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LOCAL STRUCTURE OF FEASIBLE SETS IN NONLINEAR PROGRAMMING, PART III: STABILITY AND SENSITIVITY.

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ABSTRACT

This paper continues the local analysis of nonlinear programming problems begun in Parts I and II. In this part we exploit the tools developed in the earlier parts to obtain detailed information about local optimizers in the nondegenerate case. We show, for example, that these optimizers obey a weak type of differentiability and we compute their derivatives in this weak sense.

Key words: Nonlinear programming, stability, sensitivity analysis, nondegeneracy.

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1. <u>Introduction</u>. This paper continues the local analysis of nonlinear programming problems begun in [7] and [8]. There, we introduced a fundamental algebraic decomposition of the space around a feasible point of the basic problem

$$\begin{aligned} \text{minimize}_x \quad & f(x,p) \\ \text{subject to} \quad & h(x,p) = 0 \qquad\qquad (1.1)(p) \\ & x \in C, \end{aligned}$$

where f and h are C^r functions $(r \geq 1)$ from $\Omega \times \Pi$ to $\mathbb{R}$ and $\mathbb{R}^m$ respectively, Ω and Π are open subsets of $\mathbb{R}^n$ and of a real Banach space P respectively, and C is a convex subset of $\mathbb{R}^n$. In [7] C was not assumed to be closed; in [8] it was assumed closed and (for most of the paper) polyhedral, and stronger results were thereby obtained. In this paper we assume throughout that C is polyhedral. The parameter p is used to study the behavior of the programming problem and its solutions under perturbations of the functions appearing in the problem.

In [8] we studied the idea of nondegeneracy, defined as follows: Suppose $p_0 \in \Pi$, and let x_0 be a feasible point for (1.1) (p_0). Denote the tangent cone to C at a point $x \in \mathbb{R}^n$ by $T_C(x)$, and the normal cone by $N_C(x)$. The feasible point x_0 is said to be <u>nondegenerate</u> if

$$h_x(x_0, p_0)\,[\text{lin } T_C(x_0)] = \mathbb{R}^m, \qquad (1.2)$$

where $\text{lin } T_C(x_0)$ is the <u>lineality space</u> of the cone $T_C(x_0)$ (the largest subspace contained in it), and where h_x denotes the partial Fréchet derivative of h with respect to the x-variables. The property of nondegeneracy is stronger than that of <u>regularity</u>, studied in [7]: x_0 is said to be <u>regular</u> if

$$h_x(x_0,p_0)[T_C(x_0)] = \mathbb{R}^m. \tag{1.3}$$

In [7], we used regularity, together with the decomposition mentioned earlier, to derive optimality conditions and to examine the structure of the feasible set $\mathcal{F}(p_0)$ near x_0; here $\mathcal{F}(p)$ is defined to be $\{x \in C \mid h(x,p) = 0\} = C \cap h(\cdot,p)^{-1}(0)$. In [8], we showed that under the stronger hypothesis of nondegeneracy, considerably more could be done. Since we shall use the results of [8] in what follows, we summarize them here.

Given a point $x_0 \in \mathcal{F}(p_0)$, denote $h_x(x_0,p_0)$ by D. Let M be the subspace of $\mathbb{R}^n$ parallel to aff C, the affine hull of C. Let $K := M \cap \ker D$, and let L and J be subspaces complementary to K in M and in $\ker D$ respectively. The regularity hypothesis (1.3) implies that $\mathbb{R}^n = J \oplus K \oplus L$. The stronger nondegeneracy hypothesis (1.2) implies that L can be chosen to lie in $\operatorname{lin} T_C(x_0)$, and we shall assume that this has been done. Let P_J, P_K, and P_L be the projectors onto J, K, and L along, in each case, the other two spaces, and let $P_0 = P_J + P_K$, the projector onto $\ker D$ along L.

In [8, Th. 2.2] we showed that there were open neighborhoods U_* of the origin in $\mathbb{R}^n$, V_* of p_0 in P, and W_* of x_0, such that for each $p \in V_*$ the function

$$\theta_p := P_0[(\cdot) - x_0] \mid W_* \cap \mathcal{F}(p)$$

was a C^r diffeomorphism of $W_* \cap \mathcal{F}(p)$ onto $U_* \cap \Lambda$, where $\Lambda := (C - x_0) \cap K$. We also exhibited the inverse ψ_p of θ_p. This diffeomorphism property was a key result of [8] since it implied that we could replace (1.1)(p), for p near p_0 and x near x_0, by the problem

$$\text{minimize}_y \; \{\tilde{\phi}(y,p) \mid y \in \Lambda\}, \qquad (1.4)(p)$$

where $\tilde{\phi}(y,p) := f[\theta_p^{-1}(y), p]$. In replacing (1.1)(p) by (1.4)(p) we have changed a problem whose feasible set is defined by nonlinear, parametrically dependent functions into one whose feasible set is a fixed, polyhedral convex set.

Note that the definition of $\tilde{\phi}$ just given makes sense only for arguments $y \in U_* \cap \Lambda$, since ψ_p is only defined there. This will be slightly inconvenient, so we shall extend $\tilde{\phi}$ to a function ϕ defined for all small $y \in \mathbb{R}^n$ and all p near p_0 in P. To do so, we recall from [7] and [8] the construction of θ_p and its inverse. We first define uniquely a particular generalized inverse D^- of D by the requirements

$$D D^- = I, \quad D^- D = P_L . \qquad (1.5)$$

Next, we observe that the equation

$$0 = D^- h[x(y,p),p] + (I - D^- D)[x(y,p) - (x_0 + y)] \qquad (1.6)$$

defines, for y near 0 and p near p_0, a C^r function $x(y,p)$. When $x(\cdot,p)$ is restricted to $U_* \cap \Lambda$ it becomes a diffeomorphism of $U_* \cap \Lambda$ onto $U_* \cap F(p)$ whose inverse is θ_p. Details of this construction are in [7] and [8]. To obtain our desired function ϕ, we need only take the composition $f(\cdot,p) \circ x(\cdot,p) \circ P_0$.

It will be convenient for later use to record some of the first and (if $r \geq 2$) second derivatives of $x(y,p)$ with respect to y and/or p.

Standard calculus applied to (1.6) yields, for any $r, s \in \mathbb{R}^n$ and $q \in P$,

$$x_y(y,p)(s) = A(y,p)^{-1} P_0 s$$

and

$$x_p(y,p)(q) = -A(y,p)^{-1} D^- h_p[x(y,p), p]q,$$

where

$$A(y,p) := P_0 + D^- h_x[x(y,p),p].$$

Note that since $P_0 = I - D^- D$, we have $A(0,p_0) = I$. We then obtain

$$x_{yy}(y,p)(r)(s) = -A(y,p)^{-1} D^- h_{xx}[x(y,p),p][x_y(y,p)r][x_y(y,p)s], \tag{1.7}$$

and

$$\begin{aligned} x_{yp}(y,p)(r)(q) = A(y,p)^{-1} D^- \{&h_{xx}[x(y,p),p][x_p(y,p)q][x_y(y,p)r] \\ &-h_{xp}[x(y,p),p][x_y(y,p)r][q]\}. \end{aligned} \tag{1.8}$$

These formulas become considerably simpler when evaluated at $(y,p) = (0,p_0)$.

The <u>reduced gradient</u> g_0 for (1.1)(p_0) at x_0 is the derivative of $\psi(\cdot,p_0)$ at 0: that is,

$$g_0 := \phi_y(0,p_0) = P_0^* f_x(x_0,p_0). \tag{1.9}$$

The first-order optimality criterion for (1.1)(p_0) is the inclusion $g_0 \in -N_C(x_0)$ [8, Prop. 3.1]. If we use (1.9) to write this as

$$f_x(x_0,p_0) - D^*[(D^-)^* f_x(x_0,p_0)] \in -N_C(x_0),$$

and if we write $\lambda(x_0,p_0) := (D^-)^* f_x(x_0,p_0)$ then we have

$$g_0 = f_x(x_0,p_0) - D^* \lambda(x_0,p_0) \in -N_C(x_0). \tag{1.10}$$

Note that the multipliers $\lambda(x_0,p_0)$ are reversed in sign from those in [8]. The purpose of the present sign convention is to facilitate the following simple geometric interpretation of g_0: if we recall that $(\ker D) \oplus L = \mathbb{R}^n$, we can see that also

$$(\ker D)^{\perp} \oplus L^{\perp} = \mathbb{R}^n. \tag{1.11}$$

The projectors on $(\ker D)^{\perp}$ and $L^{\perp}$, along the other subspace in each case, are P_L^* and P_0^* respectively, so since $P_L^{\perp} + P_0^* = I$ we have

$$P_L^* f_x(x_0,p_0) = f_x(x_0,p_0) - D^*\lambda(x_0,p_0) = g_0.$$

In other words, g_0 is the component of $f_x(x_0,p_0)$ in $L^{\perp}$ under the decomposition (1.11). If we take the rows of D as a basis for $(\ker D)^{\perp}$, then the multipliers $\lambda(x_0,p_0)$ are simply the coordinates, in that basis, of the complementary component of $f_x(x_0,p_0)$ in $(\ker D)^{\perp}$. When we remove this component from $f_x(x_0,p_0)$, we are left with the reduced gradient g_0.

