Divided Differences, Hyperbolic Equations and Lifting Distributions

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Abstract. We discuss the relationship between divided differences, fundamental function of hyperbolic equations, multivariate interpolation and polyhedral splines.

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§ 1. Introduction.

The motivation for this paper comes from two sources: the distributional definition of a polyhedral spline, [1,4] and polynomial interpolation obtained from lifting distributions [2] (see Definition 2.2). In this paper we unify these ideas, extend their range of applicability and show their relationship to the fundamental function for hyperbolic equations.

In general terms we are concerned with inversion of the Radon transform of a distribution. Specifically, we will identify certain univariate distributions of approximation theoretic interest which can be lifted. This method has led to several interesting multivariate versions of important univariate methods of approximation theory. Definitions, main results and numerous examples will be given later. For the moment, we describe two particularly important examples of lifting distributions.

Suppose $\phi \in C_0^\infty(\mathbb{R}^1)$ and $t_0, \ldots, t_n$ are any real numbers. We let $p(t_0, \ldots, t_n)\phi$ be the polynomial of degree $\le n$ which interpolates ϕ and all its derivatives at each t_i up to order $| \{j{:}t_j = t_i\} | - 1$ that is, Hermite interpolation to ϕ. This distribution $\phi \to (p(t_0, \ldots, t_n)\phi)(t)$ has a multivariate lift to any $\mathbb{R}^k$. The lifted distribution has compact support, is of order n and gives a *natural* extension of Hermite interpolation to $\mathbb{R}^k$, [8] . Next, we let $M(t \mid t_0, \ldots, t_n)$ be the univariate B-spline of degree $\le n-1$ with knots at $t_0, \ldots, t_n$. It also has a lift to any $\mathbb{R}^k$ which is called the *multivariate B-spline* and has been actively studied, [4] .

These examples are special cases of a class of univariate distributions which can be lifted by using the method used by F. John, [6] for the construction of the solution to a hyperbolic equation. This connection will be explained in Section 3. In Section 2, we will classify distributions on entire functions which can be lifted. Finally, in the last section we will give several more examples of our results.

§2. Lifting Distributions on $\mathscr{E}_k$.

We begin by recalling the following definition.

Definition 2.1. *Given a* kxn *matrix* A, *the multivariate B-spline M_A is the functional defined by*

$$M_A\psi := \int_{S_{n-1}} \psi \circ A, \quad \psi \in C(\mathbb{R}^k) \tag{2.1}$$

where $S_{n-1} := \{(t_1, \ldots, t_n) : t_\nu \geq 0, \ \sum_{\nu=1}^{n} t_\nu = 1\}$. We will use $\psi \circ f$ to indicate composition of functions, $\psi(f(x))$, and integrals are always taken relative to Lebesgue measure. When $\mathrm{vol}_k A(S_{n-1}) > 0$, M_A can be realized as a piecewise polynomial on $\mathbb{R}^k$. In this case, we sometimes denote it by $M(x \mid x^1, \ldots, x^n)$ where $x^1, \ldots, x^n$ are the columns of A. For k=1 it reduces to the univariate B-spline of Curry and Schoenberg [3]. For a function ϕ of one variable and $\zeta, z \in \mathbb{C}^k$ we set $(\phi \circ \zeta)(z) := \phi(\zeta z)$. Here $\zeta z := \sum_{i=1}^{k} \zeta_i z_i$ is the inner product of ζ with $\bar{z}$. Thus we see that the functional M_A has the striking property that it reduces to the univariate B-spline when acting on the plane wave functions $\phi \circ z$. Specifically, we have $M_A(\phi \circ z) = M_{zA}\phi$. This equation motivates the following definition. Denote by $\mathscr{E}_k$ the space of entire functions in k complex variables with the topology of uniform convergence on compact sets.

Definition 2.2. *A family of continuous linear functionals $\Lambda_z, z \in \mathbb{C}^n$, on $\mathscr{E}_1$ is liftable whenever for any* kxn *matrix* A *there exists a functional $\Lambda_A \in \mathscr{E}'_k$ ($\mathscr{E}'$ denotes the dual space of $\mathscr{E}$.) with the property that*

$$\Lambda_A(\phi \circ z) = \Lambda_{zA}\phi, \quad \phi \in \mathscr{E}_1, \ z \in \mathbb{C}^k. \tag{2.2}$$

Note that (2.2) defines the functional (if it exists) uniquely since linear combinations of the homogeneous polynomials $e_j \circ z$, $e_j(\tau) = \tau^j$, are dense in $\mathscr{E}_k$.

The concept of lifting is useful for constructing multivariate extensions for various univariate polynomial projections, [2] .

