ON THE INTEGER TRANSLATES OF A COMPACTLY SUPPORTED FUNCTION: DUAL BASES AND LINEAR PROJECTORS

by

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ABSTRACT

Given a multivariate compactly supported function ϕ, we discuss here linear projectors to the space $S(\phi)$ spanned by its integer translates. These projectors are constructed with the aid of a dual basis for the integer translates of ϕ, hence under the assumption that these translates are linearly independent. Our main result shows that the linear functionals of the dual basis are local, hence makes it possible to contruct local linear projectors onto $S(\phi)$. We then discuss, for a general compactly supported function, a scheme for the construction of such local projectors.

In the second part of the paper we apply these observations to piecewise-polynomials and piecewise-exponentials to obtain a necessary and sufficient condition for a quasi-interpolant to be a projector. The results of that part extend and refine recent constructions of dual bases and linear projectors for polynomial and exponential box splines.

AMS (MOS) Subject Classifications: primary 41A15, 41A63; secondary 41A05, 46A22, 65D07.

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On the integer translates of a compactly supported function: dual bases and linear projectors

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1. Introduction

Let ϕ be a complex-valued compactly supported continuous function defined on $\mathbb{R}^s$, and E^α the **shift** operator

$$E^\alpha f = f(\cdot + \alpha).$$

Associated with ϕ is the semi-discrete convolution operator

$$\phi* : c \mapsto \phi * c := \sum_{\alpha \in \mathbb{Z}^s} c(\alpha) E^{-\alpha}\phi, \tag{1.1}$$

which acts from the domain

$$C := \{c : \mathbb{Z}^s \to \mathbb{C}\}$$

of all complex-valued sequences defined on the multi-integers to the range

$$S(\phi) := \operatorname{ran} \phi*$$

of the functions spanned by the integer translates of ϕ.

In multivariate spline approximation, $S(\phi)$ is of potential use as a space of approximants for a larger function space (e.g., $C(\mathbb{R}^s)$). The operator $\phi*$ is exploited in the derivation of explicit approximation schemes from $S(\phi)$. Such a scheme may result in a linear projector onto $S(\phi)$ (cf. [B$_1$] and [BF] for the construction of linear projectors for *univariate* and *tensor product* splines). Since the construction of linear projectors in the multivariate case is usually quite involved, one is satisfied with the so-called quasi-interpolation schemes, which yield the same approximation order.

In case the integer translates of ϕ are globally linearly independent (i.e., $\ker(\phi*) = 0$), a natural way to define a linear projector onto $S(\phi)$ is with the aid of a linear functional λ that satisfies

$$\lambda(E^\alpha \phi) = \delta_{\alpha,0}, \ \alpha \in \mathbb{Z}^s.$$

For then, the functionals $\Lambda := \{\lambda E^\alpha\}_\alpha$ form a dual basis for $\Phi := \{E^{-\alpha}\phi\}_\alpha$, and a linear projector Ψ can than be defined in the usual way:

$$\Psi = \sum_{\alpha \in \mathbb{Z}^s} E^{-\alpha}\phi \, \lambda E^\alpha.$$

For the analysis of the approximation properties of the projector, its **localness** is important: a projector Ψ is termed "local" if for every compact $A \subset \mathbb{R}^s$ there exists a compact $B \subset \mathbb{R}^s$ such that $\Psi(f)_{|A}$ is determined by $f_{|B}$. The existence of a local projector is guaranteed in case the generator λ of the dual basis Λ is local. The construction of local linear projectors is facilitated if one assumes that the integer translates of ϕ are **locally linearly independent** i.e., the condition

$$(\phi * c)_{|A} = 0 \text{ and } \operatorname{supp} E^{-\alpha}\phi \cap A \neq \emptyset \Longrightarrow c(\alpha) = 0, \tag{1.2}$$

for every open $A \subset \mathbb{R}^s$. It is one of the main themes of this paper to show that local projectors can be constructed even under the weaker assumption of global linear independence. As a matter of fact, that observation neither makes use of the shift-invariance of the space $S(\phi)$ nor of the fact that ϕ is a function:

(1.3) Theorem. *Let $\Phi = \{\phi_\alpha\}_{\alpha\in\mathbb{Z}^s}$ be a locally finite collection of (globally) linearly independent compactly supported distributions in $\mathcal{D}'(\mathbb{R}^s)$. Then each functional λ_α in the (algebraic) dual basis $\Lambda = \{\lambda_\alpha\}_{\alpha\in\mathbb{Z}^s}$ of Φ is local. Precisely, for every $\alpha \in \mathbb{Z}^s$ there exists a compact $B_\alpha \subset \mathbb{R}^s$ such that $\lambda_\alpha(f) = 0$ whenever $\operatorname{supp} f \cap B_\alpha = \emptyset$.*

In section 2 we prove (1.3)Theorem and employ this result to show that the assumption of linear independence of the integer translates of ϕ is already sufficient to allow the existence of a dual basis based on a linear functional λ of point-evaluation at a finite set of $\mathbb{R}^s$. The proof of the theorem gives also information about the diameter of $\operatorname{supp}\lambda$ which in the case of local linear independence coincides with the standard result. Finally, we comment about the connection of these results to the area of cardinal interpolation by translates of a compactly supported function.

