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ON DIGRAPHS WITH THE
ODD CYCLE PROPERTY

by

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On Digraphs with the Odd Cycle Property

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Abstract

We say that a digraph D has the odd cycle property if there exists an edge subset S such that every cycle of D has an odd number of edges from S. We give necessary and sufficient conditions for a digraph to have the odd cycle property. We also consider the analoguous problem for graphs.

1. Introduction

A signed digraph (D,S), is a digraph D whose edges are associated with signs, positive or negative, where S is the set of negative edges. Several problems related to the parity of the number of negative edges in the cycles of signed graphs and digraphs have been studied. For instance, balanced signed digraphs and graphs have been studied extensively. See for example [2]. A signed digraph (graph) is balanced if all its cycles have an even (possibly zero) number of negative edges. Another, more recent example is the study of signed digraphs, all of whose cycles have an odd number of negative edges. This problem has been considered in the study of qualitative matrices, see for example [1,4,6,7]. Let $A = (a_{ij})$ be a real $n \times n$ matrix with $a_{ii} < 0$. We say that A is sign-nonsingular (or an L-matrix) if A is nonsingular and every matrix B whose sign-pattern is the same as the sign-pattern of A, is nonsingular. We associate with A a signed digraph $(D(A),S)$ with vertex set $\{1,2,\ldots,n\}$ corresponding to the rows of A and with edge set $\{(i,j): a_{ij} \neq 0\}$. The set S of negative edges consists of those edges (i,j) for which $a_{ij} < 0$. It has been shown that A is sign-nonsingular if and only if every elementary cycle in $D(A)$ has an odd number of negative edges [1]. The time complexity of recognizing digraphs with the above property is still open. However, the time-complexity of several related problems have been determined in [4,5,8].

It seems natural to consider the following question: Given a digraph D, is there a subset S of the edge set such that every cycle of D has an odd number of edges from S? If such a set S exists, we say that D has the odd cycle property. We study this question together with the analogous question for graphs. Harary, Lundgren and Maybee [3] have independently considered this question and have presented three families of digraphs with the odd cycle property. They have however left open the problem of characterizing general digraphs with the odd cycle property. We give here a necessary and sufficient condition for a digraph to have the odd cycle property. We also consider the special class of symmetric digraphs, which is one of the families treated in [3]. Our characterization of symmetric digraph with the odd cycle property is essentially the same as that of [3] but our method of proof is different.

2. Basic definitions.

As the term is used here, a walk in a digraph [resp. graph] is a sequence W of vertices $(i_0, i_1, \ldots i_k)$ such that $k \geq 1$ and (i_{j-1}, i_j) (resp. $[i_{j-1}, i_j]$) is an edge, $j = 1, \ldots, k$. The walk is a path if no vertex is repeated. The sequence W is a closed walk if $i_0 = i_k$ and it is a cycle if no other vertex is repeated. Let P be the walk $(u = x_0, x_1, \ldots, x_t = v)$ and let Q be the walk $(v = y_0, y_1, \ldots, y_s)$. We denote by $P \cup Q$ the walk $(x_0, x_1, \ldots, x_t, y_1, \ldots, y_s)$. We say that two paths P and Q from u to v are vertex disjoint if $V(P) \cap V(Q) - \{u,v\} = \phi$. Here we use the common notation $V(P)$ and $E(P)$ to denote the vertex set and edge set of P respectively. In a diagraph, every closed walk can be viewed as a union of cycles. However, this is not the case for an undirected graph. For example, if $[u,v]$ is an edge of a graph, the closed walk (u,v,u) is not a union of cycles. Let D be a digraph. The multiset of edges of a closed walk W is written as a union of the edge sets of several cycles, where the union is understood to be in the multiset sense; for example, $\{e\} \cup \{e\} = \{2 \cdot e\}$. We say that a digraph has the unique parity property if for every closed walk W in D, $E(W) = \bigcup_{i=1}^{k} E(C_i)$ and $E(W) = \bigcup_{j=1}^{\ell} E(D_j)$ imply that $k \equiv \ell \pmod 2$, where C_i and D_i are cycles in D $(i=1,\ldots,k;\ j=1,\ldots,\ell)$. In the remainder of the paper we use the notation $x \equiv y$ to abbreviate $x \equiv y \pmod 2$. Let D be a digraph with the unique parity property. For every closed walk W we define the parity of W, denoted by $N(W)$, as follows

$$N(W) = \begin{cases} 1 & \text{if} \quad E(W) = \bigcup_{i=1}^{k} E(C_i) \quad \text{and} \quad k \equiv 1 \\ 0 & \text{if} \quad E(W) = \bigcup_{j=1}^{\ell} E(D_j) \quad \text{and} \quad \ell \equiv 0 \end{cases}.$$

