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by

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CONVERGENCE OF ITERATES OF AN INEXACT MATRIX SPLITTING ALGORITHM FOR THE SYMMETRIC MONOTONE LINEAR COMPLEMENTARITY PROBLEM

O.L. MANGASARIAN†

Abstract. **Convergence of iterates** is established for a symmetric regular matrix splitting algorithm for the solution of the symmetric monotone linear complementarity problem where the subproblems are solved **inexactly**. The notable iterate convergence recently established by Luo and Tseng for exact subproblem solution is extended here to inexact subproblem solution for a symmetric matrix splitting. A principal application of the present result is to iterate convergence for the inexact block Jacobi method for which Pang and Yang established convergence of a **subsequence** of the iterates.

Key words. Iterative matrix splitting, linear complementarity problems

Abbreviated title. Inexact LCP algorithm

1. Introduction.

We consider the classical symmetric linear complementarity problem (LCP) of finding an x in the n-dimensional real space R^n such that

$$Mx + q \geq 0, \quad x \geq 0, \quad x(Mx+q) = 0 \tag{1.1}$$

where M is a given $n \times n$ real symmetric positive semidefinite (spsd) matrix and q is a given vector in R^n. This problem is equivalent to

$$\min_{x \geq 0} f(x) := \min_{x \geq 0} \frac{1}{2} xMx + qx \tag{1.2}$$

Many iterative methods for solving this problem [1,3,5,9,10,11,13] can be modeled as follows. Split the matrix M [10] as follows

$$M = B + C \tag{1.3}$$

and consider the sequence of (simpler) LCP's

$$Bx^{i+1} + Cx^i + q \geq 0, \; x^{i+1} \geq 0, \; x^{i+1}(Bx^{i+1} + Cx^i + q) = 0, \; i = 0, 1, \ldots \tag{1.4}$$

Convergence of a **subsequence** of the iterates $\{x^i\}$ for a variety of iterative methods [1,3,4,5,9,10,11,13] can be established under the simple assumption of a **regular splitting**, that is

$$M = B + C, \quad B - C \text{ positive definite} \tag{1.5}$$

Recently Luo and Tseng [4] were the first to establish that the **whole** sequence $\{x^i\}$ generated by (1.4) converges for a regular splitting (1.5) for a spsd M when $f(x)$ is bounded below on the nonnegative orthant R^n_+. Their proof is rather complex and

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requires that the subproblems (1.4) be solved exactly. By contrast our Algorithm 2.1 below requires only the approximate solution of (1.4) in a precisely defined and implementable way. However, our proof, which is considerably shorter, requires that B be symmetric. It is unclear whether the symmetry of B is the inevitable price one has to pay in order to allow inexactness in the solution of the subproblems (1.4). This was the case also in [13] where convergence is established for a subsequence of the iterates of a two stage procedure for solving the symmetric LCP and where the inner iteration constituted an approximate solution of (1.4), with a symmetric B. An open question therefore remains: Can the symmetry assumption on B be removed from our principal result, Theorem 2.4, while maintaining inexactness of the subproblem solution?

A word about our notation now. For a vector x in the n-dimensional real space R^n, x_+ will denote the vector in R^n with components $(x_+)_i := max\{x_i, 0\}$, $i = 1, \ldots, n$. A symmetric positive definite $n \times n$ real matrix induces an elliptic norm $\|\cdot\|_B$ on R^n, defined by $(xBx)^{\frac{1}{2}}$ for x in R^n. When $B = I$, we have the Euclidean or 2-norm $(xx)^{\frac{1}{2}}$, which we denote simply as $\|\cdot\|$. The one-norm of x, $\sum_{i=1}^{n} |x_i|$ will be denoted by $\|\cdot\|_1$. For an $m \times n$ real matrix A signified by $A \in R^{m\times n}$, A_i denotes the ith row, while A' denotes the transpose. A vector of ones in a real space of any dimension will be denoted by e without a superscript. The identity matrix of any order will be denoted by I. The nonnegative orthant in R^n will be denoted by R^n_+. The projection of a point x in R^n on a closed convex S set in R^n employing the norm $\|\cdot\|_B$ is defined as

$$arg \min_{p\in S} (p-x)B(p-x)$$

and is denoted by $p(x)$.

