Pi$, (3.11) can hold only if $M_K * (e_\alpha q) = 0$. This finishes the proof. ♠

Theorem 3.2 reduces the computation of the approximation order of $S(M_\Xi)$ to the identification of those subsets $Y \subset \Xi$ for which $1 \in \Pi(M_Y)$. For an integral Ξ, we have already mentioned a simple criterion for this to hold, viz., $1 \in \Pi(M_Y)$ if and only if Y is of rank s. However, for a general Ξ the situation appears to be much more involved. We know of such a condition only when $s = 1$ ($1 \in \Pi(M_\Xi)$ if and only if one of the entries of Ξ is integral). The following example, whose computational details are omitted, indicates some of the difficulties that occur even in the bivariate setting. In what follows, we define

$$\mathbb{K}'_U(\Xi) := \{ \, Y \in \mathbb{K}_U(\Xi) \ : \ Y \backslash \xi \notin \mathbb{K}_U(\Xi) \text{ for any } \xi \in Y \, \},$$

and note that

$$D_U(\Xi) = \cap_{Y \in \mathbb{K}'_U(\Xi)} \ker p_Y(D).$$

Example 3.12. Let $s = 2$, $n = 8$ and

$$\Xi = \begin{pmatrix} 1/2 & 0 & 1/2 & 1 & 1/2 & 1/2 & 1/2 & 1 \\ 0 & 1/2 & 1/2 & 1/2 & 1 & 1 & -1/2 & 0 \end{pmatrix}.$$

Using the fact that for any $\xi \in \Xi$ and $\alpha \in \mathbb{Z}^s$, $\widehat{M_\xi}(2\pi\alpha) = 0$ if and only if $\xi \cdot \alpha \in \mathbb{Z} \backslash \{0\}$, it can be verified that for $Y \subset \Xi$, M_Y provides a partition of unity if and only if Y contains one of the following matrices

$$Y_1 := \begin{pmatrix} 1/2 & 0 & 1/2 & 1 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 & 1/2 & 1 & -1/2 \end{pmatrix},$$

$$Y_2 := \begin{pmatrix} 1/2 & 1 \\ 1 & 0 \end{pmatrix}.$$

It follows that the (six) elements of $\mathbb{K}'_U(\Xi)$ are all 2×2, so Theorem 3.2 guarantees that $\Pi(M_\Xi) = \Pi_1$. The matrix Ξ is irredundant in the sense that removal of any column from Ξ leads to a box spline M with $\Pi(M) \neq \Pi_1$, yet one can remove five (!) columns from Ξ to get

$$\Xi' := \begin{pmatrix} 1/2 & 1/2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

whose corresponding $\Pi(M_{\Xi'})$ is of dimension $2 = \dim \Pi(M_\Xi) - 1$. This is in stark contrast with the integral case, where removal of any direction from Ξ results in a corresponding polynomial space which is (strictly) smaller. ♠

However, the approximation order of $S(M_\Xi)$, for a general Ξ, can be computed as follows. First, for every $\alpha \in \mathbb{C}^s$, let

$$K_\alpha(\Xi) := \{\xi \in \Xi : \ \xi \cdot \alpha \in \mathbb{Z}\backslash 0\}.$$

Then we have

Theorem 3.13. *Let M_Ξ be a box spline, and let $\{K_\alpha(\Xi) : \ \alpha \in \mathbb{C}^s\}$ be as above. Then the approximation order from the space $S(M_\Xi)$ is the number*

$$\min\{\#K_\alpha(\Xi) : \ \alpha \in \mathbb{Z}^s\backslash 0\}. \tag{3.14}$$

Proof: From (1.2) it follows that

$$\widehat{M}_\xi(2\pi\alpha) = 0 \iff \xi \in K_\alpha(\Xi),$$

hence also

$$M_\xi * e_{2\pi\alpha} = 0 \iff \xi \in K_\alpha(\Xi).$$

Therefore, for every α, $M_{\Xi\backslash K_\alpha(\Xi)} * e_{2\pi\alpha} \neq 0$, which implies, by Theorem 2.9, that $K_\alpha(\Xi) \in \mathbb{K}_U(\Xi)$ for every $\alpha \in \mathbb{Z}^s\backslash 0$. Furthermore, if $K \in \mathbb{K}_U(\Xi)$, then $1 \notin \Pi(M_{\Xi\backslash K})$, hence, by Theorem 2.9, there exists an $\alpha \in \mathbb{Z}^s\backslash 0$ such that $M_{\Xi\backslash K} * e_{2\pi\alpha} \neq 0$, which implies that $K_\alpha(\Xi) \subset K$. We conclude that

$$\mathbb{K}'_U(\Xi) \subset \{\ K_\alpha(\Xi) \ : \ \alpha \in \mathbb{Z}^s\backslash 0\ \} \subset \mathbb{K}_U(\Xi),$$

hence

$$D_U(\Xi) = \cap_{\alpha \in \mathbb{Z}^s\backslash 0} \ker p_{K_\alpha(\Xi)}(D).$$