Since W has the unique parity property, $N(W)$ is well defined. We observe that W_1 and W_2 are two closed walks having a common vertex, then with $W = W_1 \cup W_2$,

$$N(W) \equiv N(W_1) + N(W_2) \quad .$$

The digraph D_1 of Fig. 1 has the unique parity property and $N(W) = 0$ for every closed walk of D_1. However, the digraph D_2 does not have the unique parity property since with $W = 3,1,2,4,3,4,1,2,3$, $E(W)$ is a multiset and $E(W) = E(C_1) \cup E(C_2) = E(D_1) \cup E(D_2) \cup E(D_3)$ where

$$C_1 = 3,1,2,4,3 \ ; \quad C_2 = 3,4,1,2,3 \ ;$$

$$D_1 = 1,2,3,1 \ ; \quad D_2 = 1,2,4,1 \quad \text{and} \quad D_3 = 3,4,3$$

but $2 \not\equiv 3$. We also note that D_1 has the odd cycle property while D_2 does not.

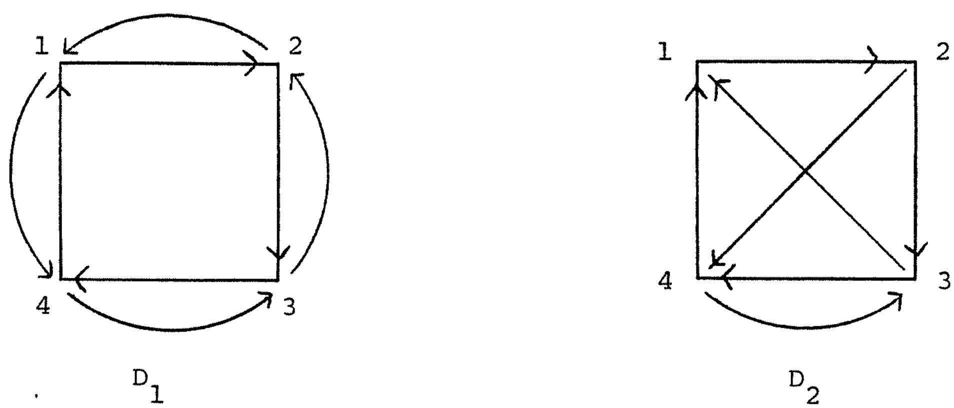

Figure 1

Denote by $c(D)$ the number of strongly connected components of the digraph D. The condensation digraph of D is a multi-digraph whose vertices are the strongly connected components of D and for every edge (u,v) of D with $u \in D_i$ and $v \in D_j$ there is an edge in the condensation graph from D_i to D_j. The condensation digraph is clearly acyclic.

3. Digraphs with the odd cycle property

Theorem 3.1: Let $D = (V,E)$ be a strongly connected digraph. Then D has the odd cycle property if and only if it has the unique parity property.

Proof: Suppose D has the odd cycle property. Then there exists a subset $S \subset E$ such that for every cycle C, $|E(C) \cap S| \equiv 1$. Let W be a closed walk in D with $E(W) = E(C_1) \cup \ldots \cup E(C_k)$ and $E(W) = E(D_1) \cup \ldots \cup E(D_\ell)$ where C_i and D_j are cycles of D $(i=1,\ldots,k;\ j=1,\ldots,\ell)$. Then

$$E(W) \cap S = (S \cap E(C_1)) \cup \ldots \cup (S \cap E(C_k)) =$$

$$= (S \cap E(D_1)) \cup \ldots \cup (S \cap E(D_\ell)) \quad .$$

Hence

$$|E(W) \cap S| = \sum_{i=1}^{k} |S \cap E(C_i)| \equiv k$$

and

$$|E(W) \cap S| = \sum_{j=1}^{\ell} |S \cap E(D_j)| \equiv \ell \ .$$

Thus $k \equiv \ell$ and D has the unique parity property.

