SCHEDULING WITH EARLINESS AND TARDINESS PENALTIES

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Abstract. We show that the deterministic non preemptive scheduling problem with earliness and tardiness penalties can be solved in polynomial time for certain forms of an objective function provided that a certain optimization problem can be solved. We give instances where this problem has a solution and show that this generalizes several results from the literature. These results do not require symmetric penalization and the penalty functions need only be lower semicontinuous.

1 Introduction

The theory of deterministic scheduling has been mainly concerned with minimizing a regular measure of performance, i.e. with minimizing functions which are nondecreasing in job completion times. However, in some practical situations production schedules must be evaluated with respect to both earliness and tardiness costs. If that is the case, one usually has to deal with a nonregular measure of performance. Examples of such practical situations are mentioned in [MM72, Sid77, Kan81, GTW88]. They include the production of perishable goods, organization of computer files, material requirements planning and situations in which equal treatment of jobs is desirable.

We consider single machine problems with n jobs numbered by the natural numbers $1, \ldots, n$. Each job is to be processed on a continuously available machine and at any time the machine can handle at most one job. Every job, j, requires a positive processing time p_j and has a specified release time ρ_j and a specified deadline δ_j such that

$$-\infty \leq \rho_j < \delta_j \leq +\infty$$

If we set $N := \{1, 2, \ldots, n\}$, $N_0 := \{0, 1, 2, \ldots, n\}$ and

$$\sigma(t) := \begin{cases} j & \text{whenever job } j \text{ is being executed at time } t \\ 0 & \text{otherwise} \end{cases}$$

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then we can present every schedule as a piecewise constant function

$$\sigma: (-\infty, \infty) \to N_0$$

with the following properties:

$$\text{the measure of the set } \{t \mid \sigma(t) = j\} \text{ is equal to } p_j \text{ for all } j \in N \tag{1}$$

$$\sigma(t) \neq j \text{ whenever } j \in N \text{ and } t \notin [\rho_j, \delta_j] \tag{2}$$

In order to avoid unnecessary complications, we shall confine ourselves to representation by functions which are lower semicontinuous with a finite number of discontinuities. Of course, not every such function represents a feasible schedule. In general, feasibility is characterized by various systems of conditions reflecting additional capacity and technological restrictions. Here we direct our attention to the problems in which no preemption is allowed, i.e. in addition to (1) and (2) we require that

$$\sigma(t') = \sigma(t'') = j \implies \sigma(t) = j \text{ for all } t' < t < t'' \text{ and } j \in N \tag{3}$$

In the sequel, we shall let X denote the set of feasible schedules, that is those schedules satisfying (1) – (3).

Regardless of whether preemption is permitted or not, each feasible schedule σ determines uniquely the start time

$$S_j(\sigma) := \inf \{t \mid \sigma(t) = j\}$$

and the completion time

$$C_j(\sigma) := \sup \{t \mid \sigma(t) = j\}$$

of each job j in the schedule σ. It is useful to note that if α is a real number and if σ_α is defined by

$$\sigma_\alpha(t) := \sigma(t + \alpha)$$

then

$$S_j(\sigma_\alpha) = S_j(\sigma) - \alpha, \quad C_j(\sigma_\alpha) = C_j(\sigma) - \alpha$$

for each $j \in N$.

In dealing with situations where penalties are incurred both when jobs are completed late or started early, it is usually assumed that each job j has a target start time r_j and a due date d_j satisfying

$$\rho_j \leq r_j < d_j \leq \delta_j$$

Quantitative measures for calculating schedules considered here are based on the following constructs. The lateness of j in σ, $L_j(\sigma) := C_j(\sigma) - d_j$, the promptness of j in σ, $\mathcal{E}_j(\sigma) := r_j - S_j(\sigma)$, the tardiness of j in σ, $T_j(\sigma) := \max\{0, C_j(\sigma) - d_j\}$ and the earliness of j in σ, $E_j(\sigma) := \max\{0, r_j - S_j(\sigma)\}$.

