A NOTE ON THE HAMILTONIAN CIRCUIT PROBLEM ON DIRECTED PATH GRAPHS

Giri Narasimhan

Computer Sciences Technical Report #839

April 1989

A NOTE ON THE HAMILTONIAN CIRCUIT PROBLEM ON DIRECTED PATH GRAPHS†

Giri Narasimhan

Computer Sciences Department

University of Wisconsin-Madison

Madison, Wisconsin-53706

Abstract

Bertossi and Bonuccelli [2] proved that the Hamiltonian Circuit problem is NP-Complete even when the inputs are restricted to the special class of Undirected Path graphs. The corresponding problem on Directed Path graphs was left as an open problem. We use a characterization of Directed Path graphs due to Monma and Wei [8] to prove that the Hamiltonian Circuit problem and the Hamiltonian Path problem are NP-Complete on Directed Path graphs.

Keywords: Hamiltonian circuit, intersection graph, undirected path graph, directed path graph, NP-completeness, hamiltonian path

1. Introduction

The Hamiltonian Circuit problem on general graphs is a well-known NP-Complete problem. It is known to be NP-Complete even when the inputs are restricted to several interesting special classes of graphs. For example, it is NP-Complete on planar cubic 3-connected graphs [4], bipartite graphs [1], edge graphs [3], and chordal graphs [5] (see [7] for a summary of these results). Bertossi and Bonuccelli [2] proved that the problem of finding a hamiltonian circuit is also NP-Complete on several interesting classes of intersection graphs. They proved that the hamiltonian circuit problem is NP-Complete on undirected path graphs, double interval graphs, and rectangle graphs, all three of which are generalizations of interval graphs. They left unsolved the problems of determining whether the hamiltonian circuit problem is NP-Complete on directed path graphs and circular-arc graphs.

In this paper we answer one of the open questions posed by them. We prove that the hamiltonian circuit problem on directed path graphs (HCDPG problem) is NP-Complete too. Our proof uses techniques similar to those used in [2]. We also use a theorem due to Monma and Wei [8],

† **To be published in** *Information Processing Letters*

which characterizes directed path graphs in terms of its cliques. The reduction is carried out from the hamiltonian circuit problem on bipartite graphs with maximum degree 3 (HCBGD3 problem), which was proved to be NP-Complete by Itai, et al. [6].

2. Directed Path Graphs

We will use the same notation and terminology used in [2]. A *Hamiltonian circuit* of a graph is an ordering of the vertices such that every two cyclically consecutive vertices are joined by an edge. The *Intersection graph* for a family of sets is obtained by associating each set with a vertex of the graph and joining two vertices by an edge exactly when their corresponding sets have a nonempty intersection. We will be interested in classes of intersection graphs arising from families of simple vertex paths in a tree. Here we assume a vertex path to consist of the set of vertices on it. *Undirected Path graphs* are intersection graphs of paths in a tree. *Directed Path graphs* are intersection graphs of directed paths in a directed tree.

The transformation from an instance of HCBGD3 problem to an instance of HCDPG problem is described below. Given a bipartite graph $B(M, N, E)$ with maximum degree 3, an instance of the HCBGD3 problem consists of determining whether the graph B has a hamiltonian circuit. Without loss of generality, we assume that the vertex sets M and N both have n vertices and that B has no vertex with degree 1, since otherwise B has no hamiltonian circuit. Let $M = \{m_1, m_2, \ldots, m_n\}$ and $N = \{n_1, n_2, \ldots, n_n\}$. We show how to construct an instance of the HCDPG problem by showing how to construct a directed path graph $G(V, A)$ such that B has a hamiltonian circuit if and only if G has a hamiltonian circuit. We describe G by describing all its maximal cliques. Note that describing all the maximal cliques of a graph fully defines the graph itself. Corresponding to vertex $m_i \in M$, construct a clique $K_i = X_i \cup \{A_{ij} : (m_i, n_j) \in E\}$. Corresponding to each vertex $n_j \in N$ with degree 3, construct cliques $K'_j = Y_j \cup \{A_{ij} : (m_i, n_j) \in E\}$ and $K''_j = Z_j \cup \{A_{ij} : (m_i, n_j) \in E\}$. Corresponding to each vertex $n_j \in N$ with degree 2, construct a clique $K'_j = Y_j \cup \{A_{ij} : (m_i, n_j) \in E\}$. Finally construct one large clique $K_0 = \{A_{ij} : (m_i, n_j) \in E\}$.

