FACTORIZATION THEOREMS FOR UNIVARIATE SPLINES ON REGULAR GRIDS

by

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ABSTRACT

For a given univariate compactly supported distribution ϕ, we investigate here the space $S(\phi)$ spanned by its integer translates, the subspace $H(\phi)$ of all exponentials in $S(\phi)$ and the kernel K_ϕ of the associated semi-discrete convolution $\phi *$. The paper addresses a variety of results including a complete structure of $H(\phi)$ and K_ϕ and a characterization of splines of minimal support. The main result shows that each ϕ can be expressed as $\phi = p(\nabla)\tau * M$, where $p(\nabla)$ is a finite difference operator, τ is a distribution of smaller support satisfying $H(\tau) = K_\tau = \{0\}$, and M is a spline which depends on $H(\phi)$ but not on ϕ itself, and which in the generic case (termed here "regular") is the exponential B-spline associated with $H(\phi)$.

The approach chosen is direct and avoids entirely the Fourier analysis arguments. The fact that a distribution is examined, rather than a function, is essential for the methods employed.

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Factorization Theorems for Univariate Splines on Regular Grids

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1. Introduction

The theory of multivariate splines on regular grids has been rapidly devolped in the recent years, primarily due to the introduction of box splines by de Boor, DeVore and Höllig [BD], [BH]. A remarkable variety of techniques has been used to investigate the various aspects of these splines; starting with Fourier analysis methods (e.g., Poisson summation formula) through partial differential and difference operators, spectral analysis, ideals of polynomials, disrtibution theory and more.

In this note we revisit the **univariate** case. Expoiting some of the techniques known in the multivariate situation together with some new ones, we obtain here factorization results for univariate splines which illuminate the crucial role played here by polynomial and exponential B-splines. We then use these results to analyze the question of splines of minimal support and singular exponential B-splines. It should be emphasized that analogous results are (unfortunately) non-valid in the multivariate situation; yet the fact that the multivariate box splines are expressed as convolution of measures supported on lines makes it possible to apply some of the observations here to that important case. A typical example of that sort of applications has been given in [R_1; §6].

Some of the results here might have been known to others (see e.g., [SF; p.825]), yet, it seems that this is the first systematic analysis of the possible factorizations of univariate splines. In this analysis, as happened in the analysis of the multivariate exponential box splines [BR], we found it often more efficient (as well as more explicit) to apply appropriate differential and difference operators rather than the alternative (and more standard) techniques of the Fourier tranform.

Several observations from multivariate splines on regular grids had guided us here. Nevertheless, considering the fact that a potential reader of this note may be unfamiliar with the theory of multivariate splines, proofs are provided for most of these results, thus making the paper essentially self-contained.

2. The model

The model we investigate here can be presented as follows: let ϕ be a compactly supported measurable function (or even a distribution from $\mathcal{D}'(\mathrm{I\!R})$). Each function gives rise to the semi-discrete convolution operator $\phi*$ which assigns to each c in the space

$$C := \{c | \ c : \mathbb{Z} \to \mathbf{C}\} \tag{2.1}$$

of all complex-valued sequences, a function (or distribution) $\phi * c$ defined by

$$\phi * c := \sum_{\alpha \in \mathbb{Z}} c(\alpha) E^{\alpha} \phi, \tag{2.2}$$

with E the **shift** operator

$$E : f \mapsto f(\cdot - 1).$$

The **range** and the **kernel** of $\phi*$ are denoted here by $S(\phi)$ and K_ϕ respectively. Another important space in our discussion is the space $H(\phi)$ of all **exponentials** in $S(\phi)$; here and hereafter "an exponential" is a function of the form

$$\sum_{j=1}^{n} e_{\lambda_j} p_j, \tag{2.3}$$

where, for $j = 1, ..., n$, $p_j \in \pi$:=the space of all polynomials, $\lambda_j \in \mathbf{C}$ and e_λ is the exponential

$$e_\lambda : x \mapsto e^{\lambda x}.$$

The spaces K_ϕ, $H(\phi)$ and the preimage H_ϕ (in C) of $H(\phi)$ are of use in the evaluation of the **approximation** properties of $S(\phi)$ and approporiate scaled-versions of it and hence for a given ϕ it is important to identify these spaces. In this paper we also reverse the question and seek a characterization of ϕ in terms of $H(\phi)$ and K_ϕ. These issues will be examined later on in this paper. Here, as a preparation, we discuss some basics concerning the above model.

In the following π_k denotes the space of all polynomials of degree $\leq k$. Difference operators are applied either to functions and distributions or to sequences; yet, the *kernel* of a difference operator is always regarded in the sequence space C.

We first note that the spaces $S(\phi)$, $H(\phi)$, K_ϕ and H_ϕ are **shift-invariant** i.e, invariant under integer translates. (For the two latter spaces these are the only admissible translates, as being sequence spaces). Moreover, since ϕ is of compact support and $H(\phi)$ consists of entire functions, $H(\phi)$ is finite-dimensional. Although being less evident, the finite-dimensionality holds also for K_ϕ.

(2.4) Proposition[R$_2$;Cor.2.4]. *Let ϕ be a compactly supported distribution. Then the space K_ϕ constitutes the kernel of a finite difference operator $p(\nabla)$, hence admits a representation*

$$K_\phi = \bigoplus_{j=1}^{m} (e_{\lambda_j}\pi_{k_j})_{|\mathbb{Z}}, \tag{2.5}$$

where $(k_j)_{j=1}^m$ are some non-negative integers and for $1 \le j < l \le m$

$$\lambda_j - \lambda_l \notin 2\pi i \mathbb{Z}.$$

Furthermore,

$$\dim K_\phi \le \operatorname{diam} \operatorname{supp} \phi. \tag{2.6}$$

Proof: We first treat the case when ϕ is a function: Assume that $\operatorname{diam}\operatorname{supp}\phi \le n$. Then one easily checks that the only sequence c in K_ϕ with $c(0) = c(1) = \ldots = c(n-1) = 0$ is the trivial one, and hence (2.6) holds. Thus, K_ϕ is finite-dimensional and shift-invariant, hence is annihilated by $p(\nabla) := q(E)$, q being the characteristic polynomial of $E_{|K_\phi}$, and $\dim K_\phi = \deg q$. Since $\deg q$ also coincides with the dimension of $\ker p(\nabla)$ (in C), we conclude that $\ker p(\nabla) = K_\phi$. The rest of the proposition is merely a standard way of writing the kernel of a difference operator.

