ON MGR[ν] MULTIGRID METHODS

by

David Kamowitz
and
Seymour V. Parter

Computer Sciences Technical Report #575

January 1985

ON MGR[ν] MULTIGRID METHODS*

by

David Kamowitz
and
Seymour V. Parter

*Supported by the Air Force Office of Scientific Research under contract #AFOSR-82-0275 and the Los Alamos National Laboratory.

ABSTRACT

The MGR[ν] algorithm of Ries, Trottenberg and Winter with $\nu = 0$ and the Algorithm 2.1 of Braess are essentially the same multigrid algorithm for the discrete Poisson equation: $-\Delta_h U = f$. In this report we consider the extension to the general diffusion equation $-\nabla \cdot p\nabla u = f$, $p = p(x,y) \geq p_0 > 0$. In particular, for the two-grid scheme we reobtain the basic result: Let ε^0 be the error before a single multigrid cycle and let ε^1 be the error after this cycle. Then $\|\varepsilon^1\|_{L_h} \leq \frac{1}{2}(1+Kh)\|\varepsilon^0\|_{L_h}$. Computational results indicate that other constant coefficient results carry over as well.

I. Introduction

Multigrid methods are proving themselves as (very) successful tools for the solution of the algebraic equations associated with discretization of Elliptic Boundary-Value problems - see [1], [4], [5], [6], [10]. Nevertheless, it seems we are just beginning to "understand" this powerful idea. Hence, there is a need for continued probing, experimentation and new proofs - less for the sake of proof and more for the sake of insight.

In [2] Braess proposed and analyzed a class of multigrid methods. In particular, he considered a particular algorithm for the Poisson Equation - "Algorithm 2.1". He shows that the contraction number ρ for a two-grid method is given by

$$\rho \leq \frac{1}{2} ! \tag{1.1}$$

This result holds whenever Ω is a polygonal domain whose sides have slope ± 1, 0 or ∞ and the discretization satisfies an additional condition (see ΩI of section 2). In [9] Ries, Trottenberg and Winter discuss the class of MGR[ν] methods for the Poisson Equation in a square. Using Fourier Analysis they obtain an explicit formula for the corresponding contraction numbers $\rho[\nu]$. In particular, they obtain - for two grids

$$\rho[0] = \frac{1}{2}, \quad \rho[1] = \frac{2}{27}, \quad \rho[\nu] = \frac{1}{2} \frac{(2\nu)^{2\nu}}{(2\nu+1)^{2\nu+1}} . \tag{1.2}$$

As it happens MGR[0] is the same as "Algorithm 2.1" and the results of [2] and [9] are consistent. The results of [9] are more precise for more restricted problems.

In this paper we consider the more general diffusion equation

$$
(1.3)\qquad \begin{aligned} -\nabla \cdot p(x,y)\nabla u &= f \quad \text{in} \quad \Omega , \\ u &= 0 \quad \text{on} \quad \partial\Omega \\ p(x,y) \geq p_0 &> 0 \quad \text{and} \end{aligned}
$$

where Ω may either be a general bounded piecewise smooth domain or Ω is a polygonal domain whose sides have slope ± 1, 0, or ∞. We employ the usual five-point difference analog of (1.3) and seek to solve the (large) system of linear algebraic equations. We consider a class of linear multi-grid methods which include the MGR[ν] methods when $p(x,y) \equiv 1$. Our basic result is the following: Consider the two-grid method. Let ε^0 be the error before a single multigrid cycle and let ε^1 be the error after this cycle. Then

$$
\| \varepsilon^1 \|_{L_h} \leq \frac{1}{2}(1+Kh) \| \varepsilon^0 \|_{L_h} .
$$

where $\| \ \|_{L_h}$ denotes the energy norm and K is a constant determined by p_0 and $\|\nabla p\|_\infty$, the ∞ norm of the gradient of $p(x,y)$. Moreover, the proof clearly indicates why one should expect improvement when further smoothing steps are introduced — see the remark at the end of section 3. This result, (Theorem 3.1) for the two-grid iterative scheme is valid in quite general domains provided that we use a modification of "approximation of degree 0" (see [7]) to describe the boundary conditions. Thus we extend the results of Braess [2], Ries, Trottenberg and Winter [9] to include a variable diffusion coefficient $p(x,y)$ and more general regions.

In section 2 we formulate the problem and the basic two-grid method of solution. In section 3 we prove the basic estimate. The proof proceeds from a fundamental insight of McCormick and Ruge [8]. Section 4 describes the results of some computational experiments which lead one to believe that the results of [9] are essentially correct for the variable coefficient case as well. These computations were carried out on the CRAY I at the Los Alamos National Laboratory. Finally, an appendix gives the basic "energy" estimate required in section 3.

2. The Problem

Given a (small) value $h > 0$ let $\{(x_k, y_j) = (kh, jh);\ k, j = 0, \pm 1, \pm 2, \ldots\}$ be the associated mesh points in the $x - y$ plane. Let

$$R_E := \{(x_k, y_j);\ k+j \equiv 0 \pmod 2\}, \tag{2.1a}$$

$$R_0 := \{(x_k, y_j);\ (k+j) \equiv 1 \pmod 2\}. \tag{2.1b}$$

Let Ω be a bounded domain in the plane with a piecewise smooth boundary $\partial\Omega$. We define the set of "interior" mesh points, Ω_h. We assume that h is less than $\frac{1}{4}$ the length of each smooth section of $\partial\Omega$ and that the radius of curvature at each point of a smooth arc satisfies

$$R \geq 10\, h\,.$$

These restrictions are required to avoid pathological geometric problems which vanish as $h \to 0$.

Definition:*

(i) If $(x_k, y_j) \in R_0 \cap \Omega$ we say that $(x_k, y_j) \in \Omega_h$ if the four neighbors $\{(x_{k+1}, y_j), (x_{k-1}, y_j), (x_k, y_{j-1}), (x_k, y_{j+1})\}$ *and the line segments from* (x_j, y_j) *to each of its neighbors* all lie in $\bar{\Omega}$, the closure of Ω.

(ii) If $(x_k, y_j) \in R_E \cap \Omega$ we say that $(x_k, y_j) \in \Omega_h$ if the eight neighbors $\{(x_{k+1}, y_j), (x_{k-1}, y_j), (x_k, y_{j-1}), (x_k, y_{j+1}), (x_{k+1}, y_{j+1}), (x_{k+1}, y_{j-1}), (x_{k-1}, y_{j+1}), (x_{k-1}, y_{j-1})\}$ *and the line segments from* (x_j, y_j) *to each of its neighbors* all lie in $\bar{\Omega}$.

(*) We must consider the line segments from (x_k, y_j) to the neighbors only in the case of reentrant corners or cusps.

When $\partial\Omega$ has a cusp or a corner at a point (x,y) we require that

$$(x,y) = (x_k,y_j) \in R_E .$$

The points $(x_k,y_j) \in \overline{\Omega}/\Omega_h$ are the boundary points of Ω_h. That is

$$\partial\Omega_h := \{(x_k,y_j) \in \overline{\Omega}/\Omega_h\} .$$

A true multigrid scheme requires the use of many coarser grids. In such general regions the treatment of the boundary conditions on succeeding coarser grids gets complicated. In truth, the multigrid literature has barely touched on this question. This description of Ω_h and $\partial\Omega_h$ includes the case studied by Braess [2] where Ω is a polygonal domain whose sides have slope ± 1, 0 or ∞ and the corners all belong to the coarsest (and hence, the finest) grid. For this case we note that (see Figure 1):

(ΩI.a) $$\partial\Omega_h \subset \partial\Omega$$

and

(ΩI.b) *if* σ *is a side of* Ω *with slope* ± 1, *then all the points of* $\partial\Omega_h$ *which also lie on* $\partial\Omega$ *belong to* R_E .