The optimality condition (1.10) implies that, for each $c \in C$, $\langle g_0, c-c_0 \rangle \geqq 0$. The set C_0 consisting of those $c \in C_0$ for which this inequality holds as an equation ($\langle g_0, c-c_0 \rangle = 0$) is a face of C. In fact, this face has special relevance to local optimization. If we define $\Lambda_0 := (C_0 - x_0) \cap K$, then we showed in [8, Th. 3.4] that for y near 0 and p near p_0, the local minimizers of the problem

$$\text{minimize}_y \ \{\phi(y,p) \mid y \in \Lambda\} \tag{1.12)(p}$$

are exactly the same as those of the more tightly constrained problem

$$\text{minimize}_y \ \{\phi(y,p) \mid y \in \Lambda_0\}. \tag{1.13)(p}$$

In fact, we can simplify (1.12)(p) and (1.13)(p) even more if we recall that C and hence C_0 are polyhedral. Therefore, near x_0 the sets $C - x_0$ and $C_0 - x_0$ coincide with their tangent cones $T_C(x_0)$ and $T_{C_0}(x_0)$, which we shall denote by T and T_0 respectively. It follows that Λ and Λ_0 coincide, near x_0, with $T \cap K$ and $T_0 \cap K$ respectively. The cone $T \cap K$ is actually $T_{\mathcal{F}(p_0)}(x_0)$, as shown in [7, Th. 3.1].

It then follows from our earlier diffeomorphism result that if we choose U_*, V_* and W_* to be small enough, $W_* \cap \mathcal{F}(p)$ will be diffeomorphic to $U_* \cap T \cap K$ via θ_p, and $W_* \cap \mathcal{F}(p) \cap C_0$ will be differmorphic to $U_* \cap T_0 \cap K$. Thus, we may replace (1.12)(p) and (1.13)(p) by

$$\min \quad \{\phi(y,p) \mid y \in T \cap K\} \qquad (1.14)(p)$$

and

$$\min \quad \{\phi(y,p) \mid y \in T_0 \cap K\} \qquad (1.15)(p)$$

respectively.

By this diffeomorphism technique we have replaced the problem of studying local minimizers of (1.1)(p) by that of studying local minimizers of the C^r function $\phi(\cdot,p)$ on the polyhedral convex cone $T_0 \cap K$ (which does not depend on p). We call $T_0 \cap K$ the <u>critical cone</u> for (1.1)(p_0) at x_0. We shall see that this transformation enables us to use simple geometric properties of the critical cone to gain optimality and sensitivity information about (1.1)(p) in an easy and natural way. In the process we shall recover several criteria that have previously been proved for particular cases of (1.1)(p).

With the reduced gradient g_0 defined in (1.9) we can associate the cone $(g_0) := \{\lambda g_0 | \lambda \geqq 0\}$. Using this notation it is easy to show that $T_0 = T \cap (g_0)^\circ$. As the inner product $\langle g_0, \cdot \rangle$ is non-negative on T because $(-g_0 \in N_C(x_0) = T^\circ)$, this also shows that T_0 is the face of T defined by $T_0 = \{t \in T | \langle g_0, t \rangle = 0\}$. We shall use these observations in Section 2.

The remainder of this paper is organized as follows: in Section 2, we study the critical cone $T_0 \cap K$. We show how to determine when this cone is actually a subspace, and how to compute its affine hull even when it is not a subspace. We also relate these results to particular criteria that have appeared in the literature for special cases of (1.1)(p_0). In Section 3, we show that strong convexity of $\phi(\cdot, p_0)$ on aff $(T_0 \cap K)$ ensures existence, local uniqueness, and Lipschitz continuity of a minimizer of (1.1)(p) for p near p_0. We show that this criterion generalizes earlier work of Kojima [2] and the author [4], and we provide a convenient test to determine when this strong convexity holds. Finally, we show that the minimizer will exhibit a weak kind of differentiability, which we call Bouligand differentiability. This concept is explained, and some of its properties are derived, in the Appendix. The final result of Section 3 shows how to compute the Bouligand derivative of the minimizer.

2. Properties of the critical cone. In this section we study various aspects of the critical cone $T_0 \cap K$ identified in Section 1. We show that the problem (1.1)(p), for p near p_0 and x near x_0, behaves essentially like an unconstrained minimization problem when $T_0 \cap K$ is a subspace, and we observe that in some familiar special cases of (1.1)(p) this will occur precisely when certain well-known conditions (strict complementary slackness or dual nondegeneracy) hold. Then we examine the more general situation when $T_0 \cap K$ is not a subspace. We show that in a particular case frequently seen in the literature, the critical cone $T_0 \cap K$ is the linear image of a certain cone occurring in the second-order optimality conditions. Finally, for this case we show how to compute the affine hull of $T_0 \cap K$, since that subspace will play an important part in the results of Section 3.

We have already observed that for all p near p_0 and for all y near 0, if y is to be a local minimizer of (1.12)(p), then y must lie in $T_0 \cap K$, and that therefore we can restrict our attention to the problem of minimizing ϕ on $T_0 \cap K$. If $T_0 \cap K$ is a subspace, let k be its dimension and let Q be an injective linear transformation from $\mathbb{R}^k$ onto $T_0 \cap K$. The problem of minimizing $\phi(y,p)$ in y on $T_0 \cap K$ is then evidently equivalent to that of minimizing $\phi(\cdot,p) \circ Q$ on $\mathbb{R}^k$, so that in this case our nonlinear programming problem has been reduced to a simple unconstrained minimization. It is therefore of interest to be able to determine whether in a particular problem $T_0 \cap K$ is in fact a subspace, and we show in Proposition 2.2 how to do this. In order to prove that proposition, we need the following lemma.

LEMMA 2.1: *Let* W *be a closed convex cone in* $\mathbb{R}^n$ *and let* $w \in \mathbb{R}^n$. *Then* $(w)^\circ \cap W$ *is a subspace if and only if* $-w \in \mathrm{ri}\, W^\circ$. *In the latter case, we actually have* $(w)^\circ \cap W = \mathrm{lin}\, W$.

Proof (only if): Suppose $(w)^\circ \cap W$ is a subspace. If $-w \notin \mathrm{ri}\, W^\circ$, then by the proper separation theorem [9, Th. 11.3] there is some $v \in \mathbb{R}^n$ with $\langle v,-w \rangle \geqq 0$ and $\langle v,y \rangle \leqq 0$ for each $y \in W^\circ$ (so that $v \in W^{\circ\circ} = W$). Further, either $\langle v,-w \rangle > 0$ or $\langle v,y_0 \rangle < 0$ for some $y_0 \in W^\circ$.

As $v \in W$ with $\langle v,w \rangle \leqq 0$, it follows that $v \in (w)^\circ \cap W$. But the latter set is a subspace by hypothesis, so $-v \in (W^\circ) \cap W$. Thus $\langle -v,w \rangle \leqq 0$ so in fact $\langle v,-w \rangle = 0$, which implies $\langle v,y_0 \rangle < 0$ for some $y_0 \in W^\circ$. This contradicts the fact that $-v \in W$ and hence $\langle -v,y_0 \rangle \leqq 0$. Thus $-w \in \mathrm{ri}\, W^\circ$.

(if): Designate by B_0 the intersection of the unit ball B with aff W°. Since $-w$ ri W, there is some $\varepsilon > 0$ with $-w + \varepsilon B_0 \subset W^\circ$. Thus

$$\varepsilon B_0 = w + (-w + \varepsilon B_0) \subset (w) + W^\circ$$

The right-hand side is a cone, so we actually have aff $W^\circ \subset (w) + W^\circ$. However, $(w) \subset -W^\circ$ by hypothesis, so $(w) + W^\circ \subset$ aff W°, so in fact $(w) + W^\circ =$ aff W°. It follows that

$$(w)^\circ \cap W = [(w)+W^\circ]^\circ = (\mathrm{aff}\, W^\circ)^\circ = \mathrm{lin}\, W,$$

as required. This completes the proof.