For a general ϕ, we convolve ϕ with an infinitely differentiable molifier σ to obtain an infinitely differentiable function $\phi * \sigma$. By the first part of the proof, $K_{\phi*\sigma}$ is finite-dimensional, hence so is K_ϕ, since $K_\phi \subset K_{\phi*\sigma}$. Also, σ can be chosen to be of an arbitrary small support, hence $\operatorname{diam}\operatorname{supp}\phi*\sigma < \operatorname{diam}\operatorname{supp}\phi + \varepsilon$, and thus (2.6) follows. The rest of the proof is identical with that of the first part. ♠

We now use the above proposition to derive similar results on the preimage H_ϕ of $H(\phi)$.

(2.7) Corollary. *The space H_ϕ is also the kernel of a finite difference operator, hence admits a representation similar to (2.5). Furthermore,*

$$\dim H_\phi \le \operatorname{diam}\operatorname{supp}\phi.$$

Proof: Let $p(D)$ be a differential operator with constant coefficients which annihilates $H(\phi)$. Define $\psi := p(D)\phi$. Then for every $c \in H_\phi$, $p(D)(\phi * c) = 0$, therefore $H_\phi \subset K_\psi$. Application of (2.4)Proposition thus yields that H_ϕ is finite-dimensional. Using the fact that $\operatorname{supp}\psi \subset \operatorname{supp}\phi$, the rest of the proof follows that of (2.4)Proposition. ♠

Occasionally, we generate sequences in C by restricting functions to $\mathbb{Z}$, and then apply the operation $\phi*$ to these sequences. In this case it is convenient to use the notations

$$(2.8) \qquad f_| := f_{|\mathbb{Z}}, \qquad \phi *' f := \phi * (f_|).$$

The following simple observation is one of the most useful results in the theory of multivariate splines on regular grids:

(2.9) Proposition[B; §1]. *Assume ϕ is a compactly supported function, and let F be a shift-invariant subspace of $S(\phi)$. Then F is an invariant subspace of $\phi*'$, i.e., $\phi *' F \subset F$.*

Proof: Let $f \in F$. Note that the operator $f*'$ is well-defined on the domain of all compactly supported functions. Also, since $f \in S(\phi)$, $f = \phi * c$ for some sequence c. Therefore

$$\phi *' f = \phi *' (\phi * c) = (\phi *' \phi) * c = c * (\phi *' \phi) = (\phi * c) *' \phi = f *' \phi.$$

Now, the claim follows from the fact that $f *' \phi \in F$, by the shift-invariance of F. ♠

This last proposition yields significant results about H_ϕ:

(2.10) Corollary. *Every shift-invariant subspace of $H(\phi)$ is an invariant subspace of $\phi*'$. In particular $H(\phi)_| \subset H_\phi$.* ♠

In the rest of this preliminary section we discuss a special type of compactly supported functions: the exponential B-splines. Let H be a finite-dimensional exponential space which is also **D-invariant**, i.e., closed under differentiation. Such a space H forms the kernel of the differential operator $p_H(D)$, with p_H being the characteristic polynomial of $D_{|H}$ and D being the usual differentiation. The differential operator $p_H(D)$ admits a discrete analog, i.e., a difference operator $p_H(\nabla)$: if q is a polynomial of the form $q = \prod_j(\cdot - \lambda_j)$, then

$$(2.11) \qquad q(\nabla) := \prod_j (1 - e^{\lambda_j} E).$$

The exponential B-spline associated with H, B_H, is now defined as the (unique) compactly supported function which satisfies the equation

$$(2.12) \qquad p_H(D) B_H = p_H(\nabla).$$

Application of Fourier transform to both sides of (2.12) yields the Fourier transform $\widehat{B}_H$ of B_H:

$$\widehat{B}_H(x) = \prod_{j=1}^{deg\ p} \int_0^1 e^{(\lambda_j - ix)t}\, dt, \tag{2.13}$$

where $(\lambda_1, ..., \lambda_{degp})$ are the roots of p_H (counting multiplicities).

(2.12) is somewhat a non-standard way to define a B-spline. As a matter of fact, one of our goals in this note is to demonstrate the fundamental importance of this identity, and therefore used it to define B_H. Dedicated to this goal, we will make no use of (2.13) in the sequel, and derive all the relevent properties of B_H directly from (2.12).

First, (with $n := \dim H$), note that since $p_H(\nabla)$ is supported on $\{0, ..., n\}$, (2.12) proves the fact that B_H is locally in $\ker p_H(D) = H$, with knots $\{0, ..., n\}$ and support $[0, n]$. With somewhat more effort, one checks that $B_H \in C^{n-2} \backslash C^{n-1}$ in a neighbourhood of each of the knots.

We also note that in case $H = H_1 \bigoplus H_2$ for some D-invariant exponential spaces H_1 and H_2, B_H is factored to

$$B_H = B_{H_1} * B_{H_2}. \tag{2.14}$$

Indeed, since $p_{H_1}(D)p_{H_2}(D)$ annihilates H, we must have $p_H(D) = p_{H_1}(D)p_{H_2}(D)$, and therefore also $p_{H_1}(\nabla)p_{H_2}(\nabla) = p_H(\nabla)$. Thus, by (2.12)

$$p_H(D)(B_{H_1} * B_{H_2}) = p_{H_1}(D)p_{H_2}(D)(B_{H_1} * B_{H_2}) = p_{H_1}(\nabla)p_{H_2}(\nabla) = p_H(\nabla),$$

and (2.14) follows from (2.12).

For an exponential B-spline B_H, the following improvement of (2.10)Corollary is available

(2.15) Proposition. *The operator B_H* maps $\ker p_H(\nabla)$ onto $H(B_H)$. Moreover,*

$$\ker p_H(\nabla) = H_{B_H}.$$

Proof: Let $c \in \ker p_H(\nabla)$. Then by (2.12)

$$p_H(D)(B_H * c) = p_H(\nabla) * c = p_H(\nabla)c = 0,$$

thus $\phi*$ carries $\ker p_H(\nabla)$ *into* H, hence by the definition of $H(B_H)$, into $H(B_H)$. This means that

$$\ker p_H(\nabla) \subset H_{B_H}.$$

Since $\dim\ker p_H(\nabla) = \dim H$ and, by (2.7)Corollary, $\dim H_{B_H} \le \operatorname{diam}\operatorname{supp} B_H = \dim H$, the claim follows. ♠

For the choice $\phi = B_H$, the last corollary identifies the difference operator mentioned in (2.7)Corollary.