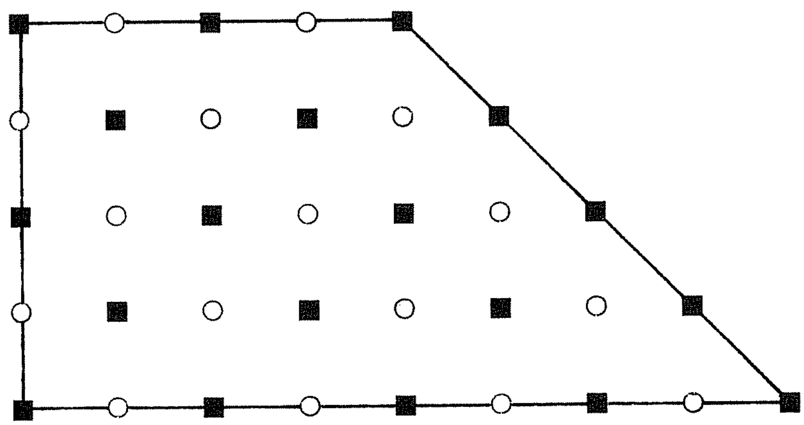
For any function $F(x,y)$ defined on the (x,y) plane we write

(2.2a) $$F_{k,j} := F(x_k,y_j) ,$$

(2.2b) $$F_{k+\frac{1}{2},j} := F((k+\tfrac{1}{2})h,y_j) ,$$

(2.2c) $$F_{k,j+\frac{1}{2}} := F(x_k,(j+\tfrac{1}{2})h) .$$

To obtain an approximate solution of (3.1) we seek a grid function $\{U_{kj}\}$ defined on the mesh points and satisfying the system of equations: for $(x_k,y_j) \in \Omega_h$



Key

■ points of $R_E \cap \overline{\Omega}$,

○ points of $R_0 \cap \overline{\Omega}$.

Figure 1

$$(2.3a)\quad \begin{cases} \frac{1}{h^2}\{p_{k-\frac12,j}[U_{k,j}-U_{k-1,j}] - p_{k+\frac12,j}[U_{k+1,j}-U_{k,j}]\} + \\ \\ \frac{1}{h^2}\{p_{k,j-\frac12}[U_{k,j}-U_{k,j-1}] - p_{k,j+\frac12}[U_{k,j+1}-U_{k,j}]\} = f_{kj}, \end{cases}$$

and, for $(x_k,y_j) \notin \Omega_h$

$$(2.3b)\qquad U_{kj} = 0 .$$

We rewrite (2.3) as

$$(2.4a)\qquad [L_hU]_{kj} = f_{kj}, \quad (x_k,y_j) \in \Omega_h ,$$

$$(2.4b)\qquad U_{kj} = 0 , \quad (x_k,y_j) \notin \Omega_h .$$

When Ω is a polygonal domain whose sides have slope ± 1, 0 or ∞ this fine-difference discretization is an $O(h^2)$ scheme. Unfortunately, in the general case, the error estimate is $O(h)$.

We turn to the question of the solution of these linear algebraic equations via a "two-grid" method. Let

$$(2.5)\qquad \Omega_E := R_E \cap \Omega_h , \quad \Omega_0 := R_0 \cap \Omega_h .$$

Our two grids are Ω_h and Ω_E . Let S_h and S_E be the spaces of grid functions defined on $R_E \cup R_0$ and R_E which vanish outside Ω_h and Ω_E , respectively. Our first step is to set up "communication" between these two spaces. To be specific, we construct linear "interpolation" and "projection" operators I_h^E , I_E^h so that

(2.6a) $$I_h^E : S_h \to S_E \,, \qquad \text{(Projection)},$$

(2.6b) $$I_E^h : S_E \to S_h \,. \qquad \text{(Interpolation)}.$$

Define the *interpolation* operator I_E^h by

(2.7a) $$[I_E^h U]_{kj} = U_{kj} \,, \quad \text{if} \quad (x_k, y_j) \in R_E \,,$$

and, if $(x_k, y_j) \in \Omega_0$, then

(2.7b) $$[I_E^h U]_{kj} = \frac{1}{c_{kj}} \{ p_{k-\frac{1}{2},j} U_{k-1,j} + p_{k+\frac{1}{2},j} U_{k+1,j} + p_{k,j-\frac{1}{2}} U_{k,j-1} + p_{k,j+\frac{1}{2}} U_{k,j+1} \}$$

where

(2.7c) $$c_{kj} = \{ p_{k+\frac{1}{2},j} + p_{k-\frac{1}{2},j} + p_{k,j-\frac{1}{2}} + p_{k,j+\frac{1}{2}} \} \,.$$

Finally, if $(x_k, y_j) \in R_0/\Omega_0$ then (of course)

(2.7d) $$[I_E^h U]_{kj} = 0 \,.$$

Observe that (2.7a) implies that I_E^h is of full rank, i.e.,

$$\dim \mathit{Range} \; I_E^h = \dim S_E \,.$$

The *projection* operator I_h^E is defined by

(2.8) $$I_h^E = \frac{1}{2} (I_E^h)^T \,.$$

Let

(2.9) $$R := \mathit{Range} \; I_E^h \,.$$

The choice of interpolation operator I_E^h enables us to characterize $\mathcal{R}$ as follows:

<u>Lemma 2.1</u>: Let I_E^h be defined by (2.7). Then a function $U = U(h) \in S_h$ is in $\mathcal{R}$ if and only if

$$(2.10) \qquad [L_h U]_{kj} = 0 \quad \forall (k,j) \text{ with } (x_k, y_j) \in \Omega_0 . \quad \blacksquare$$

To describe the two-grid method for solving (2.4), we consider a smoothing operator G. That is, given $u^0 \in S_h$ we construct $\tilde{u}$ via

$$(2.11a) \qquad \tilde{u} = Gu^0 = u^0 + B(f - L_h u^0) = G_0 u^0 + Bf$$

$$(2.11b) \qquad G_0 = (I - BL_h)$$

where B is a given matrix and

$$(2.11c) \qquad \| G_0 \|_{L_h}^2 = \sup_{\psi \neq 0} \frac{\langle L_h G_0 \psi, G_0 \psi \rangle}{\langle L_h \psi, \psi \rangle} \leq 1 .$$

An important smoother G is the odd-even Gauss-Seidel scheme which is defined in terms of two half-step operators H^0 and H^E by $G = H^E H^0$. That is, define H^0 - relaxation on the odd points -

$$(2.12a) \qquad (H^0 u)_{kj} = u_{kj} , \quad (x_k, y_j) \in R_E \cup R_0 / \Omega_0 ,$$

and

$$(2.12b) \qquad [L_h (H^0 u)]_{kj} = f_{kj} , \quad (x_k, y_j) \in \Omega_0 .$$

Similarly define H^E, relaxation on the even points, by

(2.13a) $$(H^E u)_{kj} = u_{kj}\ , \quad (x_k,y_j) \in R_0 \cup R_E/\Omega_E\ ,$$

(2.13b) $$[L_h(H^E u)]_{kj} = f_{kj}\ , \quad (x_k,y_j) \in \Omega_E\ .$$

Let $\nu \geq 0$ be an integer. We obtain the generalized MGR[ν] two-grid iterative scheme by choosing

(2.14) $$G = (H^E H^0)^\nu\ .$$

Observe that the solutions of (2.12b) and (2.13b) are explicit. This follows immediately from the odd-even structure of the difference equations (2.4).

Algorithm 2.1:

Step 1: Given $u^0 \in S_h$, form $\tilde{u} = Gu^0$.

Step 2: Construct $\hat{u}$ via

$$\hat{u} = H^0\tilde{u}\ .$$

Step 3: Set $r = f - L_h\hat{u}$, $r_E = I_h^E r$.

Step 4: Solve $L_E\phi = r_E$ where L_E is the "coarse grid operator" to be described later.

Step 5: Set $u^1 = \hat{u} + I_E^h\phi$.