Using Lemma 2.1 we can now develop a convenient criterion for $T_0 \cap K$ to be a subspace. In keeping with our convention for $T_C(x_0)$ we write N for $N_C(x_0)$.

PROPOSITION 2.2: $T_0 \cap K$ *is a subspace if and only if* $-g_0 \in \mathrm{ri}\, N$, *and in that case one has* $T_0 \cap K = \mathrm{lin}\,(T \cap K)$.

PROOF: We apply Lemma 2.1 with $w = g_0$ and $W = T \cap K$. As $T_0 = T \cap (g_0)^\circ$, we see that the statement that $T_0 \cap K$ is a subspace is equivalent to saying $(w)^\circ \cap W$ is a subspace. By Lemma 1, this is equivalent to $-w \in \text{ri } W^\circ$: that is, to $-g_0 \in \text{ri } (T \cap K)^\circ = (\text{ri } N) + K^\perp$. Lemma 2.1 also tells us that then $T_0 \cap K = \text{lin } (T \cap K)$. Hence the proof of the proposition amounts to proving that $-g_0 \in (\text{ri } N) + K^\perp$ if and only if $-g_0 \in \text{ri } N$. The "if" part is obvious. For the "only if" part, suppose $-g_0 = r + v$, $r \in \text{ri } N$, $v \in K^\perp$. We know $L \subset \text{lin } T$, so $\text{aff } N \subset L^\perp$: hence $r \in L^\perp$. As $g_0 = P_0^* f_x(x_0, p_0)$, $g_0 \in \text{im } P_0^* = L^\perp$. It follows that both $-g_0$ and r belong to $L^\perp$: hence $v \in K^\perp \cap L^\perp = (K+L)^\perp = M^\perp = \text{lin } N$. But $N = N + \text{lin } N$, so $\text{ri } N = \text{ri } N + \text{lin } N$. As $r \in \text{ri } N$ and $v \in \text{lin } N$, we have $-g_0 \in \text{ri } N$, which proves the proposition.

Proposition 2.2, and the discussion preceding it, makes precise the idea that a nonlinear optimization problem may be "locally essentially unconstrained," and it provides a test for determining just when this property holds. In the rest of this section we show that in two familiar special cases of (1.1)(p), this test reduces to properties already familiar in the literature.

Example I: Standard linear programming. Here we are concerned with the problem

$$\min \quad \{\langle c,x \rangle \mid Ax = b,\ x \geqq 0\},$$

so we can set $f(x) = \langle c,x \rangle$, $h(x) = Ax - b$, and $C = \mathbb{R}^n_+$. We suppress the perturbation parameter p since it plays no role in this example. If we suppose that x_0 is a basic feasible point corresponding to the basis B

and the partition [B N] of A, then we pointed out in [8] that nondegeneracy in the sense used here corresponds to the requirement that $x_B > 0$ (that is, to primal nondegeneracy in the usual linear programming sense). In this case we take $L = \text{lin}\, T_C(x_0) = \mathbb{R}^B \times \{0\}^N$, where the superscripts indicate that individual factors are to be taken to be $\mathbb{R}$ or $\{0\}$ according as the particular index is in B or N. We then have

$$D^- = \begin{bmatrix} B^{-1} \\ \\ 0 \end{bmatrix}, \quad P_L = \begin{bmatrix} I & B^{-1}N \\ \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad P_0 = \begin{bmatrix} 0 & -B^{-1}N \\ \\ 0 & I \end{bmatrix}.$$

The multipliers are $\lambda = (D^-)^* f_x(x_0) = [(B^{-1})^* 0]c = c_B B^{-1}$, and the reduced gradient is

$$g_0 = P_0^* f_x(x_0) = \begin{bmatrix} 0 & 0 \\ \\ -(B^{-1}N)^* & I \end{bmatrix} c = [0_B, c_N - c_B B^{-1} N],$$

where we have abused notation slightly in order to write the multipliers and the reduced gradient in familiar forms.

In this case we have $T = \mathbb{R}^B \times (\mathbb{R}_+)^N$, so $N = \{0\}^B \times (\mathbb{R}_-)^N$. Hence, $-g_0$ will belong to N whenever $c_N - c_B B^{-1} N \geqq 0$, the familiar linear programming optimality criterion. The problem will be "essentially unconstrained" near x_0 whenever $-g_0 \in \text{ri}\, N$: that is, when $c_N - c_B B^{-1} N > 0$. This is the criterion usually referred to in the linear programming literature as "dual nondegeneracy."

It is of interest here to compute $T_0 \cap K$ to see what kind of subspace we are dealing with. In this case $\text{aff}\, C = \mathbb{R}^n$, so J has dimension zero, and thus $K = J + K = \text{im}\, P_0$. Using the expressions for P_0 and T given above, we see that the cone $T \cap K$ is given by

$$T \cap K = \left\{ \begin{bmatrix} -B^{-1}Ns \\ s \end{bmatrix} \middle| \, s \geqq 0 \right\} .$$

The cone $T_0 \cap K$ consists of those elements v of $T \cap K$ for which $\langle g_0 , v \rangle \leqq 0$; that is, for which $\langle C_N - C_B B^{-1} N , s \rangle \leqq 0$. However, if $T_0 \cap K$ is to be a subspace then as already pointed out we have $C_N - C_B B^{-1} N > 0$, and as $s \geqq 0$ this implies $s = 0$. Hence in the case of standard linear programming, $T_0 \cap K$ is a subspace if and only if it is just the origin. This should not come as any particular surprise, since we know from linear programming theory that dual nondegeneracy implies a unique "corner solution," and the solution of an unconstrained minimization problem with a linear objective function will be unique if and only if the dimension of the space over which the minimization is done is actually zero.

Example II: Nonlinear programming with inequality and equality constraints. We consider next the nonlinear optimization problem

$$\begin{aligned} \text{minimize} \quad & d(z) \\ \text{subject to} \quad & c_E(z) = 0 \\ & c_A(z) \leqq 0 \\ & c_I(z) \leqq 0 , \end{aligned} \tag{2.1}$$

where $z \in \mathbb{R}^k$ and d, c_E, c_A, and c_I are C^r functions $(r \geqq 1)$ from an open set $\Omega \subset \mathbb{R}^k$ into $\mathbb{R}, \mathbb{R}^e, \mathbb{R}^a$, and $\mathbb{R}^i$ respectively. Let $z_0 \in \Omega$ and suppose that $c_E(z_0) = 0$, $c_A(z_0) = 0$ and $c_I(z_0) < 0$: thus c_A and c_I identify the inequality constraints that are respectively active and inactive at z_0. We shall write d_z for $d_z(z_0)$.

To convert (2.1) to the form (1.1)(p), we introduce slack variables s_A and s_I. We let $x := (z, s_A, s_I)$ and $f(x) := d(z)$,

$$h(x) := \begin{bmatrix} c_E(z) \\ c_A(z) + s_A \\ c_I(z) + s_I \end{bmatrix},$$

and $C := \mathbb{R}^k \times \mathbb{R}^a_+ \times \mathbb{R}^i_+$. Then an equivalent formulation of (2.1) is

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & h(x) = 0 \\ & x \in C. \end{array} \tag{2.2}$$

Note that we have suppressed the perturbation parameter in (2.1) and (2.2); in the analysis that we shall do here it would simply remain constant, so there is no point in writing it out.

Now suppose that $x_0 = (z_0, -c_A(z_0), -c_E(z_0))$ is a nondegenerate feasible point of (2.2). Write G_E for $c_E'(z_0)$, and define G_A and G_I similarly. We shall determine the various elements of the reduced problem at z_0 in terms of these matrices. In particular, we shall compute $T_0 \cap K$ and its affine hull.