The construction of local projectors is then discussed in section 3. There we take ϕ to be an arbitrary function whose translates are linearly independent, and, based on a new extended notion of a quasi-interpolant, provide a necessary and sufficient condition for a quasi-interpolant to be a linear projector. With the aid of this observation, we then describe a general scheme for the derivation of local projectors onto $S(\phi)$.

In the two last sections we examine the piecewise-exponential case (which contains the piecewise-polynomial case). Section 4 is devoted to a brief discussion of some known methods for constructing quasi-interpolants for piecewise-exponentials. These results, together with observations from section 3, are used in the last section where we show that for piecewise-exponentials a slightly stronger sense of local linear independence is sufficient to imply that every quasi-interpolant is also a linear projector. We conclude that section with a review of the constructions of linear projectors in [DM$_{1,2}$] and [J$_{1,2}$] providing thereby new proofs and extensions to these results.

2. Linear projectors are local

In these section we prove (1.3)Theorem and discuss some of its applications.

The following is an equivalent form of (1.3)Theorem, which is slightly more convenient for the proof employed:

(2.1) Theorem. *Let $\Phi = \{\phi_\alpha\}_{\alpha\in\mathbb{Z}^s}$ be a locally finite collection of globally linearly independent compactly supported distributions in $\mathcal{D}'(\mathbb{R}^s)$. Then there exists a ball*

$$B := \{x : \|x\| \le L\} \tag{2.2}$$

such that if $f = \sum_{\alpha\in\mathbb{Z}^s} c(\alpha)\phi_\alpha$ satisfies $\operatorname{supp} f \cap B = \emptyset$ then $c(0) = 0$.

Indeed, (1.3)Theorem readily follows from (2.1)Theorem: assume (without loss) that $\alpha = 0$ in (1.3)Theorem. For every $f = \sum_{\alpha\in\mathbb{Z}^s} c(\alpha)\phi_\alpha \in \operatorname{span}\Phi$, $\lambda_0(f) = c(0)$. So, if we assume (2.1)Theorem and choose B_0 of (1.3)Theorem to be B in (2.2), then whenever $\operatorname{supp} f \cap B = \emptyset$, (1.3)Theorem implies $\lambda_0(f) = c(0) = 0$.

We postpone the proof of (2.1)Theorem to the end of this section, and discuss now some of its applications.

Suppose that $\Phi \subset C(\mathbb{R}^s)$† and choose $\alpha \in \mathbb{Z}^s$. Let B_α be the ball associated with α (by (1.3)Theorem). Define

$$\nu_\alpha := \{\beta \in \mathbb{Z}^s : \operatorname{supp} \phi_\beta \cap B_\alpha \neq \emptyset\}. \tag{2.3}$$

Since the elements of Φ are locally finite, the set ν_α is finite. Defining

$$S(\Phi) = \operatorname{span} \Phi, \quad S_\alpha(\Phi) := \operatorname{span}\{\phi_{\beta|B_\alpha} : \beta \in \nu_\alpha\},$$

(1.3)Theorem implies that any extension $\mu_\alpha \in C(\mathbb{R}^s)^*$ of the restricted linear functional $\lambda_{\alpha|S_\alpha(\Phi)}$ is also an extension of λ_α, provided that $\operatorname{supp} \mu_\alpha \subset B_\alpha$. Now, $S_\alpha(\Phi)$ is a finite-dimensional subspace of $C(B_\alpha)$, B_α being compact, and hence there are various available ways to represent the restriction of λ_α to $S_\alpha(\Phi)$; e.g., one may choose a set $b_\alpha \subset B_\alpha$ of cardinality $\#\nu_\alpha$, which is **total** for $S_\alpha(\Phi)$ (i.e., no element in $S_\alpha(\Phi)\backslash 0$ vanishes on b_α). Then there is a unique linear combination $\mu_\alpha = \sum_{x \in b_\alpha} c(x)\delta_x$ satisfying

$$\mu_\alpha(f) = \lambda_\alpha(f), \ \forall f \in S(\Phi), \tag{2.4}$$

where δ_x is the functional of point-evaluation at x. Thus we conclude

(2.5) Corollary. *Assume that $\Phi \subset C(\mathbb{R}^s)$. Then there exists a projector*

$$\Psi : C(\mathbb{R}^s) \to S(\Phi) : f \mapsto \sum_{\alpha \in \mathbb{Z}^s} \mu_\alpha(f)\phi_\alpha, \tag{2.6}$$

such that each μ_α is supported on a finite set.