2. Iterate convergence of a symmetric matrix splitting.

Before stating our algorithm, it is useful to note that the linear complementarity problem (1.1) with any matrix M is equivalent to

$$x = \big(x - (Mx+q)\big)_+ \tag{2.1}$$

This equivalence can be easily checked component wise. Hence the subproblems (1.4) are equivalent to

$$x^{i+1} = \big(x^{i+1} - (Bx^{i+1} + Cx^i + q)\big)_+ \tag{2.2}$$

We shall assume that the associated quadratic function f is bounded below and hence the LCP (1.1) is solvable. Let X^* denote its closed convex solution set. Let $X^* \cap X_\alpha \neq \phi$ where for a positive number α,

$$X_\alpha := \{x | x \in R^n_+,\ ex \le \alpha\} \tag{2.3}$$

We are now ready to state our algorithm.

2.1 ALGORITHM. *Given x^i determine x^{i+1} such that for some "error" sequence $\{h^i\} \subset R^n$ satisfying $\sum_{i=1}^{\infty} \|h^i\| < \infty$:*

$$x^{i+1} = \big(h^{i+1} + x^{i+1} - (Bx^{i+1} + Cx^i + q)\big)_+,\quad i = 0, 1, \ldots \tag{2.4}$$

where $B + C$ s a regular splitting of M with $B' = B$ and

$$(2.5) \qquad \sum_{i=0}^{\infty} \|h^{i+1}\| \cdot (\|x^{i+1} - x^i\| + \|x^i - p^i\|) < \infty,$$

and $p^i := p(x^i)$ is the projection of x^i on $X^* \cap X_\alpha$ using the norm $\|\cdot\|_B$.

Note first that (2.4) which is equivalent to the LCP

$$(2.4a) \quad Bx^{i+1} + Cx^i + q - h^{i+1} \geq 0,\ x^{i+1} \geq 0,\ x^{i+1}(Bx^{i+1} + Cx^i + q - h^{i+1}) = 0$$

is solvable for all values of x^i and h^{i+1} in R^n because B is positive definite, which is in the class of matrices Q for which the linear complementarity problem is solvable for all values of problem data.

Before establishing the convergence of the iterates generated by Algorithm 2.1 we make a few remarks. The assumption that $X^* \cap X_\alpha \neq \phi$ does not imply the boundedness of X^* but merely that its intersection with the simplex X_α is nonempty. The positive number α is some upper bound on the 1-norm of a solution to the LCP (1.1) with least 1-norm. In general α is unknown, but is chosen sufficiently large to insure that $X^* \cap X_\alpha \neq \phi$. If α is not chosen large enough and nonconvergence of the iterates $\{x^i\}$ to a solution of the LCP (1.1) occurs, then this is easily detected and α can be increased. Note also that if there exists an $\hat{x} \geq 0$ which is not a solution of the LCP such that $M\hat{x} + q > 0$, then [6, Theorem 2.2] X^* is indeed bounded and α may be taken as:

$$(2.6) \qquad \alpha = \hat{x}(M\hat{x} + q) \Big/ \min_{1 \leq k \leq n} (M_k\hat{x} + q_k)$$

However, we do not asume the existence of such an $\hat{x}$. The size of α enters Algorithm 2.1 only in ensuring that condition (2.5) is satisfied. This is discussed further in Remark 2.5 below. Lemma 2.6 below gives a precise way for implementing (2.5). The plausibility of (2.5) can be demonstrated as follows. Since B is positive definite, the subproblem (2.4) is solvable for all values of h^{i+1}. Denote the explicit dependence of x^{i+1} on h^{i+1} by writing $x^{i+1}(h^{i+1})$. By [7, Theorem 3.3], $x^{i+1}(h^{i+1})$ is Lipschitzian in h^{i+1} with a Lipschitz constant ν depending on B only. Hence

$$\|h^{i+1}\| \cdot \|x^{i+1}(h^{i+1}) - x^i\| \leq \nu\|h^{i+1}\|^2 + \|h^{i+1}\| \cdot \|x^{i+1}(0) - x^i\|$$

which ensures the smallness of the first term of (2.5) by picking $\|h^{i+1}\|$ sufficiently small. The existence of an upper bound on $\|x^i - p^i\|$ in terms of x^i [8, Theorem 2.11] ensures the smallness of the second term of (2.5). See Remark 2.5 and Lemma 2.6 for details.

We shall need the following simple but useful two lemmas, the first due to Cheng [1] which is a special case of a more general lemma [14, Lemma 2, p. 44].