Given these definitions, we are interested in finding a schedule $\sigma \in X$ to minimize some objective or cost function defined in terms of these constructs. Several of the objective functions that may occur in practice include:

(a) $\sigma \mapsto \max\{\max_j g(E_j(\sigma)), \max_j h(T_j(\sigma))\}$

(b) $\sigma \mapsto \max\{\sum_j g(E_j(\sigma)), \sum_j h(T_j(\sigma))\}$

(c) $\sigma \mapsto \max_j g(E_j(\sigma)) + \max_j h(T_j(\sigma))$

(d) $\sigma \mapsto \sum_j g(E_j(\sigma)) + \sum_j h(T_j(\sigma))$

(e) $\sigma \mapsto \max_j\{g(E_j(\sigma)) + h(T_j(\sigma))\}$

(f) $\sigma \mapsto \sum_j \max\{g(E_j(\sigma)), h(T_j(\sigma))\}$

where $g, h: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \bigcup\{+\infty\}$ are nondecreasing. Note that we allow the possibility of an infinite penalty on earliness or tardiness. An analogous set of objective functions might be defined by replacing E_j and T_j by $\mathcal{E}_j$ and L_j respectively, and defining g and h on $\mathbb{R}$.

We now present some examples from the literature which can be reformulated as special cases of the above problems.

Example 1 Sidney[Sid77] proposed an algorithm for minimizing (a) under the following additional assumptions:

$$\rho_j = -\infty \text{ and } \delta_j = +\infty \text{ for all } j \in N \tag{4}$$

$$\text{functions } g \text{ and } h \text{ are continuous} \tag{5}$$

the target start times and due dates have the property that if $r_i < r_j$ then $d_i \leq d_j$

Sidney's algorithm has a time complexity of $O(n^2)$, under the assumption that the equation

$$g(x) = h(\Delta - x) \tag{6}$$

can be solved for $x \in [0, \Delta]$ sufficiently fast for $\Delta > 0$. Subsequently, Lakshminarayan et al[LLPR78] developed an algorithm that can be implemented to run in $O(n \log n)$ time. Furthermore, assuming (4) and (5) only, Achuthan et al[AGS81] established that the problem can be solved via the maximum tardiness problem with general release times. In fact, they showed that their approach applies to all problems with objective function (a) whose feasible sets have the following translation property: if σ is a feasible schedule and $\alpha \in \mathbb{R}$, then σ_α defined by $\sigma_\alpha(t) := \sigma(t + \alpha)$ is also feasible. In all these results, continuity of g and h is assumed to guarantee that (6) has a solution in $[0, \Delta]$ for each $\Delta > 0$. However many standard penalty functions are discontinuous at certain points. The results presented in [Vla83] allow the continuity assumption (5) to be weakened to one-sided continuity.

Example 2 Kanet[Kan81] considered the problem of minimizing the function

$$\sigma \mapsto \frac{1}{n} \sum_j |L_j(\sigma)|$$

subject to (4) and

$$d_i = d_2 = \cdots = d_n \geq \sum_j p_j$$

Introducing $r_j := d_j - p_j$ for all $j \in N$ and assuming that preemption is not allowed, we have

$$\max\{0, d_j - C_j(\sigma)\} = \max\{0, r_j - S_j(\sigma)\}$$

for each $j \in N$ and each $\sigma \in X$. Moreover

$$|C_j(\sigma) - d_j| = \max\{0, d_j - C_j(\sigma)\} + \max\{0, C_j(\sigma) - d_j\}$$

It follows that Kanet's problem can be considered as a special case of minimizing function (d). A slightly different version of Kanet's problem has been studied by Sundararaghavan and Ahmed[SA84].