Now let D have the unique parity property. Let $e \in E$ and $D_e = (V, E-\{e\})$. Harary et al [3, Theorem 7], have shown that if for every edge e, D_e is not strongly connected then D has the odd cycle property. They refer to such digraphs as ministrong. Thus ministrong digraphs have the odd cycle property and this is independnet of the unique parity property. We may now assume there exists an edge $e = (u,v)$ such that $D' = D_e$ is strongly connected. We proceed by induction on the number of edges in D. When $|E| = 2$, either D has no cycles or D consists of a single cycle of length 2 and with S being any one any one edge of D, we see that D has the odd cycle property. Since D has the unique parity property, so does D'. Hence by induction there is a subset $S' \subset E - \{e\}$ such that every cycle of D' has an odd number of edges from S'. If every two paths P_1 and P_2 from v to u satisfy $|E(P_1) \cap S'| \equiv |E(P_2) \cap S'|$, then let

$$(3.2) \qquad S = \begin{cases} S' & \text{if} \quad |E(P_1) \cap S| \equiv 1 \\ S' \cup \{e\} & \text{if} \quad |E(P_1) \cap S| \equiv 0 \ . \end{cases}$$

It follows that every cycle of D has an odd number of edges from S. We show that indeed, if P_1 and P_2 are paths from v to u then $|E(P_1) \cap S'| \equiv |E(P_2) \cap S'|$. Since D' is strongly connected there exists a path P from u to v in D'. Consider the walk $W = P_1 \cup P \cup P_2 \cup (u,v)$ in D. Then since both $P_1 \cup (u,v)$ and $P_2 \cup (u,v)$ are cycles, we have

$$N(W) \equiv N(P \cup P_1) + N(P_2 \cup \{e\}) \equiv N(P \cup P_1) + 1$$

and

$$N(W) \equiv N(P \cup P_2) + N(P_1 \cup \{e\}) \equiv N(P \cup P_2) + 1 \ .$$

Hence $N(P \cup P_1) \equiv N(P \cup P_2)$. But since D' has the odd cycle property, $N(W) \equiv |E(W) \cap S'|$ for every closed walk W in D'. Hence

$$N(P \cup P_1) \equiv |(E(P) \cup E(P_1)) \cap S'| \equiv |E(P) \cap S'| + |E(P_1) \cap S'|$$

and

$$N(P \cup P_2) \equiv |(E(P) \cup E(P_2)) \cap S'| \equiv |E(P) \cap S'| + |E(P_2) \cap S'| \ .$$

Thus $|E(P_1) \cap S'| \equiv |E(P_2) \cap S'|$ and hence D has the odd cycle property.

Since every closed walk in a digraph is contained in one of its strongly connected components, we get:

Corollary 3.3: A digraph D has the odd cycle property if and only if D has the unique parity property.

The unique parity property is not easy to check. However, for some special classes of digraphs, the odd cycle property can be more easily checked. For example the class of ministrong digraphs can be recognized in polynomial time. More information on the structure of ministrong digraph is given in the next proposition.

Proposition 3.4: Let $D = (V,E)$ be a ministrong digraph and let $e \in E$. Then the condensation digraph of $D_e = (V,E-\{e\})$ is a directed path.