Step 6: Set $u^1 \to u^0$ and return to step 1.

We now describe two choices of the coarse grid operator L_E.

Case 1: Let

$$(2.15a)\qquad a_{k-\frac12,j-\frac12} = \frac{1}{h^2}\left[\frac{p_{k-\frac12,j}p_{k-1,j-\frac12}}{c_{k-1,j}} + \frac{p_{k,j-\frac12}p_{k-\frac12,j-1}}{c_{k,j-1}}\right],$$

$$(2.15b)\qquad b_{k+\frac12,j-\frac12} = \frac{1}{h^2}\left[\frac{p_{k,j-\frac12}p_{k+\frac12,j-1}}{c_{k,j-1}} + \frac{p_{k+\frac12,j}p_{k+1,j-\frac12}}{c_{k+1,j}}\right],$$

$$(2.15c)\qquad d_{kj} = [a_{k-\frac12,j-\frac12} + a_{k+\frac12,j+\frac12} + b_{k+\frac12,j-\frac12} + b_{k-\frac12,j+\frac12}] .$$

Then, if $(k+j) \equiv 0 \pmod 2$,

$$(2.16)\qquad \begin{aligned}[L_E^{(1)}U]_{kj} = &- a_{k+\frac12,j+\frac12}U_{k+1,j+1} - a_{k-\frac12,j-\frac12}U_{k-1,j-1}\\ &- b_{k+\frac12,j-\frac12}U_{k+1,j-1} - b_{k-\frac12,j+\frac12}U_{k-1,j+1} + d_{kj}U_{kj} .\end{aligned}$$

Case 2: (The Standard Case): if $k+j \equiv 0 \pmod 2$ then

$$(2.17a)\qquad \begin{aligned}[L_E^{(2)}U]_{kj} = \frac{1}{2h^2}\{&-p_{k+\frac12,j+\frac12}U_{k+1,j+1} - p_{k+\frac12,j-\frac12}U_{k+1,j-1}\\ &-p_{k-\frac12,j-\frac12}U_{k-1,j-1} - p_{k-\frac12,j+\frac12}U_{k-1,j+1} + S_{kj}U_{kj}\}\end{aligned}$$

where

$$(2.17b)\qquad S_{kj} = \{p_{k+\frac12,j+\frac12} + p_{k+\frac12,j-\frac12} + p_{k-\frac12,j+\frac12} + p_{k-\frac12,j-\frac12}\} .$$

Remark: The operator $L_E^{(2)}$ is the skewed version of L_h on the Ω_E grid. The operator $L_E^{(1)}$ is a multiple of the part of $\hat{L}_E = I_h^E L_h I_E^h$ which is based on the same skewed points. The reasons for studying $L_E^{(1)}$ will become clearer in section 3.

3. Analysis of the Algorithm

We begin our analysis with an observation which is (by now) well-known among multigrid theorists (see [8]). Let

$$\hat{L}_E := I_h^E L_h I_E^h . \tag{3.1}$$

Note that with our choice of I_h^E, I_E^h the operator $\hat{L}_E$ is essentially $\frac{1}{2}$ of L_h restricted to $\mathcal{R}$. Hence $\hat{L}_E$ is nonsingular. An easier calculation shows that these choices imply that $\hat{L}_E$ is positive definite. Consider Steps 4-5 of the two-grid iteration. Suppose we replace L_E by $\hat{L}_E$, i.e., suppose we find the function ψ which satisfies

$$\hat{L}_E \psi = r_E ,$$

and set

$$u^1 = \hat{u} + I_E^h \psi .$$

We claim that

$$L_h u^1 = f ,$$

i.e. u^1 is the desired solution! To see this we set

$$\hat{\varepsilon} = U - \hat{u} \tag{3.2a}$$

where U is the exact solution of (2.4). Observe that Step 2 implies that if $k + j \equiv 1 \pmod 2$, then

$$(L_h \hat{\varepsilon})_{kj} = (L_h U - L_h \hat{u})_{kj} = (f - L_h \hat{u})_{kj} = 0 .$$

Hence Lemma 2.1 asserts that there is a function $V \in S_E$ such that

$$\hat{\varepsilon} = I_E^h V . \tag{3.2b}$$

But

$$\hat{L}_E V = I_h^E(L_h I_E^h V) = I_h^E L_h \hat{\varepsilon} = r_E .$$

So that

$$\psi = V$$

and, hence,

(3.3) $$\hat{u} - I_E^h \psi = \hat{u} - \hat{\varepsilon} = U!!$$

Unfortunately we have chosen Step 4 with L_E and <u>not</u> $\hat{L}_E$. This choice was not merely pique on our part (or the part of Braes and Ries, Trottenberg and Winter). The point is -- having chosen L_E as a five point star we can now proceed to replace Step 4 with a new two grid step -- i.e. we can build a true multigrid scheme. Since $\hat{L}_E$ involves a nine-point stencil this would be more difficult with $\hat{L}_E$ as our coarse grid operator.

In any case, the problem of <u>Step 4</u> is just

$$L_E \phi = \hat{L}_E \psi ,$$

where, as we see from Lemma 2.1, $I_E^h \psi$ is the L_h projection of $\tilde{\varepsilon} = U - \tilde{u}$ onto $\mathcal{R}$. Hence,

(3.4a) $$\|\hat{\varepsilon}\|_{L_h} = \|I_E^h \psi\|_{L_h} \leq \|\tilde{\varepsilon}\|_{L_h} .$$

We will give a complete description of $\hat{L}_E$ in the appendix. For now, we write

(3.5) $$\hat{L}_E = \frac{1}{2} L_E + \frac{1}{2} \tilde{L}_E$$

where $\tilde{L}_E$ is defined by this equation. Observe that both $\hat{L}_E$ and L_E (either $L_E^{(1)}$ or $L_E^{(2)}$) are symmetric, positive definite operators. Hence the associated $\tilde{L}_E$ is a symmetric operator. Our main estimate is

Lemma 3.1: For $L_E = L_E^{(1)}$ or $L_E = L_E^{(2)}$, there is a constant K, depending only on $\|\nabla p\|_\infty$, the maximum norm of the first derivatives of the diffusion coefficient $p(x,y)$, and p_0 such that, for all $\phi \in S_E$, $\phi \not\equiv 0$ we have

$$-Kh \le \frac{\langle \tilde{L}_E\phi,\phi\rangle}{\langle L_E\phi,\phi\rangle} \le (1+Kh) \,. \tag{3.6}$$

Proof: See Theorem A of the Appendix. ∎

Consider the eigenvalue problem

$$(\lambda L_E - \hat{L}_E)\,\psi = 0\,, \quad \psi \not\equiv 0\,. \tag{3.7}$$

Using (3.5) we see that this problem is equivalent to

$$[(2\lambda-1)L_E - \tilde{L}_E]\psi = 0\,, \quad \psi \not\equiv 0\,. \tag{3.8}$$

From Lemma 3.1 we find

$$\frac{1-Kh}{2} \le \lambda \le (1+Kh)\,. \tag{3.9}$$

Theorem 3.1: Let

$$\varepsilon^0 = U - u^0\,, \quad \varepsilon^1 = U - u^1$$

then

$$\|\varepsilon^1\|_{L_h} \le \frac{1}{2}(1+Kh)\,\|\varepsilon^0\|_{L_h}\,. \tag{3.10}$$

Proof: We have

$$\tilde{\varepsilon} = U - \tilde{u} = G_0 \varepsilon^0 .$$

Using (2.11c) we see that

$$\| \varepsilon \|_{L_h} \leq \| \varepsilon^0 \|_{L_h} . \tag{3.11}$$

From (3.2a), (3.3b) and Step 5 of the multigrid algorithm we have

$$\varepsilon^1 = \hat{\varepsilon} - L_E^h \phi = I_E^h (\psi - \phi) . \tag{3.12a}$$