As provided out in [8], nondegeneracy of x_0 means that the matrix $\begin{bmatrix} G_E \\ G_A \end{bmatrix}$ has full row rank. Thus, we can write

$$\begin{bmatrix} G_E \\ G_A \end{bmatrix} = [0 \ R] \, Q^T,$$

where Q is a $k \times k$ orthogonal matrix and R is a nonsingular, upper triangular $(e+a) \times (e+a)$ matrix. If we now write $Q = [Q_1 \; Q_2]$, where Q_1 is $k \times (k-e-a)$ and Q_2 is $k \times (e+a)$, then

$$\begin{bmatrix} G_E \\ \\ G_A \end{bmatrix} = RQ_2^T$$

Referring to the definition of C in (2.2), we can see that $T_C(x_0) = \mathbb{R}^k \times \mathbb{R}^a \times \mathbb{R}^i$, and thus $\operatorname{lin} T_C(x_0) = \mathbb{R}^k \times \{0\}^a \times \mathbb{R}^i$. Choose $L = (\operatorname{im} Q_2) \times \{0\}^a \times \mathbb{R}^i$, so that $L \subset \operatorname{lin} T_C(x_0)$. In this case $\operatorname{aff} C = \mathbb{R}^{k+a+i}$, so J is the zero subspace and $K = \ker D$, where

$$D = \begin{bmatrix} G_E & 0 & 0 \\ G_A & I & 0 \\ G_I & 0 & I \end{bmatrix}.$$

To show that the subspaces K and L satisfy $K \oplus L = \mathbb{R}^{k+a+i}$, suppose first that $y \in K \cap L$. Partition y as (y_1, y_2, y_3), with $y_1 \in \mathbb{R}^k$, $y_2 \in \mathbb{R}^a$, and $y_3 \in \mathbb{R}^i$. As $y \in \ker D$, we have

$$\begin{aligned} G_E y_1 & & &= 0 \\ G_A y_1 &+ y_2 & &= 0 \\ G_I y_1 & &+ y_3 &= 0, \end{aligned} \tag{2.3}$$

but as $y \in L$ we also have $y_1 \in \operatorname{im} Q_2$ (so that, say, $y_1 = Q_2 w$) and $y_2 = 0$. Using $y_2 = 0$ in (2.3) we find that

$$0 = \begin{bmatrix} G_E \\ \\ G_A \end{bmatrix} y_1 = (RQ_2^T)(Q_2 w) = Rw,$$

since $Q_2^T Q_2 = I$. As R is nonsingular we must have $w = 0$ and thus $y_1 = 0$. Using this in (2.3) we find that $y_3 = 0$. Hence K and L are independent.

As $\operatorname{im} Q_2$ has dimension $e+a$, we see that $\dim L = e+a+i$. But D has full row rank, so its kernel K has dimension $(k+a+i) - (e+a+i) = k-e$. As K and L are independent we have $\dim(K+L) = k+a+i$, and hence $\mathbb{R}^{k+a+i} = K \oplus L$.

To compute D^- we follow the procedure suggested in [7] , defining a $(k+a+i) \times (e+a+i)$ matrix E by

$$E = \begin{bmatrix} Q_2 & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix},$$

so that

$$DE = \begin{bmatrix} R & 0 \\ G_I Q_2 & I \end{bmatrix}, \quad (DE)^{-1} = \begin{bmatrix} R^{-1} & 0 \\ -G_I Q_2 R^{-1} & I \end{bmatrix}.$$

Thus

$$D^- = E(DE)^{-1} = \begin{bmatrix} Q_2 R^{-1} & 0 \\ 0 & 0 \\ -G_I Q_2 R^{-1} & I \end{bmatrix}.$$

It follows that the multipliers at x_0 are

$$f_x(x_0) D^- = [d_z Q_2 R^{-1}, 0] . \tag{2.4}$$

For future reference we partition R^{-1} into $[Z_E\ Z_A]$, where Z_E is $(e+a)\times e$ and Z_A is $(e+a)\times a$. With this partitioning we can display the multipliers for each type of constraint by rewriting (2.4) as

$$f_x(x_0)D^- = [d_Z Q_2 Z_E, d_Z Q_2 Z_A, 0] =: (\lambda_E, \lambda_A, \lambda_I) .$$

To find the projectors on L and on $\ker D$, we compute

$$P_L = D^- D = \begin{bmatrix} Q_2 Q_2^T & Q_2 Z_A & 0 \\ 0 & 0 & 0 \\ G_I Q_1 Q_1^T & -G_I Q_2 Z_A & I \end{bmatrix},$$

where we have used $Q_1 Q_1^T = I - Q_2 Q_2^T$ and $R^{-1} = [Z_E\ Z_A]$.

Then

$$P_0 = I - P_L = \begin{bmatrix} Q_1 Q_1^T & -Q_2 Z_A & 0 \\ 0 & I & 0 \\ -G_I Q_1 Q_1^T & G_I Q_2 Z_A & 0 \end{bmatrix},$$

and the reduced gradient at x_0 is

$$f_x(x_0)P_0 = [d_Z Q_1 Q_1^T, -d_Z Q_2 Z_A, 0] . \tag{2.5}$$

Note that in the second position of the reduced gradient we have the negatives of the multipliers for the active inequality constraints.

We can now compute the cones $T\cap K$ and $T_0\cap K$. For $T\cap K$, recall that $T_C(x_0) = \mathbb{R}^k \times \mathbb{R}_+^a \times \mathbb{R}^i$ and that (2.3) describes the vectors (y_1, y_2, y_3) in $\ker D$. Putting these together we see that

$$T\cap K = \{(y_1, -G_A y_1, -G_I y_1) \mid G_E y_1 = 0, G_A y_1 \leq 0\} . \tag{2.6}$$

Up to now we have assumed only that x_0 was a nondegenerate feasible point. To find $T_0 \cap K$ we assume that x_0 is a stationary point, so that the reduced gradient belongs to $-N_C(x_0)$. Recalling that $N_C(x_0) = \{0\}^k \times \mathbb{R}^a_- \times \{0\}^i$ and using (2.5), we see that stationarity implies

$$d_z Q_1 = 0 , \quad d_z Q_2 Z_A \leqq 0 . \tag{2.7}$$

The points y of $T_0 \cap K$ are those in $T \cap K$ that make a non-positive inner product with the reduced gradient. Using (2.5) and (2.6) we see that this implies $(-d_z Q_2 Z_A)(-G_A y_1) \leqq 0$. However, as $d_z Q_2 Z_A = \lambda_A \leqq 0$ by (2.7) and $G_A y_1 \leqq 0$ since $y \in T \cap K$, we see that λ_A and $G_A y_1$ must have complementary supports. Hence we have

$$T_0 \cap K = \begin{bmatrix} I \\ -G_A \\ -G_I \end{bmatrix} (U_0) , \tag{2.8}$$

where we have defined the polyhedral convex cone U_0 by

$$U_0 := \{y_1 \in \mathbb{R}^k \mid G_E y_1 = 0,\ G_A y_1 \leqq 0 \text{ and } (G_A y_1)j = 0 \text{ if } (\lambda_A)j \neq 0\} . \tag{2.9}$$

This cone U_0 is the familiar cone appearing in the second-order optimality conditions (see, e.g., [3, §10.3]).

From Proposition 2.2 we know that $T_0 \cap K$ will be a subspace if and only if the reduced gradient belongs to $-\text{ri}\ N_C(x_0)$: in this case, this means precisely that every component of λ_A is negative. This property is usually referred to as "strict complementary slackness" in the nonlinear programming literature, so we have shown that the problem (2.1) reduces locally to an essentially unconstrained minimization if at the stationary point x_0 one has (1) strict complementary slackness and (2) linear independence of the gradients of the active (binding) constraints.

As a final step in the analysis of (2.1) we compute the affine hull of $T_0 \cap K$, since we shall need this information in the next section. For this purpose, let us suppose that the rows of G_A have been ordered so that all of those corresponding to negative components of λ_A are placed before all corresponding to zero components of λ_A. Then we can partition G_A accordingly, as $\begin{bmatrix} G_- \\ G_0 \end{bmatrix}$, and we can partition the component matrix Z_A of R^{-1} as $[Z_- \; Z_0]$. Now recall that

$$I = RQ_2^T Q_2 R = \begin{bmatrix} G_E \\ G_- \\ G_0 \end{bmatrix} Q_2 \, [Z_E \; Z_- \; Z_0] \, .$$

It follows that $\begin{bmatrix} G_E \\ G_- \end{bmatrix} (Q_2 Z_0) = 0$ and $G_0(Q_2 Z_0) = I$. Referring to (2.8) we see that each column of $-Q_2 Z_0$ belongs to the cone U_0 defined earlier. However, these same columns form a basis for $\ker \begin{bmatrix} G_E \\ G_- \end{bmatrix}$, so $\ker \begin{bmatrix} G_E \\ G_- \end{bmatrix} \subset$ aff U_0. However, it follows from (2.9) that $U_0 \subset \ker \begin{bmatrix} G_E \\ G_- \end{bmatrix}$, so actually aff $U_0 = \ker \begin{bmatrix} G_E \\ G_- \end{bmatrix} = \text{im}\,(Q_2 Z_0)$. We then have

$$\text{aff } T_0 \cap K = \begin{bmatrix} I \\ -G_A \\ -G_I \end{bmatrix} \left(\ker \begin{bmatrix} G_E \\ G_- \end{bmatrix}\right) . \tag{2.10}$$

3. Stability and sensitivity analysis of an optimizer. In Section 2 we studied the critical cone $T_0 \cap K$ associated with a local minimizer x_0. The results of that section made no use of the perturbation parameter p, which remained fixed at p_0.