Proof: First we show that the condensation digraph is simple, namely it has no multiple edges. For suppose G_i and G_j are two connected components of D_e and $(x,y) \neq (x',y')$ are edges of D_e with $x,x' \in G_i$ and $y,y' \in G_j$. Then $D_{(x,y)}$ is strongly connected, a contradiction. Now choose $e = (u,v) \in E$ such that $m = c(D_e) \leq c(D_f)$ for every $f \in E$. Let $G_1,\ldots,G_m$ be the strongly connected components of D_e where G_1 contains v and G_m contains u. Then $G_1 \neq G_m$. Note that since D is strongly connected, there is a path in the condensation digraph of D_e from G_1 to every component G_j, and there is no path from G_m to any other component. Let P be a path in the condensation digraph from G_1 to G_m. Suppose that for some G_i in P $(1 \leq i < m)$ and for G_j not in P there exists an edge $e' = (u',v') \in E - \{e\} - E(P)$ with $u' \in G_i$ and $v' \in G_j$. Since e is an edge of $D_{e'}$, the vertices belonging to the same strongly connected components in P, all belong to the same connected component of $D_{e'}$. Since $G_1 \neq G_m$ it follows that $c(D_{e'}) < c(D_e)$, a contradiction to the minimality of $c(D_e)$. Thus the vertices of P are all the strongly connected components of D_e and we may assume $P = (G_1,G_2,\ldots,G_m)$. Moreover, if there exists an edge $f \in E - \{e\}$ joining a vertex of G_i to a vertex of G_j with

$j \neq i+1$ then G_f is strongly connected, a contradiction. It follows that the condensation digraph of D_e is a path $P = (G_1, G_2, \ldots, G_m)$.

We conclude this section with a more structural characterization of another special class of digraphs. As noted before, Harary et al [3] have independently considered this class of digraphs. A digraph $D = (V,E)$ is <u>symmetric</u> if $(u,v) \in E$ implies $(v,u) \in E$. The graph G obtained by identifying the edges (u,v) and (v,u) to form an unordered pair $[u,v]$ is the <u>underlying</u> graph of D. We denote by D_G the symmetric digraph whose underlying graph is G.

Let G be a graph and C a cylce in G. A <u>diagonal path</u> of C is an (elementary) path $P = (x_1, \ldots, x_k)$ such that $x_1, x_k \in V(C)$ and $x_i \notin V(C)$ $i \neq 1,k$. We start with two Lemmas.

Lemma 3.5: <u>Let</u> G <u>be a graph and suppose</u> D_G <u>has the odd cycle property with the edge subset</u> S. <u>If</u> P_1 <u>and</u> P_2 <u>are two vertex disjoint directed paths of odd length from</u> u <u>to</u> v <u>in</u> D_G. <u>Then</u> $|E(P_1) \cap S| \equiv |E(P_2) \cap S|$.

Proof. Let P_i be the directed path $(u = x_{i1}, x_{i2}, \ldots, x_{ik_i} = v)$ and let $\overleftarrow{P}_i$ be the directed path $(v = x_{ik_i}, \ldots, x_{i1} = v)$, $(i=1,2)$. Then $P_1 \cup \overleftarrow{P}_2$ is a cycle in D_G and hence $|E(P_1 \cup \overleftarrow{P}_2) \cap S| \equiv 1$. Since P_i has odd length $|E(P_i \cup \overleftarrow{P}_i) \cap S| \equiv 1$ for $i = 1,2$. It follows that $|E(P_1) \cap S| \equiv |E(P_2) \cap S|$.

Let P be the directed path $(x_1, x_2, \ldots x_k)$. Let $x = x_i \in V(P)$ and $y = x_j \in V(P)$ with $i < j$. We denote by xPy the directed path $(x = x_i, x_{i+1}, \ldots, x_j = y)$ and by $y\overleftarrow{P}x$ the directed path $(y = x_j, x_{j-1}, \ldots, x_i = x)$.

Lemma 3.6. <u>Let</u> $G = (V,E)$ <u>be a bipartite graph in which every diagonal path of every cycle has odd length</u>. <u>Let</u> $e = [u,v]$ <u>be an edge of</u> G <u>and</u>

$G_e = (V, E-\{e\})$. <u>Suppose</u> P <u>and</u> Q <u>are two directed paths from</u> u <u>to</u> v <u>in</u> D_{G_e}. <u>Then there exist</u> $t \geq 0$ <u>and vertices</u> $x_{11}, x_{21}, \ldots x_{t1}$ <u>such that</u> <u>with</u> $x_{01} = u$ <u>and</u> $x_{t+1,1} = v$ (3.7) <u>holds</u>.

$$(3.7) \qquad P = \bigcup_{i=0}^{t} x_{i1}Px_{(i+1)1}; \qquad Q = \bigcup_{i=0}^{t} x_{i1}Qx_{(i+1)1}$$

and $x_{i1}Px_{(i+1)1}$ and $x_{i1}Qx_{(i+1)1}$ are vertex disjoint directed paths of odd length from x_{i1} to $x_{(i+1)1}$.