But

$$L_E \phi = \hat{L}_E \psi . \tag{3.12b}$$

So that

$$\psi - \phi = (I - L_E^{-1} \hat{L}_E) \psi . \tag{3.13}$$

Thus

$$\begin{aligned} \langle L_h \varepsilon^1, \varepsilon^1 \rangle_h = \langle L_h I_E^h (\psi - \phi), I_E^h (\psi - \phi) \rangle_h &= 2 \langle I_h^E L_h I_E^h (\psi - \phi), (\psi - \phi) \rangle_E \\ &= 2 \langle \hat{L}_E (\psi - \phi), (\psi - \phi) \rangle_E \\ &= 2 \langle \hat{L}_E (I - L_E^{-1} \hat{L}_E) \psi, (I - L_E^{-1} \hat{L}_E) \psi \rangle_E . \end{aligned} \tag{3.14a}$$

Hence

$$\| \varepsilon^1 \|_{L_h}^2 = 2 \langle [I - \hat{L}_E^{1/2} L_E^{-1} \hat{L}_E^{1/2}] \hat{L}_E^{1/2} , [I - \hat{L}_E^{1/2} L_E^{-1} \hat{L}_E^{1/2}] \hat{L}_E^{1/2} \psi \rangle_E . \tag{3.14b}$$

Since the eigenvalues of $L_E^{-1}\hat{L}$ are also the eigenvalues of the symmetric operator $\hat{L}^{\frac{1}{2}}L_E^{-1}\hat{L}_E^{\frac{1}{2}}$, (3.9) implies that the eigenvalues μ of the symmetric operator $(I-\hat{L}_E^{\frac{1}{2}}L_E^{-1}\hat{L}_E^{\frac{1}{2}})$ satisfy

$$-Kh \leq \mu \leq \frac{1}{2}(1+Kh) .$$

Thus, (3.14b) implies

$$\begin{aligned} \|\varepsilon^1\|_{L_h}^2 &\leq 2\mu^2 \langle \hat{L}_E^{\frac{1}{2}}\psi, \hat{L}_E^{\frac{1}{2}}\psi\rangle_E = 2\mu^2 \langle \hat{L}_E\psi,\psi\rangle_E = 2\mu^2 \langle L_h I_E^h\psi, (I_h^E)^T\psi\rangle_h \\ &= 2\mu^2\,\frac{1}{2}\langle L_h I_E^h\psi, I_E^h\psi\rangle_h \\ &= \mu^2\langle L_h\hat{\varepsilon},\hat{\varepsilon}\rangle_h = \mu^2\|\hat{\varepsilon}\|_{L_h}^2 . \end{aligned}$$

This result, together with (3.4a) and (3.11) implies the Theorem. ∎

<u>Remark</u>: This analysis is based on (3.6), an estimate of the ratio of two approximations to

$$\int\int p(x,y)|\nabla\phi|^2 dxdy .$$

One expects that, the smoother ϕ is, the nearer this ratio is to 1. From (3.8) we see that: the nearer this ratio is to 1, the nearer λ is to 0. Thus, we expect that if more smoothing steps have been applied to ε^0, I_E^hV will be smoother. Hence V will be smoother and $|\lambda|$ will be smaller.

4. Experimental Results

We have developed an experimental program implementing Algorithm 2.1 on the CRAY I at the Los Alamos National Laboratory. Our purpose was to verify (a modified) formula (1.2) for the variable coefficient case. We were also interested in the differences in the results which arise from the choice of L_E, i.e. $L_E^{(1)}$ or $L_E^{(2)}$. The region Ω is the unit square.

The computer program runs in an interactive fashion and allows the user to provide a number of parameters. These include N, the number of points on a side of Ω_h, the fine grid in the unit square, and ν, the number of smoothing iterations. Starting with a particular choice of p(x,y) and u(x,y) we construct the right hand side of (1.3). Using the initial guess U^0 with interior points of Ω_E equal to 5 and interior points of Ω_0 equal to -5, Algorithm 2.1 is repeated until the discrete L_2 norm of the residual is less than 10^{-8}.

Experiments were done with the coarse grid operator chosen to be $L_E^{(1)}$ and $L_E^{(2)}$. The calculation of $L_E^{(1)}$ was complicated by the fact that for points of Ω_E for which $L_E^{(1)}$ refers to points of $\partial\Omega_h$, formula (2.15c) does not apply. The reason for this is because the computation of either $a_{k\pm\frac{1}{2}j\pm\frac{1}{2}}$ or $b_{k\pm\frac{1}{2}j\mp\frac{1}{2}}$ involves referring to points outside of Ω_h. Of course since $U_{kj} = 0$ if $U_{kj} \in \partial\Omega_h$ we may set $a_{k\pm\frac{1}{2}j\pm\frac{1}{2}}$ and $b_{k\pm\frac{1}{2}j\mp\frac{1}{2}}$ to zero when $U_{k\pm 1 j\pm 1}$ and $U_{k\pm 1 j\mp 1}$ are in $\partial\Omega_h$. However, we still need a value for d_{kj} for the two nearest interior points. For the four corner points, we set d_{kj} to be the value of d_{kj} of the nearest interior point. As the mesh gets finer, this approximation to the true d_{kj} improves. However, in almost all the experiments the rate of convergence using $L_E^{(1)}$ was not quite as good as the rate obtained using $L_E^{(2)}$.

The tables below list the functions $p(x,y)$ and the true solutions $u(x,y)$ used for the experiments. For each problem the numerical results obtained using both $L_E^{(1)}$ and $L_E^{(2)}$ are displayed. N corresponds to the number of interior points on a side of Ω_h and ν corresponds to the number of smoothing iterations. The smoother used was the odd-even Gauss-Seidel scheme as described in section 2. $\sigma(\nu)$ in the tables corresponds to the theoretical rate given in equation (1.2) of section 1. The theoretical rate has only been proven to be valid, when $\nu > 0$, in the constant coefficient case. However, as can be seen from the numerical results it appears to be valid in the variable coefficient case as well.

In conclusion, the numerical results demonstrate the validity of Theorem 3.1 for the case $\nu = 0$ and support extending equation (1.2) of section 1 to the variable coefficient case.

Table I, Experimental Results

Problem 1, $p(x,y) = 1$, $u(x,y) = 0$

$L_E = L_E^{(1)}$

$N \setminus \nu$	0	1	2	3
15	.4858	.0646	.0344	.0200
31	.4844	.0696	.0375	.0252
63	.4836	.0708	.0386	.0263
$\sigma(\nu)$	.5000	.0741	.0410	.0283

$L_E = L_E^{(2)}$

$N \setminus \nu$	0	1	2	3
15	.4858	.0646	.0344	.0200
31	.4844	.0696	.0375	.0252
63	.4836	.0708	.0386	.0263
$\sigma(\nu)$	.5000	.0741	.0410	.0283

Problem 2, $p(x,y) = 1$, $u(x,y) = \sin \pi x \sin \pi y$

$L_E = L_E^{(1)}$

$N \setminus \nu$	0	1	2	3
15	.4858	.0646	.0344	.0200
31	.4844	.0696	.0375	.0252
63	.4836	.0708	.0386	.0263
$\sigma(\nu)$	.5000	.0741	.0410	.0283

$L_E = L_E^{(2)}$

$N \setminus \nu$	0	1	2	3
15	.4858	.0696	.0344	.0200
31	.4844	.0696	.0375	.0252
63	.4836	.0708	.0386	.0263
$\sigma(\nu)$	.5000	.0741	.0410	.0283

Problem 3, $p(x,y) = 1$, $u(x,y) = x(1-x)y(1-y)$

$L_E = L_E^{(1)}$

N \ ν	0	1	2	3
15	.4858	.0646	.0344	.0200
31	.4844	.0696	.0375	.0252
63	.4836	.0708	.0386	.0263
σ(ν)	.5000	.0741	.0410	.0283