Here we allow p to vary near p_0, and we study the questions of existence, uniqueness, and stability of a local minimizer near x_0 when the problem is perturbed. We show that, under appropriate conditions, not only does a local minimizer exist near x_0 for each p near p_0, but when regarded as a function of p this optimizer is Lipschitzian. In fact, it is almost differentiable in the sense that it can be well approximated by a simpler function that can, in principle, be computed using information available at the solution. The "almost" results from the fact that this approximating function is not affine (unless $T_0 \cap K$ is a subspace): in general, the portion of its graph near p_0 is a cone rather than an affine set.

Recall that we have transformed the problem (1.1)(p) into the considerably simpler problem (1.15)(p), whose feasible set is the polyhedral convex cone $T_0 \cap K$. Of course, without making some assumptions about the functions involved in the problem we cannot expect to find any interesting results about stability. Thus, we consider next what kinds of assumptions we should make about (1.15)(p) in order to ensure that its minimizer is unique and well behaved for p near p_0. Then, of course, any information we gain about that minimizer can be immediately translated into corresponding information about a minimizer of (1.1)(p) near x_0.

We shall use the concept of strong convexity in connection with (1.15) (p). This idea is defined as follows:

DEFINITION 3.1: Let γ be a function from $\mathbb{R}^n$ to $(-\infty, +\infty]$. γ is strongly convex with modulus $\rho > 0$ if for each $x_0, x_1 \in \mathbb{R}^n$ and each $\lambda \in (0,1)$,

$$\gamma[(1-\lambda)x_0 + \lambda x_1] \leqq (1-\lambda)\gamma(x_0) + \lambda\gamma(x_1) - \tfrac{1}{2}\rho\lambda(1-\lambda)\|x_0 - x_1\|^2 .$$

We say γ is strongly convex on a subset U of $\mathbb{R}^n$ if the function equal to γ on U and to $+\infty$ off U is strongly convex.

The following lemma characterizes strong convexity when the function in question is restricted to an affine set and when enough differentiability is present.

LEMMA 3.2: Let γ be a C^2 function from a neighborhood N of x_0 in $\mathbb{R}^n$ to $\mathbb{R}$. Let A be an affine set containing x_0 and let S be the subspace parallel to A. Then γ is strongly convex on a neighborhood of x_0 in A, if and only if $\gamma''(x_0)$ is positive definite on the subspace S.

PROOF (only if): Suppose γ is strongly convex with modulus ρ on a neighborhood of x_0 in A. We can take this neighborhood to be $M \cap A$, where $M \subset N$ is a neighborhood of x_0 in $\mathbb{R}^n$. Let $s \in S$; we can assume with no loss of generality that s is small enough so that $x_0 + s$ and $x_0 - s$ belong to M. As $s \in S$, these points also lie in A, hence in $M \cap A$. Thus we have

$$\tfrac{1}{2}\rho\|s\|^2 = \tfrac{1}{2}\rho(\tfrac{1}{2})(\tfrac{1}{2})\|2s\|^2 \leqq \tfrac{1}{2}\gamma(x_0+s) + \tfrac{1}{2}\gamma(x_0-s) - \gamma(x_0) .$$

But

$$\gamma(x_0+s) = \gamma(x_0) + \gamma'(x_0)s + \tfrac{1}{2}\langle s,\gamma''(x_0)s\rangle + o(\|s\|^2) ,$$

$$\gamma(x_0-s) = \gamma(x_0) - \gamma'(x_0)s + \tfrac{1}{2}\langle s,\gamma''(x_0)s\rangle + o(\|s\|^2) ,$$

so

$$\tfrac{1}{2}\rho\|s\|^2 \leqq \tfrac{1}{2}\langle s,\gamma''(x_0)s\rangle + o(\|s\|^2) .$$

Since this inequality holds for all small $s \in S$, it follows that for all $s \in S$, $\langle s,\gamma''(x_0)s\rangle \geqq \rho\|s\|^2$.

(if): Suppose $\gamma''(x_0)$ is positive definite on S, and let $s \in S$. Since γ is C^2, there is an open convex neighborhood M of x_0 in $\mathbb{R}^n$ such that if $X \in M$ then $\gamma''(x)$ is positive definite on s with modulus ρ. Define an auxiliary function θ from the neighborhood $\{t|w+ts \in M\cap A\}$ of the origin in $\mathbb{R}$ to $\mathbb{R}$ by

$$\theta(t) : \gamma(w+ts) - \gamma'(w)(w+ts) .$$

Then we have

$$\gamma(w+ts) - \gamma(w) - \gamma'(w)ts = \theta(t) - \theta(0) = \int_0^t \theta'(v)dv$$

$$= \int_0^t [\gamma'(w+vs) - \gamma'(w)]\, s\, dv = \int_0^t \int_0^v \langle s,\gamma''(w+ys)s\rangle\, dy\, dv$$

$$\geqq \rho\|s\|^2 \int_0^t \int_0^v dy\, dv = \tfrac{1}{2}\rho t^2\|s\|^2 . \tag{3.1}$$

Now let x_1 and x_2 be any two points in $M\cap A$, and let $\lambda \in (0,1)$. Apply (3.1) with $s = x_1 - x_2$ and with $w = (1-\lambda)x_1 + \lambda x_2$, taking the indicated choices for t to obtain the inequalities shown:

$$t = \lambda : \gamma(x_1) - \gamma(w) - \lambda\gamma'(w)(x_1-x_2) \geqq \tfrac{1}{2}\rho\lambda^2\|x_1-x_2\|^2 ; \qquad (3.2)$$

$$t = \lambda-1 : \gamma(x_2) - \gamma(w) + (1-\lambda)\gamma'(w)(x_1-x_2) \geqq \tfrac{1}{2}\rho(1-\lambda)^2\|x_1-x_2\|^2 . \qquad (3.3)$$

Multiplying (3.2) by $(1-\lambda)$ and (3.2) by λ and adding, we obtain

$$(1-\lambda)\gamma(x_1) + \lambda\gamma(x_2) - \gamma[(1-\lambda)x_1+\lambda x_2] \geqq \tfrac{1}{2}\rho\lambda(1-\lambda)\|x_1-x_2\|^2 ,$$

which completes the proof.

Our approach to stability analysis of (1.15)(p) will be the following: we consider a local minimizer x_0, and we make the assumption that $\phi(\cdot,p_0)$ is strongly convex on aff $(T_0 \cap K)$. Since aff $(T_0 \cap K)$ is itself a subspace, we see from Lemma 3.2 that our assumption is equivalent to the assumption that $\phi_{yy}(x_0,p_0)$ is positive definite on aff $(T_0 \cap K)$. As we shall see later, this latter assumption is well known in the special case of nonlinear programming with inequality and equality constraints, being exactly the "strong second order sufficient condition" used by Kojima [2] and the author [4].

With this assumption, we consider the nonlinear generalized equation

$$0 \in \phi_y(y,p) + N_{T_0 \cap K}(y) , \qquad (3.4)(p)$$

which expresses the condition for a stationary point of (1.15)(p) to exist. We can use (3.4)(p) since our assumption will guarantee that not only $\phi(\cdot,p_0)$, but also $\phi(\cdot,p)$ for p near p_0, is strongly convex. Hence the stationary-point condition (3.4)(p) is equivalent to the minimization condition in (1.15)(p).

Applying to (3.4)(p) the analytical machinery devised in [4], we can prove that for each p near p_0 there is a unique minimizer $Y(p)$ of (1.15)(p) near 0, and that the function Y is Lipschitzian. Of course, this immediately implies the existence of a Lipschitzian function $X(p)$ yielding the unique local minimizer of (1.1)(p) near x_0.

Finally, we shall use some additional results about generalized equations to show that the functions Y and X are almost differentiable, in the sense that they can be well approximated near p_0 by relatively simple functions that are in principle computable. However, the graphs of these simple functions are not subspaces (as would be the case with derivatives) but rather cones.