Not all distributions are liftable. As a simple example consider the divided difference of $\phi \in \mathscr{E}_1$ at $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$

$$[z]\phi := [z_1, \ldots, z_n]\phi. \tag{2.3}$$

From the Hermite-Genocchi formula

$$[z]\phi = \int_{S_{n-1}} \phi^{(n-1)} \circ z \tag{2.4}$$

it follows that

$$M_z \phi^{(n-1)} = [z]\phi. \tag{2.5}$$

This distribution is not liftable even if n=2. To see this, suppose otherwise that Λ_A lifts the divided difference. Let $\phi = e_1$ and observe that (2.2) implies

$$\Lambda_A(\phi \circ \zeta) = [\zeta A]\phi = 1$$

for any $\zeta \in \mathbb{C}^k$. On the other hand, by linearity,

$$\Lambda_A(\phi \circ (2\zeta)) = \Lambda_A(2(\phi \circ \zeta)) = 2\Lambda_A(\phi \circ \zeta)$$

which yields a contradiction. To present our first result we begin by pointing out that $\Lambda_z \in \mathscr{E}'_1$, $z \in \mathbb{C}^n$ can be represented in the form

$$\Lambda_z \phi = \sum_{j=0}^{\infty} \frac{1}{j!} a_j(z) \phi^{(j)}(0). \tag{2.6}$$

Alternatively, we have

$$\Lambda_z\phi = \frac{1}{2\pi R}\int_{\sigma_1(R)} G(z,\xi)\phi(\xi)dm_\xi, \quad R > R_z \tag{2.7}$$

where

$$G(z,\xi) = \sum_{j=0}^{\infty} \frac{a_j(z)}{\xi^j} \tag{2.8}$$

and $\sigma_k(R) = \{\zeta : \zeta = (\zeta_1, \ldots, \zeta_k) : |\zeta|^2 := |\zeta_1|^2 + \cdots + |\zeta_k|^2 = R^2\}$. For instance, for the divided difference

$$[z]\phi = \sum_{j=0}^{\infty} \frac{1}{j!} ([z]\omega_j)\phi^{(j)}(0)$$

and

$$[z]\phi = \frac{1}{2\pi R}\int_{\sigma_1(R)} \frac{\xi}{(\xi - z_1)\ldots(\xi - z_n)}\phi(\xi)dm_\xi, \quad z = (z_1, \ldots, z_n),$$

where $R > R_z := \max_{1 \le i \le n} |z_i|$. In general, (2.7) gives a representation for $\Lambda_z\phi$ on $A(\sigma_1(R))$, i.e. functions analytic on $\{\xi : |\xi| < R\}$.

Theorem 2.1. *Λ_z, $z \in \mathbb{C}^k$ is liftable if and only if the coefficients a_j in* (2.4) *are homogeneous polynomials of degree* j *which satisfy*

$$|a_j(z)| \le c(|z| R_\Lambda)^j \tag{2.9}$$

for some constants c, $R_\Lambda > 0$. In this case, for $\psi \in \mathscr{E}_k$

$$\Lambda_A\psi = \sum_{j=0}^{\infty} \frac{1}{j!} (a_j(\nabla_k A)\psi)(0), \qquad \nabla_k\psi := \left(\frac{\partial\psi}{\partial z_1}, \ldots, \frac{\partial\psi}{\partial z_k}\right) \tag{2.10}$$

or

$$\Lambda_A\psi = \frac{(k-1)!}{2\pi^k R^{2k-1}} \int_{\sigma_k(R)} L_k\left(\frac{\bar{\zeta}A}{R^2}\right)\psi(\zeta)dm_\zeta \tag{2.11}$$

where

$$L_k(z) = \frac{(-1)^{k-1}}{(k-1)!}\partial_t^{k-1}\left(\frac{G(z,t)}{t}\right)\Big|_{t=1} \tag{2.12}$$

Proof: Assume first that Λ_z is liftable. From (2.2) with $k = n$, $\phi = e_j$ and $A = I$, the identity matrix we obtain

$$a_j(z) = \Lambda_z e_j = \Lambda_I(e_j \circ z) = \sum_{|\alpha|=j} z^\alpha \binom{j}{\alpha} \Lambda_I E_\alpha$$

where $E_\alpha(\zeta) = \zeta^\alpha = \zeta_1^\alpha \ldots \zeta_k^\alpha$, $\zeta = (\zeta_1, \ldots, \zeta_k)$, $\alpha = (\alpha_1, \ldots, \alpha_k)$. This shows that a_j is a homogeneous polynomial of degree j. The estimate (2.9) follows from the continuity of Λ_z

$$|\Lambda_I(e_j \circ z)| \leq |\Lambda_I| \max_{|\zeta| \leq R_\Lambda} |\zeta \bullet z|^j$$

where R_Λ depends on Λ_I.