$L_E = L_E^{(2)}$

N \ ν	0	1	2	3
15	.4858	.0646	.0344	.0200
31	.4844	.0696	.0375	.0252
63	.4836	.0708	.0386	.0263
σ(ν)	.5000	.0741	.0410	.0283

Problem 4, $p(x,y) = e^{xy}$, $u(x,y) = xe^{xy} \sin \pi x \sin \pi y$

$L_E = L_E^{(1)}$

N \ ν	0	1	2	3
15	.4863	.0760	.0437	.0292
31	.4841	.0742	.0425	.0303
63	.4842	.0720	.0401	.0283
σ(ν)	.5000	.0741	.0410	.0283

$L_E = L_E^{(2)}$

N \ ν	0	1	2	3
15	.4858	.0643	.0342	.0199
31	.4840	.0697	.0373	.0252
63	.4841	.0709	.0384	.0264
σ(ν)	.5000	.0741	.0410	.0283

Problem 5, $p(x,y) = \frac{1}{(3-x)(3-y)}$, $u(x,y) = e^{xy} \sin \pi x \sin \pi y$

$L_E = L_E^{(1)}$

N \ ν	0	1	2	3
15	.4841	.0708	.0393	.0270
31	.4819	.0713	.0398	.0276
63	.4820	.0709	.0386	.0268
σ(ν)	.5000	.0741	.0410	.0283

$L_E = L_E^{(2)}$

N \ ν	0	1	2	3
15	.4839	.0643	.0339	.0199
31	.4819	.0694	.0373	.0250
63	.4820	.0706	.0381	.0261
σ(ν)	.5000	.0741	.0410	.0283

Problem 6, $p(x,y) = e^{x}(1+\frac{1}{2}\sin \pi y)$, $u(x,y) = e^{xy} \sin \pi x \sin \pi y$

$L_E = L_E^{(1)}$

N \ ν	0	1	2	3
15	.4879	.1084	.0727	.0565
31	.4854	.0901	.0582	.0442
63	.4851	.0784	.0473	.0350
σ(ν)	.5000	.0741	.0410	.0283

$L_E = L_E^{(2)}$

N \ ν	0	1	2	3
15	.4869	.0686	.0377	.0255
31	.4851	.0710	.0390	.0270
63	.4850	.0715	.0390	.0270
σ(ν)	.5000	.0741	.0410	.0283

Problem 7, $p(x,y) = e^{-xy}$, $u(x,y) = (1-e^{x})(x-1)y \cos \frac{\pi y}{2}$

$L_E = L_E^{(1)}$

N \ ν	0	1	2	3
15	.4857	.0797	.0482	.0351
31	.4842	.0739	.0431	.0312
63	.4836	.0714	.0399	.0278
σ(ν)	.5000	.0741	.0410	.0283

$L_E = L_E^{(2)}$

N \ ν	0	1	2	3
15	.4853	.0650	.0347	.0207
31	.4841	.0697	.0376	.0253
63	.4835	.0708	.0386	.0263
σ(ν)	.5000	.0741	.0410	.0283

Problem 8, $p(x,y) = e^{(\sin \frac{\pi x}{2} \cos \pi y)}$, $u(x,y) = e^{-xy}x(x-1)y(y-1)$

$L_E = L_E^{(1)}$

N \ ν	0	1	2	3
15	.4849	.0772	.0451	.0296
31	.4839	.0751	.0437	.0313
63	.4842	.0721	.0404	.0289
σ(ν)	.5000	.0741	.0410	.0283

$L_E = L_E^{(1)}$

N \ ν	0	1	2	3
15	.4843	.0645	.0342	.0202
31	.4837	.0697	.0373	.0253
63	.4842	.0710	.0385	.0264
σ(ν)	.5000	.0741	.0410	.0283

Appendix

In this section we determine $\hat{L}_E$ and the quadratic forms

$$\langle \hat{L}_E\psi,\psi\rangle, \quad \langle \tilde{L}_E\psi,\psi\rangle, \quad \langle L_E\psi,\psi\rangle .$$

Further, in Lemmas A.4, A.5, and A.6 we establish the local estimates which enable us to prove Theorem A.

Let $u \in S_E$ and $(x_k,y_j) \in \Omega_E$. Then

$$[\hat{L}_E u]_{kj} = \frac{1}{2}[L_h I_E^h u]_{kj} . \tag{A.1}$$

For any $v \in S_h$, $[L_h v]_{kj}$ involves the four values $v_{k\pm1,j}$, $v_{k,j\pm1}$. Therefore we consider the four squares I, II, III, IV (see fig. 2) with vertices

$$\text{I: } \{(x_k,y_j),\ (x_{k+1},y_{j+1}),\ (x_{k+2},y_j),\ (x_{k+1},y_{j-1})\} , \tag{A.2a}$$

$$\text{II: } \{(x_k,y_j),\ (x_{k+1},y_{j+1}),\ (x_k,y_{j+2}),\ (x_{k-1},y_{j+1})\} , \tag{A.2b}$$

$$\text{III: } \{(x_k,y_j),\ (x_{k-1},y_{j+1}),\ (x_{k-2},y_j),\ (x_{k-1},y_{j-1})\} , \tag{A.2c}$$

$$\text{IV: } \{(x_k,y_j),\ (x_{k-1},y_{j-1}),\ (x_k,y_{j-2}),\ (x_{k+1},y_{j-1})\} . \tag{A.2d}$$

In each square the value of $[I_E^h u]$ at the center point is a weighted average (given by (2.7b), (2.7c)) of the values of u at the corners. Thus, in general $\hat{L}_E$ is a 9-point operator based on the 9 vertices of these four squares. Since $\hat{L}_E$ is a symmetric operator it takes the form

$$\begin{aligned} [\hat{L}_E u]_{kj} = {} & E_{kj}u_{kj} - \alpha_{k+1,j}u_{k+2,j} - \alpha_{k-1,j}u_{k-2,j} \\ & -\beta_{k,j+1}u_{k,j+2} - \beta_{k,j-1}u_{k,j-2} \\ & -\gamma_{k+\frac12,j+\frac12}u_{k+1,j+1} - \gamma_{k-\frac12,j-\frac12}u_{k-1,j-1} \\ & -\sigma_{k-\frac12,j+\frac12}u_{k-1,j+1} - \sigma_{k+\frac12,j-\frac12}u_{k+1,j-1} . \end{aligned} \tag{A.3}$$

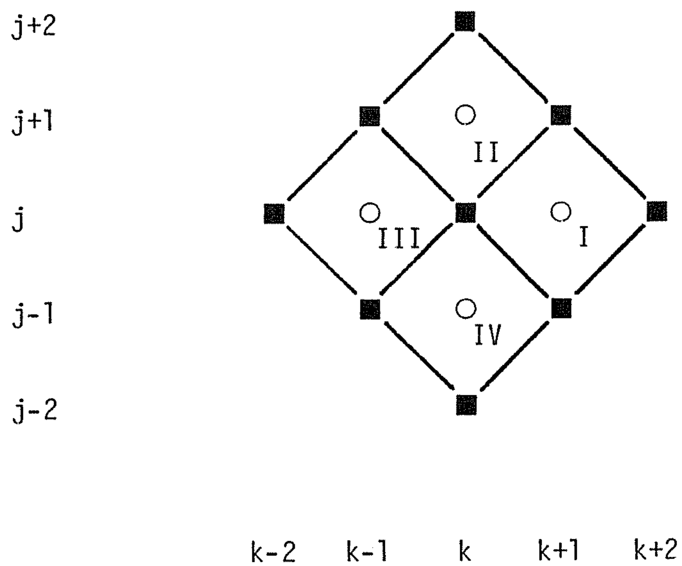


Figure 2

Lemma A.1: Let

(A.4a)
$$E_{kj} = E^0_{kj} + \tilde{E}_{kj}$$

where

(A.4b)
$$E^0_{kj} = \alpha_{k+1,j} + \alpha_{k-1,j} + \beta_{k,j-1} + \beta_{k,j+1} + \gamma_{k+\frac{1}{2},j+\frac{1}{2}} + \gamma_{k-\frac{1}{2},j-\frac{1}{2}} + \sigma_{k+\frac{1}{2},j-\frac{1}{2}} + \sigma_{k-\frac{1}{2},j+\frac{1}{2}} .$$