Example 3 Gupta and Sen[GS84] addressed the problem of scheduling n jobs on one machine so as to minimize the function

$$\sigma \mapsto \max_j L_j(\sigma) - \min_j L_j(\sigma)$$

subject to the following additional conditions

$$\rho_1 = \rho_2 = \cdots = \rho_n = 0, \quad \delta_1 = \delta_2 = \cdots = \delta_n = \infty$$

and

no idle time is allowed between jobs

Again introducing $r_j := d_j - p_j$, we have

$$\begin{array}{rcl} \min_j L_j(\sigma) & = & \min_j[C_j(\sigma) - d_j] \quad = \quad -\max_j[d_j - C_j(\sigma)] \\ & = & -\max_j[r_j + p_j - C_j(\sigma)] \quad = \quad -\max_j[r_j - S_j(\sigma)] \end{array}$$

It follows that

$$\max_j L_j(\sigma) - \min_j L_j(\sigma) = \max_j L_j(\sigma) + \max_j \mathcal{E}_j(\sigma)$$

Therefore, this problem can be considered as a special case of minimizing the function obtained from (c) by replacing E_j and T_j by $\mathcal{E}_j$ and L_j respectively.

Example 4 Garey et al[GTW88] studied the problems of minimizing the functions

$$\sigma \mapsto \max_j |S_j(\sigma) - r_j|$$

and

$$\sigma \mapsto \sum_j |S_j(\sigma) - r_j|$$

under the assumption that for each $j \in N$

$$p_j = d_j - r_j$$

Assuming this condition and no preemption we see that

$$|S_j(\sigma) - r_j| = E_j(\sigma) + T_j(\sigma)$$

and therefore, the first problem is a special case ($g(x) = h(x) = x$) of minimizing function (c) and the second problem is a special case of minimizing function (d).

2 The general problem

For the remainder of this paper we shall concern ourselves with the following problem

$$\min_{\sigma \in X} f(\sigma) := \psi(g(\max_j E_j(\sigma)), h(\max_j T_j(\sigma))) \tag{7}$$

where $\psi : \mathbb{R}_+ \bigcup \{+\infty\} \times \mathbb{R}_+ \bigcup \{+\infty\} \to \mathbb{R} \bigcup \{+\infty\}$. We assume the standard conventions for ordering with respect to $+\infty$, see for example [Rud76]. We also assume that ψ is isotonic, that is

$$u \geq x, v \geq y \implies \psi(u, v) \geq \psi(x, y)$$

The general problem includes as special cases the objective functions (a) and (c).

We are concerned with single machine problems; however, some results can be extended to problems with several machines. Given p_j, ρ_j , δ_j, r_j, d_j satisfying

$$p_j > 0, \quad \rho_j = -\infty, \quad \delta_j = +\infty, \quad r_j < d_j$$

we let X denote the set of feasible schedules satisfying (1) – (3) and let Y be the subset of X given by the additional condition

$$(j \in N, t < r_j) \implies \sigma(t) \neq j$$

Furthermore, let Δ stand for the minimum value of the function

$$\sigma \mapsto \max_j L_j(\sigma)$$

over the set Y. It is standard practice in the literature to refer to this problem as $1|r_j|L_{max}$, see for instance [LLRKS89], (single machine, general release times, maximum lateness). For the rest of this paper we let η be a schedule from Y such that

$$\max_j L_j(\eta) = \Delta \tag{8}$$

The following two lemmas are useful for our development.

Lemma 5 *If $\Delta > 0$, then for each $x \in [0, \Delta]$ there is $\sigma_x \in X$ such that*

$$\max_j E_j(\sigma_x) \leq x$$

and

$$\max_j T_j(\sigma_x) \leq \Delta - x$$

Proof For each $x \in [0, \Delta]$ define σ_x by

$$\sigma_x(t) := \eta(t + x) \tag{9}$$

Obviously

$$\begin{aligned} E_j(\sigma_x) &= \max\{0, r_j - S_j(\sigma_x)\} \\ &= \max\{0, r_j - S_j(\eta) + x\} \\ &= x + \max\{-x, r_j - S_j(\eta)\} \end{aligned}$$

and

$$\begin{aligned} T_j(\sigma_x) &= \max\{0, C_j(\sigma_x) - d_j\} \\ &= \max\{0, C_j(\eta) - x - d_j\} \\ &= -x + \max\{x, C_j(\eta) - d_j\} \end{aligned}$$

for each $j \in N$. Since $0 \leq x \leq \Delta$ and $S_j(\eta) \geq r_j$ and $C_j(\eta) - d_j \leq \Delta$, we have