In the special case of interest, viz. when $\phi_\alpha = E^{-\alpha}\phi$, we have

$$\sum_{x \in b} c(x)\phi(x+\alpha) = \mu(E^\alpha \phi) = \begin{cases} 1, & \alpha = 0, \\ 0, & \alpha \neq 0. \end{cases}$$

(with $\phi := \phi_0$, $\mu := \mu_0 = \sum_{x \in b} c(x)\delta_x$, and $b := b_0$). This proves the following

(2.7) Corollary. *Let ϕ be a compactly supported continuous function whose integer translates are globally linearly independent. Then there exists a finite linear combination ψ of (real) translates of ϕ satisfying*

$$\psi_{|\mathbb{Z}^s} = \delta_\alpha.$$

Such ψ is usually referred to as "a fundamental solution". In cardinal interpolation, one looks for a fundamental solution spanned by (infinitly many) *integer* translates of ϕ. We note that the existence of the latter fundamental solution does not require the global linear independence of the translates, yet if one imposes decay conditions on ψ then global linear independence is, in general, an insufficient condition for the existence of such ψ (cf. [Ri] for details).

† $C(\mathbb{R}^s)$ is chosen only for convenience; any superspace $V \supset S(\phi)$ in which point-evaluation is well defined will do.

Proof of (2.1)Theorem: Assume, on the contrary, that such a ball B does not exist. For every positive integer n, let B_n be the open ball centered at the origin with radius n and define

$$\nu_n := \{\alpha \in \mathbb{Z}^s : \operatorname{supp}\phi_\alpha \cap B_n \neq \emptyset\}. \tag{2.8}$$

Let M_n be the linear space of all sequences c defined on ν_n and satisfying

$$\operatorname{supp}(\sum_{\alpha\in\nu_n} c(\alpha)\phi_\alpha) \cap B_n = \emptyset. \tag{2.9}$$

Since Φ is locally finite, every ν_n is finite, and hence every M_n is finite-dimensional. On the other hand, no M_n is trivial, since by our assumption each M_n must contain at least one element satisfying $c(0) \neq 0$. To obtain the desired contradiction, we will show that there is a non-trivial sequence in $\mathcal{C}$ whose restriction to each ν_n lies in M_n. Since the union of the sets $\{\nu_n\}_n$ is $\mathbb{Z}^s$, such sequence c induces a non-trivial vanishing combination of Φ thus contradicting the linear independence of the elements in Φ.

For this purpose, define $M_0 := \mathbf{C}$ and for all non-negative integers $m \geq n > 0$, let R_n^m be the restriction map from M_m to M_n (with $R_0^n : c \mapsto c(0)$ and R_0^0 the identity mapping). Clearly,

$$R_0^m \neq 0, \quad \forall\, m. \tag{2.10}$$

Defining $K_n \subset M_n$ by

$$K_n := \bigcap_{m\geq n} \operatorname{ran} R_n^m,$$

we note that the condition

$$K_n \neq 0$$

is necessary (but apparently not sufficient) for the existence of a sequence c whose restriction to each ν_n lies in M_n. Indeed, we claim

(2.11) Lemma. *For each $n \geq 0$, $K_n \neq 0$.*

Proof: Since for every $n \leq k \leq m$

$$R_n^m = R_n^k R_k^m, \tag{2.12}$$

$\{\operatorname{ran} R_n^m\}_{m\geq n}$ is a decreasing sequence of finite-dimensional vector spaces. Therefore, for all sufficiently big k and m, $R_n^m = R_n^k$, and hence $K_n = \operatorname{ran} R_n^m$ for all sufficiently big m. Finally, since $R_0^m = R_0^n R_n^m$, then (2.10) implies that $R_n^m \neq 0$, hence so is K_n. ♠

In the following we will prove the existence of a sequence $c \in \mathcal{C}$ whose restriction to each ν_n is in K_n, hence in M_n. For this we first note that the proof of the previous lemma shows that $K_n = \operatorname{ran} R_n^m$, for sufficiently big m. Therefore, we can find m such that

$$K_j = \operatorname{ran} R_j^m, \; j = n, n+1,$$

and thus

$$K_n = R_n^{n+1} R_{n+1}^m M_m = R_n^{n+1} K_{n+1}. \tag{2.13}$$

We can now complete the proof of (2.1)Theorem as follows: let n be arbitrary. By (2.11)Lemma there exists a non-trivial $c_n \in K_n$. Invoking (2.13) we may choose $c_{n+1} \in K_{n+1}$ whose restriction to ν_n is c_n. Again, (2.13) can be employed to provide $c_{n+2} \in K_{n+2}$ whose restriction to ν_{n+1} is c_{n+1}. Proceeding in this manner we obtain $(c_m \in K_m)_{m=n}^{\infty}$ such that $R_n^m c_m = c_n$. This gives rise to a sequence $c \in C$ satifying

$$\operatorname{supp}\left(\sum_{\alpha\in\nu_m} c(\alpha)\phi_\alpha\right) \cap B_m = \emptyset, \ \forall m,$$

while c is non-trivial since its restriction c_n to ν_n is non-trivial. We therefore conclude that the distributions in Φ are globally linearly dependent, in contradiction to the assumptions the theorem. ♠

3. Linear projectors and quasi-interpolation

Throughout this section we assume that ϕ is a compactly supported continuous function whose integer translates are globally linearly independent, and $A \subset \mathbb{R}^s$ is an open bounded set which satisfies for every $c \in C$ the condition

$$(\phi * c)_{|A} = 0 \implies c(0) = 0. \tag{3.1}$$

The existence of such an A was proved in (1.3)Theorem.