Then

(A.5)
$$\begin{aligned}\langle \hat{L}_E\psi,\psi\rangle &= \sum \alpha_{k+1,j}[\psi_{k+2,j} - \psi_{kj}]^2 \\ &+ \sum \beta_{k,j+1}[\psi_{k,j+2} - \psi_{kj}]^2 + \sum \gamma_{k+\frac{1}{2},j+\frac{1}{2}}[\psi_{k+1,j+1} - \psi_{kj}]^2 \\ &+ \sum \sigma_{k+\frac{1}{2},j-\frac{1}{2}}[\psi_{k+1,j-1} - \psi_{kj}]^2 + \sum \tilde{E}_{kj}\psi^2_{kj} .\end{aligned}$$

Proof: This follows from summation by parts. ∎

Similar calculations yield

Lemma A.2: Using the definitions (2.15), (2.17) we have

(A.6)
$$\begin{aligned}\langle L^{(1)}_E\psi,\psi\rangle &= \sum a_{k+\frac{1}{2},j+\frac{1}{2}}[\psi_{k+1,j+1} - \psi_{kj}]^2 \\ &+ \sum b_{k+\frac{1}{2},j-\frac{1}{2}}[\psi_{k+1,j-1} - \psi_{kj}]^2 ,\end{aligned}$$

and

(A.7)
$$\begin{aligned}\langle L^{(2)}_E\psi,\psi\rangle &= \frac{1}{2h^2}\sum p_{k+\frac{1}{2},j+\frac{1}{2}}[\psi_{k+1,j+1} - \psi_{kj}]^2 \\ &+ \frac{1}{2h^2}\sum p_{k+\frac{1}{2},j-\frac{1}{2}}[\psi_{k+1,j-1} - \psi_{kj}]^2 .\end{aligned}$$ ∎

We now compute the contribution of each square to $[\hat{L}_E u]_{kj}$ and the quadratic forms. In evaluating $[L_h I_E^h u]_{kj}$ we have five terms. The four terms

$$-p_{k\pm\frac{1}{2},j}(I_E^h u)_{k\pm 1,j}, \qquad -p_{k,j\pm\frac{1}{2}}(I_E^h u)_{k,j\pm 1}$$

are clearly associated with squares $\binom{\mathrm{I}}{\mathrm{III}}$ and $\binom{\mathrm{II}}{\mathrm{IV}}$, respectively. It is convenient to agree that

$$p_{k\pm\frac{1}{2},j}u_{kj} \text{ is associated with square } \binom{\mathrm{I}}{\mathrm{III}},$$

and

$$p_{k,j\pm\frac{1}{2}}u_{kj} \text{ is associated with square } \binom{\mathrm{II}}{\mathrm{IV}}.$$

Let $E_{kj}(R)$, $\alpha_{k\pm 1,j}(R)$, $\beta_{k,j\pm 1}(R)$, $\gamma_{k\pm\frac{1}{2},j\pm\frac{1}{2}}(R)$, $\sigma_{k\pm\frac{1}{2},j\mp\frac{1}{2}}(R)$ denote the contributions of square R to the corresponding coefficients E_{kj}, $\alpha_{k\pm 1,j}$, $\beta_{k,j\pm 1}$, $\gamma_{k\pm\frac{1}{2},j\pm\frac{1}{2}}$, $\sigma_{k\pm\frac{1}{2},j\mp\frac{1}{2}}$ of $\hat{L}_E$.

Consider square I. We must distinguish two cases, either $(x_{k+1},y_j) \in \Omega_h$ or $(x_{k+1},y_j) \notin \Omega_h$. The following geometric lemma is essential to understanding the computations in the latter case.

<u>Lemma A.3</u>: Suppose $(x_{k+1},y_j) \notin \Omega_h$. Then either $(x_{k+2},y_j) \notin \overline{\Omega}$ or the line segment from (x_{k+1},y_j) to (x_{k+2},y_j) is not entirely in $\overline{\Omega}$. Further $(x_{k+1},y_{j+1}), (x_{k+1},y_{j-1}) \notin \Omega_E$.

<u>Proof</u>: (see Figure 3). Since $(x_k,y_j) \in \Omega_E$ the points (x_{k+1},y_{j+1}), (x_{k+1},y_j), $(x_{k+1},y_{j-1}) \in \overline{\Omega}$. On the other hand, $(x_{k+1},y_j) \notin \Omega_h$ implies that either (x_{k+2},y_j) or the line segment from (x_{k+1},y_j) to $(x_{k+2},y_j) \notin \overline{\Omega}$.

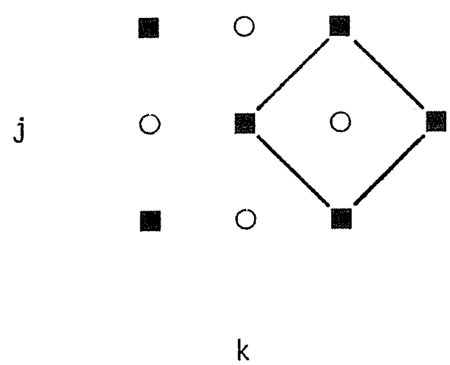


Figure 3

Because of our assumptions on the length of a smooth side of Ω and the lower bound on the radius of curvature, if $(x_{k+2},y_j) \notin \overline{\Omega}$ then clearly (x_{k+1},y_{j+1}) and $(x_{k+1},y_{j-1}) \notin \Omega_E$. If $(x_{k+2},y_j) \in \overline{\Omega}$ then a portion of $\partial\Omega$ crosses the line segment $x_{k+1} \leq x < x_{k+2}$, $y = y_j$. If that portion of the boundary continues smoothly near (x_{k+1},y_j), then the line segments from (x_{k+1},y_{j+1}) to (x_{k+2},y_j) are not entirely in $\overline{\Omega}$. Finally, if there is a non-convex corner (x,y) near (x_{k+1},y_j) that corner $(x,y) \in R_E$. Hence that corner must be (x_{k+1},y_{j+1}) or (x_{k+1},y_{j-1}) which is therefore not in Ω_E. The other one is not in Ω_E because h is less than $\frac{1}{4}$ the length of smooth segments of $\partial\Omega$. ∎

We return to the calculation of $E_{kj}(I)$, $\alpha_{k+1,j}(I)$, $\gamma_{k+\frac{1}{2},j+\frac{1}{2}}(I)$, and $\sigma_{k+\frac{1}{2},j-\frac{1}{2}}(I)$. Square I does not contribute to the other coefficients.