Note that the cliques mentioned above are the only maximal cliques (or maximal completely connected subgraphs) in G. Hence it is clear that

$$V(G) = \{X_1, \ldots, X_n\} \cup \{Y_1, \ldots, Y_n\} \cup \{Z_j : \text{degree}(n_j) = 3\} \cup \{A_{ij} : (m_i, n_j) \in E\}$$

We first prove that the resulting graph G is a directed path graph. Then we show that the transformation described above is a polynomial-time reduction from the HCBGD3 problem to the HCDPG problem.

Claim 1: The graph G constructed above is a directed path graph.

To prove the above claim we use a result due to Monma and Wei [8], which characterizes directed path graphs in terms of its maximal cliques. They proved the following theorem called the *Clique Tree Theorem* for directed path graphs, which we state without proof.

Theorem 1 [8]: *Clique Tree Theorem for directed path graphs*

Let $G = (V, A)$ be a graph and let $\mathcal{K}$ be the set of all maximal cliques of G. For each vertex $v \in V$, let $\mathcal{K}_v$ be the set of cliques of $\mathcal{K}$ containing the vertex v. Then G is a directed path graph if and only if there exists a directed tree T with vertex set $\mathcal{K}$, such that for every $v \in V$, $T(\mathcal{K}_v)$ is a directed path in T. Here $T(\mathcal{K}_v)$ denotes the subgraph of T with vertices or edges corresponding to $\mathcal{K}_v$.

Proof of Claim 1: Let $\mathcal{K}$ be the set of all maximal cliques of G. Hence,

$$\mathcal{K} = \{K_0\} \cup \{K_i, K_i' : i = 1, \ldots, n\} \cup \{K_j'' : \text{degree}(n_j) = 3\}$$

Let T be a directed graph with vertex set $\mathcal{K}$. Hence T has a vertex for each maximal clique of G. Let the directed edges of T be as follows:

$$\begin{aligned} &(K_0, K_i), \quad \forall i, \quad 1 \leq i \leq n \\ &(K_j', K_0), \quad \forall j, \quad 1 \leq j \leq n \\ &(K_j'', K_j'), \quad \forall j, \quad 1 \leq j \leq n, \quad \text{degree}(n_j) = 3 \end{aligned}$$

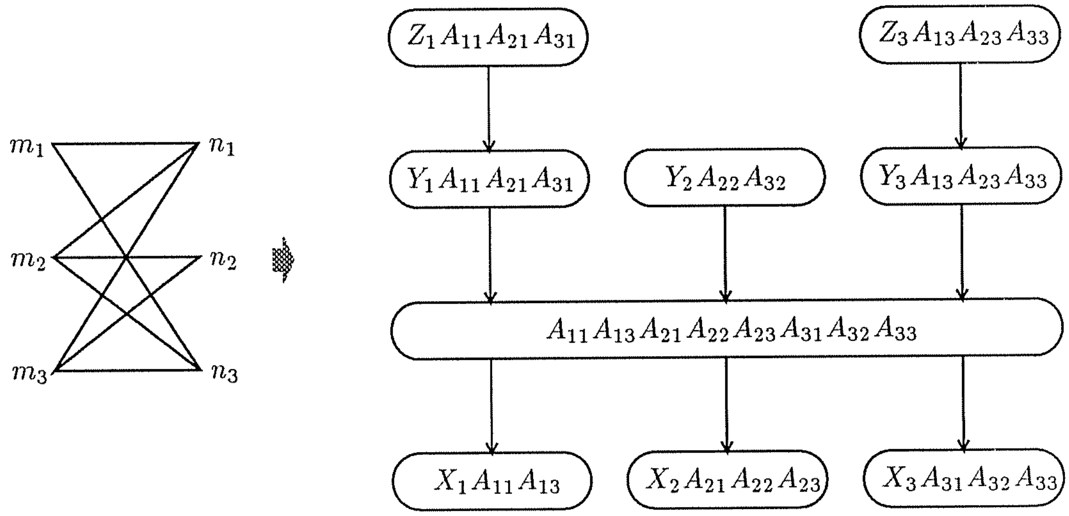
T is clearly a directed tree. We now illustrate the construction using the same bipartite graph example from [2]. Figure 1 below shows the bipartite graph B with maximum degree 3, and the clique tree T corresponding to the directed path graph G.

