A NOTE ON CONVERGENCE OF THE MULTIGRID V-CYCLE*,†

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ABSTRACT

Several recent papers have discussed the convergence of the multigrid V-cycle. In particular there are several results for the symmetric case: where the number of smoothings before the fine-to-coarse transfer and after the coarse-to-fine transfer are the same. In most instances, the smoother $H = (I - E^{-1}A)$ has been limited to the case where E is positive definite and the eigenvalues h of H satisfy $0 \leq h \leq 1$. In this note we extend these results to asymmetric V-cycles and the case where $-b \leq h \leq 1$ with $0 < b < 1$.

1. Introduction

There have been several recent papers and reports [1], [2], [4], [5], [6], [7], [8] giving proofs for the multigrid V-cycle. A careful reading shows that the formulation of the problem and the basic results of Yserentant [8], Bank and C. C. Douglas [1], and Braess and Hackbusch [2] are quite similar. All of these authors deal with smoothing steps of the form

(1) $$H = I - E^{-1}A$$

where A is the positive definite matrix of the problem and E is a positive definite matrix with

(2) $$\langle Au,u \rangle \leq \langle Eu,u \rangle .$$

As a consequence all the eigenvalues, say h_s, of H satisfy

(3) $$0 \leq h_s \leq 1 .$$

However, other analyses (see [3]) of certain special cases, e.g., damped Jacobi smoothing iterations, yield multigrid convergence results when the eigenvalues of H satisfy

(4) $$-h_0 \leq h_s \leq 1 , \quad 0 \leq h_0 < 1 .$$

Note that the weaker assumption

$$-1 \leq h_s \leq 1$$

(which is all that we can say for undamped Jacobi) is not sufficient to establish *multigrid convergence*. By multigrid convergence we mean:

there exists a constant ρ, $0 \leq \rho < 1$, independent of h, that bounds some norm of the multigrid process, i.e.,

$$\| MG \| \leq \rho < 1 .$$

When we say a method fails as a multigrid iterative method we mean

$$\lim_{h \to 0} \| MG \| \geq 1 .$$

In this report we employ both the basic insights of S. McCormick [6], [7] and a basic estimate of Yserentant [8] (which is essentially repeated in Bank and C. C. Douglas [1]) to study a more general class of smoothers

(6a) $$H = (I - \omega E^{-1} A) ,$$

with

(6b) $$0 < \omega < 2 .$$

The significance of this is that when $1 < \omega < 2$ some of the eigenvalues of H become negative. We obtain multigrid convergence for the V-cycle based on these smoothers with bounds of the form

(7)
$$\rho = \frac{c-\omega}{c+\omega(2m-1)} , \quad 0 < \omega \leq 1 ,$$
$$\rho = \frac{c-(2-\omega)}{c+(2-\omega)(2m-1)} , \quad 1 \leq \omega < 2 .$$

Note that ρ tends to 1 as $\omega \to 0+$, $2-$. When $0 < \omega \leq 1$ this is a slight improvement of the results of Yserentant [8] which are a bit better than the results of Bank and C. C. Douglas [1].

While the specific extensions of known results is interesting in itself, it is our view that one should not lose sight of the importance of employing the results of McCormick [6], [7] together with the estimates of [8] and [1].

Finally, it should be mentioned that these results are not sharp. In [3] Kamowitz and Parter computed the exact α_k of McCormick's Theory (see section 3). If that theory - as represented by Theorem 3.1 and Theorem 3.2 of section 3 - were sharp, we would obtain results of the form

(8a) $$\alpha_k = \frac{c}{c+k}$$

and

(8b) $$\| M\backslash_j \| \to \alpha_k^{\frac{1}{2}} \quad \text{as} \quad j \to \infty .$$

However, the results of [3] indicate that neither (8a) nor (8b) hold. Thus, there is still a need for a theory that yields sharp results for particular multigrid schemes. On the other hand, it is comforting that we now have theories which yield multigrid convergence theorems consistent with computational experience.

In section 2 we describe the problem and prove a basic result relating the three multigrid schemes $M/_j$, $M\backslash_j$, MV. In particular, we have

(9a) $$\| M/_j \|_A = \| M\backslash_j \|_A ,$$

(9b) $$\| MV \|_A = \| M/_j \|_A^2 .$$

This is a result of McCormick [6]. We include our organization of the proof only because it seems somewhat more transparent. In section 3 we collect some basic facts from the papers mentioned above. Section 4 is devoted to the multigrid convergence theorem.