For the converse we suppose a_j in the representation (2.6) of Λ_z are homogeneous polynomials of degree j satisfying (2.9). We define $\Lambda_A\psi$ by (2.10) and observe that from the representation

$$\psi(\zeta) = \frac{(k-1)!}{2\pi^k R^{2k-1}} \int_{\sigma_k(R)} K\left(\frac{\bar{z}\zeta}{R^2}\right) \psi(z) dm_z, \quad \psi \in \mathscr{E}_k$$

where

$$K(\xi) = (1-\xi)^{-k} = \sum_{j=0}^{\infty} \binom{j+k-1}{j} \xi^j$$

(the Cauchy-Szegö kernel for $H_2(\sigma_k(R))$) that

$$\Lambda_A \psi = \frac{(k-1)!}{2\pi^k R^{2k-1}} \int_{\sigma_k(R)} \sum_{j=0}^{\infty} \binom{j+k-1}{j} a_j \left(\frac{\bar{z}A}{R^2}\right) \psi(z) dm_z. \tag{2.13}$$

Choosing $R > R_\Lambda \mid A \mid$ we get

$$\mid \Lambda_A \psi \mid \leq c \sum_{j=0}^{\infty} \binom{j+k-1}{j} \left(\frac{\mid A \mid R_\Lambda}{R}\right)^j \max_{\mid \zeta \mid = R} \mid \psi(\zeta) \mid$$

which shows that $\Lambda_A \in \mathscr{E}_k$. Since

$$a_j(\nabla_k)(\phi \circ z) = a_j(zA)(\phi^{(j)} \circ z) \tag{2.14}$$

it is easy to check that (2.2) holds. To verify the last assertion we see from (2.13) that

$$L_k(\zeta) = \sum_{j=0}^{\infty} \binom{k+j-1}{j} a_j(\zeta).$$

Since

$$G(z, \xi) = \sum_{j=0}^{\infty} \frac{a_j(z)}{\xi^j}$$

(2.12) becomes clear.

Remark 2.1. Note that when $\Lambda_z \in \mathcal{E}'_1, z \in \mathbb{C}^n$ can be lifted, $G(t, \xi)$ is an *analytic and homogeneous* function on the domain $\{(z, \xi) : R \mid z \mid < \mid \xi \mid \}$. Moreover, for $\xi \in \sigma_1(R)$ it has the form

$$G(z, \xi) = L_1\left(\frac{\bar{\xi}z}{R^2}\right)$$

Observe also that $z \to \Lambda_z\phi$ must be entire when ϕ is entire.

Example 2.1. For the multivariate B-spline, the kernel $G(z, \xi)$ is the Stieltjes transform

$$G(z, \xi) = \xi \int_{S_{n-1}} \frac{dx}{\xi - xz} .$$

Using (2.4) we get

$$L_n(z) = \frac{1}{(n-1)!} \frac{1}{\prod_{i=1}^{n} (1 - zx^i)}$$

which can be used to represent $M(x \mid x^1, \ldots, x^n)$ on $A(\sigma_n(R))$.

Next we give two applications of Theorem 2.1. For this purpose, we let $\pi_m(\mathbb{R}^k)$ be the space of polynomials of total degree $\leq m$ on $\mathbb{R}^k$.

Definition 2.3. *A family of projections $P_z, z \in \mathbb{C}^n$ from $\mathcal{E}_1$ onto $\pi_m(\mathbb{R}^1)$ is liftable if and only if for any integer* k *and any* kxn *matrix $A \in \mathbb{C}^{kn}$ there exists a projection P_A onto $\pi_m(\mathbb{R}^k)$ for which*

$$P_A(\phi \circ z) = (P_{zA}\phi) \circ z, \quad z \in \mathbb{C}^k, \phi \in \mathscr{E}_1. \tag{2.15}$$

Theorem 2.2. *Suppose that* $P_z, z \in \mathbb{C}^n$ *is a family of projections from* $\mathscr{E}_1$ *onto* $\pi_m(\mathbb{R}^k)$. *Then the following are equivalent:*

i) P_z *is liftable*

ii) *The family of functionals* $\Lambda_{(z,\xi)}\phi := (P_z\phi)(\xi)$, $(z, \xi) \in \mathbb{C}^n x \mathbb{C}$ *is liftable*

iii) $(P_z\phi)(\xi) = \sum_{j=0}^{m} (\Lambda_z^j \phi^{(j)})\xi^j$ *where the functionals* Λ_z^j, $j = 0, 1, \ldots, m$ *are liftable.*

Proof:

(i) $\rightarrow$ (ii): Let $B = (A, \zeta)$ be a k x n+1 matrix whose last column is ζ . Define

$$\Lambda_B\psi = (P_A\psi)(\zeta), \quad \psi \in \mathscr{E}_k,$$

then it follows from (2.15) that for $\phi \in \mathscr{E}_1$

$$\begin{aligned} \Lambda_B(\phi \circ z) &= (P_A(\phi \circ z))(\zeta) \\ &= (P_{zA}\phi)(z\zeta) \\ &= \Lambda_{(zA, z\zeta)}\phi \\ &= \Lambda_{zB}\phi \end{aligned}$$

which shows that $\Lambda_{(z,\xi)}$ is liftable.