Let v be a vertex of G. If v is either X_i, Y_j or Z_j, then $T(\mathcal{K}_v)$ consists of only one vertex and hence is a directed path of length 0. If $v = A_{ij}$ and degree$(n_j) = 3$, then $T(\mathcal{K}_v)$ consists of the directed path $\langle K_i, K_0, K_j', K_j'' \rangle$. If $v = A_{ij}$ and degree$(n_j) = 2$, then $T(\mathcal{K}_v)$ consists of the directed path $\langle K_i, K_0, K_j' \rangle$. Hence for each vertex $v \in V(G)$, $T(\mathcal{K}_v)$ is a directed path in T. Hence by theorem 1, G is a directed path graph. ∎

In Claim 2 we prove that the HCBGD3 problem polynomially reduces to the HCDPG problem.

Claim 2: The HCBGD3 problem can be reduced in polynomial time to the HCDPG problem.

Proof of Claim 2: Consider the transformation of an instance of a HCBGD3 problem to an



B **T**

Figure 1: A bipartite graph B, and the constructed clique tree T.

instance of the HCDPG problem as described above. Now, the claim is that there exists a hamiltonian circuit in B iff there exists one in G. Verifying one direction of the claim is easy. If B has a hamiltonian circuit C_B, we obtain a hamiltonian circuit C_G for G as follows. If m_i, n_j, m_k are three consecutive vertices in C_B, we obtain C_G by substituting the sequence $\langle m_i, n_j, m_k \rangle$ either with the sequence $\langle X_i, A_{ij}, Y_j, A_{hj}, Z_j, A_{kj}, X_k \rangle$ if n_j has degree 3 (in this case m_h is the third vertex joined to n_j by an edge in E), or with the sequence $\langle X_i, A_{ij}, Y_j, A_{kj}, X_k \rangle$. This results in a hamiltonian circuit for G since all the vertices are covered and no vertex is covered more than once.

Conversely, let C_G be a hamiltonian circuit for G. The sequence of vertices in C_G must consist of sequences of the form $\langle X_i, A_{ij}, Y_j, A_{hj}, Z_j, A_{kj}, X_k \rangle$ if n_j has degree 3, or of the form $\langle X_i, A_{ij}, Y_j, A_{kj}, X_k \rangle$ if n_j has degree 2. If this were not true then Z_j (if vertex n_j has degree 3), and/or Y_j would get excluded from the sequence of vertices forming the hamiltonian circuit, since they are not adjacent to any other A-vertices. It is now clear that substituting either of these sequences by the sequence $\langle m_i, n_j, m_k \rangle$ would obtain a hamiltonian circuit in B. ∎

Theorem 2: The Hamiltonian Circuit problem on directed path graphs is NP-Complete.

Proof of Theorem 2: It trivially follows from lemmas 1 and 2. ∎

Just as in [2], this proof can easily be extended to prove that the hamiltonian path problem on directed path graphs is also NP-Complete.

3. Conclusion

The hamiltonian circuit problem remains open on the class of circular-arc graphs. It is also open for a subclass of the class of directed path graphs, namely *Rooted Directed Path graphs*, which are the intersection graphs of directed paths in a rooted directed tree.

5. References

[1] T. Akiyama, T. Nishizeki, and N. Saito, NP-Completeness of the Hamiltonian Cycle Problem for Bipartite Graphs, *J. Information Processing,* **3** (1980), pp. 73-76.

[2] A. A. Bertossi, and M. A. Bonuccelli, Hamiltonian Circuits in Interval Graph Generalizations, *Information Proc. Letters,* **23** (1986), pp. 195-200.

[3] K. S. Booth, and G. S. Lueker, Testing for the Consecutive Ones Property, Interval Graphs, and Graph Planarity using PQ-Tree Algorithms, *J. Comput. System Sci.,* **13** (1976), pp. 335-379.

[4] E. J. Cockayne, R. M. Dawes, and S. T. Hedetniemi, Total Domination in Graphs, *Networks,* **10** (1980), pp. 211-219.

[5] C. J. Colburn, and L. K. Stewart, Dominating Cycles in Series-Parallel Graphs, *Ars Combinatorica,* **19A** (1985), pp. 107-112.

[6] A. Itai, C. H. Papadimitriou, and J. L. Szwarcfiter, Hamilton paths in Grid Graphs, *SIAM J. of Comput.,* **11** (1982), pp. 676-686.

[7] D. S. Johnson, The NP-Completeness Column: an Ongoing Guide, *J. of Algorithms,* **6** (1985), pp. 434-451.

[8] C. L. Monma, and V. K. Wei, Intersection Graphs of Paths in a Tree, *J. of Combin. Theory, Series B,* **41** (1986), pp. 141-181.