<u>Case 1: $(x_{k+1},y_j) \in \Omega_h$:</u>

A straight forward calculation yields

(A.8a) $$E_{kj}(I) = \frac{1}{2h^2}\left[p_{k+\frac{1}{2},j} - \frac{(p_{k+\frac{1}{2},j})^2}{c_{k+1,j}}\right],$$

(A.8b) $$\alpha_{k+1,j}(I) = \frac{1}{2h^2}\,\frac{p_{k+\frac{1}{2},j}\,p_{k+\frac{3}{2},j}}{c_{k+1,j}},$$

(A.8c) $$\gamma_{k+\frac{1}{2},j+\frac{1}{2}}(I) = \frac{1}{2h^2}\,\frac{p_{k+\frac{1}{2},j}\,p_{k+1,j+\frac{1}{2}}}{c_{k+1,j}},$$

(A.8d) $$\sigma_{k+\frac{1}{2},j-\frac{1}{2}}(I) = \frac{1}{2h^2}\,\frac{p_{k+\frac{1}{2},j}\,p_{k+1,j-\frac{1}{2}}}{c_{k+1,j}}.$$

Case 2: $(x_{k+1}, y_j) \notin \Omega_h$:

In this case we have

$$E_{kj}(I) = \frac{1}{2h^2} p_{k+\frac{1}{2},j} , \tag{A.9a}$$

$$\gamma_{k+\frac{1}{2},j+\frac{1}{2}}(I) = \frac{1}{2h^2} \frac{p_{k+\frac{1}{2},j}\, p_{k+1,j+\frac{1}{2}}}{c_{k+1,j}} , \tag{A.9b}$$

$$\sigma_{k+\frac{1}{2},j-\frac{1}{2}}(I) = \frac{1}{2h^2} \frac{p_{k+\frac{1}{2},j}\, p_{k+1,j+\frac{1}{2}}}{c_{k+1,j}} . \tag{A.9c}$$

$$\alpha_{k+1,j}(I) = E_{kj}(I) - \gamma_{k+\frac{1}{2},j+\frac{1}{2}}(I) - \sigma_{k+1,j-\frac{1}{2}}(I) . \tag{A.9d}$$

Observe that $\alpha_{k+1,j} > 0$, and since $u_{k+2,j} = u_{k+1,j+1} = u_{k+1,j-1} = 0$, the choices of $\alpha_{k+1,j}(I)$, $\gamma_{k+\frac{1}{2},j+\frac{1}{2}}(I)$ and $\sigma_{k+\frac{1}{2},j-\frac{1}{2}}(I)$ do not effect the value of $\hat{L}_E$.

Consider Square II.

Case 1: $(x_k, y_{j+1}) \in \Omega_h$:

In this case we obtain

$$E_{kj}(II) = \frac{1}{2h^2} \left[p_{k,j+\frac{1}{2}} - \frac{(p_{k,j+\frac{1}{2}})^2}{c_{k,j+1}} \right] , \tag{A.10a}$$

$$\beta_{k,j+1}(II) = \frac{1}{2h^2} \frac{p_{k,j+\frac{1}{2}}\, p_{k,j+3/2}}{c_{k,j+1}} , \tag{A.10b}$$

(A.10c) $$\gamma_{k+\frac{1}{2},j+\frac{1}{2}}(II) = \frac{1}{2h^2}\frac{p_{k,j+\frac{1}{2}}\,p_{k+\frac{1}{2},j+1}}{c_{k,j+1}},$$

(A.10d) $$\sigma_{k-\frac{1}{2},j+\frac{1}{2}}(II) = \frac{1}{2h^2}\frac{p_{k,j+\frac{1}{2}}\,p_{k-\frac{1}{2},j+1}}{c_{k,j+1}},$$

<u>Case 2: $(x_k, y_{j+1}) \notin \Omega_h$:</u>

Using arguments similar to those used in case 2 of square I we have

(A.11a) $$E_{kj}(II) = \frac{1}{2h^2}\,p_{k,j+\frac{1}{2}}\,.$$

For $\gamma_{k+\frac{1}{2},j+\frac{1}{2}}(II)$ and $\sigma_{k-\frac{1}{2},j+\frac{1}{2}}(II)$ we may use the formulae of (A.10c) and (A.10d). Finally

(A.11b) $$\beta_{k,j+1}(II) = E_{kj}(II) - \gamma_{k+\frac{1}{2},j+\frac{1}{2}}(II) - \sigma_{k-\frac{1}{2},j+\frac{1}{2}}(II)\,.$$

Because $\hat{L}_E$ is symmetric it is not necessary to compute the contributions from squares III and IV. We now make a similar decomposition of the coefficients of $L_E^{(1)}$. We set

(A.12a) $$a_{k+\frac{1}{2},j+\frac{1}{2}}(I) = \frac{1}{h^2}\frac{p_{k+\frac{1}{2},j}\,p_{k+1,j+\frac{1}{2}}}{c_{k+1,j}},$$

(A.12b) $$a_{k+\frac{1}{2},j+\frac{1}{2}}(II) = \frac{1}{h^2}\frac{p_{k,j+\frac{1}{2}}\,p_{k+\frac{1}{2},j+1}}{c_{k,j+1}},$$

(A.12c) $$b_{k+\frac{1}{2},j-\frac{1}{2}}(I) = \frac{1}{h^2}\frac{p_{k+\frac{1}{2},j}\,p_{k+1,j-\frac{1}{2}}}{c_{k+1,j}},$$

(A.12d) $$b_{k-\frac{1}{2},j+\frac{1}{2}}(II) = \frac{1}{h^2}\frac{p_{k,j+\frac{1}{2}}\,p_{k-\frac{1}{2},j+1}}{c_{k,j+1}}.$$

(A.13) $$d_{kj}(I) = a_{k+\frac{1}{2},j+\frac{1}{2}}(I) + b_{k+\frac{1}{2},j-\frac{1}{2}}(I) .$$

From (A.8c), (A.8d), (A.12a) and (A.12c) we have

(A.14a) $$\frac{1}{2}\, a_{k+\frac{1}{2},j+\frac{1}{2}}(I) = \gamma_{k+\frac{1}{2},j+\frac{1}{2}}(I) ,$$

(A.14b) $$\frac{1}{2}\, b_{k+\frac{1}{2},j-\frac{1}{2}}(I) = \sigma_{k+\frac{1}{2},j-\frac{1}{2}}(I) .$$

From (A.10c), (A.10d), (A.12b), (A.12d) we see that

(A.14c) $$\frac{1}{2}\, a_{k+3/2,j-\frac{1}{2}}(I) = \gamma_{k+3/2,j-\frac{1}{2}}(I) ,$$

(A.14d) $$\frac{1}{2}\, b_{k+\frac{1}{2},j-\frac{1}{2}}(I) = \sigma_{k+\frac{1}{2},j-\frac{1}{2}}(I) .$$

Let $\langle \hat{L}_E U,U\rangle_I$ and $\langle L_E^{(1)}U,U\rangle_I$ denote the contribution of square I to the quadratic forms $\langle \hat{L}_E U,U\rangle$ and $\langle L_E^{(1)}U,U\rangle$ respectively. Then, using Lemma A.1 and (A.14) we see that

(A.15a) $$\begin{cases} \langle \hat{L}_E\psi,\psi\rangle_I = \alpha_{k+1,j}(I)[\psi_{k+2,j} - \psi_{k,j}]^2 + \\ \beta_{k+1,j}(I)[\psi_{k+1,j+1} - \psi_{k+1,j-1}]^2 + 2a_{k+\frac{1}{2},j+\frac{1}{2}}(I)[\psi_{k+1,j+1} - \psi_{k,j}]^2 \\ + 2a_{k+3/2,j-\frac{1}{2}}(I)[\psi_{k+2,j} - \psi_{k+1,j-1}]^2 + 2b_{k+\frac{1}{2},j-\frac{1}{2}}(I)[\psi_{k+1,j-1} - \psi_{kj}]^2 \\ + 2b_{k+3/2,j+\frac{1}{2}}(I)[\psi_{k+1,j+1} - \psi_{k+2,j}]^2 , \end{cases}$$

and

$$\text{(A.15b)}\quad \begin{cases} \langle L_E^{(1)}\psi,\psi\rangle_I = a_{k+\frac{1}{2},j+\frac{1}{2}}(I)[\psi_{k+1,j+1}-\psi_{kj}]^2 \\ + a_{k+3/2,j-\frac{1}{2}}(I)[\psi_{k+2,j}-\psi_{k+1,j-1}]^2 + b_{k+\frac{1}{2},j-\frac{1}{2}}(I)[\psi_{k+1,j-1}-\psi_{k,j}]^2 \\ + b_{k+3/2,j+\frac{1}{2}}(I)[\psi_{k+1,j+1}-\psi_{k+2,j}]^2 . \end{cases}$$