(ii) $\Rightarrow$ (iii) . By Theorem 2.1 we have

$$\Lambda_{(z,\xi)}\phi = \sum_{j=0}^{m} a_j(z, \xi)\phi^{(j)}(0)$$

where $a_j(z, \xi)$ is a homogeneous polynomial satisfying (2.8). Therefore,

$$a_j(z, \xi) = \sum_{\ell=0}^{j} q_{\ell, j}(z)\xi^{j-\ell}$$

where each $q_{\ell, j}(z)$, $0 \le \ell \le j$ is homogeneous of degree e. Furthermore, if we define

$$\Lambda_z^j \phi = \sum_{j=\ell}^{m} q_{j-\ell, j}(z)\phi^{(\ell)}(0)$$

then the representation claimed in (iii) holds. Moreover, each Λ_z^j is liftable because the estimate (2.8) for $q_{\ell, j}(z)$ follows from the estimate which holds for $a_j(z; \xi)$.

(iii) $\Rightarrow$ (i) Define for any kxn matrix A

$$(P_A\psi)(\zeta) = \sum_{j=0}^{M} \Lambda_A^j \left((\zeta\nabla_k)^j \psi\right) \tag{2.16}$$

where Λ_A^j lifts Λ_z^j to $\mathscr{E}_k$. Then for any $\phi \in \mathscr{E}_1$ and $\zeta, z \in \mathbb{C}^k$ we have

$$\begin{aligned} P_A(\phi \circ z)(\zeta) &= \sum_{j=0}^{m} \Lambda_A^j((\zeta z)^j \phi^{(j)} \circ z) \\ &= \sum_{j=0}^{m} (\zeta z)^j \Lambda_{zA}^j \phi^{(j)} \\ &= (P_{zA}\phi)(\zeta z) \\ &= ((P_{zA}\phi) \circ z)(\zeta) \end{aligned} .$$

Thus $P_A\phi$ lifts P_z to $\mathscr{E}_k$. Furthermore, the linear functionals $\Lambda_A^{j;\alpha}\psi$, $|\alpha| = j$, $0 \le j \le m$ defined by

$$\Lambda_A^j (\zeta\nabla_k)^j \psi = \sum_{|\alpha|=j} (\Lambda_A^{j,\alpha}\psi)\zeta^\alpha$$

determine the polynomial $P_A\psi$ uniquely by the conditions

$$\Lambda_A^{j,\alpha}\psi = \Lambda_A^{j,\alpha}P_A\psi, \quad |\alpha| = j, \quad 0 \le j \le m. \tag{2.17}$$

To see this, we will show that for any $p \in \pi_m(\mathbb{R}^k)$ such that

$$\Lambda_A^j((\zeta\nabla_k)^j p) = 0, \quad 0 \le j \le m, \quad \zeta \in \mathbb{C}^k,$$

p must be identically zero. For this purpose we decompose p into its homogeneous parts, that is, $p = p_0 + p_1 + \cdots + p_m$ where each p_j is a homogeneous polynomial of degree j. Thus we have for all $\zeta \in \mathbb{C}^k$

$$0 = \Lambda_A^m((\zeta\nabla_k)^m p_m) = m!p_m(\zeta)\Lambda_A^m 1 = p_m(\zeta).$$

Similarly, each $p_j = 0$, $0 \le j \le m$, from which we conclude that p=0 .

Example 2.2. Let $z = (z_0, \dots, z_n) \in \mathbb{C}^{n+1}$ and $P_z\phi$ be Hermite interpolation to $\phi \in \mathscr{E}_1$. We introduce $P_z^r\phi = (P_z\phi^{(-r)})^{(r)}$. Then using the Newton form for $P_z\phi$ we have

$$\begin{aligned} P_z^r\phi &= \sum_{j=r}^{n} ([z_0, \dots, z_j]\phi^{(-r)})((\,\bullet - z_0)\dots(\,\bullet - z_{j-1}))^{(r)} \\ &= \sum_{j=r}^{n} (\int_{\mathbb{R}^1} M(t \mid z_0, \dots, z_{j-1})\phi^{(j-r)}(t)dt)((\,\bullet - z_0)\dots(\,\bullet - z_{j-1}))^{(r)}. \end{aligned}$$

Setting

$$q_j(t, z) = \frac{d^r}{dt^r}(t - z_0)\dots(t - z_{j-1})$$

then the left of P_z^r to $\mathscr{E}_k$ is

$$P_A^r\psi(x) = \sum_{j=r}^{n} \int_{\mathbb{R}^k} M(y \mid x^0, \dots, x^j)q_j(x\nabla_k, x^0\nabla_k, \dots, x^{j-1}\nabla_k)\psi(y)dy$$

where $x^{\circ}, \ldots, x^{n}$ are the columns of A. The case r=0 was studied in [8] and the case r=k-1 appears in [5].

We now give another application of Theorem 2.1.

For every $z \in \mathbb{C}^{n}$, let $p_r(\xi, z)$ and $q_s(\xi, z)$ be polynomials of degree r, s, respectively in $\xi \in \mathbb{C}^{1}$ normalized so that $q_s(\xi, z) = \xi^{s} + \cdots$. We define the kernel

$$G(z, \xi) = \frac{p_r(\xi, z)}{q_s(\xi, z)} \tag{2.18}$$

and observe that for $r \leq s$, $G(z, \xi)$ is analytic in ξ at ∞. Thus we may expand it in a neighborhood of ∞,

$$G(z, \xi) = \sum_{j=0}^{\infty} \frac{a_j(z)}{\xi^{j}}, \qquad | \xi | \geq R_z \tag{2.19}$$

and conclude that the functionals

$$\Lambda_z \phi = \sum_{j=0}^{\infty} \frac{a_j(z)}{j!} \phi^{(j)}(0), \quad \phi \in \mathcal{E}_1 \tag{2.20}$$

are in $\mathcal{E}'_1$, for each $z \in \mathbb{C}^{n}$.