3. Factorization theorems for univariate splines

In this section we show that a compactly supported function ϕ is characterized, up to convolution by a compactly supported distribution, by its kernel K_ϕ and the space of exponentials $H(\phi)$.

For later reference, we first record the following simple lemma:

(3.1) Lemma. *Let ϕ be a compactly supported distribution and p a polynomial. Then*

(a) $$S(p(D)\phi) = p(D)S(\phi), \quad H(p(D)\phi) = p(D)H(\phi).$$

(b) $$S(p(\nabla)\phi) = S(\phi), \quad H(p(\nabla)\phi) = H(\phi), \quad p(\nabla)K_{p(\nabla)\phi} = K_\phi.$$

Proof: (a) Since the summation in the semi-discrete convolution $\phi * c$ is locally finite, it commutes with the differential operator $p(D)$, and therefore $S(p(D)\phi) = p(D)S(\phi)$. Now, for an exponential f, the only solutions for the equation $p(D)? = f$ are exponentials, whence $H(p(D)\phi) = p(D)H(\phi)$.

(b) Since $p(\nabla)$ maps C onto itself

$$S(p(\nabla)\phi) = p(\nabla)\phi * C = \phi * p(\nabla)C = S(\phi),$$

and therefore also $H(p(\nabla)\phi) = H(\phi)$. As for the last equality

$$p(\nabla)K_{p(\nabla)\phi} = \{p(\nabla)c \mid p(\nabla)\phi * c = 0\} = \{p(\nabla)c \mid \phi * p(\nabla)c = 0\} = K_\phi.$$ ♠

The following proposition provides the main tool for the analysis made in this paper.

(3.2) Proposition. *Let ϕ be a distribution with support $[a,b]$. Set $n := \dim K_\phi$. Then there exist a difference operator $p(\nabla)$ and a compactly supported distribution τ such that $\ker p(\nabla) = K_\phi$, $\operatorname{supp}\tau \subset [a, b-n]$, $K_\tau = \{0\}$ and*

$$\phi = p(\nabla)\tau. \tag{3.3}$$

Moreover, $H(\phi) = H(\tau)$, and if ϕ is a function so is τ.

Proof: By (2.4)Proposition, there exists a difference operator $p(\nabla)$ satisfying $\ker p(\nabla) = K_\phi$. Since $p(\nabla)$ is a finite difference operator, it can be factored into linear factors: $p(\nabla) = \prod_{j=1}^{n} \nabla_j$. We may assume that $p(\nabla)$ and each of his linear factors are normalized in such a way that each ∇_j can be written in the form

$$\nabla_j = 1 - e^{\lambda_j} E. \tag{3.4}$$

To prove the proposition we proceed by induction on n: define

$$\tau_1 = \sum_{\alpha=0}^{\infty} e^{\lambda_1 \alpha} E^\alpha \phi;$$

then clearly $\nabla_1 \tau_1 = \phi$, and τ_1 is a function in case ϕ is a function. We contend that also $\operatorname{supp} \tau_1 \subset [a, b-1]$. To prove that latter claim, note first that, by the definition of τ_1, $\operatorname{supp} \tau_1 \subset [a, \infty)$. Now, let f be a test function with $\operatorname{supp} f \subset (b-1, \infty)$. Then for $\alpha \leq -1$

$$\operatorname{supp} f \cap \operatorname{supp} E^\alpha \phi \subset (b-1, \infty) \cap [a+\alpha, b+\alpha] = \emptyset,$$

hence $E^\alpha \phi(f) = 0$. Therefore,

$$\tau_1(f) = (\sum_{\alpha=0}^{\infty} e^{\lambda_1} E^\alpha \phi)(f) = (\sum_{\alpha=-\infty}^{\infty} e^{\lambda_1 \alpha} E^\alpha \phi)(f) = 0,$$

proving that $\operatorname{supp} \tau_1 \subset [a, b-1]$, as claimed.

Proceeding in this manner we obtain τ supported in $[a, b-n]$ such that $\phi = p(\nabla)\tau$. Then, by (3.1)Lemma, $H(\phi) = H(p(\nabla)\tau) = H(\tau)$ and $K_\tau = p(\nabla) K_{p(\nabla)\tau} = p(\nabla) K_\phi = \{0\}$. ♠

An analogous result can be obtained with respect to the space $H(\phi)$, provided that this space is **regular** in the following sense.

(3.5) Definition. *A finite-dimensional exponential space H is termed regular (with respect to $\mathbb{Z}$) if it is D-invariant and satisfies*

$$H_| = \ker p_H(\nabla).$$

A non-regular finite-dimensional D-invariant exponential space is termed "singular".

We will elaborate on the regularity concept in the next section, revealing its various meanings. As an immediate illustration for the usefulness of this notion, note that for a regular H (2.15)Proposition implies that $H_| = H_{B_H}$. The regularity notion also plays an significant role in the following

(3.6) Proposition. *Let ϕ be a function supported in $[a,b]$ and H an n-dimensional regular exponential space satisfying $H \subset H(\phi)$. Then there exists a compactly supported distribution τ such that* $\operatorname{supp}\tau \subset [a, b-n]$, $H(\tau) = p_H(D)H(\phi)$ and $\phi = B_H * \tau$.

Proof: Since H is D-invariant, it is also shift-invariant (even translation-invariant, as the kernel of $p_H(D)$). Therefore, an application of (2.10)Corollary yields that $\phi *' H \subset H$. It follows that

$$(p_H(D)\phi) *' H = p_H(D)(\phi *' H) \subset p_H(D)H = \{0\},$$

and hence $H_| \subset K_{p_H(D)\phi}$. Since H is regular, we conclude that $\ker p_H(\nabla) \subset K_{p_H(D)\phi}$. Thus, (3.2)Proposition ensures the existence of a distribution τ with support in $[a, b-n]$ such that $p_H(D)\phi = p_H(\nabla)\tau$. Convolving both sides of this last equation with B_H and using (2.12) we get

$$p_H(\nabla)(\tau * B_H) = p_H(\nabla)\tau * B_H = p_H(D)\phi * B_H = \phi * p_H(D)B_H = p_H(\nabla)\phi.$$

Since no compactly supported distribution can be annihilated by a finite difference operator, we conclude $\phi = \tau * B_H$. The identity $H(\tau) = p_H(D)H(\phi)$ follows from (3.1)Lemma. ♠

We combine the two last propositions in the following

(3.7) Theorem. *Let ϕ be a function supported in $[a,b]$, and assume that $H := H(\phi)$ is regular. Then there exists a difference operator $p(\nabla)$ and a compactly supported distribution τ satisfying*

(a) $$\phi = p(\nabla)\tau * B_H;$$

(b) $$H(\tau) = \{0\};$$

(c) $$\ker p(\nabla) = K_\phi, \quad K_\tau = \{0\};$$

(d) $$\operatorname{supp}\tau \subset [a, b-n-m],$$

with n and m the dimensions of $H(\phi)$ and K_ϕ resp.