Lemma A.4: Let $\psi \in S_E$ then

$$\text{(A.16a)}\quad \begin{aligned} &[\psi_{k+1,j+1}-\psi_{k+1,j-1}]^2 + [\psi_{k+2,j}-\psi_{k,j}]^2 = [\psi_{k+1,j+1}-\psi_{k+2,j}]^2 \\ &+ [\psi_{k+2,j}-\psi_{k+1,j-1}]^2 + [\psi_{k+1,j+1}-\psi_{k,j}]^2 + [\psi_{k,j}-\psi_{k+1,j-1}]^2 \\ &- [\psi_{k+1,j-1}-\psi_{k,j}+\psi_{k+1,j+1}-\psi_{k+2,j}]^2 . \end{aligned}$$

Hence

$$\text{(A.16b)}\quad \begin{aligned} &[\psi_{k+1,j+1}-\psi_{k+1,j-1}]^2 + [\psi_{k+2,j}-\psi_{k,j}]^2 \le [\psi_{k+2,j}-\psi_{k+1,j+1}]^2 \\ &+ [\psi_{k+1,j+1}-\psi_{kj}]^2 + [\psi_{k+2,j}-\psi_{k+1,j-1}]^2 + [\psi_{k+1,j-1}-\psi_{kj}]^2 . \end{aligned}$$

Proof: One can verify (A.16a) by a direct computation. A detailed proof is given in Braess [3, p. 512] .

Lemma A.5: Suppose $(x_{k+1},y_j) \in \Omega_h$. Then there is a constant K depending only on $\|\nabla p\|_\infty$ and p_0 such that

$$\text{(A.17)}\quad 0 \le \langle \tilde{L}_E^{(1)}\psi,\psi\rangle_I \le (1+Kh)\,\langle L_E^{(1)}\psi,\psi\rangle_I .$$

Proof: Note that

(A.18) $$\frac{1}{2}\langle \tilde{L}_E^{(1)}\psi,\psi\rangle_I = \langle \hat{L}_E\psi,\psi\rangle_I - \frac{1}{2}\langle L_E^{(1)}\psi,\psi\rangle_I .$$

Therefore, using (A.15) we see that

(A.19) $$\begin{cases} \langle \tilde{L}_E^{(1)}\psi,\psi\rangle_I = 2\alpha_{k+1,j}(I)[\psi_{k+2,j}-\psi_{k,j}]^2 \\ \qquad\qquad + 2\beta_{k+1,j}(I)[\psi_{k+1,j+1}-\psi_{k+1,j-1}]^2 . \end{cases}$$

Thus, we have established the left hand inequality of (A.17). Using (A.12), i.e., the definitions of $a_{k+\frac{1}{2},j+\frac{1}{2}}(I)$, $b_{k+\frac{1}{2},j-\frac{1}{2}}(I)$ etc. and (A.8b), the definition of $\alpha_{k+1,j}(I)$, and (A.10b), the definition of $\beta_{k+1,j}(I)$, we see that there is a constant K, depending only on $\|\nabla p\|_\infty$ and p_0 such that

(A.20a) $$\frac{\alpha_{k+1,j}(I)}{a_I} \le \frac{1}{2}(1+Kh)$$

(A.20b) $$\frac{\beta_{k+1,j}(I)}{a_I} \le \frac{1}{2}(1+Kh)$$

where

(A.20c) $$a_I = \text{any of } \{a_{k+\frac{1}{2},j+\frac{1}{2}},\ a_{k+\frac{3}{2},j-\frac{1}{2}},\ b_{k+\frac{1}{2},j-\frac{1}{2}},\ b_{k+\frac{3}{2},j+\frac{1}{2}}\} .$$

Therefore (A.19) and (A.16b) and (A.15b) yield the right hand inequality of (A.17). ∎

Corollary:

(A.21) $$\frac{1}{2}\langle L_E^{(1)}\psi,\psi\rangle_I \le \langle \hat{L}_E\psi,\psi\rangle_I \le (1+Kh)\langle L_E^{(1)}\psi,\psi\rangle_I . \quad \blacksquare$$

Lemma A.6: Suppose $(x_{k+1},y_j) \notin \Omega_h$. Then the conclusion of Lemma A.5 and its corollary hold.

Proof: Calculations similar to those of Lemma A.5 now yield

(A.22) $$0 \le \langle \tilde{L}_E^{(1)}\psi,\psi\rangle_I = \alpha_{k+1,j}[\psi_{k+2,j} - \psi_{kj}]^2 = \alpha_{k+1,j}[\psi_{kj}]^2 .$$

Thus, as before, the left hand inequality of (A.17) holds. In this case (A.9), the definition of $\alpha_{k+1,j}$, implies

(A.23) $$\frac{\alpha_{k+1,j}}{a_I} \le (1+Kh) .$$

Using Lemma A.3 we see that

$$\psi_{k+2,j} = \psi_{k+1,j+1} = \psi_{k+1,j-1} = 0 .$$

Using (A.15b), we see that

$$\langle L_E^{(1)}\psi,\psi\rangle_I = [a_{k+\frac{1}{2},j+\frac{1}{2}}(I) + b_{k+\frac{1}{2},j-\frac{1}{2}}(I)][\psi_{k,j}]^2 ,$$

hence (A.22) and (A.23) imply the right-hand side of (A.17). Thus, the lemma is proven. $\blacksquare$

Theorem A: In either case, $L_E = L_E^{(1)}$ or $L_E^{(2)}$ and with the associated $\tilde{L}_E$, there is a constant K, depending only on $\|\nabla p\|_\infty$ and p_0 such that, if $\phi \not\equiv 0$ and $\phi \in S_E$ we have

$$-Kh \leq \frac{\langle \tilde{L}_E\phi,\phi \rangle}{\langle L_E\phi,\phi \rangle} \leq (1+Kh) \ .$$

Proof: The arguments which give Lemma A.5 and Lemma A.6 extend to all the squares, II, III and IV. Thus, those lemmas imply that the theorem holds for $L_E = L_E^{(1)}$. The case of $L_E = L_E^{(2)}$ follows from (3.5) and the observation that

$$(1-Kh)\langle L_E^{(1)}\psi,\psi \rangle \leq \langle L_E^{(2)}\psi,\psi \rangle \leq (1+Kh)\ \langle L_E^{(1)}\psi,\psi \rangle \ . \qquad \blacksquare$$

REFERENCES:

[1] R. E. Alcouffe, A. Brandt, J. E. Dendy Jr. and J. W. Painter: The multigrid method for the diffusion equation with strongly discontinuous coefficients. SIAM J. Sci. Stat. Comput. 2, 430-454 (1981).

[2] D. Braess: The contraction number of a multigrid method for solving the poisson equation, Numer. Math. 37, 387-404 (1981).

[3] D. Braess: The convergence rate of a multigrid method with Gauss-Seidel relaxation for the Poisson equation. Math of Comp. 42, 505-519 (1984).

[4] A. Brandt: Multilevel adaptive solutions to boundary-value problems, Math. Comp. 31, 333-390 (1977).

[5] J. E. Dendy, Jr.: Black box multigrid, J. Comp. Phys. 48, 366-386 (1982).

[6) P. O. Frederickson: Fast approximate inversion of large sparse linear systems, Math. Report 7075, Lakehead University, Ontario, Canada, 1975.

[7] G. E. Forsythe and W. R. Wasow: Finite Difference Methods for Partial Differential Equations, Wiley Publishers, New York, 1960.

[8] S. F. McCormick and J. W. Ruge: Multigrid methods for variational problems, SIAM J. Num. Anal. 19, 924-929 (1982).

[9] M. Ries, U. Trottenberg and G. Winter: A note on MGR methods, Linear Algebra Appl. 49, 1-26 (1983).

[10] J. C. South (Jr.) and A. Brandt: Application of a multi-level grid method to transonic flow calculations. Transonic Flow Problems in Turbomachinery. T. C. Adamson and M. F. Platzer Editors, Hemisphere, Washington D.C.