To begin the detailed analysis we make the assumption, for the remainder of this section, that x_0 is a nondegenerate local minimizer of (1.1)(p_0) and that f and h are C^2 on a neighborhood of (x_0,p_0). Of course, this then implies that 0 is a local minimizer of (1.15)(p_0), and that ϕ is C^2 on a neighborhood of $(0,p_0)$. We also assume that $\phi(\cdot,p_0)$ is strongly convex on a neighborhood of 0 in aff $(T_0 \cap K)$. By Lemma 3.2, this is equivalent to assuming that $\phi_{yy}(0,p_0)$ is positive definite on aff $(T_0 \cap K)$. Since ϕ is C^2, there are neighborhoods U_0 of 0 in $\mathbb{R}^n$ (open and convex) and V_0 of p_0 in π, such that if $(y,p) \in U_0 \times V_0$, than $\phi_{yy}(y,p)$ is positive definite on aff $(T_0 \cap K)$ with, say, constant $\rho > 0$: i.e., for each $s \in$ aff $(T_0 \cap K)$, $\langle s, \phi_{yy}(y,p)s \rangle \geq \rho \|s\|^2$. Now Lemma 3.2 (in particular, the proof of the "if" direction) implies that for each $p \in V_0$, $\phi(\cdot,p)$ is strongly convex on U_0. This tells us that $\phi(\cdot,p)$ has a local minimizer $Y(p)$ on $U_0 \cap (T_0 \cap K)$ if and only if the stationary point condition (3.4)(p) holds for $y = Y(p)$; further, if such a minimizer exists it is the unique global minimizer of $\phi(\cdot,p)$ on $U_0 \cap (T_0 \cap K)$. Thus we have now to investigate the solvability of (3.4)(p).

Fortunately, quite a lot is known about inclusions like (3.4)(p); these <u>generalized equations</u> exhibit solvability and regularity properties analogous to those of conventional nonlinear equations. A survey of this area is given in [6], and many details and proofs are in [4].

From results in [4] and [6] we can see that the key to analyzing the behavior of (3.4)(p) is the linearization given by

$$0 \in \phi(0,p_0) + \phi_{yy}(0,p_0)y + N_{T_0 \cap K}(y) \,. \tag{3.5}$$

For example, [6, Th. 4.4] or [4, Th. 2.1, Cor. 2.2] show that if the inverse of the operator on the right in (3.5) is (locally) single-valued and Lipschitzian near 0, then for any p near p_0 (3.4)(p) will have a locally unique solution $Y(p)$ that is Lipschitzian in p. In fact, under our assumptions such a solution will even be globally unique.

To investigate the inverse of the operator in (3.5), note that y solves

$$w \in \phi_y(0,p_0) + \phi_{yy}(0,p_0)y + N_{T_0 \cap K}(y) \tag{3.6}$$

if and only if y solves the convex quadratic programming problem

$$\text{minimize} \quad \{\langle \phi_y(0,p_0)-w,y\rangle + \tfrac{1}{2}\langle y,\phi_{yy}(0,p_0)y\rangle \mid y \in T_0 \cap K\} \,. \tag{3.7}$$

Since $\phi_{yy}(0,p_0)$ is positive definite on $\operatorname{aff}(T_0 \cap K)$, we can see that for each $w \in \mathbb{R}^n$ (3.7) has a unique global optimizer $y(w)$, and such an optimizer is then also the unique solution of (3.6). Further, if w_1 and

w_2 are two points in $\mathbb{R}^n$, then

$$w_i - \phi_y(0,p_0) - \phi_{yy}(0,p_0)y(w_i) \in N_{T_0 \cap K}(y(w_i)), \; i = 1,2 .$$

The definition of normal cone then yields

$$\langle w_1 - \phi_y(0,p_0) - \phi_{yy}(0,p_0)y(w_1), \; y(w_1) - y(w_2) \rangle \geqq 0 ,$$

$$\langle w_2 - \phi_y(0,p_0) - \phi_{yy}(0,p_0)y(w_2), \; y(w_1) - y(w_2) \rangle \leqq 0 .$$

Subtracting the second inequality from the first, we find that

$$\langle w_1 - w_2, \; y(w_1) - y(w_2) \rangle \geqq \langle \phi_{yy}(0,p_0)[y(w_1)-y(w_2)], \; y(w_1) - (w_2) \rangle$$
$$\geqq \rho \, \| y(w_1) - y(w_2) \|^2 ,$$

so that we have

$$\| y(w_1) - y(w_2) \| \leqq \rho^{-1} \| w_1 - w_2 \| ,$$

and therefore the inverse operator we are considering is in fact Lipschitzian. (Another way to reach this conclusion would have been to show that the operator in (3.6) is strongly monotone and then to use general results about such operators. In this case the direct argument seemed simpler.)

We are now able to prove the following theorem about minimizers of (1.1)(p) and (1.15)(p):

THEOREM 3.3: *Suppose that* x_0 *is a nondegenerate local minimizer of* (1.1)(p_0), *that* f *and* h *are* C^2 *on a neighborhood of* (x_0,p_0), *and that* $\phi_{yy}(0,p_0)$ *is positive definite on* aff $(T_0 \cap K)$.

Then there exist neighborhoods U *of* 0 *in* $\mathbb{R}^n$, V *of* p_0 *in* P, *and* W *of* x_0 *in* $\mathbb{R}^n$, *and Lipschitzian functions* $Y : V \to U$ *and* $X : V \to W$, *such that for each* $p \in V$, $Y(p)$ *is the unique minimizer of* (1.15)(p) *in* U, *and* $X(p)$ *is the unique minimizer of* (1.1)(p) *in* W.

PROOF: Our analysis just prior to the statement of the theorem showed that the generalized equation (3.4)(p_0) was regular at the origin in the sense of [6] (i.e., its linearization has a Lipschitzian inverse). Applying [6, Th. 4.4], we see that there are neighborhoods U of 0 in $\mathbb{R}^n$ and V of p_0 in P (which we can take to be contained in $U_0 \cap U_*$ and $V_0 \cap V_*$ respectively), and a Lipschitzian function $Y : V \to U$ such that, for each $p \in V$, $Y(p)$ is the unique solution of (3.4)(p) in U. However, our previous remarks show that this is equivalent to saying that $Y(p)$ is the unique minimizer of (1.15)(p) on U. Finally, we set $X = \psi_p \circ Y$; since $Y(p)$ remains in $U_* \cap \Lambda$ we know from our earlier diffeomorphism results that $X(p)$ will then be the unique minimizer of (1.1)(p) on $W := \psi_p(U)$. As W is a neighborhood of x_0, we have completed the proof.

At this point two remarks are in order. First, the review in Section 1 showed that the multipliers λ were C^r functions of x: thus, as $X(p)$ varies with p the corresponding multipliers will be Lipschitzian in p, and possibly smoother: in fact they will share whatever smoothness properties X may have (up to C^r). Second, there is indeed a situation in which X will be at least C^2: it is precisely the case that we studied in Section 2, in which $T_0 \cap K$ is a subspace. We saw that this case would arise whenever $-g_0$ belonged to the relative interior of $N_C(x_0)$.

To see why X is in fact C^r in this case, we return to the generalized equation (3.4)(p). If $T_0 \cap K$ is a subspace, say of dimension k, we can write (3.4)(p) equivalently as

$$\begin{aligned} \phi_y(y,p) &\in (T_0 \cap K)^{\perp}, \\ y &\in T_0 \cap K. \end{aligned} \tag{3.8}$$

If we let Q be an injective linear transformation from $\mathbb{R}^k$ to $\mathbb{R}^n$ with $T_0 \cap K = \text{im } Q$, then we can reformulate (3.8) as

$$Q^T \phi_y(Qz,p) = 0 , \tag{3.9}$$

where we have replaced y by Qz. The expression (3.9) is a system of nonlinear equations, and we can analyze its solution using the implicit-function theorem. To do so, we examine its first derivative in z at $z = 0$ (hence $y = 0$), which is $Q^T \phi_{yy}(0,p_0)Q$. This is a positive definite (hence nonsingular) linear transformation from $\mathbb{R}^k$ to $\mathbb{R}^k$, since we have assumed that $\phi_{yy}(0,p_0)$ is positive definite on aff $(T_0 \cap K) = \text{im } Q$. The implicit-function theorem then tells us that there is a C^r solution $z(p)$ of (3.9) for p near p_0, which is unique in some neighborhood of the origin in $\mathbb{R}^k$. Putting $Y(p) = Qz(p)$, we obtain a C^r solution of the minimization problem (1.15)(p) and thence a C^r solution $X(p)$ of (1.1)(p). Derivatives of X and Y can then be computed, using Q and the derivatives of ϕ.

Returning to the general case, we investigate the positive definiteness condition on $\phi_{yy}(0,p_0)$ in the special case of nonlinear inequality and equality constraints. This case was dealt with in detail in Section 2. There, we found that

$$\text{aff } (T_0 \cap K) = \begin{bmatrix} I \\ -G_A \\ -G_I \end{bmatrix} \left(\ker \begin{bmatrix} G_E \\ G_- \end{bmatrix}\right) . \tag{2.10}$$

Since we are interested only in $\phi_{yy}(0,p_0)$, we shall suppress the perturbation parameter p in writing the problem functions. Using the notation of Section 2 we have

$$f_{xx}(x_0)rs = d_{zz}(z_0)r_z s_z \; ; \; h_{xx}(x_0)rs = \begin{bmatrix} (c_E)_{zz}(z_0)r_z s_z \\ (c_A)_{zz}(z_0)r_z s_z \\ (c_I)_{zz}(z_0)r_z s_z \end{bmatrix} \tag{3.10}$$

where r_z and s_z denote the portions of r and s corresponding to z (the first k components). Incidentally, the form of (3.10) shows that the introduction of slack variables to convert (2.1) to (2.2) made no essential difference in the derivatives of the problem functions; this point is of interest when considering the computational effectiveness of such a procedure.