Proof: By (3.2)Proposition there exists a difference operator $p(\nabla)$ and a compactly supported function τ_1 such that $\ker p(\nabla) = K_\phi$, $\phi = p(\nabla)\tau_1$, $H(\phi) = H(\tau_1)$, $K_{\tau_1} = \{0\}$ and $\operatorname{supp}\tau_1 \subset [a, b-m]$. Application of (3.6)Proposition to τ_1 implies the existence of a distribution τ with $\operatorname{supp}\tau \subset [a, b-m-n]$ such that $H(\tau) = \{0\}$ and $\tau_1 = \tau * B_H$. Consequently $\phi = p(\nabla)\tau * B_H$. Since trivially $K_\tau \subset K_{\tau_1}$, the condition $K_{\tau_1} = \{0\}$ implies $K_\tau = \{0\}$. ♠

4. Regularity

We examine here the notion of "regularity" as defined in (3.5), showing among other things that it was an essential condition in (3.6)Proposition and (3.7)Theorem.

To analyze the regularity concept, we first define the **spectrum** of a (finite-dimensional) exponential space. For this purpose, let $\mathrm{cl}_D(H)$ denote the smallest D-invariant exponential space containing H. Then

$$\text{(4.1)} \qquad \operatorname{spec} H := \operatorname{spec} \mathrm{cl}_D(H) := \{\lambda |\ e_\lambda \in \mathrm{cl}_D(H)\} = \text{ the roots of } p_{\mathrm{cl}_D(H)}.$$

The charaterization of the regularity condition is done in the following

(4.2) Proposition. *Let H be an n-dimensional exponential space. Then the following conditions are equivalent*

(a) H is regular.

(b) H is D-invariant, and restriction to $\{0, 1, ..., n-1\}$ is a 1-1 operation on H.

(c) H is D-invariant, and restriction to $\mathbb{Z}$ is a 1-1 operation on H.

(d) H is shift-invariant and satisfies

$$\text{(4.3)} \qquad \lambda - \mu \notin 2\pi i \mathbb{Z} \backslash 0, \ \forall \lambda, \mu \in \operatorname{spec} H.$$

Proof: To prove the implication (a) $\Longrightarrow$ (b), we claim that restricting elements from $\ker p_H(\nabla)$ to the pointset $\{0, 1, ..., n-1\}$ is injective, i.e., no sequence $c \in \ker p_H(\nabla)\backslash 0$ satisfies

$$\text{(4.4)} \qquad c(0) = c(1) = ... = c(n-1) = 0,$$

and prove this statement by induction on n. The claim is trivial for $n = 1$, since in this case $\ker p_H(\nabla)$ is spanned by one exponential e_λ. Assume therefore that $n > 1$, and write $p_H = q_1 q_2$ where $\deg q_1 = 1$. Suppose that $c \in \ker p_H(\nabla)$ satisfies (4.4), and define $c_1 = q_2(\nabla)c$. Since $q_1(\nabla)c_1 = 0$ and $c_1(n-1) = 0$, the proof for the case $n = 1$ implies that $c_1 = 0$. Hence $q_2(\nabla)c = 0$, and by the induction hypothesis $c = 0$.

The implication (b) $\Longrightarrow$ (c) is trivial, while the implication (c) $\Longrightarrow$ (d) is simple: First, the shift-invariance is obvious. Second, if (4.3) is violated by some λ, μ then the difference $e_\lambda - e_\mu$ vanishes identically on $\mathbb{Z}$, and (c) is thus violated as well.

We complete the proof by showing that (d) $\Longrightarrow$ (c) $\Longrightarrow$ (a). Assuming (d), we first prove that H is D-invariant. Let $f = \sum_{j=1}^{m} e_{\lambda_j} p_j \in H$. For $j = 1, ..., m$, define $q_j := \cdot - \lambda_j$. Note that $q_j(\nabla)(e_{\lambda_k} p_k) = e_{\lambda_k} \widetilde{p}_k$, with

$$\text{(4.5)} \qquad \deg \widetilde{p}_k \leq \deg p_k,$$

where, in case $k = j$, $\deg \widetilde{p}_k = \deg p_k - 1$. Due to (4.3), the numbers $(e^{\lambda_j})_{j=1}^m$ are pairwise different, hence it follows that equality holds in (4.5) if (and only if) $k \neq j$. Fixing $1 \leq j \leq m$ and $0 \leq l \leq \deg p_j$, we define

$$\nabla := q_j(\nabla)^{\deg p_j - l} \prod_{\substack{k=1 \\ k \neq j}}^{m} q_k(\nabla)^{deg\ p_k + 1},$$

and conclude that $\nabla f = e_{\lambda_j} q_l$, with $\deg q_l = l$. Since H is shift-invariant, $\nabla f \in H$ hence

$$e_{\lambda_j} \pi_{deg\ p_j} \subset H,$$

and since j was arbitrary

$$\bigoplus_{j=1}^{m} e_{\lambda_j} \pi_{deg\ p_j} \subset H,$$

proving that H is D-invariant.

To obtain (c), it remains to prove that no $f \in H \backslash 0$ vanishes identically on $\mathbb{Z}$. For this purpose, let $f \in H \backslash 0$. Then, one can use the previous part of the proof to construct a difference operator ∇ such that $\nabla f = e_\lambda$ for some $\lambda \in \operatorname{spec} H$. Since e_λ does not vanish identically on $\mathbb{Z}$, so does f.