LEMMA 2.2. *[1, Lemma 2.1] Let $\{e^i\}$ and $\{\varepsilon^i\}$ be two sequences of nonnegative real numbers with $\sum_{i=0}^{\infty} \varepsilon^i < \infty$ and $0 \leq e^{i+1} \leq e^i + \varepsilon^i$ for $i = 0, 1, \ldots$ Then the sequence $\{e^i\}$ converges.*

LEMMA 2.3. *[5, Lemma 2.2] For $x \in R^n$ and $y \in R^n_+$ it follows that $(x - x_+)(y - x_+) \leq 0$.*

We are ready now to state and prove our principal convergence result. We remark that our proof is motivated by Polyak's convergence proof of the gradient projection algorithm [14, pp. 207-208]. However, a number of new ideas were needed such as introducing the inexactness h^{i+1} and the manner in which it is introduced and decreased, introducing the truncation X_α and projecting on its intersection with the solution set, and the use of the matrix B in the projection norm.

THEOREM 2.4. *Let the LCP (1.1) be solvable for some symmetric positive semidefinite M. Then the iterates $\{x^i\}$ of Algorithm 2.1 converge to a solution x^* of the LCP (1.1).*

Proof. By (2.4) x^{i+1} is a projection on R^n_+ and hence it follows by Lemma 2.3 above that

$$\left(h^{i+1} - (Bx^{i+1} + Cx^i + q)\right)(p^i - x^{i+1}) \leq 0$$

Since $(p^i - x^{i+1})(Mp^i + q) \leq 0$ it follows that

$$(p^i - x^{i+1})\left(B(x^i - x^{i+1}) - M(x^i - p^i)\right) \leq h^{i+1}(x^{i+1} - p^i) \tag{2.7}$$

From the identity

$$0 = \|x^i - p^i\|_B^2 - \|(x^i - x^{i+1}) - (p^i - x^{i+1})\|_B^2$$

and the symmetry of B we have

$$2(x^i - x^{i+1})B(p^i - x^{i+1}) = -\|x^i - p^i\|_B^2 + \|x^i - x^{i+1}\|_B^2 + \|x^{i+1} - p^i\|_B^2 \tag{2.8}$$

From the symmetry and positive semidefiniteness of M we have for $a,\ b \in R^n$

$$aMa + bMa \geq -bMb/4$$

Hence

$$\begin{aligned}(x^{i+1} - p^i)M(x^i - p^i) = (x^{i+1} - x^i)M(x^i - p^i) + (x^i - p^i)M(x^i - p^i) \geq \\ -(x^{i+1} - x^i)M(x^{i+1} - x^i)/4\end{aligned} \tag{2.9}$$

Use of (2.8) and (2.9) in inequality (2.7) multiplied by 2 and invoking the positive definiteness of $B - C$ gives

$$\|x^{i+1} - p^i\|_B^2 \leq \|x^i - p^i\|_B^2 + 2h^{i+1}(x^{i+1} - p^i) \tag{2.10}$$

or equivalently (adding and subtracting p^{i+1} within the first term)

$$\begin{aligned}\|x^{i+1} - p^{i+1}\|_B^2 + 2(x^{i+1} - p^{i+1})B(p^{i+1} - p^i) + \|p^{i+1} - p^i\|_B^2 \leq \\ \|x^i - p^i\|_B^2 + 2h^{i+1}(x^{i+1} - p^i)\end{aligned} \tag{2.11}$$

Since

$$p^{i+1} = arg \min_{X^* \cap X_\alpha} (p - x^{i+1})\frac{B}{2}(p - x^{i+1})$$

it follows by the Minimum Principle that

$$(p - p^{i+1})B(p^{i+1} - x^{i+1}) \geq 0 \qquad \forall p \in X^* \cap X_\alpha$$

Hence the second term in (2.11) is nonnegative and can be dropped. Dropping also the third term in (2.11) which is also nonnegative gives

$$\|x^{i+1} - p^{i+1}\|_B^2 \le \|x^i - p^i\|_B^2 + 2\|h^{i+1}\|(\|x^{i+1} - x^i\| + \|x^i - p^i\|)$$