Using the derivative formulas from Section 1, one finds

$$\phi_{yy}(0)rs = f_{xx}(x_0)(P_0 r)(P_0 s) - f_x(x_0)D^- h_{xx}(x_0)(P_0 r)(P_0 s) \; . \tag{3.11}$$

Recalling that $f_x(x_0)D^- = (\lambda_E,\lambda_A,\lambda_I)$, and using (3.10), one obtains from (3.11) an expression in the original problem functions from (2.1) , namely

$$\begin{aligned}\phi_{yy}(0)rs = d_{zz}(z_0)(P_0 r)_z(P_0 s)_z - \langle \lambda_E,(c_E)_{zz}(z_0)(P_0 r)_z(P_0 s)_z\rangle \\ - \langle \lambda_A,(c_A)_{zz}(z_0)(P_0 r)_z(P_0 s)_z\rangle \; ,\end{aligned} \tag{3.12}$$

since we know $\lambda_I = 0$.

Now if r and s lie in aff $(T_0 \cap K)$ they surely lie in $\ker h_x(x_0)$, so the multiplication by P_0 will be superfluous. Thus we can see from (3.12) that if $r, s \in$ aff $(T_0 \cap K)$ then

$$\phi_{yy}(0)rs = d_{zz}(z_0)r_z s_z - \langle \lambda_E,(c_E)_{zz}(z_0)r_z s_z\rangle - \langle \lambda_A,(c_A)_{zz}(z_0)r_z s_z\rangle . \tag{3.13}$$

The expression on the right of (3.13) is simply the quadratic form defined by the second derivative of the standard Lagrangian of (2.1) at z_0, evaluated at the pair (r_z, s_z). Referring to (2.10), we see that r and s belong to aff $(T_0 \cap K)$ if and only if r_z and s_z belong to

$$\ker \begin{bmatrix} G_E \\ G_- \end{bmatrix};$$ that is, to the subspace orthogonal to the gradients of (i) the equality constraints, and (ii) those inequality constraints having nonzero multipliers. Hence, our requirement that $\phi_{yy}(0)$ be positive definite on aff $(T_0 \cap K)$ reduces, in this special case, to the requirement that the second derivative of the standard Lagrangian be positive definite on the subspace just described. This is, as we noted above, precisely the "strong second-order sufficient condition" used by Kojima [2] and the author [4] in their analyses of this particular case.

Although we have shown that the functions X and Y are Lipschitzian, it is possible to gain some additional information about them by showing that they are in fact differentiable in a certain sense, weaker than that of Fréchet differentiability. We call this weak form of differentiability Bouligand differentiability (B-differentiability). It is defined, and some of its properties are derived, in the Appendix. In the remainder of this section we shall use these results without further comment, and we shall also assume that the space P is finite-dimensional: say, $\mathbb{R}^k$.

We are going to prove that the functions X and Y of Theorem 3.3 are B-differentiable, and to exhibit their B-derivatives. In order to do this, we shall require a lemma in which we use the idea of a polyhedral function or multifunction (multivalued function). This simply means a function or multifunction whose graph is the union of finitely many polyhedral convex sets. Such multifunctions are treated in some detail in [5].

LEMMA 3.4: *Let* y *be a single-valued polyhedral function from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$ *with* $y(0) = 0$. *Then there is a neighborhood* N *of the origin such that for any* $z \in N$ *and any* $\lambda \in [0,1]$, $y(\lambda z) = \lambda y(z)$.

PROOF: By [5, Lemma 1] there is some convex neighborhood N of the origin such that if any component of graph y intersects $N \times \mathbb{R}^m$, then that component actually contains $(0,0)$. Now let $z \in N$ and $\lambda \in [0,1]$. The pair $(z,y(z))$ belongs to some component of graph y, and as $z \in N$ that component also contains $(0,0)$. Thus it must also contain $(\lambda z, \lambda y(z)) = (1-\lambda)(0,0) + \lambda(z,y(z))$. Hence $(\lambda z, \lambda y(z))$ belongs to graph y, and by single-valuedness we then have $y(\lambda z) = \lambda y(z)$, which proves Lemma 3.4.

The main result about B-differentiability of X and Y is the following.

THEOREM 3.5: *Assume the hypotheses of Theorem 3.3, and assume further that* $P = \mathbb{R}^k$. *Then the functions* X *and* Y *of Theorem 3.3 are B-differentiable at* p_0, *and one has for small* $q \in \mathbb{R}^k$

$$DY(p_0)(q) = y \circ [-\phi_{yp}(0,p_0)](q) , \tag{3.14}$$

and

$$DX(p_0)(q) = DY(p_0)(q) - D^- h_p(x_0,p_0)(q) , \tag{3.15}$$

where $y(w)$ *is the single-valued Lipschitzian function defined by (3.6) or (3.7).*

PROOF: We first obtain the expression for $DY(p_0)$. Recall that

$$\phi_y(0,p_0+q) = \phi_y(0,p_0) + \phi_{yp}(0,p_0)(q) + o(q) .$$

We also know from [6, Th. 4.5] or [4, Th. 2.3] that since $Y(p)$ solves (3.4)(p),

$$Y(p_0+q) = y[\phi_y(0,p_0) - \phi_y(0,p_0+q)] + o(q) ,$$

where y is defined by (3.6) or (3.7). As y is shown (in the discussion following (3.7)) to be Lipschitzian, we have

$$\begin{aligned} Y(p_0+q) &= y[-\phi_{yp}(0,p_0)(q) + o(q)] \\ &= y\circ[-\phi_{yp}(0,p_0)](q) + o(q) . \end{aligned}$$

Lemma 3.4 shows that there is a function v defined from $\mathbb{R}^n$ to $\mathbb{R}^n$, whose graph is a cone, and a neighborhood N of the origin such that $v(w) = y(w)$ whenever $w \in N$. It follows that for small q, $v\circ[-\phi_{yp}(0,p_0)](q) = y\circ[-\phi_{yp}(0,p_0)](q)$. Thus, for such q we have (since $y(p_0) = 0$)

$$Y(p_0+q) = Y(p_0) + v\circ[-\phi_{yp}(0,p_0)](q) + o(q) .$$

Since the graphs of v and $-\phi_{yp}(0,p_0)$ are cones, so is that of their composition. Applying Theorem A.2 to the Lipschitzian function Y, we conclude that Y is B-differentiable at p_0, and that

$$DY(p_0) = v\circ[-\phi_{yp}(0,p_0)] .$$

However, for small q, $-\phi_{yp}(0,p_0)(q)$ belongs to the region near 0 where v agrees with y; hence we have (3.14).

To establish (3.15) we note that for p near p_0 $X(p) = x[y(p),p]$, where the function x is as defined in Section 1. Hence, using the chain rule for B-derivatives (Corollary A.4), we find that X is B-differentiable at p_0 with

$$\begin{aligned} DX(p_0)(q) &= x_y(0,p_0)\circ DY(p_0)(q) + x_p(0,p_0)(q) \\ &= DY(p_0)(q) - D^{-}h_p(x_0,p_0)(q) , \end{aligned}$$

which is (3.15). This completes the proof.

We note in closing that Aubin [1] derived very general results about solvability and sensitivity of convex optimization problems. He used contingent derivatives, which are multivalued generalizations of the Bonligand derivatives employed here. We have chosen to use a direct approach in analyzing (1.15)(p), rather than to attempt to apply Aubin's results, because with the direct approach we can apply special information that we have about (1.15)(p) (e.g., strong convexity) to prove sharper results than would be true for general convex optimization.

APPENDIX: BOULIGAND DERIVATIVES

This appendix presents results about Bouligand derivatives that are needed in the last part of Section 3. Only those results that are needed here will be presented.

In [1], Aubin defined contingent derivatives and analyzed some of their properties. These contingent derivatives are obtained by considering the contingent cone (originally introduced by Bouligand) to the graph by a multivalued function, at a point in that graph, to be the graph of a certain operator. This operator is the contingent derivative of the multivalued function at that point of its graph. Of course, in general the contingent derivative will itself be multivalued.