We assume without loss of generality that r=s and for some $z \in \mathbb{C}^{n}, p_r(\xi, z), q_r(\xi, z)$ have no common factors as functions of ξ.

Theorem 2.3. *Suppose* p_r, q_s *are as above. Then the functional* $\Lambda_z \phi$ *can be lifted if and only if* $p_r(z, \xi)$ *and* $q_s(z, \xi)$ *are homogeneous polynomials of degree* r, s, *respectively, in the variables* $(z, \xi) \in \mathbb{C}^{n+1}$.

Proof:

Expanding p_r, q_r as

$$p_r(\xi, z) = \sum_{j=0}^{r} p_j(z)\xi^j, \qquad q_r(\xi, z) = \sum_{j=0}^{r} q_j(z)\xi^j$$

we see that (2.19) is equivalent to the equations

$$\begin{aligned} p_r &= a_0 q_r \\ p_{r-1} &= a_0 q_{r-1} + a_1 q_r \\ &\vdots \\ p_0 &= a_0 q_0 + \cdots + a_r q_r \end{aligned} \tag{2.21}$$

and

$$a_\nu q_0 + \cdots + a_{\nu+r} q_r = 0, \quad \nu = 1, 2, 3, \ldots . \tag{2.22}$$

First assume $p_r(\xi, z), q_r(\xi, z)$ are homogeneous of degree r in (ξ, z), so that $p_j(z), q_j(z)$ are homogeneous of degree r-j, j=0,1,...,r. Since $q_r(z) = 1$ we see from (2.21) and (2.22) by induction that each a_j is a homogeneous polynomial of degree j, j=0, 1, ... To prove the estimate (2.8) we choose $R > 0$ such that $| q_j(z) | \le (R | z |)^{r-j}$, $j = 0, 1, \ldots, r$, and again use induction. Thus we suppose

$$| a_\nu(z) | \le c(R | z |)^\nu, \qquad \nu \le \mu \tag{2.23}$$

with $\mu \ge r$. Then it follows from (2.22) that

$$
\begin{aligned}
| a_{\mu+1}(z) | &\leq \sum_{\rho=r}^{r} | a_{\mu+1-\rho}(z) q_{r-\rho}(z) | \\
&\leq \sum_{\rho=1}^{r} c(R | z |)^{\mu+1-\rho} (R | z |)^{\rho} \\
&\leq c(R | z |)^{\mu+1}.
\end{aligned}
$$

We now conclude from Theorem 2.1 that Λ_z is liftable. Conversely, suppose that Λ_z is liftable. We then solve any r consecutive equations from (2.22) to obtain for any $\nu \geq 1$ that

$$
H_\nu(z) q_\ell(z) = B_{\nu\ell}(z), \quad \ell = 0, 1, \ldots, r-1
$$

where $H_\nu(z)$ is the Hankel determinant ,

$$
\begin{vmatrix}
a_\nu(z) & \cdots & a_{\nu+r+1}(z) \\
\vdots & & \vdots \\
a_{\nu+r-1}(z) & \cdots & a_{\nu+2(r-1)}(z),
\end{vmatrix}
$$

a homogeneous polynomial in z, and $B_{\nu\ell}(z)$ is some homogeneous polynomial of degree $\deg H_\nu + r - \ell$. The only way $H_\nu(z) = 0$, for all $\nu = 1, 2, \ldots$ is for $a_1, a_2, \ldots$ to satisfy some nontrivial difference equation

$$
\mathrm{v}_0 a_\nu + \mathrm{v}_1 a_{\nu+1} + \cdots + \mathrm{v}_{r-1} a_{\nu+r-1} = 0
$$

of degree m-1. If this were the case it would follow that $p_r(\xi; z)/q_r(\xi; z)$ can be represented as a rational function with a denominator of degree $<$ m . Hence, by hypothesis for at least some z there is a μ such that $H_\mu(z) \neq 0$. But $q_\ell(z)$ is finite for all z and so H_μ must divide each $B_{\mu\ell}$, $\ell = 0, 1, \ldots, r-1$, which shows q_ℓ is a homogeneous polynomial of degree $r - \ell$. Finally using equations (2.20) we easily see that each p_ℓ is a homogeneous polynomial of degree $r - \ell$ thereby completing the proof of the theorem.