Proof: Let P be a directed path. We use the notation $x <_p y$ if both x and y are vertices of P and x precedes y on P. Let $x_{11} \neq u$ be the first intersection of P and Q along P, and let y be the next intersection along P. That is $y >_p x_{11}$. If $y >_Q x_{11}$ as well, we denote y by x_{21} otherwise we denote y by x_{12}. Using this scheme we label all vertices in $V(P) \cap V(Q)$ to obtain the following

$$x_{11} <_p x_{12} <_p \cdots <_p x_{1n_1} <_p x_{21} <_p \cdots <_p x_{2n_2} <_p \cdots$$

$$<_p x_{t1} <_p \cdots <_p x_{tn_t}, \quad \text{where} \quad t \geq 0 \quad \text{and} \quad n_i \geq 1.$$

Moreover, $x_{i1} <_Q x_{(i+1)1}$ but $x_{i1} >_Q x_{ij}$ for $j = 2, \ldots, n_i$ (See Fig. 3)

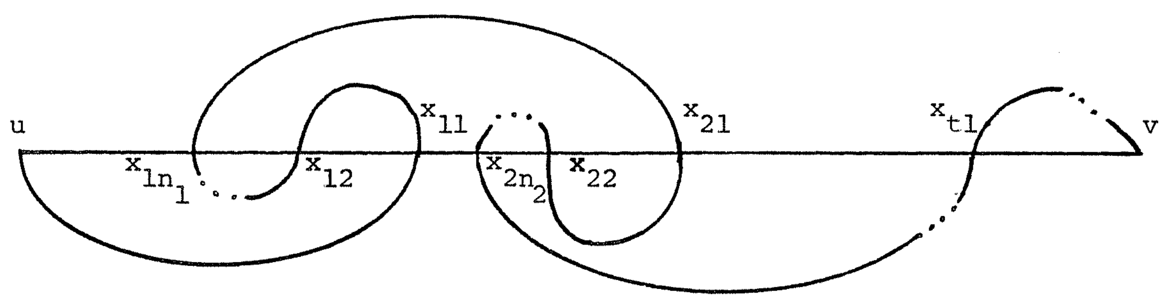

Figure 3

(The path P is a straight line, and the other path is Q) .

Note that with $j > i$, $2 \leq k \leq n_j$ and $1 \leq \ell < n_i$ it is possible that $x_{jk} <_Q x_{i\ell}$, but $x_{j1} >_Q x_{i\ell}$. We show that with $x_{11}, x_{21}, \ldots, x_{t1}$ (3.6) is satisfied. Since $x_{i1} <_P x_{(i+1)1}$ for $i = 0,\ldots,t$, $P = \bigcup_{i=0}^{t} x_{i1}Px_{(i+1)1}$. Similarly since $x_{i1} <_Q x_{(i+1)1}$ for $i = 0,\ldots,t$, $P = \bigcup_{i=0}^{t} x_{i1}Qx_{(i+1)1}$. Next we note that $x_{i1}Px_{(i+1)1} = (x_{i1}Px_{i2}) \cup \ldots \cup x_{in_i}Px_{(i+1)1}$ and $x_{ij} <_Q x_{i1} <_Q x_{(i+1)1}$. It follows that $x_{i1}Px_{(i+1)1}$ and $x_{i1}Qx_{(i+1)1}$ are vertex disjoint. It remains to show that both have odd length. For $i = 1,\ldots,t$, let C_i be a cycle in the undirected graph G defined by

$$C_i = (uPx_{i1}) \cup (x_{i1}Qv) \cup (v,u) \qquad i = 0,\ldots,t \ .$$

Clearly uPx_{i1} and $x_{i1}Q_v$ are paths in G. Moreover, since for $j \leq i - 1$ and $k \leq n_j$, $x_{jk} \leq_Q x_{j1} <_Q x_{i1}$, the vertex x_{jk} is not a vertex of any segment $x_{\ell 1}Qx_{(\ell+1)1}$ for $\ell \geq i$. It follows that the above two paths are vertex disjoint and hence C_i is a cycle. It also follows that $x_{i1}Px_{(i+1)1}$ is a diagonal path of C_i and hence is of odd length. But $x_{i1}Qx_{(i+1)1}$ and $x_{i1}Px_{(i+1)1}$ are vertex disjoint paths joining x_{i1} and $x_{(i+1)1}$ in G and hence $(x_{i1}Px_{(i+1)1}) \cup (x_{(i+1)1}Qx_i)$ is a cycle in G. Since G is bipartite, the path $x_{i1}Qx_{(i+1)1}$ has odd length.