2. The Problem

We consider a definite dimensional linear vector space S_M with inner product $\langle , \rangle$. Consider the problem

$$A_M U^{(M)} = f^{(M)} \tag{2.1}$$

where A_M is a symmetric positive definite operator.

Consider a sequence of finite dimensional spaces

$$\{S_j, \; j=0,1,\ldots,M\} \tag{2.2a}$$

with

$$\dim S_{j-1} < \dim S_j , \quad j = 1,2,\ldots,M . \tag{2.2b}$$

Consider linear operators I_{j-1}^{j}, I_j^{j-1} which enable us to communicate between these spaces, where

$$I_j^{j-1}: \; S_j \to S_{j-1} \quad \text{(projection)} , \tag{2.3a}$$

$$I_{j-1}^{j}: \; S_{j-1} \to S_j \quad \text{(interpolation)} . \tag{2.3b}$$

In this note we also require that

$$I_j^{j-1} = (I_{j-1}^{j})^* . \tag{2.3c}$$

For each space S_j we define

$$A_j = I_{j+1}^{j} A_{j+1} I_j^{j+1} , \quad j = 0,1,\ldots,(M-1) . \tag{2.4}$$

Finally, we require "smoothing" operators $G_j(u,f)$. In this note we consider a special class of smoothing operators which are a slight extension of the smoothing operators discussed in [1], [2], [8]. In particular let ω be a fixed constant with

(2.5) $$0 < \omega < 2 .$$

Let E_j be a symmetric positive definite linear operator defined on S_j which satisfies

(2.6a) $$\langle Av,v \rangle \leq \langle E_j v,v \rangle ; \ \forall v \in S_j .$$

Let $u,f \in S_j$. Then

(2.6b) $$G_j(u,f) = (I-\omega E_j^{-1} A_j)u + \omega E_j^{-1} f .$$

We are now (as in [6]) in a position to define three multigrid iterative schemes for the solution of (2.1). These schemes are defined recursively as follows.

1) <u>The Symmetric Scheme: $MV(j,u^j,f^j)$.</u>

If $j = 0$ then

(2.7a) $$MV(0,u^0,f^0) = U^0$$

where U^0 is the solution of

(2.7b) $$A_0 U^0 = f^0 .$$

If $1 \leq j \leq M$ perform the following:

(i) do m times:

$$G_j(u^j,f^j) \to u^j$$

(ii) set: $r_j = f^j - A_ju^j$, $f^{j-1} = I_j^{j-1}r_j$, $u^{j-1} = 0$

(iii) $$u^j \leftarrow u^j + I_{j-1}^j MV(j-1,u^{j-1},f^{j-1})$$

(iv) do m times:

$$G_j(u^j,f^j) \to u^j .$$

Return to step (i).

As McCormick [6] has pointed out, this $MV(j,u^j,f^j)$ iterative scheme is closely related to the following "one-sided" schemes.

2) <u>The coarse-to-fine cycle: $M/_j(u^j,f^j)$.</u>

Once more, if $j = 0$ then

$$M/_0(u^0,f^0) = U^0$$

the solution of (2.7b). If $1 \leq j \leq M$ perform the following:

(i) set: $r_j = f^j - A_ju^j$, $f^{j-1} = I_j^{j-1}r_j$, $u^{j-1} = 0$

(ii) $u^j \leftarrow u^j + I_{j-1}^j M/_{j-1}(u^{j-1},f^{j-1})$

(iii) do m times

$$G_j(u^j,f^j) \to u^j ,$$

return to step (i).

3) The fine-to-coarse cycle: $M\backslash_j(j^j, f^j)$.

If $j = 0$, then

$$M\backslash_0(u^0, f^0) = U^0 ,$$

the solution of (2.7b). If $1 \leq j \leq M$ perform the following:

(i) do m times

$$G_j(u^j, f^j) \to u^j$$

(ii) set: $r_j = f^j - A_j u^j$, $f^{j-1} = I_j^{j-1} r_j$, $u^{j-1} = 0$

(iii) $u^j \leftarrow u^j + I_{j-1}^j M\backslash_{j-1}(u^{j-1}, f^{j-1})$

Return to step (i).