Given a linear functional $\lambda : C(\mathbb{R}^s) \to \mathbb{C}$, we examine here conditions that guarantee that the operator

$$Q_\lambda := \phi * \Lambda(\cdot)$$

is a projector. Here

$$\Lambda : C(\mathbb{R}^s) \to C : f \mapsto (\lambda E^\alpha f)_{\alpha\in\mathbb{Z}^s}.$$

Our aim is to generate a space $F \subset S(\phi)$ (which replaces the missing polynomial space usually associated with a piecewise-polynomial ϕ) that is of help in the identification of a projector Q_λ. We use the notation $f_| := f_{|\mathbb{Z}^s}$, and let

$$\phi *'$$

stand for the semidiscrete convolution operator from $C(\mathbb{R}^s)$ to $S(\phi)$ defined by

$$\phi *' f := \phi * f_| = \sum_{\alpha\in\mathbb{Z}^s} f(\alpha) E^{-\alpha}\phi. \tag{3.2}$$

Finally,

$$\phi\, '*' f$$

denotes the discrete convolution $\phi_| * f_|$.

(3.3) Theorem. *Let F be a subspace of $S(\phi)$ satisfying*

$$F_{|A} = S(\phi)_{|A}, \tag{3.4}$$

and assume that $\operatorname{supp}\lambda \subset A$. *Then the following conditions are equivalent:*

(a)
$$Q_\lambda(f) = f, \ \forall f \in F.$$

(b) Q_λ is a projector, i.e.,
$$Q_\lambda(f) = f, \ \forall f \in S(\phi).$$

Proof: The implication (b) $\Longrightarrow$ (a) is trivial. For the converse, it is necessary and sufficient to prove that
$$\lambda(E^{-\alpha}\phi) = \delta_\alpha(0).$$

Fix α and let $f \in F$ be such that $f_{|A} = (E^{-\alpha}\phi)_{|A}$; then, since $Q_\lambda(f) = f$,
$$Q_\lambda(f)_{|A} = E^{-\alpha}\phi_{|A}.$$

Now, $E^{-\alpha}\phi = \phi * \delta_\alpha$, while $Q_\lambda(f) = \phi * \Lambda(f)$, and therefore, by (3.1), we must have $\lambda(f) = \Lambda(f)(0) = \delta_\alpha(0)$. On the other hand, λ is supported on A and therefore since $E^{-\alpha}\phi$ and f coincide on A we conclude $\lambda(E^{-\alpha}\phi) = \lambda(f) = \delta_\alpha(0)$. ♠

The assumption $\operatorname{supp}\lambda \subset A$ in the theorem is essential, as shown by the following simple example:

Example. Let ϕ be the univariate hat function supported on $[0,2]$. Let $F = \pi_1$, $\lambda = \frac{1}{2}(\delta_{\frac{1}{2}} + \delta_{1\frac{1}{2}})$. Then Q_λ reproduces π_1, and $F = \pi_1$ also satisfies (3.4) with A being any subset of $[0,1]$ or $[1,2]$. Yet, λ is supported in no one of these A's and therefore Q_λ is not guaranteed to be a projector. Indeed,
$$\lambda(E^\alpha\phi) = \begin{cases} \frac{1}{2}, & \alpha = 0, \\ \frac{1}{4}, & \alpha = \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

Needless to say, there exist projectors whose corresponding λ is supported in no admisible A; e.g., $\lambda = \frac{1}{2}(-\delta_{\frac{1}{3}} + 2\delta_{\frac{2}{3}} + 2\delta_{\frac{4}{3}} - \delta_{\frac{5}{3}})$.

We now employ the above theorem, to show that, with an appropriate choice of the space F, the task of constructing linear projectors is reduced to the construction of the so-called quasi-interpolants. For that, assume that F is a shift-invariant (i.e., closed under integer translates) subspace of $S(\phi)$. Then it follows ([B_2],[Ro]) that F is an invariant subspace for $\phi*'$. If we further assume that F is finite-dimensional and $\phi*'$ is 1-1 on F, then $\phi*'$ induces an automorphism on F. We may then follow [BH], call this automorphism T and define a functional on F by
$$[0]T^{-1} : f \mapsto T^{-1}f(0).$$

For $\mu \in F^*$ rather than $[0]T^{-1}$, the linear independence of the integer translates of ϕ would imply that Q_μ is not the identity mapping on F and therefore no extension of such μ would yield a projector. On the other hand, for $\mu = [0]T^{-1}$ it follows [BH], that $\mu(E^\alpha f) = T^{-1}f(\alpha)$ and therefore (with Q_μ defined only on F)

$$Q_\mu(f) = f, \ \forall f \in F.$$

In case $f \mapsto f_{|A}$ is 1-1 on F, this functional can be extended to a functional $\lambda : C(\mathbb{R}^s)$ that is supported on A and then (3.3)Theorem would imply that Q_λ is a projector.