Finally, to establish that (c) $\Longrightarrow$ (a), assume that H is D-invariant. We then use (2.12) to conclude that for every function f

$$p_H(D) f * B_H = p_H(\nabla) f.$$

Also, for a continuous function f, $p_H(\nabla) f = 0$ only if $p_H(\nabla)(f_|) = 0$. We thus conclude that $H_| \subset \ker p_H(\nabla)$. Since both H and $\ker p_H(\nabla)$ are n-dimensional, then $H_| = \ker p_H(\nabla)$ if and only if restriction to $\mathbb{Z}$ is 1-1 on H, i.e., (a) and (c) are equivalent. ♠

Needless to say, a D-invariant exponential space H is not necessarily regular (take $H = \operatorname{span}\{1, e_{2\pi i}\}$). Nevertheless we have

(4.6) Proposition. *Let ϕ be a compactly supported function. Then $H(\phi)$ is regular if (and only if) it is D-invariant.*

Proof: The "only if" statement is trivial. Suppose therefore that $H(\phi)$ is D-invariant, and assume to the contrary that (4.3) is violated. Then, there exist $e_{\lambda_1}, e_{\lambda_2} \in H(\phi)$ with $\lambda_1 - \lambda_2 \in 2\pi i \mathbb{Z} \backslash 0$. By (3.2)Proposition there exists a compactly supported τ with $H(\tau) = H(\phi)$ and $K_\tau = \{0\}$. Now, the spaces $H_j := \operatorname{span}\{e_{\lambda_j}\}$, $j = 1, 2$, are shift-invariant subspaces of $H(\tau)$, hence by (2.10)Corollary

$$\tau *' H_j \subset H_j,\ j = 1, 2.$$

On the other hand, $H_{1|} = H_{2|}$ and thus

$$\tau *' H_1 = \tau *' H_2 \subset H_1 \cap H_2 = \{0\}.$$

It follows therefore that $e_{\lambda_1|} \in K_\tau$, contradicting the fact that $K_\tau = \{0\}$. ♠

The last proposition should be regarded as a "negative" result: no singular D-invariant exponential space is $H(\phi)$ for a compactly supported ϕ. This striking result is in full contrast to the multivariate situation: there, **every** finite-dimensional D-invariant exponential space is $H(\phi)$ for some compactly supported ϕ. (cf. the example in [BR; Ex.7.1]). The difference between the univariate and multivariate situations is primarily due to the non-existence of a multivariate analog of (3.2)Proposition.

Note that in view of (4.6)Proposition the regularity assumption which was imposed on $H(\phi)$ in (3.6)Proposition and (3.7)Theorem can be replaced by D-invariance.

Application of (4.6)Proposition to exponential B-splines yields:

(4.7) Corollary. *Assume H is a singular finite-dimensional D-invariant exponential space. Then $H(B_H)$ is a proper subspace of H.* ♠

A slightly stronger version of the above corollary was obtained (by other means) in [BR] (see there Corollary 7.4). Characterizing the structure of all spaces of the form $H(\phi)$, we provide in the last section a farmost extension of (4.6)Proposition and (4.7)Corollary.

5. Splines of minimal support: the regular case

Let H be an n-dimensional shift-invariant exponential space. We seek a function ϕ of minimal support among all these satisfying

$$H \subset H(\phi).$$

Such a spline will be referred to as "a minimal support spline". Throughout this section, we assume the regularity condition

$$\lambda - \mu \notin 2\pi i\mathbb{Z}\backslash 0, \ \forall \lambda, \mu \in \operatorname{spec} H. \tag{5.1}$$

This assumption greatly simplifies the analysis of the minimal support question, as may be expected from the results of the previous sections.

We start the discussion here with the following existence result:

(5.2) Proposition. *Let H be an n-dimensional regular exponential space. Then there exists a unique function ϕ supported on $[0,n]$ that satisfies*

$$\phi *' f = f, \ \forall f \in H. \tag{5.3}$$

Proof: The proof is obtained by specializing [R3; Thm.2.1] to the present situation, and is only sketched here. For details we refer to [R3].

Since H is regular, then, by (4.2)Proposition, there exists a basis $\{f_j\}_{j=0}^{n-1}$ for H satisfying

$$f_j(k) = \delta_{j,k}, \ 0 \le j, k \le n-1.$$

Define $\phi = \sum_{j=0}^{n-1} E^{-j}(f_{j|_{[n-1,n)}})$. Then $\phi *' f_j = f_j$ and hence (5.3) holds. If ψ is a distribution that satisfies (5.3) as well, then $H_| \subset K_{\phi-\psi}$. Since $H_|$ is n-dimensional, then (3.2)Proposition implies that

$$\phi - \psi = p(\nabla)\tau,$$

for some difference operator $p(\nabla)$, and with $\operatorname{supp}\tau \subset \{0\}$. Hence, if $\tau \ne 0$, ψ is not a function.

♠

(5.4) Theorem. *Let H be a regular n-dimensional exponential space. Then ϕ is a spline of minimal support with respect to H if and only if*

$$\phi = p(D)E^a B_H, \tag{5.5}$$

where $a \in \mathbb{R}$, $\deg p < \dim H$ and p vanishes nowhere on $\operatorname{spec} H$.

Proof: Set $n := \dim H$.

First, assume that ϕ is of minimal support with respect to H. Since, by (5.2)Proposition, there exists a spline of minimal support $[0,n]$, we conclude that

$$\operatorname{supp}\phi = [a, a+n],$$

for some $a \in \mathbb{R}$. By appealing to (3.7)Theorem, we obtain that

$$\phi = B_H * \tau,$$

where $\operatorname{supp}\tau = \{a\}$, i.e., $\tau = p(D)E^a$ for some polynomial p. Writing

$$p = p_H q + r,$$

with $\deg r < \deg p_H = \dim H = n$, we see that

$$\phi = q(D)p_H(D)E^a B_H + r(D)E^a B_H. \tag{5.6}$$

From (2.12) we conclude that the first term in the right hand of (5.6) is a distribution with support $\subset \{a, a+1, ..., a+n\}$, while, since $\deg r \leq n-1$, the second term is a well-defined function. Thus the fact that ϕ is assumed to be a *function* implies that $q = 0$. Hence $\phi = p(D)E^a B_H$, with $\deg p < n$. Finally, invoking (3.1)Lemma we obtain

$$H = H(\phi) = p(D)H(B_H) \subset p(D)H \subset H. \tag{5.7}$$

Therefore, $p(D)$ is 1-1 on H, which is equivalent to p vanishes nowhere on $\operatorname{spec} H$.