Let U^j be the solution of

(2.8a) $$A_j U^j = f^j$$

and let

(2.8b) $$E^j = U^j - u^j ,$$

(2.8c) $$\varepsilon^j = A^{\frac{1}{2}} E^j .$$

Following Bank and C. C. Douglas [1] we describe the "error propogator" as follows: Let $\varepsilon_0^{(j)}$ be the error at the start of a multigrid cycle (for a problem in S_j) and $\varepsilon_1^{(j)}$ be the error at the end of that multigrid cycle.

We have

Lemma 2.1: Let

$$G_j = I - \omega A_j^{\frac{1}{2}} E_j^{-1} A_j^{\frac{1}{2}} .$$

Case 1: The symmetric multigrid scheme: $MV(j,u^j,f^j)$

Let

(2.9a) $$Q_0 = 0 .$$

For $j = 1,2,\ldots,M$ we set

(2.9b) $$C_j = \{I - A_j^{\frac{1}{2}} I_{j-1}^{j} A_{j-1}^{-\frac{1}{2}} (I - Q_{j-1}) A_{j-1}^{-\frac{1}{2}} I_j^{j-1} A_j^{\frac{1}{2}}\}$$

(2.9c) $$Q_j = G_j^m C_j G_j^m .$$

Then

(2.10) $$\varepsilon_1^j = Q_j \varepsilon_0^j .$$

Case 2: The coarse-to-fine cycle: $M/_j(u^j,f^j)$

Let

(2.11a) $$Q/_0 = 0 .$$

For $j = 1,2,\ldots,M$ we set

(2.11b) $$C/_j = \{I - A_j^{\frac{1}{2}} I_{j-1}^{j} A_{j-1}^{-\frac{1}{2}} (I - Q/_{j-1}) A_{j-1}^{-\frac{1}{2}} I_j^{j-1} A_j\},$$

(2.11c) $$Q/_j = G_j^m (C/_j) .$$

Then

(2.12) $$\varepsilon_1^j = (Q/_j)\varepsilon_0^j$$

Case 3: The fine-to-coarse cycle: $M\backslash_j(u^j,f^j)$

Let

(2.13a) $$Q\backslash_0 = 0\ .$$

For $j = 1,2,\ldots,M$ we set

(2.13b) $$C\backslash_j = \{I-A_j^{\frac{1}{2}}I_{j-1}^{j}A_{j-1}^{\frac{1}{2}}(I-Q\backslash_{j-1})A_{j-1}^{-\frac{1}{2}}I_j^{j-1}A_j^{\frac{1}{2}}\}\ ,$$

(2.13c) $$Q\backslash_j = (C\backslash_j)G_j^m\ ,$$

then

(2.14) $$\varepsilon_1^j = (Q\backslash_j)\varepsilon_0^j\ .$$

Proof: Direct Computation.

A basic result of McCormick [6] is

Theorem 2.1: We have

(2.15) $$C\backslash_j = C^*/_j\ ,\quad Q\backslash_j = Q^*/_j$$

and

(2.16) $$Q_j = (Q/_j \circ Q\backslash_j)\ .$$

Hence,

(2.17a) $$\| Q\backslash_j \|_{\ell_2} = \| Q/_j \|_{\ell_2}$$

and

(2.17b) $$\| Q_j \|_{\ell_2} = \| Q/_j \|^2_{\ell_2} .$$

Note: Using the notation of [6] and section 3 we have

(2.18) $$\| \varepsilon^j \|^2_{\ell_2} = \| E^j \|^2_A .$$

Proof: Since

$$Q\backslash_0 = Q/_0 = 0$$

and (2.3c) holds, then

$$C/_1 = (C\backslash_1)^* .$$

Since $G^*_j = G_j$ we have

$$(Q/_1)^* = Q\backslash_1 .$$

A straightforward inductive argument then gives us (2.15). A direct computation yields (2.16).

3. Background Results

In this section we collect some results of McCormick [6], [7], Bank and C. C. Douglas [1] and Yserentant [8].

Let

(3.1) $$R_j := \mathit{Range}\ I_{j-1}^{j} ,$$

(3.2) $$N_j := \mathit{Nullspace}\ I_j^{j-1} A_j .$$

Let $\langle\, ,\rangle_A$ denote the "A inner product", i.e.,

(3.3) $$\langle u,v\rangle_A = \langle A_j u,v\rangle , \quad u,v \in S_j .$$

Then, using (2.3c) we see that

(3.4) $$S_j = R_j \oplus N_j$$

and R_j and N_j are A-orthogonal. Let

(3.5a) $\mathcal{T}_j :=$ A-orthogonal projection onto N_j .