$$\max\{-x, r_j - S_j(\eta)\} \leq 0, \quad \max\{x, C_j(\eta) - d_j\} \leq \Delta$$

for each $j \in N$. Therefore

$$\max_j E_j(\sigma_x) \leq x \text{ and } \max_j T_j(\sigma_x) \leq \Delta - x$$

for each $j \in N$, as required. □

Lemma 6 *If $\Delta > 0$, then for each $\sigma \in X$ we have*

$$\max_j \mathcal{E}_j(\sigma) + \max_j L_j(\sigma) \geq \Delta$$

Proof Suppose the claim is false. Then there is some $\bar{\sigma} \in X$ such that

$$\max_j \mathcal{E}_j(\bar{\sigma}) + \max_j L_j(\bar{\sigma}) < \Delta$$

For $\hat{\sigma}$ defined by

$$\hat{\sigma}(t) := \bar{\sigma}(t - \max_k \mathcal{E}_k(\bar{\sigma}))$$

we have

$$S_j(\hat{\sigma}) = S_j(\bar{\sigma}) + \max_k \mathcal{E}_k(\bar{\sigma}) \geq S_j(\bar{\sigma}) + \mathcal{E}_j(\bar{\sigma}) = r_j$$

and so $\hat{\sigma} \in Y$. Moreover

$$\begin{aligned}
\max_j L_j(\hat{\sigma}) &= \max_j [C_j(\hat{\sigma}) - d_j] \\
&= \max_j [C_j(\bar{\sigma}) + \max_k \mathcal{E}_k(\bar{\sigma}) - d_j] \\
&= \max_j L_j(\bar{\sigma}) + \max_k \mathcal{E}_k(\bar{\sigma}) \\
&< \Delta
\end{aligned}$$

However, the inequality $\max_j L_j(\hat{\sigma}) < \Delta$ for $\hat{\sigma} \in Y$ contradicts the definition of Δ. □

The following theorem is the key result of this paper and suggests a method for solving the general problem (7) via the maximum lateness problem.

Theorem 7 *If* $\Delta > 0$ *and*

$$x^* \in \arg\min_{x \in [0,\Delta]} \psi(g(x), h(\Delta - x)) \tag{10}$$

then for η *defined by (8)*

$$\sigma_{x^*}(t) := \eta(t + x^*)$$

minimizes $f(\sigma)$ *over* $\sigma \in X$.

Proof The proof is in two parts. First of all, consider σ_x defined by (9). Now

$$f(\sigma_x) = \psi(g(\max_j E_j(\sigma_x)), h(\max_j T_j(\sigma_x)))$$

Invoking Lemma 5 and Lemma 6 we see that for all $\sigma \in X$

$$\max_j E_j(\sigma_x) + \max_j T_j(\sigma_x) \leq \Delta \leq \max_j \mathcal{E}_j(\sigma) + \max_j L_j(\sigma)$$

However, $\sigma_x \in X$, $E_j(\sigma) \geq \mathcal{E}_j(\sigma)$ and $T_j(\sigma) \geq L_j(\sigma)$ so that for each $x \in \Delta$

$$\max_j E_j(\sigma_x) + \max_j T_j(\sigma_x) = \Delta$$

Moreover

$$\max_j T_j(\sigma_x) = \max_j \max\{0, L_j(\sigma_x)\} = \max\{0, \max_j L_j(\sigma_x)\}$$

and

$$\begin{aligned} \max_j L_j(\sigma_x) &= \max_j[C_j(\sigma_x) - d_j] &&= \max_j[C_j(\eta) - x - d_j] \\ &= -x + \max_j L_j(\eta) &&= \Delta - x \end{aligned}$$

Therefore, $\max_j T_j(\sigma_x) = \max\{0, \Delta - x\} = \Delta - x$, for each $x \in [0, \Delta]$. It now follows that $\max_j E_j(\sigma_x) = x$ and hence that

$$f(\sigma_x) = \psi(g(x), h(\Delta - x)), \quad \text{for all } x \in [0, \Delta] \tag{11}$$