We summarize all these observations in the following

(3.5) Corollary. *Let F be a finite-dimensional shift-invariant subspace of $S(\phi)$ which satisfies (3.4). Assume that the operator $T := \phi *' {}_{|F}$ is injective and consider the following conditions for a linear functional $\lambda \in C(\mathbb{R}^s)^*$:*
(a) Q_λ is a projector.
(b) $\lambda(f) = [0]T^{-1}f, \ \forall f \in F$.
Then (a) $\Longrightarrow$ (b), and the converse implication holds provided that λ is supported in A.

The construction of a quasi-interpolant using the inverse of the map T appears first in [BH] in the context of the approximation order for box splines (hence for a polynomial F), without however making the connection to linear projectors. The discussion above emphasizes the fact (which is by now well-known) that this is (essentially) the only way to construct quasi-interpolants. In particular, other constructs (cf. e.g., [SF], [DM$_1$], [CD], [CL] for piecewise-polynomials and [DR] for piecewise-exponentials) are as a matter of fact special ways to extend $[0]T^{-1}$.

Our next task is to prove the existence of a space F which satisfies all these conditions. This purpose can be accomplished without appeal to linear independence. First, we associate with A the set

$$\nu(A) := \nu_\phi(A) := \{\alpha \in \mathbb{Z}^s : \ A - \alpha \cap \operatorname{supp} \phi \neq \emptyset\}, \tag{3.6}$$

which consists of all integers α whose corresponding $E^{-\alpha}\phi$ has some support on A. Furthermore, we assume without loss that $\phi_| \neq 0$; otherwise ϕ can be replaces by one of its non-integer translates.

We now look for a shift-invariant space P which on the one hand interpolates correctly on $\nu(A)$ (that is, $\dim P = \#\nu(A)$ and no $p \in P\backslash 0$ vanishes identically on $\nu(A)$), while on the other hand has trivial intersection with $\ker \phi\, '*'$. Any space satisfying these two conditions will do here. In particular, we may first choose P to be a homogeneous translation-invariant polynomial space that interpolates correctly on $\nu(A)$, (cf. [BR$_1$] for construction of such P of least degree). If $\phi\, '*'$ is not 1-1 on P, it may be replaced by a space $P_\theta := e_\theta P$, where the exponential $e_\theta : x \mapsto e^{\theta\cdot x}$ is chosen such that the **discrete** convolution $\phi\, '*' e_\theta \neq 0$. This readily implies that the operator $\phi\, '*'$ is 1-1 on $e_\theta \pi$ (and hence so is $\phi *'$). Since for every $\theta \in \mathbf{C}^s$, the space P_θ also interpolates correctly on $\nu(A)$, P_θ satifies the required conditions.

Define now

$$F := \phi *' P_\theta.$$

Since P_θ is translation-invariant, hence shift-invariant, so is F. F is also finite-dimensional; in fact, since $\phi *'$ is 1-1 on P_θ, $\dim F = \#\nu(A)$. Moreover, by the definition of $\nu(A)$, $S(\phi)_{|A} = \operatorname{span}\{E^{-\alpha}\phi_{|A}\}_{\alpha\in\nu(A)}$, and since P_θ interpolates correctly on $\nu(A)$, we have

$$F_{|A} = S(\phi)_{|A}.$$

Finally, the discrete convolution ϕ '$*'$' is injective on P_θ, thus induces an automorphism on that space and consequently F coincides with P_θ on $\mathbb{Z}^s$. We then conclude

(3.7) Theorem. *Let ϕ be a compactly supported continuous function, and assume that $\phi_| \neq 0$. Let A be an open subset of IR^s. Then there exists a shift-invariant subspace $F \subset S(\phi)$ satisfying*
(a) $\dim F = \#\nu(A)$;
(b) $\phi *'{}_{|F}$ *is an automorphism;*
(c) $F_{|A} = S(\phi)_{|A}$.
(d) $F_|$ *is, up to multiplication by an exponential, a homogeneous (sequence) polynomial space.*

The following scheme sketches the various steps required in the construction of linear projectors using the approach above.