(3.5b) $\mathcal{S}_j :=$ A-orthogonal projection onto R_j .

Let $Q\backslash_j^{(k)}$ denote $Q\backslash_j$ when G_j^m is replaced by G_j^k, i.e., "k" is the number of smoothing steps - not "m".

Let

(3.6) $$H_j^{(k)} = A_j^{-\frac{1}{2}} G_j^k A_j^{\frac{1}{2}} .$$

A basic result of McCormick [6], [7] is

Theorem 3.1: Let α_k, $0 < \alpha_k < 1$ be a fixed number which satisfies

(3.7) $$\alpha_k \| T_j u \|_A^2 + \| S_j u \|_A^2 \geq \| H_j^{(k)} u \|_A , \quad j = 1,2,\ldots,M .$$

Then

$$\| Q^{k}/_{j} \| \leq \alpha_k^{\frac{1}{2}} .$$

Thus the "coarse-to-fine" multigrid V-cycle $M/_j^{(k)}$ with k smoothing steps in each cycle satisfies

(3.8a) $$\| M/_j^{(k)} \|_A \leq \alpha_k^{\frac{1}{2}} ,$$

(3.8b) $$\| E_1^j \|_A \leq \alpha_k^{\frac{1}{2}} \| E_0^j \|_A .$$

Corollary: Let (3.7) hold. Then

(3.9a) $$\| M\backslash_j^{(k)} \|_A \leq \alpha_k^{\frac{1}{2}} ,$$

and

(3.9b) $$\| MV(j,\ldots) \|_A \leq \alpha_k .$$

Proof: Apply Theorem 2.1.

Another result of McCormick (see Theorem 3.4 of [7]) is

Theorem 3.2: Let α_1 be the smallest number satisfying (3.7) with "k" replaced by "1". Let $c = \frac{\alpha_1}{1-\alpha_1}$ so that

(3.10a) $$\alpha_1 = \frac{c}{c+1} .$$

Then (3.7) holds for $k \geq 1$ with α_k given by

(3.10b) $$\alpha_k = \frac{c}{c+k} .$$

In words, if $\alpha_1^{1/2}$ is an upper bound for $\| Q\backslash_j^{(1)} \|$, then $\alpha_k^{1/2}$ is an upper bound for $\| Q\backslash_j^{(k)} \|$.

Following Bank and C. C. Douglas [1] and Yserentant [8] we consider the generalized eigenvalue problem

(3.11) $$A_j U_j^{(k)} = \lambda_{k,j} E_j U_j^{(k)}$$

(we will sometimes dispense with the subscript "j"). We order the eigenvalues

(3.12a) $$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{\dim S_j}$$

and normalize the eigenvectors $U^{(k)}$ so that

(3.12b) $$\langle U^{(k)}, AU^{(s)} \rangle = \lambda_k \delta_{ks} ,$$

(3.12c) $$\langle U^k, EU^s \rangle = \delta_{ks} .$$

For $u \in S_j$ let

(3.13) $$\| u \|_E := \langle u, E_j u \rangle^{1/2} .$$

With this notation we see that: if $u \in S_j$ and

(3.14) $$u = \sum c_s U^s$$

then

(3.15a) $$Au = \sum c_s \lambda_s U^s$$

(3.15b) $$\| u \|_E^2 = \sum |c_s|^2$$

(3.15c) $$\| u \|_A^2 = \sum |c_s|^2 \lambda_s$$

(3.16a) $$H_j^\sigma u = \sum c_s (1-\omega\lambda_s)^\sigma U^s$$

(3.16b) $$\| H_j^\sigma u \|_A^2 = \sum |c_s|^2 |1-\omega\lambda_s|^{2\sigma} \lambda_s \ .$$

Note: In (3.16a) and (3.16b) the exponent σ may be any non-negative number.

The basic assumption of [7] and [8] is

Assumption A: There is a constant $c > 0$, independent of j, such that, for every $u \in S_j$

(3.17) $$\| T_j u \|_E^2 \leq c \| T_j u \|_A^2 , \quad j = 1,2,\ldots,M \ .$$

We close this section with a basic estimate due to Yserentant (see Lemma of [8]).

Theorem 3.3: Let Assumption A hold. Let

(3.18) $$0 < \omega \leq 1 \ .$$

Then

(3.19) $$\| T_j u \|_A^2 \leq \frac{c}{\omega} \{ \| u \|_A^2 - \| H_j^{\frac{1}{2}} u \|_A^2 \} \ .$$

Remark: Yserentant [8] proves this estimate within the finite element setting. However, a quick check of his proof shows that it applies in our setting as well.