It follows from (2.5) of Algorithm 2.1 and Cheng's Lemma that the sequence $\{\|x^i - p^i\|_B\}$ converges, and so does $\{\|x^i - p^i\|\}$ converge to β, say. Then for any $\delta > 0$ we have

$$-\delta \le \|x^i - p^i\| - \beta \le \delta \qquad \forall i \ge i(\delta)$$

and hence

$$\|x^i\| \le \|p^i\| + \beta + \delta \qquad \forall i \ge i(\delta)$$

Since $\{p^i\} \subset X_\alpha$, it follows that $\{p^i\}$ is bounded and so is $\{x^i\}$. Now

$$\begin{aligned}
f(x^i) - f(x^{i+1}) &= -(Mx^i + q)(x^{i+1} - x^i) - \|x^{i+1} - x^i\|^2_{M/2} \\
&= \left(x^{i+1} + h^{i+1} - \left[h^{i+1} + x^{i+1} - B(x^{i+1} - x^i) - (Mx^i + q)\right]\right)\cdot \\
&\quad\ (x^i - x^{i+1}) + \|x^{i+1} - x^i\|^2_{\frac{B-C}{2}} \\
&\ge h^{i+1}(x^i - x^{i+1}) + \|x^{i+1} - x^i\|^2_{\frac{B-C}{2}} \\
&\quad\ \text{(By Lemma 2.3, because } x^{i+1} \text{ is the projection on } R^n_+ \\
&\quad\ \text{of the term in the square bracket.)} \\
&\ge -\|h^{i+1}\| \cdot \|x^{i+1} - x^i\| + \gamma\|x^{i+1} - x^i\|^2 \\
&\quad\ \text{(where } \gamma \text{ is the smallest eigenvalue of } \frac{B-C}{2}\text{)}
\end{aligned}$$

Hence

$$\text{(2.12)} \qquad f(x^i) - f(x^{i+1}) \ge \gamma\|x^{i+1} - x^i\|^2 - \|h^{i+1}\| \cdot \|x^{i+1} - x^i\|$$

Let $\bar{x} \in X^*$, then

$$0 \le f(x^{i+1}) - f(\bar{x}) \le \gamma\|x^{i+1} - x^i\|^2 + f(x^{i+1}) - f(\bar{x}) \le f(x^i) - f(\bar{x}) + \|h^{i+1}\| \cdot \|x^{i+1} - x^i\|$$

By (2.5) we have that $\sum_{i=0}^{\infty} \|h^{i+1}\| \cdot \|x^{i+1} - x^i\| < \infty$ and $\{\|h^{i+1}\| \cdot \|x^{i+1} - x^i\|\} \to 0$. Hence again by Cheng's Lemma, the sequence $\{f(x^i) - f(\bar{x})\}$ converges, and so does the sequence $\{f(x^i)\}$. It follows from (2.12) that

$$0 = \lim_{i\to\infty} \left(f(x^i) - f(x^{i+1}) + \|h^{i+1}\| \cdot \|x^{i+1} - x^i\|\right) \ge \gamma \lim_{i\to\infty} \|x^{i+1} - x^i\|^2 \ge 0$$

Hence $\lim_{i\to\infty} \|x^{i+1} - x^i\| = 0$. Now, since B is positive definite, the single-valued map $x^{i+1} = T(x^i,\, h^{i+1})$ defined by (2.4) or equivalently by (2.4a) is Lipschitzian [7, Theorem 3.3], with Lipschitz constant dependent on B only. Thus $\lim_{i\to\infty} \|T(x^i,\, h^{i+1}) - x^i\| = 0$. Since $\{h^{i+1}\} \to 0$ and T is continuous, it follows that for an accumulation point x^* of the bounded sequence $\{x^i\}$, that $\{x^{i_j}\} \to x^*$ and $T(x^*,\, 0) = x^*$. The condition $T(x^*,\, 0) = x^*$ is equivalent to $x^* \in X^*$.

We now repeat the argument which begins this proof until we reach (2.10) but with x^* replacing p^i. Thus replacing p^i by x^* in (2.10) gives

$$\|x^{i+1} - x^*\|_B^2 \leq \|x^i - x^*\|_B^2 + 2\|h^{i+1}\| \cdot \|x^{i+1} - x^*\| \tag{2.13}$$

Now employing the Lipschitz continuity of the projection operator p with Lipschitz constant μ and the fact that p^{i+1} is in X_α we have that

$$\begin{aligned}\|x^{i+1} - x^*\| = \|x^{i+1} - x^i - p^{i+1} + p^i + x^i - p^i + p^{i+1} - x^*\| \leq \\ (1+\mu)\|x^{i+1} - x^i\| + \|x^i - p^i\| + \alpha + \|x^*\|\end{aligned} \tag{2.14}$$