(3.8) Scheme. *Let ϕ be a compactly supported function whose integer translates are globally linearly independent. Check whether $\phi_| = 0$; if so replace ϕ by a translate of it. Find a subset $A \subset \mathrm{IR}^s$ satisfying (3.1); then*
(a) Compute the finite set $\nu(A)$.
(b) Find a polynomial space P which interpolates correctly on $\nu(A)$. For that you may apply the algorithm given in [BR₁; §4].
(c) Find an exponential e_θ such that $\sum_{\alpha\in\mathbb{Z}^s} e_\theta(\alpha)\phi(-\alpha) \neq 0$. Define $F = \phi *' e_\theta P$. At this point you may wish to replace F by a shift-invariant subspace of it which still satisfies (3.4).
(d) For a given basis $f_1, \ldots, f_n$ for F, find the basis $g_1, \ldots, g_n$ for F that satisfies

$$\phi *' g_j = f_j,\ j = 1, \ldots, n.$$

(This step can also be executed with discrete convolution, i.e., with $\phi_|$ replacing ϕ).
(e) Define a linear functional on F by $\mu(f_j) = g_j(0)$, $j = 1, \ldots, n$, and extend μ to your favorite choice of a functional λ defined on some superspace of $S(\phi)$ and supported in A. (In case the extension is to $C(\mathrm{IR}^s)$, you may choose λ to be supported on an appropriate finite subset of A with cardinality $\leq \#\nu(A)$).
(f) The resulting Q_λ is a linear projector.

4. Piecewise-polynomials and piecewise-exponentials: quasi-interpolation

In this section we review some methods concerning the construction of quasi-interpolants for piecewise-exponentials (and in particular piecewise-polynomials). Most of the results here are known, and the approach taken follows that of [BR₂]. The discussion also makes use of various observations from [BAR], [B₂] and [Ro].

Let H be a finite-dimensional exponential space. This means, by definition, that each function in H admits the form

$$(4.1) \qquad \sum_{j=1}^{n} e_{\theta_j} p_j, \ \theta_j \in \mathbb{C}^s, \ p_j \in \pi, \ j = 1, ..., n.$$

We refer to the elements of H simply as "exponentials". Furthermore, we assume hereafter that H is translation-invariant, which implies that a basis for H is given in terms of functions of the form

$$e_\theta p, \ p \in \pi.$$

The **spectrum** of H is the set of frequencies of all exponentials in H:

$$(4.2) \qquad \operatorname{spec} H := \{\theta \in \mathbb{C}^s : \ e_\theta \in H\}.$$

Now let ϕ be a (compactly supported) piecewise-H function for which

$$(4.3) \qquad H \subset S(\phi).$$

A **quasi-interpolant** here means any linear map Q from some superspace V of $S(\phi)$ into $S(\phi)$ which satisfies

$$(4.4) \qquad Q(f) = f, \ \forall f \in H.$$

Under the regularity assumption

$$(4.5) \qquad \widehat{\phi}(-i\theta) \neq 0, \ \forall \theta \in \operatorname{spec} H,$$

the operator $T := \phi *' |_H$ is an automorphism.

(4.6) Proposition. *For $\lambda \in V^*$, the condition*

$$\lambda(f) = [0]T^{-1}(f), \ \forall f \in H,$$

is sufficient for Q_λ to be a quasi-interpolant. Furthermore, this condition is also necessary in case $\phi *$ is injective.

This last proposition shows that a careful study of the map T^{-1}, hence also of T, is essential for construction of a quasi-interpolant. This study is facilitated by the following

(4.7) Result[BR$_2$]. *Let $e_\theta p$ be a function in $S(\phi)$ (with $\theta \in \mathbb{C}^s$ and $p \in \pi$). Then*

$$\phi *' (e_\theta p) = \phi * (e_\theta p),$$

where the right hand side convolution is the usual convolution between functions (or distributions).

The above result suggests that in order to find a preimage of $e_\theta p \in H$, we may solve the convolution equation

$$\phi * (e_\theta ?) = e_\theta p.$$

Dividing both sides by e_θ and applying Fourier transform we get

$$(E^{-i\theta}\widehat{\phi})\widehat{?} = \widehat{p}.$$

Since $\operatorname{supp}\widehat{p} = 0$ and $\widehat{\phi}(-i\theta) \neq 0$, we may divide both sides of (4.7) by $E^{-i\theta}\widehat{\phi}$ to conclude

$$\widehat{?} = \frac{\widehat{p}}{E^{-i\theta}\widehat{\phi}} = \sum_{\alpha\geq 0} \frac{D^\alpha\psi(\theta)}{\alpha!}\widehat{D^\alpha p}, \quad \psi(x) := \frac{1}{\widehat{\phi}(-ix)},$$

hence

$$? = \sum_{\alpha\geq 0} \frac{D^\alpha\psi(\theta)}{\alpha!} D^\alpha p.$$

With $\frac{D^\alpha\psi(\theta)}{\alpha!}$ denoted by $a_{\theta,\alpha}$, we conclude

(4.8) Proposition. *For $e_\theta p \in H$*

$$[0]T^{-1}(e_\theta p) = [0]\sum_{\alpha\geq 0} a_{\theta,\alpha} D^\alpha p. \tag{4.9}$$

Combining the last proposition with (4.6)Proposition we conclude

(4.10) Corollary. *Assume that the integer translates of ϕ are globally linearly independent. Then the condition*

$$\lambda(e_\theta p) = [0]\sum_{\alpha} a_{\theta,\alpha} D^\alpha p, \ \forall e_\theta p \in H$$

is necessary and sufficient for Q_λ to be a quasi-interpolant.