For the converse, we first deduce from (5.7) that

$$H(B_H) = H. \tag{5.8}$$

Now, assume that ϕ has the representation (5.5). Since $B_H \in C^{n-2}(\mathbb{R})$, and its restriction to each $[\alpha, \alpha+1]$ (with $\alpha \in \mathbb{Z}$) lies in H, then ϕ is a well-defined function with (possible) discontinuities at $\{a, a+1, ..., a+n\}$. To prove that ϕ is of minimal support with respect to H, we only need to show that $H \subset H(\phi)$. Now, $p(D)E^a$ induces an endomorphism on H, and since p vanishes nowhere on $\operatorname{spec} H$, this endomorphism is injective, i.e., an automorphism. Consequently, by (5.8) and (3.1)Lemma, $H = p(D)E^a H = p(D)E^a H(B_H) = H(\phi)$, and ϕ is indeed of minimal support with respect to H. ♠

(5.8) together with (4.7)Corollary can be summerized as follows

(5.9) Corollary. *Let H be a finite-dimensional D-invariant exponential space. Then $H = H(B_H)$ if and only if H is regular.* ♠

The exponential B-spline B_H is well-known to be the unique piecewise-H function in $C^{\dim H-2}$ with support $[0, \dim H]$. The following is a closely related uniqueness property of B_H:

(5.10) Corollary. *Let H be a regular n-dimensional exponential space. Let $\phi \in C^{n-2}(\mathbb{R})$ be of minimal support with respect to H. Then*

$$\phi = cE^a B_H,$$

for some $a \in \mathbb{R}$ and $c \in \mathbb{C}\backslash 0$.

Proof: By (5.4)Theorem, $\phi = p(D)E^a B_H$. Since the smoothness of B_H at each $j = 0, 1, ..., n$ is *exactly* $n-2$, it follows that $p(D)E^a B_H \in C^{n-\deg p-2}(\mathbb{R})\backslash C^{n-\deg p-1}(\mathbb{R})$. The assumption $\phi \in C^{n-2}(\mathbb{R})$ thus implies $\deg p = 0$. ♠

(5.11) Corollary. *Let H be a regular n-dimensional exponential space. Then the unique function ϕ of support $[0,n]$ that satisfies*

$$\phi *' f = f, \ \forall f \in H, \tag{5.12}$$

has the form $\phi = p(D)B_H$, with p the unique polynomial of degree $< n$ satisfying

$$B_H *' p(D)f = f, \ \forall f \in H. \tag{5.13}$$

Proof: We first note that for an arbitrary compactly supported function ψ and an exponential $e_\lambda p \in H(\psi)$, (2.10)Corollary implies that $(e_\lambda p)_| \in H_\psi$. Application of the observations made in [B; §2] allows us to conclude that for every differential operator $p(D)$

$$p(D)[\psi *' (e_\lambda p)] = \psi *' [p(D)(e_\lambda p)]. \tag{5.14}$$

The corollary now easily follows from (5.4)Theorem. Indeed, this result implies that $\phi = p(D)B_H$ for some (and clearly unique) polynomial of degree $< n$. Then for every $f = e_\lambda p \in H$ we have by (5.14) (with $\psi = B_H$)

$$f = \phi *' f = p(D)[B_H *' f] = B_H *' p(D)f. \tag{5.15}$$

(5.13) now follows from (5.15) and the fact that functions of the form $e_\lambda p$ span H. ♠

The above corollary is essentially well-known. It is the proof provided (that avoids completely the standard Fourier analysis arguements) that is new here.

6. The singular case

In this section we generalize the factorization and minimal support results of the previous sections to singular exponential spaces. Quite surprisingly, most of the results obtained in the regular situation remains valid in the singular case as well, with B_H being replaced by a factor of it. In particular, we obtain a characterization of all spaces of the form $H(\phi)$, and determine completely H_ϕ in terms of $H(\phi)$ and K_ϕ. Throughout this section we assume that H is a fixed n-dimensional D-invariant exponential space.

To simplify the analysis we induce on $\operatorname{spec} H$ the equivalence relation

$$\lambda \sim \mu \iff \lambda - \mu \in 2\pi i \mathbb{Z}. \tag{6.1}$$

From each equivalence class [], we choose a representer λ which has maximal multiplicity as a root of p_H (in comparison with the other elements in the same equivalence class). This gives rise to the set $\mathrm{spec}_r\, H$ of all representers just chosen, and its complement $\mathrm{spec}_s\, H$. Since H has the form $H = \bigoplus_{\lambda \in specH} e_\lambda \pi_{k_\lambda}$, we may decompose H into

$$H_r := \bigoplus_{\lambda \in spec_r H} e_\lambda \pi_{k_\lambda}, \tag{6.2}$$

and

$$H_s := \bigoplus_{\lambda \in spec_s H} e_\lambda \pi_{k_\lambda}. \tag{6.3}$$

Then $H = H_r \bigoplus H_s$.

We make use of this decomposition in the following

(6.4)Theorem. *Let ϕ be a compactly supported distribution, $K_\phi = \{0\}$, $\mathrm{cl}_D(H(\phi)) = H$. Then*

(a) $H_\phi = H_{r|}$.

(b) For $e_\lambda p \in H_r$

$$\phi *' (e_\lambda p) = \sum_{\mu \in [\lambda] \cap specH} e_\mu p_\mu, \tag{6.5}$$

where p_μ is a polynomial of degree $\deg p - m_\lambda + m_\mu$ (unless this number is negative, which means $p_\mu = 0$). Here m_λ and m_μ are the multiplicities of the roots λ and μ in the polynomial p_H.

Proof: We divide the proof into two claims.

Claim 1. For $\lambda \in \mathbb{C}$ and $p \in \pi$, $(e_\lambda p)_| \in H_\phi$ only if $[\lambda] \cap \mathrm{spec}\, H \neq \emptyset$ and $\deg p < m_\mu$, μ being the representer of $[\lambda]$ in $\mathrm{spec}\, H_r$; i.e., $H_\phi \subset H_{r|}$.

Proof of Claim 1. If, for some $\lambda \in \mathbb{C}$ and $p \in \pi$, $(e_\lambda p)_| \in H_\phi$ then

$$\phi *' (e_\lambda p) = \sum_{\mu \in spec\ H} e_\mu q_\mu, \tag{6.6}$$

where $\deg q_\mu < m_\mu$. Defining $\nabla := 1 - e^\lambda E$, we observe that, with $t := \deg p + 1$, $\nabla^t(e_\lambda p) = 0$, and therefore by applying ∇^t to both sides of (6.6), and using the fact that exponentials of different frequencies are always linearly independent, we obtain $\nabla^t(e_\mu q_\mu) = 0$ for all $\mu \in \mathrm{spec}\, H$. Since ∇ is degree-preserving on the polynomial part of $e_\mu q_\mu$ unless $\mu \in [\lambda]$, we conclude that $q_\mu = 0$ for $\mu \in \mathrm{spec}\, H \backslash [\lambda]$. Assuming $[\lambda] \cap \mathrm{spec}\, H = \emptyset$, we are led to $\phi *' (e_\lambda p) = 0$, contradicting the assumption $K_\phi = \{0\}$; therefore $[\lambda] \cap \mathrm{spec}\, H \neq \emptyset$, and since H_ϕ is a sequence space, we can always choose λ to satisfy $\lambda \in \mathrm{spec}\, H_r$.