For the second part of the proof we take $\sigma \in X$. It follows from Lemma 6 that

$$\max_j T_j(\sigma) \geq \Delta - \max_j E_j(\sigma)$$

However $\max_j T_j(\sigma) \geq 0$ so we obtain

$$\begin{aligned} \max_j T_j(\sigma) &\geq \max\{\Delta - \Delta, \Delta - \max_j E_j(\sigma)\} \\ &= \Delta + \max\{-\Delta, -\max_j E_j(\sigma)\} \\ &= \Delta - \min\{\Delta, \max_j E_j(\sigma)\} \\ &= \Delta - D(\sigma) \end{aligned}$$

where $D(\sigma) := \min\{\Delta, \max_j E_j(\sigma)\}$. Note that $D(\sigma) \in [0, \Delta]$ and that

$$\max_j E_j(\sigma) \geq \min\{\Delta, \max_j E_j(\sigma)\} = D(\sigma)$$

It now follows from the last two inequalities, the isotonicity of ψ, and g and h nondecreasing that

$$\begin{aligned} f(\sigma) &= \psi(g(\max_j E_j(\sigma)), h(\max_j L_j(\sigma))) \\ &\geq \psi(g(D(\sigma)), h(\Delta - D(\sigma))) \\ &\geq \min_{x \in [0,\Delta]} \psi(g(x), h(\Delta - x)) \\ &= f(\sigma_{x^*}) \end{aligned}$$

the last equality by (11). This is the required result. □

Theorem 7 suggests the following procedure for solving the general problem given by (7).

Step 1 Determine Δ and find a schedule $\eta \in Y$ such that $\max_j L_j(\eta) = \Delta$.

Step 2 If $\Delta \leq 0$, then stop (η is optimal); otherwise, find x^* satisfying (10) and go to Step 3.

Step 3 Construct σ_{x^*} from η and stop (σ_{x^*} is optimal).

Note that the problem (7) is already NP–hard for $\psi(x,y) = \max\{x,y\}$ and $g(x) = h(x) = x$, (see [AGS81]) and also that $1|r_j|L_{max}$ is NP–hard. However, Step 1 can be realized in polynomial time whenever $1|r_j|L_{max}$ can be solved in polynomial time. This is the case, for example, if the data satisfy the condition

$$r_i < r_j \implies d_i \leq d_j$$

To show this consider the maximum lateness problem with release times r_j, due dates d_j and processing times p_j. Without loss of generality let us assume that

$$r_1 \leq r_2 \leq \cdots \leq r_n, \quad d_1 \leq d_2 \leq \cdots \leq d_n \tag{12}$$

First, modify r_j by setting

$$\begin{aligned} r'_1 &:= r_1 \\ r'_j &:= \max\{r_j, r'_{j-1} + p_{j-1}\}, \quad j = 2,3,\ldots,n \end{aligned}$$

Second, define η by

$$\eta(t) := \begin{cases} j & \text{whenever } r'_j < t \leq r'_j + p_j,\ j \in N \\ 0 & \text{otherwise} \end{cases}$$

It is clear that η is the best schedule among those in which the jobs are processed in the order given by the natural ordering of N. Let such a set of schedules be denoted by $\bar{Y}$. We now establish that such a set of schedules contains an optimal schedule.

Lemma 8 *If (12) is satisfied, then for each $\sigma \in Y$, there is a $\eta \in \bar{Y}$ such that*

$$\max_j L_j(\eta) \leq \max_j L_j(\sigma)$$

Proof If $\sigma \in \bar{Y}$ then we take $\eta := \sigma$. If $\sigma \in Y \setminus \bar{Y}$, then there exists $j \in \{2,3,\ldots,n\}$ and $k < j$ satisfying

$$S_j(\sigma) < S_k(\sigma)$$

and

$$S_j(\sigma) + p_j < t < S_k(\sigma) \implies \sigma(t) = 0$$

Define $\bar{\sigma}$ by setting

$$S_k(\bar{\sigma}) := S_j(\sigma), \quad C_j(\bar{\sigma}) := C_k(\sigma)$$

Since $C_j(\bar{\sigma}) > C_k(\bar{\sigma})$ and $d_j \geq d_k$ by assumption, we have

$$\begin{aligned}
\max\{L_k(\bar{\sigma}), L_j(\bar{\sigma})\} &= \max\{C_k(\bar{\sigma}) - d_k, C_j(\bar{\sigma}) - d_j\} \\
&\leq \max\{C_j(\bar{\sigma}) - d_k, C_j(\bar{\sigma}) - d_k\} \\
&= C_j(\bar{\sigma}) - d_k \\
&= C_k(\sigma) - d_k \\
&\leq \max\{L_k(\sigma), L_j(\sigma)\}
\end{aligned}$$