Theorem 3.7: *Let $G = (V,E)$ be a graph. Then D_G has the odd cycle property if and only if G is bipartite and every diagonal path of every cycle in G has odd length.*

Proof: Suppose first that D_G has the odd cycle property. Let $C = (v_1,\ldots,v_k,v_1)$ be a cycle in G. Then the edge set of the closed walk D_C in D_G can be written both as a union of k cycles of length 2 and as a union of 2 cycles of length k. It follows from Corollary 3.3 that $k \equiv 2$ and hence G is bipartite. Now let $P = (u_1,\ldots,u_{s+1})$ be a diagonal path, of C; that is $u_1 = v_1$ and $u_{s+1} = v_i$ for some i, $2 \leq i \leq k$. Denote by $\vec{C}$ the directed cycle $(v_1,\ldots v_k,v_1)$ and by $\vec{P}$ the directed path $(u_1,\ldots,u_{s+1})$. Consider the following directed paths in D_G:

$$Q_1 = v_1\vec{C}v_i \ ; \quad Q_2 = v_i\overleftarrow{C}v_1 \ ; \ P_1 = v_1\vec{P}v_i \quad \text{and} \quad P_2 = v_i\overleftarrow{P}v_1 \ .$$

Then with $W = Q_1 \cup Q_2 \cup P_1 \cup P_2$,

$$N(W) = N(Q_1 \cup Q_2) + N(P_1 \cup P_2) \equiv s + 1$$

and

$$N(W) = N(Q_1 \cup P_2) + N(Q_2 \cup P_1) \equiv 2 \ .$$

Hence $s \equiv 1$; that is, every diagonal path in G has odd length.

Conversely, suppose G is bipartite and every diagonal path in G has odd length. Then the same holds for every subgraph of G. We proceed by induction on the number of edges in G. When $|E| = 2$ then D_G has at most one cycle which is of length 2 and hence D_G has the odd cycle property. Let $e = [u,v] \in E$ and let $G_e = (V,E - \{e\})$. By the inductive hypotheses, D_{G_e} has the odd cycle property with some $S' \subset E - \{e\}$. Let P and Q be two directed paths in D_{G_e} from u to v. Then by Lemma (3.6), there exist vertices $x_{11}, x_{21},\ldots,x_{t1}$ of P such that (3.7) is satisfied. Moreover, by Lemma (3.5), the two vertex disjoint paths $P_i = x_{i1}Px_{(i+1)1}$ and $Q_i = x_{i1}\,Q\,x_{(i+1)1}$ satisfy $|E(P_i) \cap S'| \equiv |E(Q_i) \cap S'|$. It now follows that $|E(P \cap S')| \equiv |E(Q \cap S')|$. Similarly, if P and Q are two directed paths from v to u in D_{G_e} then $|E(P \cap S')| \equiv |E(Q \cap S')|$. Let P be a directed path from u to v in D_{G_e} and Q a directed path from v to u in D_{G_e}. Since G is bipartite and $[u,v]$ is an edge of G both P and Q have odd length. Hence $P \cup \overleftarrow{P}$ is the union of an odd number of cycles of length 2 in D_{G_e}. Moreover since D_{G_e} has the odd cycle property $|E(P) \cap S'| \equiv |E(\overleftarrow{P}) \cap S'|$. But since as shown above $|E(\overleftarrow{P}) \cap S'| \equiv |E(Q) \cap S'|$ we get $E(P) \cap S'| \not\equiv |E(Q) \cap S'|$. Now let

$$S = \begin{cases} S' \cup (v,u) & \text{if } k \text{ is even } (\ell \text{ is odd}) \\ S' \cup (u,v) & \text{if } k \text{ is odd } (\ell \text{ is even}) \end{cases}$$

Then with S as above, D_G has the odd cycle property and the proof is complete.