Now, with $\lambda \in \operatorname{spec} H_r$, the above argument shows that

$$(6.7) \qquad \phi *' (e_\lambda p) = \sum_{\mu \in [\lambda] \cap spec\ H} e_\mu p_\mu .$$

If $\deg p \geq m_\lambda$, then also $\deg p \geq m_\mu$, for all $\mu \in [\lambda] \cap \operatorname{spec} H$, and hence ∇^{t-1} annihilates the right hand side of (6.7). On the other hand, $\nabla^{t-1}(e_\lambda p) = c e_\lambda$, for some non-vanishing constant c, hence $e_{\lambda|} \in K_\phi$, which again contradicts the assumption $K_\phi = \{0\}$. Therefore, we must have $\deg p < m_\lambda$ and hence $e_\lambda p \in H_r$. Since, by (2.7)Corollary, H_ϕ is spanned by sequences of the form $(e_\lambda p)_|$, the claim has been established.

Claim 2. $H_\phi = H_{r|}$. Moreover, for every $\lambda \in \operatorname{spec} H_r$,

$$(6.8) \qquad \phi *' e_\lambda ()^{m_\lambda - 1} = \sum_{\mu \in [\lambda] \cap spec\ H} e_\mu q_\mu ,$$

with $\deg q_\mu = m_\mu - 1$, i.e., the highest degree possible.

Proof of Claim 2. Fix $\lambda \in \operatorname{spec} H_r$, and let $m := m_\lambda - 1$. Then, for each $0 \leq j \leq m$ there exists an exponential f_j in $H(\phi)$ of the form $e_\lambda p_j + \sum_{\mu \neq \lambda} e_\mu q_\mu$, with $\deg p_j = j$. By the proof of Claim 1, one may choose f_j to be of the form $e_\lambda p_j + \sum_{\mu \in [\lambda] \setminus \lambda} e_\mu q_\mu$, and in this case $(\phi *)^{-1} f_j \in (e_\lambda \pi_m)_|$. Since $f_0, \ldots, f_m$ are linearly independent, then $\{(\phi *)^{-1} f_j\}_{j=0}^m$ are linearly independent as well, and therefore span $(e_\lambda \pi_m)_|$. We conclude that $H_{r|} \subset H_\phi$, which together with Claim 1, shows that $H_{r|} = H_\phi$.

Finally, note that the shifts, hence the derivatives, of $\phi *' e_\lambda ()^m$ span $\phi *' (e_\lambda \pi_m)$; thus, if in (6.8) for some $\mu \in [\lambda]$, $\deg q_\mu < m_\mu - 1$, then one concludes that $e_\mu ()^{m_\mu - 1} \notin \operatorname{cl}_D(\phi *' (e_\lambda \pi_m))$. In addition, for $\nu \in \operatorname{spec} H_r \setminus \lambda$, (6.7) shows that again $e_\mu ()^{m_\mu - 1} \notin \operatorname{cl}_D(\phi *' (e_\nu \pi_{m_\nu - 1}))$, and therefore, since $H_\phi = H_{r|}$, $e_\mu ()^{m_\mu - 1} \notin \operatorname{cl}_D(H(\phi)) = H$, in contradiction to the multiplicity of μ in p_H. This completes the proof of Claim 2.

For the proof of the theorem, it remains to verify (6.5). Here, we apply $\nabla^{m_\lambda - j - 1}$ to (6.8) (with $\nabla = 1 - e^\lambda E$), to obtain

$$\phi *' e_\lambda p_j = \sum_{\mu \in [\lambda] \cap specH} e_\mu \widetilde{q}_\mu ,$$

where $\deg p_j = j$ and $\deg \widetilde{q}_\mu = m_\mu - (m_\lambda - j - 1) - 1 = j - m_\lambda + m_\mu$. Now, (6.5) readily follows. ♠

The restriction $K_\phi = \{0\}$, although was useful in the proof of the above theorem, can be easily removed:

(6.9) Corollary. *Let ϕ be a compactly supported distribution. Let $p(\nabla)$ be the difference operator whose kernel coincides with K_ϕ. Then (with $H = \mathrm{cl}_D(H(\phi))$)*

(a) $H_\phi = \ker(p(\nabla)p_{H_r}(\nabla))$.

(b) $\dim H(\phi) = \dim H_r$. *Moreover, the projector of H on H_r with kernel H_s induces isomorphism between $H(\phi)$ and H_r.*

Proof: By (3.2)Proposition, there exists a compactly supported σ such that $\phi = p(\nabla)\sigma$, $K_\sigma = \{0\}$ and $H(\phi) = H(\sigma)$. By (6.4)Theorem, $H_\sigma = H_{r|}$, and since H_r is regular, then $H_\sigma = \ker p_{H_r}(\nabla)$, and (a) readily follows. (b) follows from (6.4)Theorem(b) when applied to σ, and the fact that $H(\sigma) = H(\phi)$. ♠

We now apply the above results in the analysis of singular exponential B-splines:

(6.10) Theorem. *Let H be an n-dimensional singular exponential space. Set $n_r := \dim H_r$. Then there exists a function $M_H \in C^{n-2}(\mathrm{I\!R})$ with support $[0, n_r]$ satisfying*

(a) $B_H = p_{H_s}(\nabla)M_H$;

(b) $H(M_H) = H(B_H),\ K_{M_H} = \{0\}$;

(c) $H_{M_H} = H_| = H_{r|}$;

(d) $\dim H(M_H) = n_r$, *and* $\mathrm{cl}_D H(M_H) = H$.

Proof: Let us first prove that $\mathrm{cl}_D(H(B_H)) = H$. For this purpose, we write

$$H = \bigoplus_{\lambda \in specH} (e_\lambda \pi_{m_\lambda - 1}),$$

m_λ being the multiplicity of λ in p_H. For a fixed $\lambda \in \mathrm{spec}\, H$, let $H_1 := e_\lambda \pi_{m_\lambda - 1}$ and $H_2 := \bigoplus_{\mu \in specH \backslash \lambda}(e_\mu \pi_{m_\mu - 1})$. Then H_1 and H_2 are D-invariant and $H = H_1 \bigoplus H_2$, and thus, by (2.14) $B_H = B_{H_1} * B_{H_2}$. Applying (2.12), we obtain $p_{H_2}(D)B_H = p_{H_2}(\nabla)B_{H_1}$. Moreover, H_1 is regular, and therefore (by (5.9)Corollary) $H(B_{H_1}) = H_1$, and thus by (3.1)Lemma

$$p_{H_2}(D)H(B_H) = H(p_{H_2}(\nabla)B_{H_1}) = H(B_{H_1}) = H_1.$$

Consequently, $H_1 \subset \mathrm{cl}_D(H(B_H))$, and since $\lambda \in \mathrm{spec}\, H$ was arbitrary, $H \subset \mathrm{cl}_D(H(B_H))$. Combining this with the fact that B_H is locally in H, we get $H = \mathrm{cl}_D(H(B_H))$.