We note that μ may be taken as the ratio of the largest to the smallest eigenvalues of B. It follows then from (2.5), from $\sum_{i=0}^{\infty} \|h^{i+1}\| < \infty$ and (2.14) that

$$\sum_{i=0}^{\infty} \|h^{i+1}\| \cdot \|x^{i+1} - x^*\| < \infty \tag{2.15}$$

Hence by (2.13), (2.15) and Cheng's Lemma we have that the sequence $\{\|x^i - x^*\|\}$ converges. We claim now that if $\{\|x^i - x^*\|\}$ converges to a positive number δ, say, a contradiction ensues. For

$$\frac{\delta}{2} > \|x^i - x^*\| - \delta > -\frac{\delta}{2} \qquad \forall i \geq \bar{i} \text{ for some } \bar{i}$$

But since $\{x^{i_j}\} \to x^*$

$$\frac{\delta}{2} > \|x^{i_j} - x^*\| \qquad \text{for some } i_j \geq \bar{i}$$

The last two inequalities are contradictory. Hence $\|x^i - x^*\| \to 0$ and $\lim_{i\to\infty} x^i = x^* \in X^*$. ∎

We discuss now how the inexactness condition (2.5) of Algorithm 2.1 can be implemented precisely.

Remark 2.5. Since M is positive semidefinite we can, by a slight modification of [8, Theorem 2.7] replace $\|x^i - p^i\|$ in (2.5) of Algorithm 2.1 by a computable error bound multiplied by a constant $\mu(M, B, q, \alpha)$ as follows:

$$\begin{aligned}\|x^i - p^i\| \leq \mu(M,B,q,\alpha)\big[\|\big(x^i(Mx^i+q),\ -Mx^i-q,\ ex^i-\alpha\big)_+\|+ \\ \Big(\big(x^i(Mx^i+q)\big)_+ + \|(-Mx^i-q)_+\|\Big)^{\frac{1}{2}}\big] = \mu(M,B,q,\alpha)\sigma(x^i)\end{aligned}$$

where $\sigma(x^i)$ is defined by the term in the square bracket. Condition (2.5) is then implied by

$$\sum_{i=0}^{\infty} \|h^{i+1}\| \cdot \big(\|x^{i+1} - x^i\| + \sigma(x^i)\big) < \infty \tag{2.5a}$$

We give now a precise way of implementing (2.5a).

LEMMA 2.6. *Inequality (2.5a) and hence (2.5) hold by taking $\|h^{i+1}\|$ equal to the largest element of $\{\frac{\|h^i\|}{2}, \frac{\|h^i\|}{4} \cdots\}$ such that*

$$\|h^{i+1}\|(\|x^{i+1}-x^i\|+\sigma(x^i)) \le \frac{\|h^i\|}{2}(\|x^i-x^{i-1}\|+\sigma(x^{i-1})) \tag{2.5b}$$

Proof. All we need to show is that (2.5b) holds for $\|h^{i+1}\|$ sufficiently small. Since B is positive definite we have that $x^{i+1}=x^{i+1}(h^{i+1})$ is Lipschitzian [7, Theorem 3.3] with constant ν depending on B only. Hence

$$\|x^{i+1}(h^{i+1})-x^{i+1}(0)\| \le \nu\|h^{i+1}\|$$

Hence (2.5b) is implied by

$$\|h^{i+1}\|(\nu\|h^{i+1}\|+\|x^{i+1}(0)-x^i\|+\sigma(x^i)) \le \frac{\|h^i\|}{2}(\|x^i-x^{i-1}\|+\sigma(x^{i-1}))$$

that is

$$\nu\|h^{i+1}\|^2+\|h^{i+1}\|\big[\|x^{i+1}(0)-x^i\|+\sigma(x^i)\big]-\big[\frac{\|h^i\|}{2}(\|x^i-x^{i-1}\|+\sigma(x^{i-1}))\big] \le 0 \tag{2.5c}$$