4. Graphs with the odd cycle property.

A graph G has the <u>odd cycle property</u> if there is an edge subset S such that every cycle of G has an odd number of edges in S. Harary et al [2] have shown that if a signed-graph has a block with more than one cycle then this block contains a cycle with an even number of negative edges. This provides a structural characterization of graphs with the odd cycle property. We show here that an undirected graph with the odd cycle property can also be characterized by an analogue to the unique parity property of digraphs.

We have noted before that a closed cycle in an undirected graph is not always a union of cycles. However, it is true that the edge set of every closed walk W in a graph can be written as

$$E(W) = E(C_1) \cup \ldots \cup E(C_k) \cup \{2n_1 \cdot e_1, \ldots, 2n_t \cdot e_t\} ,$$

where $C_1, \ldots, C_k$ are cycles, $e_1, \ldots, e_i$ are edges and $n_1, \ldots, n_t$ are non-negative integers. We say that a graph has the <u>unique parity property</u> if for every closed walk W, $E(W) = E(C_1) \cup \ldots \cup E(C_k) \cup \{2n_1 \cdot e_1\} \cup \ldots \cup \{2n_t \cdot e_t\}$ and $E(W) = E(D_1) \cup \ldots \cup E(D_\ell) \cup \{2m_1 \cdot e_1'\} \cup \ldots \cup \{2m_s \cdot e_s'\}$ imply that $k \equiv \ell$.

Theorem 4.1: <u>Let</u> G <u>be a graph. Then the following are equivalent:</u>

(i) G <u>has the odd cycle property.</u>

(ii) G <u>has the unique parity property.</u>

(iii) <u>No cycle of</u> G <u>has a diagonal path.</u>

(iv) <u>Every block</u> G <u>is either an edge or a single cycle.</u>

Proof: (i) $\Rightarrow$ (ii) Suppose S is an edge subset so that every cycle of G has an odd number of edges from S. Let W be a walk such that

$$E(W) = E(C_1) \cup \ldots \cup E(C_k) \cup \{2n_1 \cdot e_1\} \cup \ldots \cup \{2n_t \cdot e_t\}$$

and

$$E(W) = E(D_1) \cup \ldots \cup E(D_e) \cup \{2m_1 \cdot e_1'\} \cup \ldots \cup \{2m_s \cdot e_s'\} .$$

Then $|E(W) \cap S| \equiv k$ and $|E(W) \cap S| \equiv \ell$ and hence $k \equiv \ell$.

(ii) $\Rightarrow$ (iii). Let G be a graph with the unique parity property. Let C be a cycle in G and suppose P is a diagonal path of C joining the vertices u and v of C. Let $P_1 = uCv$, $P_2 = vCu$ and $E(P) = \{e_1, \ldots, e_t\}$. Then $W = P_1 \cup P \cup P_2 \cup P$ is a closed walk in G. Moreover $E(W) = E(C) \cup \{2 \cdot e_1\} \cup \ldots \cup \{2 \cdot e_t\}$ and $E(W) = E(P_1 \cup P) \cup E(P_2 \cup P)$ where C, $P_1 \cup P$ and $P_2 \cup P$ are cycles, but $1 \not\equiv 2$, a contradiction.

(iii) $\Rightarrow$ (iv) Suppose no cycle of G has a diagonal path. Let B be a block of G with more than one edge. Then B has a cycle C. If $B \neq C$ then there exists an edge $[u,v] = e$ in B but not in C with $u \in V(C)$. Let $e' = [x,y]$ be any edge of C. Since B is a block e and e' are contained in a cycle $C' = (u,v,\ldots,u)$ in B. Let $z \neq u$ be the first vertex of C' which is in C, then $uC'z$ is a diagonal path of C, a contradiction.

(iv) $\Rightarrow$ (i) Since every cycle lies in some block and every block consists of at most one cycle, we construct S as follows. For every cycle C_i of G choose any edge e_i of C_i and let $S = \bigcup_i \{e_i\}$.

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