4. Convergence Theorems

In this section we use the results of section 3 to obtain the following basic convergence theorem:

Theorem 4.1: Assume Assumption A holds with constant c. Let

(4.1)
$$c' = \begin{cases} [\frac{c}{\omega} - 1], & 0 < \omega \leq 1, \\ [\frac{c}{2-\omega} - 1], & 1 \leq \omega < 2. \end{cases}$$

Then

(4.2a)
$$\| Q^k/_j \|^2 \leq \frac{c'}{c' + 2k} .$$

That is

(4.2b)
$$\| M/_j^{(k)} \|_A^2 \leq \frac{c'}{c' + 2k} ,$$

(4.2c)
$$\| M\backslash_j^{(k)} \|_A^2 \leq \frac{c'}{c' + 2k} ,$$

(4.2d)
$$\| MV(j,\cdot,\cdot) \|_A \leq \frac{c'}{c' + 2m} .$$

As might be expected from the form of c', the proof is slightly different for the two cases, $0 < \omega \leq 1$, $1 < \omega < 2$.

Proof for the case $0 < \omega \leq 1$.

In this we merely rewrite (3.19). Since

$$\| u \|_A^2 = \| T_j u \|_A^2 + \| S_j u \|_A^2$$

the inequality (3.19) can be written as

$$\frac{c'}{c'+1}\| T_j u\|_A^2 + \| S_j u\|_A^2 \geq \| H_j^{\frac{1}{2}} u\|_A^2 .$$

Thus, we think of $H_j^{\frac{1}{2}}$ or $G_j^{\frac{1}{2}}$ as our basic smoother. Then, applying Theorem 3.1 and Theorem 3.2 we obtain (4.2a) and (4.2b). Applying Theorem 2.1 we obtain (4.2c) and (4.2d).

The proof in the remaining case follows from the same argument and the next result.

Lemma 4.1: Let $1 \leq \omega < 2$ and let

(4.3) $$\bar{c} = \frac{c}{2-\omega} .$$

Then

(4.4) $$\| T_j u\|_A^2 \leq \bar{c}\{\| u\|_A^2 - \| H_j^{\frac{1}{2}} u\|_A^2\} .$$

Proof: Let $\bar{H}$ denote $H_j^{\frac{1}{2}}$ with $\omega = 1$. Then

(4.5a) $$\| u\|_A^2 - \| \bar{H}u\|_A^2 = \sum |c_s|^2 \lambda_s^2$$

(4.5b) $$\| u\|_A^2 - \| H_j^{\frac{1}{2}} u\|_A^2 = \sum |c_s|^2 \lambda_s (1 - |1-\omega\lambda_s|) .$$

Let $\bar{s}$ be the value of s so that

(4.6a) $$|1-\omega\lambda_s| = 1 - \omega\lambda_s , \quad 1 \leq s \leq \bar{s}$$

(4.6b) $$|1-\omega\lambda_s| = \omega\lambda_s - 1 , \quad \bar{s} < s .$$

Then

$$\| u \|_A^2 - \| H_j^{\frac{1}{2}} u \|_A^2 = \sum_1 + \sum_2$$

where

(4.7a) $$\sum_1 = \sum_{s \le \bar{s}} |c_s|^2 \lambda_s (1 - |1 - \omega\lambda_s|) = \omega \sum_{s \le \bar{s}} |c_s|^2 \lambda_s^2 \ge \sum |c_s|^2 \lambda_s^2$$

and

(4.7b) $$\sum_2 = \sum_{\bar{s} < s} |c_s|^2 \lambda_s (1 - |1 - \omega\lambda_s|) = \sum_{\bar{s} < s} |c_s|^2 \lambda_s (2 - \omega\lambda_s) .$$

Since

$$\lambda_s \le \frac{2 - \omega\lambda_s}{2 - \omega}$$

we have

(4.8) $$\frac{1}{2-\omega} \sum_2 \ge \sum_{\bar{s} < s} |c_s| \ \lambda_s^2 .$$

Hence

$$\| u \|_A^2 - \| \bar{H} u \|_A^2 \le \frac{1}{2-\omega} \{ \| u \|_A^2 - \| H_j^{\frac{1}{2}} u \|_A^2 \}$$

and the lemma follows from Theorem 3.3 with $\omega = 1$.

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