5. Piecewise-polynomials and piecewise-exponentials: linear projectors

Here we combine the results of the two previous sections in the derivation of linear projectors for the piecewise-exponential space $S(\phi)$.

Retaining the notations of section 4, (and especially the notation $[\theta]$ for point-evaluation in θ), we assume throughout this section that $H \subset S(\phi)$, and that there exists an open bounded set $A \subset \mathbb{R}^s$ for which

$$\#\nu(A) = \dim H, \tag{5.1}$$

(where $\nu(A)$ is as in (3.6)). It follows that H satisfies the condition required of F in (3.4). Furthermore, the translates of ϕ are locally linearly independent on A (in the sense of (1.2)), and in particular A satisfies (3.1). With the aid of [Ro; Lem. 2.2] we can also easily conclude that $\phi *'$ is 1-1 on H.

Therefore, (3.5)Corollary reads here:

(5.2) Corollary. *Let λ be a linear functional defined on an extension V of $S(\phi)$ and vanishing on all $f \in S(\phi)$ supported in $\mathbb{R}^s \backslash A$. If*

$$Q_\lambda(f) = f, \; \forall f \in H,$$

then Q_λ is a projector.

Equivalently, Q_λ is a projector if λ extends the linear functional $[0]T^{-1} \in H^*$. The precise values of $[0]T^{-1}$ on H were determined in (4.10)Corollary, but of course many extensions (to various V's) of $[0]T^{-1}$ are available. To draw the connections between the results here and the constructions of dual bases for a box spline space in [$\mathrm{DM}_{1,2}$] and [$\mathrm{J}_{1,2}$], we concentrate now on the case when $[0]T^{-1}$ is represented (and thus extended) with the aid of differential operators. First, we associate with every $q \in \pi$ a linear functional $q^* \in H^*$ defined by

$$q^*(f) = q(D)f(0), \; \forall f \in H.$$

Note that for $e_\theta p \in H$

$$q^*(e_\theta p) = p^* E^\theta(q) = [0] \sum_{\alpha \geq 0} \frac{(D^\alpha q)(\theta)}{\alpha!} D^\alpha p,$$

while on the other hand, by (4.10)Corollary

$$[0]T^{-1}(e_\theta p) = [0] \sum_{\alpha \geq 0} a_{\theta,\alpha} D^\alpha p.$$

Hence we conclude

(5.3) Corollary. *Let $q \in \pi$ satisfy*

$$D^\alpha q(\theta) = \alpha! \, a_{\theta,\alpha}, \quad \forall \theta \in \operatorname{spec} H, \; |\alpha| \leq \max\{\deg p : \; e_\theta p \in H\}, \tag{5.4}$$

with $\{a_{\theta,\alpha}\}$ as in (4.10)Corollary. For a space $V \supset S(\phi)$, let $\lambda \in V^*$ be an extension of $q^* \in H^*$ which vanishes on all functions in $S(\phi)$ with support in $\mathbb{R}^s \backslash A$. Then Q_λ is a projector.

In case all the integer translates of ϕ belong to H in a neighborhood of the origin, one may choose to extend q^* (at least on $S(\phi)$) to the functional $\lambda(f) = q(D)f(0), \; f \in S(\phi)$. We note that in general condition (5.4) is sufficient but not necessary for the equality $[0]T^{-1} = q^*$.

Choosing ϕ to be a polynomial or exponential box spline, (5.3)Corollary verifies [DM_1; Thm. 5.1] and [DM_2; Thm.5.1]. Note that the approach taken here avoids the application of Poisson's summation formula, hence we need not to impose any further restrictions on the polynomial q (see [J_2] for a discussion of the difficulty in the application of Poisson's summation formula). Poisson's formula is implicitly used here, since this is the key tool in the proof of (4.7)Result. Nevertheless, that latter result holds for any compactly supported distribution ϕ.

In case ϕ is not smooth enough at the origin, one may wish to represent $[0]T^{-1}$ by $q_\theta^* E^\theta$ with $\theta \in \mathbb{R}^s$ chosen such that $S(\phi)$ is locally in H in a neighborhood of θ. To find the connection between the various q_θ's, let $P \subset \pi$ be a space **dual** to H in the sense that the map

$$P \rightarrow H^* : q \mapsto q^*$$

is bijective (hence every $\mu \in H^*$ is uniquely represented by some q^* with $q \in P$). With $\{f_j\}_{j=1}^n$ and $\{p_j\}_{j=1}^n$ dual bases for H and P respectively, one has

$$f = \sum_{j=1}^{n} p_j^*(f) f_j, \ \forall f \in H.$$

Let q be the unique polynomial in P satisfying $q^* = [0]T^{-1}$ and let $\theta \in \mathbb{R}^s$. Then

$$\begin{aligned}
[0]T^{-1}(f) = q^*(f) &= (q^* E^{-\theta})(E^\theta f) \\
&= \sum_{j=1}^{n} p_j^*(E^\theta f) q^*(E^{-\theta} f_j) \\
&= [\theta](\sum_{j=1}^{n} (q(D) f_j)(-\theta) p_j(D) f).
\end{aligned}$$