Since all other completion times are the same in both schedules, it is clear that

$$\max_j L_j(\bar{\sigma}) \leq \max_j L_j(\sigma)$$

If $\bar{\sigma} \in \bar{Y}$, then we set $\eta := \bar{\sigma}$. If $\bar{\sigma} \notin \bar{Y}$, then we repeat the process above. After a finite number of such steps, we obtain $\eta \in \bar{Y}$ with the required property. □

Other cases when the maximum lateness problem can be solved in polynomial time include the cases when all r_j are equal, all d_j are equal, or all p_j are equal. Note that the first two are special cases of Lemma 8. An excellent survey paper on these results is [LLRKS89].

Observe that Theorem 7 has been established without any reference to continuity of g and h. However, to convert an optimal schedule for the maximum lateness problem into an optimal schedule for the general problem using Theorem 7 one needs to find an x^* satisfying (10). It is well known[Ber63] that a a sufficient condition for a function to have a minimizer on a compact set is lower semicontinuity, so this is all we need to assume.

Lemma 9 *If* $g, h: \mathbb{R}_+ \to \mathbb{R}_+ \bigcup\{+\infty\}$ *are lower semicontinuous and*

$$\psi: \mathbb{R}_+ \bigcup\{+\infty\} \times \mathbb{R}_+ \bigcup\{+\infty\} \to \mathbb{R} \bigcup\{+\infty\}$$

is lower semicontinuous and isotonic then $\psi(g(x), h(x))$ *is lower semicontinuous on* $x \in \mathbb{R}_+$.

Proof By taking appropriate subsequences we may assume that

$$\liminf_{x \to x^*} \psi(g(x), h(x)) = \lim_{k \in K} \psi(g(x_k), h(x_k))$$

and

$$\liminf_{k \in K} g(x) = \lim_{k \in K} g(x_k)$$

and

$$\liminf_{k \in K} h(x) = \lim_{k \in K} h(x_k)$$

where K is an indexing set for the subsequence. Therefore

$$\begin{aligned}
\liminf_{x \to x^*} \psi(g(x), h(x)) &= \lim_{k \in K} \psi(g(x_k), h(x_k)) \\
&\geq \psi(\lim_{k \in K}(g(x_k), h(x_k)))
\end{aligned}$$

by lower semicontinuity of ψ. However, by invoking lower semicontinuity of g and h and the isotonicity of ψ we see that

$$\begin{aligned}
\liminf_{x \to x^*} \psi(g(x), h(x)) &\geq \psi(\lim_{k \in K} g(x_k), \lim_{k \in K} h(x_k)) \\
&\geq \psi(g(\lim_{k \in K} x_k), h(\lim_{k \in K} x_k)) \\
&= \psi(g(x^*), h(x^*))
\end{aligned}$$

as required. □

Corollary 10 *With appropriate conventions for treating $+\infty$ and if $g, h: \mathbb{R}_+ \to \mathbb{R}_+ \bigcup\{+\infty\}$ are lower semicontinuous, and ψ is defined by*

- *(a)* $\psi(x, y) = \max\{x, y\}$
- *(b)* $\psi(x, y) = x + y$
- *(c)* $\psi(x, y) = xy$
- *(d)* $\psi(x, y) = x^2 + y^2$
- *(e)* $\psi(x, y) = \{x^p + y^p\}^{1/p}$ *for* $1 \leq p < \infty$

then

$$\arg\min_{x \in [0,\Delta]} \psi(g(x), h(\Delta - x)) \neq \emptyset$$

To conclude this paper we show that Theorem 7 is an exact generalization of the work cited in Example 1. If we let $\psi(x, y) \equiv \max\{x, y\}$ then we recover the framework of that example. The auxiliary condition to be solved here was to find an x such that $g(x) = h(\Delta - x)$. We show that condition (10) is equivalent to the existence of such an x under the assumption that g and h are continuous. The following theorem proves this and also provides a saddle point condition equivalent to (10).