We now turn to the rest of the proof: Since $H = H_r \bigoplus H_s$, then $p_H(\nabla) = p_{H_r}(\nabla)p_{H_s}(\nabla)$. On the other hand, denoting by $p(\nabla)$ the difference operator whose kernel coincides with K_{B_H}, we have by (6.9)Corollary $H_{B_H} = \ker p(\nabla)p_{H_r}(\nabla)$. We now invoke (2.15)Proposition to conclude that

$$\ker p(\nabla)p_{H_r}(\nabla) = \ker p_{H_s}(\nabla)p_{H_r}(\nabla),$$

and consequently $p(\nabla) = p_{H_s}(\nabla)$. Application of (3.2)Proposition thus yields the existence of a function M_H supported in $[0, n - \dim H_s] = [0, n_r]$ and satisfying $B_H = p_{H_s}(\nabla)M_H$, $H(B_H) = H(M_H)$ and $K_{M_H} = \{0\}$. By appealing to (6.4)Theorem we conclude that $H_{M_H} = H_{r|}$, which together with $K_{M_H} = \{0\}$ implies $\dim H(M_H) = n_r$. Finally, the fact that $M_H \in C^{n-2}(\mathrm{I\!R})$ easily follows from the fact $B_H \in C^{n-2}(\mathrm{I\!R})$ when combined with (a). ♠

The results obtained so far will be used subsequently in the derivation of a generalization of (3.7)Theorem. For this purpose we first need to extend that theorem from a function ϕ to a distribution ϕ:

(6.11) Lemma. *(3.7)Theorem holds for a distribution ϕ.*

Proof: To extend (3.7)Theorem to distributions, we need to extend (3.6)Proposition to distributions. This will be achieved the moment the regular case of (2.10)Corollary is extended to distributions. So let ϕ be a compactly supported distribution, and assume that $H(\phi)$ is regular. To show that $H(\phi)_| \subset H_\phi$, let $\{\sigma_h\}_h$ be an infinitely differentiable approximate identity (cf. [Ru; p.157]. Define $\psi_h := \phi * \sigma_h$. Since $H(\phi)$ is D-invariant, it follows that $\sigma_h * H(\phi) \subset H(\phi)$, while equality holds if and only if $\widehat{\sigma}_h$ does not vanish on $-i \operatorname{spec} H(\phi)$. Being an approximate identity, $\sigma_h \underset{h \to 0}{\to} \delta$, hence $\widehat{\sigma}_h \underset{h \to 0}{\to} 1$ and therefore, for small enough h, $\sigma_h * H(\phi) = H(\phi)$. Thus, $H(\phi) \subset H(\psi_h)$, and application of (2.10)Corollary yields that $\psi_h *' H(\phi) \subset H(\phi)$ for all small enough h. Since $\phi *' H(\phi) = \lim_{h \to 0} \psi_h *' H(\phi)$, then $\phi *' H(\phi) \subset H(\phi)$, as desired. ♠

We are now ready to state and prove the main result of this paper:

(6.12) Theorem. *(The Factorization Theorem For Univariate Splines). Let ϕ be a distribution supported in $[a, b]$. Define $H := \mathrm{cl}_D(H(\phi))$, and set $n := \dim H_r$, $m := \dim K_\phi$. Then there exists a compactly supported distribution τ and a difference operator $p(\nabla)$ satisfying $\operatorname{supp} \tau \subset [a, b-n-m]$, $K_\tau = \{0\}$, $H(\tau) = \{0\}$ and $\ker p(\nabla) = K_\phi$ such that*

$$\phi = p(\nabla)\tau * M_H,$$

where M_H is the function defined in (6.10)Theorem.

Proof: Define $\psi = p_{H_s}(D)\phi$. By (3.1)Lemma, $H(\psi) = p_{H_s}(D)H(\phi)$. Since $p_{H_s}(D)$ annihilates H_s and is 1-1 on H_r, we conlcude from (6.9)Corollary(b) that $H(\psi) = H_r$. By the previous lemma, (3.7)Theorem holds also for distributions, hence there exists τ supported in $[a, b-n-m]$, satisfying $K_\tau = \{0\}$, $H(\tau) = \{0\}$ and $\psi = p(\nabla)\tau * B_{H_r}$. Convolving ψ with B_{H_s}, we then obtain by (2.12) and (2.14)

$$p_{H_s}(\nabla)\phi = p_{H_s}(D)\phi * B_{H_s} = \psi * B_{H_s} = p(\nabla)\tau * B_{H_r} * B_{H_s} = p(\nabla)\tau * B_H.$$

Now, we may invoke (6.10)Theorem to deduce that

$$p_{H_s}(\nabla)\phi = p_{H_s}(\nabla)p(\nabla)\tau * M_H,$$

which implies

$$\phi = p(\nabla)\tau * M_H. \quad \spadesuit$$

Finally, we discuss the notion of minimal support in the singular context:

(6.13) Definition. *Let H be a singular n-dimensional space. We say that ϕ is a spline of minimal support with respect to H if*

(a) $\operatorname{diam}\operatorname{supp}\phi = \dim H_r$;

(b) $\mathrm{cl}_D(H(\phi)) = H$.

We have

(6.14) Theorem. *Let H be a singular exponential space. Then ϕ has a minimal support with respect to H if and only if*

$$\phi = p(D)E^a M_H,$$

where $\deg p < \dim H$ and p vanishes nowhere on $\operatorname{spec} H$.

Proof: By (6.10)Theorem, M_H is a spline of minimal support with respect to H. Hence if ϕ is also a spline of minimal support, then $\operatorname{diam}\operatorname{supp}\phi = \dim H_r$. Also, for every polynomial p,

$$p(D)H = p(D)\mathrm{cl}_D(H(M_H)) = \mathrm{cl}_D(p(D)H(M_H)) = \mathrm{cl}_D(H(p(D)M_H)).$$

The proof now follows that of (5.4)Theorem, with (3.7)Theorem replaced by (6.11)Theorem. $\spadesuit$

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