Defining the terms in the first and second square brackets in (2.5c) by ν^i and μ^i respectively, we have that (2.5c) is satisfied, and hence also (2.5), if we take $\|h^{i+1}\| \in [0,\ \rho^{i+1}]$ where

$$\rho = \frac{-\nu^i+\sqrt{\nu^{i2}+4\nu\mu^i}}{2\nu} \qquad \blacksquare$$

Remark 2.7. We note here that the sequence $\{x^i\}$ was determined as a function of the error sequence $\{h^i\}$ by solving the subproblems (2.4) of Algorithm 2.1. This entails then an exact solution of the equivalent linear complementarity problem (2.4a) with a prescribed error term h^{i+1}, and in a certain sense that is at cross purposes to solving the original subproblems (1.4) inexactly. To avoid this we outline here a procedure that does not require exact subproblem solution. Let $\{y^i\}$ be a sequence of points in R^n which are obtained in any way as approximate solutions to the subproblems (1.4) with y replacing x in (1.4). We show now how the error sequence $\{h^i\}$ satisfying (2.4) is computed from $\{y^i\}$. For this purpose we first define the **computable** error bound in satisfying (1.4) as follows. Let

$$e^{i+1}=e^{i+1}(y^{i+1}) := min\,\{y^{i+1},\ By^{i+1}+Cy^i+q\} \tag{2.16}$$

or equivalently

$$e^{i+1} := y^{i+1}-\left(y^{i+1}-(By^{i+1}+Cy^i+q)\right)_+ \tag{2.16a}$$

By [12, Lemma 2], the error $\|y^{i+1}-y^{i+1}(0)\|$, where $y^{i+1}(0)$ is the unique solution of (1.4) with y replacing x, is bounded by the computable e^{i+1} as follows

$$\|y^{i+1}-y^{i+1}(0)\| \le \lambda(B)\|e^{i+1}\| \tag{2.17}$$

where

$$\lambda(B) := 1+\frac{\|I-B\|}{\alpha},\ \alpha := \text{min eigenvalue }(B) > 0 \tag{2.18}$$

Note that e^{i+1} is easily computable from (2.16) and hence can be used as a simple measure of how exactly y^{i+1} satisfies (1.4). We now relate h^{i+1} to e^{i+1}. From (2.16a) we have that y^{i+1} solves the linear complementarity problem

$$(2.19)\quad w^{i+1} = By^{i+1} + Cy^i + q - e^{i+1} \geq 0,\ y^{i+1} - e^{i+1} \geq 0,\ (y^{i+1} - e^{i+1})w^{i+1} = 0$$

or equivalently

$$(2.19a)\qquad \begin{aligned} w^{i+1} = B(y^{i+1} - e^{i+1}) + C(y^i - e^i) + q + (B - I)e^{i+1} + Ce^i \geq 0, \\ y^{i+1} - e^{i+1} \geq 0, (y^{i+1} - e^{i+1})w^{i+1} = 0 \end{aligned}$$

By defining

$$(2.20)\qquad x^{i+1} := y^{i+1} - e^{i+1},\ x^i = y^i - e^i,\ h^{i+1} := (I - B)e^{i+1} - Ce^i$$

the subproblem (2.19a) reduces to (2.4a), and the error term h^{i+1} can be computed from the relation $h^{i+1} = (I - B)e^{i+1} - Ce^i$ in (2.20). Thus the smallness condition (2.5a) on the sequence $\{\|h^i\|\}$ can be translated, through the relations (2.20), (2.16a) and the nonexpansiveness of the plus function $(\cdot)_+$, into a smallness condition on the sequence $\{\|e^i\|\}$ as follows

$$(2.21)\qquad \sum_{i=0}^{\infty} (\|e^{i+1}\| + \|e^i\|)(\|y^{i+1} - y^i\| + \|y^i - y^{i-1}\| + \sigma(y^i - e^i)) < \infty$$

3. Conclusions. We have established convergence of the iterates for a symmetric regular splitting algorithm for the symmetric monotone linear complementarity problem. The principal application is probably to an inexact block Jacobi method for solving the symmetric LCP. In particular if we let

$$(3.1)\qquad M = L + D + L'$$

where D is some **block** diagonal of M and $L + L'$ is the remaining part of M, then we can take

$$(3.2)\qquad B = \lambda I + D,\ C = -\lambda I + L + L',\ B - C = 2(\lambda I + D) - M$$

This splitting is regular for

$$(3.3)\qquad \lambda > \text{max eigenvalue } (\frac{M}{2} - D)$$

The splitting (3.2) is useful in the parallel solution of linear programs where the constraints of the problem are distributed among the processors and the objective function is appropriately modified for each processor by Lagrangian and proximal terms. This will be discussed in a forthcoming paper [2].

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