Theorem 11 *Suppose $g, h: \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing. Then (10) is equivalent to*

$$\exists x^* \in [0, \Delta] \text{ with } \max\{g(x^*), h(\Delta - x^*)\} \leq \sup_{y \in [0,\Delta]} \min\{g(y), h(\Delta - y)\} \tag{13}$$

If g and h are continuous, (13) is equivalent to the existence of $z^* \in [0, \Delta]$ with $g(z^*) = h(\Delta - z^*)$.

Proof First of all, assume that x^* satisfies (10). Suppose that $g(x^*) \geq h(\Delta - x^*)$. Define $u := \inf\{y \in [0, \Delta] \mid g(y) \geq h(\Delta - y)\}$ and note that $0 \leq u \leq x^*$. It follows that

$$\begin{aligned}
\sup_{y \in [0,\Delta]} \min\{g(y), h(\Delta - y)\} &\geq \sup_{y \in [u,x^*]} \min\{g(y), h(\Delta - y)\} \\
&= \sup_{y \in [u,x^*]} g(y) \\
&\geq g(x^*)
\end{aligned}$$

as required. If now, $g(x^*) \leq h(\Delta - x^*)$ then define $v := \sup\{y \in [0,\Delta] \mid g(y) \leq h(\Delta - y)\}$ and note that $x^* \leq v \leq \Delta$. Also

$$\begin{aligned}
\sup_{y\in[0,\Delta]} \min\{g(y), h(\Delta - y)\} &\geq \sup_{y\in[x^*,v]} \min\{g(y), h(\Delta - y)\} \\
&= \sup_{y\in[x^*,v]} h(\Delta - y) \\
&\geq h(\Delta - x^*)
\end{aligned}$$

as required.

For the converse, suppose that x^* satisfies (13) but not (10). Then there is some $y^* \in [0,\Delta]$ such that

$$\max\{g(y^*), h(\Delta - y^*)\} < \max\{g(x^*), h(\Delta - x^*)\}$$

Therefore

$$\begin{aligned}
&\sup_{y\in[0,\Delta]} \min\{g(y), h(\Delta - y)\} \\
&= \max\{\sup_{y\in[0,y^*]} \min\{g(y), h(\Delta - y)\}, \sup_{y\in[y^*,\Delta]} \min\{g(y), h(\Delta - y)\}\} \\
&= \max\{\sup_{y\in[0,y^*]} g(y), \sup_{y\in[y^*,\Delta]} h(\Delta - y)\} \\
&= \max\{g(y^*), h(\Delta - y^*)\} \\
&< \max\{g(x^*), h(\Delta - x^*)\}
\end{aligned}$$

which a contradiction to (13).

We now consider the case when g and h are continuous. If $g(z^*) = h(\Delta - z^*)$ then (13) clearly holds with $x^* = z^*$. For the converse, suppose (13) holds. Then $\min\{g(y), h(\Delta - y)\}$ is a continuous function and so attains it supremum over the compact set $[0,\Delta]$, say at $\bar{y}$. If $g(\bar{y}) = h(\Delta - \bar{y})$ then we are done. Suppose $g(\bar{y}) < h(\Delta - \bar{y})$, so that $g(\bar{y}) \geq h(\Delta - x^*)$ and $g(\bar{y}) \geq g(x^*)$. But $h(\Delta - \bar{y}) > h(\Delta - x^*)$ so that $\bar{y} < x^*$ implying by g nondecreasing that $g(\bar{y}) \leq g(y^*)$. It follows that g is constant on $[\bar{y}, x^*]$. The existence of a $z^* \in [\bar{y}, x^*]$ now follows from the Mean Value Theorem. (The case when $g(\bar{y}) > h(\Delta - \bar{y})$ is proved in a similar manner). □

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