§ 3. Extension to $\mathscr{C}_0^\ell(\mathbb{R}^k)$

In this section we give conditions on the family of distributions (2.20) defined by (2.18), (2.19) to have a lift which has an extension to some $\mathscr{C}_0^{\ell}(\mathbb{R}^k)$ (all functions with ℓ continuous derivatives on $\mathbb{R}^k$ with compact support). We consider the functionals in the form

$$\begin{aligned}\Lambda_z\phi &= \frac{1}{2\pi i}\int_{|\xi|=R|z|}\frac{P(\xi,z)}{Q(\xi,z)}\phi(\xi)d\xi, \quad Q(\xi,z)=\xi^m+\cdots \\ &= \sum_{j=0}^{\infty}\frac{a_j(z)}{j!}\phi^{(j)}(0).\end{aligned} \tag{3.1}$$

where

$$\frac{\xi P(\xi,z)}{Q(\xi,z)}=\sum_{j=0}^{\infty}\frac{a_j(z)}{j!}\xi^{-j}, \qquad |\ \xi\ |>R\ |\ z\ |\,. \tag{3.2}$$

By Theorem 2.3, Λ_z is liftable, if P and Q are homogeneous polynomials of some degree m-1 and m respectively in $(\xi,z)\in\mathbb{C}^{n+1}$. We begin with

Lemma 3.1. *Let* p , q *be polynomials of degree* -1 *and* m, *respectively, with no common zeros. Suppose also that* q *is free of zeros in* $|\ \xi\ |\geq R$ *and has leading coefficient one. Then the linear functional*

$$\Lambda\phi=\frac{1}{2\pi i}\int_{|\xi|=R}\frac{p(\xi)}{q(\xi)}\phi(\xi)d\xi, \qquad \phi\in\mathscr{E}_1$$

is bounded on $\mathscr{C}^{m-1}(\mathbb{R}^1)$ *if and only if* q *has real zeros.*

Proof: Suppose that there is a constant $c>0$ such that $|\ \Lambda\phi\ |\leq c\sup_{\sigma\in\mathbb{R}^1}|\ \phi^{(m-1)}(\sigma)\ |$. Let z_0 be a zero of q so that $p(z_0)\neq 0$ and for $t\in\mathbb{R}$, set $\phi(\xi)=e^{-i\xi t Im z_0}\dfrac{q(\xi)}{\xi-z_0}e^{-\xi^2}$. Then there is a constant M > 0 such that for t sufficiently large

$$| e^{-z_0^2} p(z_0) | e^{t | Im z_0 |} = | \Lambda\phi | \leq M | t |^{m-1}$$

which proves z_0 is real. Conversely, if $x_1, \ldots, x_m \in \mathbb{R}^1$ are the zeros of q, then

$$\Lambda\phi = [x_1, \ldots, x_n](p(\bullet)\phi(\bullet))$$

where $[x_1, \ldots, x_m]\phi$ is the divided difference of ϕ at $x_1, \ldots, x_m$. Thus by equation (2.4) we have

$$\Lambda\phi = \int_{\sum_{i=1}^{m} \lambda_i = 1} (p\phi)^{(m-1)} (\sum_{i=1}^{m} \lambda_i x_i) d\lambda_1 \ldots d\lambda_m, \quad \phi \in \mathscr{E}_1$$

which gives an extension of $\Lambda\phi$ to $\mathscr{C}_0^{m-1}(\mathbb{R}^1)$.

Remark 3.1. The proof of Lemma 3.1 also shows that when $\Lambda\phi$ can be extended to *some* $\mathscr{C}_0^{\ell}(\mathbb{R}^1)$ then q must have real zeros. Moreover, if q has real zeros with multiplicites all $\leq \ell$ then $\Lambda\phi$ has an extension to $\mathscr{C}_0^{\ell-1}(\mathbb{R}^1)$.

Theorem 3.1. *Assume $\Lambda_z, z \in \mathbb{C}^n$ has the form* (3.1). *Then it can be lifted to some $\mathscr{C}^{\ell}(\mathbb{R}^k)$ if and only if $P(\xi, z)$, $Q(\xi, z)$ are homogeneous polynomials of degree* m - 1, m *respectively and $Q(\xi, x)$ has only real zeros for each $x \in \mathbb{R}^n$*, i.e.

$$Q(\xi, x) \neq 0, \quad x \in \mathbb{R}^n, \quad Im\xi \neq 0. \tag{3.3}$$

Proof: Suppose first the $\Lambda_z, z \in \mathbb{C}^n$ can be lifted for each k to some $\mathscr{C}^{\ell}(\mathbb{R}^k)$. Let Λ_A be its lift to $\mathscr{C}^{\ell}(\mathbb{R}^n)$. Then for any $x \in \mathbb{C}^n$ we have

$$\Lambda_x\phi = \Lambda_I(\phi \circ x)$$

which shows that $\Lambda_x\phi$ has an extension to $\mathscr{C}^{\ell}(\mathbb{R}^1)$. From Remark 3.1 and Theorem 2.3 $Q(\xi, x)$ has only real zeros.