Since H is translation-invariant and q does not vanish on $\operatorname{spec} H$, $q(D)$ is injective on H, hence $\{q(D)f_j\}_{j=1}^n$ is also a basis for H. We have proved

(5.5) Corollary. *Given a basis $\{g_j\}_{j=1}^n$ for H and a dual space $P \subset \pi$ for H, there exists a unique basis $\{p_j\}_{j=1}^n$ for P satisfying for all $\theta \in \mathbb{R}^s$*

$$[0]T^{-1} = [\theta](\sum_{j=1}^{n} g_j(-\theta) p_j(D)).$$

Moreover, $\{p_j\}_{j=1}^n$ is the basis for P which is dual to $\{f_j\}_{j=1}^n$, where $\{f_j\}_{j=1}^n$ are defined by

$$f_j \in H, \quad q(D) f_j = g_j, \ j = 1, \ldots, n,$$

with $q \in P$ the unique polynomial satisfying $q^* = [0]T^{-1}$.

For an exponential box spline ϕ there are two natural choices for a dual P for H (cf. [BDR; §4] for details).

For a *polynomial* H, one may write each of the polynomials $\{g_j\}_{j=1}^n$ in the above corollary in power form and then use summation by parts to obtain

(5.6) Corollary. *Assume that $H \subset \pi_k$ for some non-negative integer k. Then there exist polynomials $\{p_\alpha\}_{|\alpha| \le k}$ such that*

$$[0]T^{-1} = [\theta] \left(\sum_{|\alpha| \le k} \theta^\alpha p_\alpha(D) \right).$$

The sequence $\{p_\alpha\}_{|\alpha| \le k}$ is unique in case we impose the restriction $\{p_\alpha\} \subset P$ for a space P dual to H. We mention that in case H is invariant under the complex involution, it is self-dual and we then may choose $\{p_\alpha\} \subset H$.

(5.6)Corollary captures the construction in [J$_1$]. There ϕ was a polynomial box spline and P was chosen as a specific known dual of H.

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References

[B_1] C. de Boor, On uniform approximation by splines, J. Approx. Theory **1** (1968), 219–235.

[B_2] C. de Boor, The polynomials in the linear span of integer translates of a compactly supported function, Constructive Approx. **3** (1987), 199–208.

[BAR] A. Ben-Artzi and A. Ron, Translates of exponential box splines and their related spaces, Trans. AMS **309** (1988), 683–710.

[BDR] C. de Boor, N. Dyn and A. Ron, On two polynomial spaces associated with a box spline, CMS-TSR 89-29, University of Wisconsin-Madison, April 1989.

[BF] C. de Boor and G. Fix, Spline approximation by quasi-interpolants, J. Approx. Theory **8** (1973), 19–45.

[BH] C. de Boor and K. Höllig, B-splines from parallelepipeds, J. d'Anal. Math. **42** (1982/3), 99–115.

[BR_1] C. de Boor and A. Ron, On multivariate polynomial interpolation, CMS TSR 89-17, University of Wisconsin-Madison, November 1988.

[BR_2] C. de Boor and A. Ron, The exponentials in the span of the integer translates of a compactly supported function, a manuscript.

[CD] C. K. Chui and H. Diamond, A natural formulation of quasi-interpolation by multivariate splines, Proc. AMS **99** (1987), 643–646.

[CL] C. K. Chui and M. J. Lai, A multivariate analog of Marsden's Identity and a quasi-interpolation scheme. Const. Approx. **3** (1987), 111-122.

[DM_1] W. Dahmen and C. A. Micchelli, Translates of multivariate splines. Linear Algebra and Appl. **52/3** (1983), 217–234.

[DM_2] W. Dahmen and C.A. Micchelli, On the local linear independence of translates of a box spline, Stud. Math. **82** (1985), 243–263.

[DM_3] W. Dahmen and C. A. Micchelli, Multivariate E-splines, Advances in Math., to appear.

[DR] N. Dyn and A. Ron, Local Approximation by certain spaces of multivariate exponential-polynomials, approximation order of exponential box splines and related interpolation problems, CAT Rep. 160, Texas A&M Univ., College Station, TX 77843, Jan. 1988. Trans. AMS, to appear.

[J_1] R. Q. Jia, Subspaces invariant under translation and the dual bases for box splines, Chinese Ann. of Math., to appear.

[J_2] R. Q. Jia, A dual basis for the integer translates of an exponential box spline, preprint 1988.

[Ri] S. R. Riemenschneider, Multivariate cardinal interpolation, Approximation Theory VI, eds. C. K. Chui, L. L. Schumaker and J. D. Ward, Academic Press New York, 1989, to appear.

[Ro] A. Ron, Relations between the support of a compactly supported function and the exponential-polynomials spanned by its integer translates, Constructive Approx., to appear.

[SF] G. Strang and G. Fix, A Fourier analysis of the finite element variational method. C.I.M.E. II Cilo 1971, Constructive Aspects of Functional Analysis ed. G. Geymonet 1973, 793-840.