The required result now follows from Theorem 3.2 and the Strang-Fix Conditions. ♠

4. Approximation order from submodule-translates

The determination of the appoximation order of the box spline space $S(M_\Xi)$ admits an equivalent fomulation, which, as a matter of fact, initiated our study here. This brief section is devoted to its discussion.

Let Ξ be an $s \times n$ *integral* matrix and $\mathcal{A}$ a submodule of the $\mathbb{Z}$-module $\mathbb{Z}^s$. Suppose that $\mathcal{A} = A\mathbb{Z}^s$, where A is an $s \times s$ invertible integral matrix, and define

$$S_\mathcal{A}(M_\Xi) := \text{span } \{E^\beta M_\Xi \ : \ \beta \in \mathcal{A}\}.$$

Proceeding as we did in the introductory section, we may define the approximation order of the space $S_\mathcal{A}(M_\Xi)$ in an entirely analogous fashion. Owing to the relation, [BH],

$$M_\Xi = |\det A|\, M_{A\Xi} \circ A,$$

the approximation order of $S_\mathcal{A}(M_\Xi)$ is seen to be precisely that of the spline space $S(M_{A^{-1}\Xi})$. Therefore, Theorem 3.13 readily yields

Theorem 4.1. *Let Ξ be a $s \times n$ integral matrix, A a $s \times s$ integral matrix of rank s, and $\mathcal{A} := A\mathbb{Z}^s$. Then the approximation order from $S_{\mathcal{A}}(M_\Xi)$ is the number*

$$\min\{\#K_\alpha(A^{-1}\Xi) : \alpha \in \mathbb{Z}^s\backslash 0\} = \min\{\#K_\alpha(\Xi) : \alpha \in (A^{-1})^T\mathbb{Z}^s\backslash 0\}.$$

Example 4.2. Let

$$\Xi := \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 & 1 & \dots & 1 & 1 & \dots & 1 \\ 0 & \dots & 0 & 1 & \dots & 1 & 1 & \dots & 1 & -1 & \dots & -1 \end{pmatrix},$$

where the vectors $\xi_1 := (1,0)$, $\xi_2 := (0,1)$, $\xi_3 := (1,1)$, and $\xi_4 := (1,-1)$ occur with multiplicities n_1, n_2, n_3, and n_4, respectively. Let

$$A := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathcal{A} := A\mathbb{Z}^2,$$

so that

$$(A^{-1})^T\mathbb{Z}^2 = \{\alpha = (\alpha_1, \alpha_2) \in (1/2)\mathbb{Z}^2 : (\alpha_1 + \alpha_2) \in \mathbb{Z}\}.$$

Suppose that $\alpha \in (A^{-1})^T\mathbb{Z}^2\backslash 0$. Then $\alpha \cdot \xi_j \in \mathbb{Z}$, $j = 3, 4$, so $\#K_\alpha(\Xi) \geq n_3 + n_4$ unless one of $\alpha \cdot \xi_3$ or $\alpha \cdot \xi_4$ is zero; in which case $\#K_\alpha(\Xi) \geq \min\{n_3, n_4\}$. In fact, choosing $\alpha = (1/2, \pm 1/2)$, we see that $\min\{\#K_\alpha(\Xi) : \alpha \in (A^{-1})^T\mathbb{Z}^2\backslash 0\} = \min\{n_3, n_4\}$. As a result, Theorem 4.1 implies that the approximation order of $S_{\mathcal{A}}(M_\Xi)$ is $\min\{n_3, n_4\}$. We thus recover [JR; Theorem 3]. It is not without interest to note, that in view of (1.12), the approximation order of $S(M_\Xi)$ (as opposed to $S_{\mathcal{A}}(M_\Xi)$) is $\min\{-n_j + \sum_{k=1}^4 n_k : j = 1, 2, 3, 4.\}$.

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