For the converse we require some information on initial-value problems for hyperbolic equations. Denote by $Q(\partial_t, \nabla_n)$ the differential operator corresponding to the polynomial $Q(t, x_1, \ldots, x_n)$. Then (3.3) implies that $Q(\partial_t, \nabla_n)$ is *hyperbolic*, i.e. the initial value problem

$$\begin{aligned} &Q(\partial_t, \nabla_n)u(t,x) = 0, \;\; x \in \mathbb{R}^n, t > 0, \\ &\partial_t^\ell u(0,x) = 0, \;\; \ell = 0, 1, \ldots, m-2, \;\; x \in \mathbb{R}^n, \\ &\partial_t^{m-1} u(0,x) = \psi(x), x \in \mathbb{R}^n \end{aligned} \tag{3.4}$$

is well-posed. We write

$$u = Q^{-1}(\partial_t, \nabla_n)\psi \tag{3.5}$$

so that by standard regularity results on any finite time interval $I = [0, T]$

$$\| u(t, \bullet) \|_{\mathscr{C}_0^m(I \times \mathbb{R}^n)} \leq C \| \psi \|_{\mathscr{C}_0^\ell(\mathbb{R}^n)} \tag{3.6}$$

for some ℓ, depending on n,m. The choice $\ell = n + m + 1$, will always suffice, [7] . Recall that, for a kxn matrix A we use the notation

$$(\psi \circ A)(z) = \psi(Az), \;\; z \in \mathbb{C}^n, \psi \in \mathscr{E}_k$$

and $\nabla_k A$ means the (vector) differential operator whose columns are directional derivatives in the direction of the columns of A

Now, suppose $P(\xi, z)$, $Q(\xi, z)$ are homogeneous polynomials of degree m-1, m respectively such that Q satisfies (3.3).

For $\psi \in \mathscr{C}_0^\ell(\mathbb{R}^k)$ we define

$$\Lambda_A \psi = P(\partial_t, \nabla_n) Q^{-1}(\partial_t, \nabla_n)(\psi \circ A) \mid_{(1,0)} \tag{3.7}$$

or equivalently

$$\Lambda_A \psi = P(\partial_t, \nabla_k A) Q^{-1}(\partial_t, \nabla_k A)\psi \mid_{(1,0)}. \tag{3.8}$$

Let us prove that Λ_A lifts $\Lambda_z, z \in \mathbb{C}^n$ to $\mathscr{C}^\ell(\mathbb{R}^n)$. Thus it is sufficient to prove for any $y \in \mathbb{R}^n$,

$$\Lambda_A(g \circ y) = \Lambda_{yA} g \tag{3.9}$$

where $g(\sigma) = e^{i\sigma}, \sigma \in \mathbb{R}$. From (3.1) we get

$$\Lambda_x g = \frac{1}{2\pi i} \int \frac{P(\xi, xA)}{Q(\xi, xA)} e^{i\xi} d\xi. \tag{3.10}$$

On the other hand, the solution of the initial-value problem (3.4) with $\psi = (g \circ y) \circ A = g \circ yA$ is given by the formula

$$u(t, x) = \frac{1}{2\pi} \int \frac{e^{i(\xi t + xyA)}}{Q(i\xi, iyA)} d\xi, \tag{3.11}$$

[7] . Therefore the left hand side of (3.9) by definition (3.7), agrees with (3.10) as claimed.

§ 4. Examples.

In this section, we give several examples of Theorem 3.1. All our examples have the common feature that the polynomial $P(\xi, z)$ is *independent* of ξ so that henceforth it will be denoted by P(z). In this case, some simplification takes place. We begin by observing that the solution to (3.4) can be represented as

$$u(t, x) = t^{m-1} \Omega(\psi(\,\bullet + tx)) \tag{4.1}$$

where Ω is the distribution defined by

$$\Omega\psi = Q^{-1}(\partial_t, \nabla_n)\psi \mid_{(1,0)}. \tag{4.2}$$

Since P and Q^{-1} *commute* (3.7) and (4.1) give

$$\Lambda_A\psi = \Omega(P(\nabla_n)(\psi\circ A)) \tag{4.3}$$

or equivalently

$$\Lambda_A\psi = M_A^{\Omega}(P(\nabla_k A)\psi) \tag{4.4}$$

where M_A^{Ω} is the distribution on $\mathscr{C}_0^{\ell}(\mathbb{R}^n)$ defined by

$$M_A^{\Omega} f = \Omega(f\circ A). \tag{4.5}$$

To identify Ω we observe from (4.3) with k=n, $x \in \mathbb{R}^n$ that

$$\Lambda_I(\phi\circ x) = P(x)\Omega(\phi^{(m-1)}0x). \tag{4.6}$$

On the hand, using the lifting equation (2.2) and the defining equation (3.1) we get for $x \in \mathbb{R}^n$

$$[z_1(x), \dots, z_m(x)]\phi = \Omega(\phi^{(m-1)}0x) \tag{4.7}$$

where $z_1(x), \dots, z_m(x) \in \mathbb{R}$ are the real zeros of $Q(\xi, x)$.

We consider two classes of examples of (4.3) in detail:

In the first case, we suppose $Q(\xi, x)$ is orthogonally invariant in x for all ξ. Thus we require $Q(\xi, Ux) = Q(\xi, x)$ for every isometry U on $\mathbb{R}^n$. In this case, there is a univariate polynomial q(z) of degree m with real zeros $x_1, \dots, x_m$ symmetric about the origin such that

$$Q(\xi, x) = |\, x \,|^m q(\xi/ \,|\, x \,|\,). \tag{4.8}$$

Thus equations (4.6) and (4.7) take the equivalent form

$$[x_1, \dots, x_m]\phi = \Omega(\phi^{(m-1)}\circ e), \quad e = (1, 0, \dots, 0). \tag{4.9}$$

Since Ω is also orthogonally invariant we may express it in the form

$$\Omega f = \omega T f. \tag{4.10}$$

where

$$(Tf)(t) = \frac{1}{m_n}\int_{|x|=1} f(tx)dm_x, \qquad m_n = \int_{|x|=1} dm_x = 2\pi^{n/2} \mid \Gamma(n/2) \tag{4.11}$$

is the *spherical mean* of f, [7,8]. Combining equations (4.9) and (4.10) we get

$$\omega T(\phi \circ e) = [x_1, \ldots, x_m]\phi^{-m+1} = \int_{\mathbb{R}^1} M(t \mid x_1, \ldots, x_m)\phi(t)dt. \tag{4.12}$$

For even functions $\phi \in \mathscr{C}_0^\ell(\mathbb{R}^1)$

$$T(\phi \circ e)(t) = t^{-(n-2)}\frac{2m_{n-1}}{m_n}\int_0^t (t^2 - \sigma^2)^{(n-3)/2}\phi(\sigma)d\sigma. \tag{4.13}$$

[6]. This integral transform has an inverse given by

$$(S\phi)(t) = \frac{2^{n-1}}{(n-2)!}t\left(\frac{d}{dt^2}\right)^{n-1}\int_0^t \sigma^{n-1}(t^2 - \sigma^2)^{(n-3)/2}\phi(\sigma)d\sigma \tag{4.14}$$

[6], and therefore we get ,

$$\omega\phi = \int_0^\infty M(t \mid x_1, \ldots, x_m)(S\phi)(t)dt \tag{4.15}$$

for ϕ even. Obviously (4.15) is also valid for odd ϕ since then $S\phi \equiv 0$ which is consistent with (4.12).

Note that ω is a *positive* distribution on plane waves, that is $\omega(\phi \circ x) \geq 0$, for $x \in \mathbb{R}^n$ whenever

$\phi \geq 0$, see (4.7). In spite of this, even when ω is bounded on $\mathscr{C}(\mathbb{R}^1)$ it does not correspond to measure.

We mention two specific choices of the above computation:

i) If

$$n = 2\ell + 3, \quad \ell \geq 0$$

then

$$\omega(x) = \frac{(-1)^{\ell+1} 2^{2\ell+1} \ell!}{(2\ell+1)!} x^{2\ell+2} \left(\frac{d}{dx^2} \right)^{\ell+1} M(x \mid x_1, \ldots, x_m).$$

Thus ω has compact support and is continuous for $0 \leq \ell \leq m-3$. When n=3 and $m \geq 3$ it is also positive but not otherwise.

ii) If

$$n = 2\ell + 2, \quad \ell \geq 0$$

then

$$\omega(x) = \frac{2(-1)^{\ell+1}}{\ell!} x^{2\ell} \int_x^\infty \sigma(\sigma^2 - x^2)^{-1/2} \left(\frac{d}{d\sigma^2} \right)^{\ell+1} M(\sigma \mid x_1, \ldots, x_m) d\sigma.$$

Again, ω is continuous when $0 \leq \ell \leq m-3$ and positive for n=2, $m \geq 3$.

Our next example corresponds to the choice that $Q(\xi, x)$ is of *affine lineage.* This means that we may globally factor Q in the form

$$Q(\xi, x) = \prod_{j=1}^{m} (\xi - v^j x)$$

for some $v^1, \ldots, v^m \in \mathbb{R}^n$. In this case equation (4.7) for the fundamental distribution becomes

$$[v^1 x, \ldots, v^m x]\ \phi = \Omega(\phi^{(m-1)} \circ x).$$

Using equation (2.4) we also have

$$[v^1 x, \ldots, v^m x]\phi = \int_{S^{m-1}} \phi^{(m-1)}\left((\sum_{i=1}^{m} v^i y_i) x \right) dy_1 \ldots dy_n.$$

which becomes, in view of Definition 2.1,

$$= \int_{\mathbb{R}^n} M(y \mid v^1, \ldots, v^m) \phi^{(m-1)}(xy) dy.$$

Therefore equation (4.1) shows in this case that

$$K(t, x) = t^{m-n-1} M(x/t \mid v^1, \ldots, v^m), \quad t > 0$$

is the fundamental function for the hyperbolic equation

$$\prod_{i=1}^{m} (\frac{\partial}{\partial t} - v^i \nabla_n) u(t, x) = 0, \quad t > 0, x \in \mathbb{R}^n$$

$$\partial_t^i u(0, x) = 0, \quad i = 0, 1, \ldots, m-2$$

$$\partial_t^{m-1} u(0, x) = \psi(x).$$

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