We shall be concerned here with a special case of the contingent derivative, which we call the Bouligand derivative. This special case arises when the function involved is single-valued and Lipschitzian on a neighborhood of the point in question and the contingent derivative at that point is also single-valued. In this situation the Bouligand derivative has some strong properties not shared by contingent derivatives in general.

Throughout this appendix we assume that f is a function from an open set $\Omega \subset \mathbb{R}^m$ to $\mathbb{R}^k$, which is Lipschitzian on Ω with modulus λ. If $x_0 \in \Omega$ we consider the contingent derivative $Df(x_0)$, defined by letting graph $Df(x_0)$ be the contingent cone to graph f at $(x_0, f(x_0))$: that is, the cone K defined by

$(v,w) \in K$ if and only if there exist $\rho_n > 0$ and (v_n, w_n) with $w_n = f(x_0+v_n) - f(x_0)$, $v_n \to 0$, and $\rho_n(v_n, w_n) \to (v,w)$.

We observe that $Df(x_0)\,v$ is nonempty for any $v \in \mathbb{R}^m$. Indeed, if we consider the set $\{\tau^{-1}[f(x_0+\tau v) - f(x_0)] \mid \tau > 0\}$ we see that for small τ, no element of this set has norm greater than $\lambda\|v\|$ because f is Lipschitzian. Hence there is a sequence $\tau_n \downarrow 0$ with $\tau_n^{-1}[f(x_0+\tau_n v) - f(x_0)]$ converging to some w. Taking $v_n = \tau_n v$ and $\rho_n = \tau_n^{-1}$ in the above definition, we see that $w \in Df(x_0)v$.

Of course, in general $Df(x_0)$ contains more than one point. Our next definition deals with the special case of single-valued $Df(x_0)$.

DEFINITION A.1: If $Df(x_0)$ is single-valued (i.e., a function) we call it the <u>Bouligand derivative</u> (<u>B-derivative</u>) of f at x_0.

One of the most useful properties of the Fréchet derivative is that of approximation. The following theorem shows that the B-derivative retains the approximation property of the Fréchet derivative, that it inherits the Lipschitz modulus of f, and that it provides the best approximation to f near x_0 among all functions whose graphs are cones.

THEOREM A.2: <u>Let</u> f <u>be Lipschitzian from the open set</u> $\Omega \subset \mathbb{R}^m$ to $\mathbb{R}^k$, <u>with modulus</u> λ, <u>and let</u> $x_0 \in \Omega$.

<u>a</u>. <u>If</u> f <u>is B-differentiable at</u> x_0, <u>then</u> $Df(x_0)$ <u>is Lipschitzian on</u> $\mathbb{R}^m$ <u>with modulus</u> λ, <u>and one has</u> $f(x) = f(x_0) + Df(x_0)(x-x_0) + o(x-x_0)$.

<u>b</u>. <u>If</u> d <u>is any function on</u> $\mathbb{R}^m$ <u>such that</u> graph d <u>is a cone and</u> $f(x) = f(x_0) + d(x-x_0) + o(x-x_0)$, <u>then</u> f <u>is B-differentiable at</u> x_0 <u>and</u> $Df(x_0) = d$.

PROOF: There is no loss of generality in assuming for the proof that $x_0 = 0$ and $f(x_0) = 0$.

Suppose f is B-differentiable at x_0. We show first that if $v \in \mathbb{R}^m$, then

$$Df(0)v = \lim_{\rho \downarrow 0} \rho^{-1} f(\rho v) . \qquad \text{(A.1)}$$

Note that for small ρ, the quantity $\rho^{-1}f(\rho v)$ is bounded in norm by $\lambda\|v\|$. Therefore it has one or more cluster points; we shall show that there is only one, namely $Df(0)v$, and this will establish (A.1).

Suppose that for some sequence $\{\rho_i\}$ converging to 0, we have $\rho_i^{-1}f(\rho_i v) \to y$, where y is some element of $\mathbb{R}^k$. Then $(\rho_i v, f(\rho_i v))$ belongs to graph f for each i, and $\rho_i^{-1}(\rho_i v, f(\rho_i v)) \to (v,y)$. It follows from B-differentiability that $y = Df(0)v$, and this proves (A.1).

To show that $Df(0)$ is Lipschitzian, choose any points x_1 and x_2 in $\mathbb{R}^m$, and let $\varepsilon > 0$. Choose ρ to be a positive number so small that ρx_1 and ρx_2 belong to Ω and (by (A.1))

$$\|\rho^{-1} f(\rho x_i) - Df(0)x_i\| < \tfrac{1}{2}\varepsilon \quad (i = 1,2) .$$

Then we have

$$\begin{aligned}\|Df(0)x_1 - Df(0)x_2\| &\leqq \|Df(0)x_1 - \rho^{-1}f(\rho x_1)\| \\ &+ \rho^{-1}\|f(\rho x_1) - f(\rho x_2)\| + \|\rho^{-1}f(\rho x_2) - Df(0)x_2\| \\ &< \lambda\|x_1 - x_2\| + \varepsilon .\end{aligned}$$

As ε was arbitrary, we see that $Df(0)$ is Lipschitzian on $\mathbb{R}^m$ with modulus λ.

To show that $Df(0)$ has the approximation property we want, it suffices to prove that for any sequence $\{x_n\}$ converging to 0, with $x_n \neq 0$ for each n, there is a subsequence, say $\{x_k\}$ for k belonging to some index set I, such that

$$\|x_k\|^{-1}\|f(x_k) - Df(0)x_k\| \to 0 .$$

Given $\{x_n\}$ we select I so that $v_k/\|x_k\|$ converges to some $v \in \mathbb{R}^m$. Choose $\varepsilon > 0$ and let K be so large that if $k \geqq K$ then (by (A.1))

$$\| \|x_k\|^{-1} f(\|x_k\| v) - Df(0)v\| < \tfrac{1}{2}\varepsilon$$

and

$$\|v - \|x_k\|^{-1} x_k\| < (4\lambda)^{-1}\varepsilon .$$

Write ρ_k for $\|x_k\|$; then for $k \geqq K$ we have

$$\rho_k^{-1}\|f(x_k) - Df(0)x_k\| \leqq \rho_k^{-1}\|f(x_k) - f(\rho_k v)\|$$

$$+ \|\rho_k^{-1} f(\rho_k v) - Df(0)v\| + \|Df(0)v - Df(0)(\rho_k^{-1}x_k)\|$$

$$\leqq \lambda\rho_k^{-1}\|x_k - \rho_k v\| + \tfrac{1}{2}\varepsilon + \lambda\|v - \rho_k^{-1}x_k\| < \varepsilon ,$$

where the bounds on the first and third terms come from the Lipschitzian properties of f and $Df(0)$ respectively. It follows that $\|x_k\|^{-1}\|f(x_k) - Df(0)x_k\| \to 0$, and this completes the proof of (a).

To prove (b), let d be a function with the properties described in (b). Let $(v,w) \in$ graph d. For small $\rho > 0$, $\rho v \in \Omega$, and therefore by (b)

$$f(\rho v) = d(\rho v) + o(\rho) .$$

However, $d(\rho v) = \rho d(v) = \rho w$. Thus

$$(v,w) = \rho^{-1}(\rho v,\ f(\rho v)) + o(\rho) ,$$

and it follows that (v,w) belongs to the contingent cone of graph f at $(0,0)$. Hence graph $d \subset$ graph $Df(0)$ (the contingent derivative).

Now suppose $(v,w) \in$ graph $Df(0)$. By definition, there are sequences $\{\rho_n\} \subset \mathbb{R}$ and $\{x_n\} \subset \Omega$ with $\rho_n \downarrow 0$, $x_n \to 0$, and such that

$$(v,w) = \lim_{n\to\infty} \rho_n^{-1}(x_n , f(x_n)) .$$

Hence

$$x_n = \rho_n v + o(\rho_n) \tag{A.2}$$

and

$$f(x_n) = \rho_n w + o(\rho n) \ . \tag{A.3}$$

As f is Lipschitzian, we derive from (A.2) and (A.3) the fact that

$$f(\rho_n v) = \rho_n w + o(\rho_n) \ . \tag{A.4}$$

However, by hypothesis

$$f(\rho_n v) = d(\rho_n v) + o(\rho_n) \ . \tag{A.5}$$

From (A.4), (A.5), and the fact that $d(\rho_n v) = \rho_n d(v)$ (because graph d is a cone), we obtain

$$w = d(v) + \rho_n^{-1} o(\rho_n) \ .$$

and letting $n \to \infty$ we find that $w = d(v)$, so that graph $Df(0) \subset$ graph d. But d is single-valued, and therefore since we have shown that $d = Df(0)$, we conclude that f is B-differentiable at 0 with B-derivative equal to d. This proves Theorem A.2.

This theorem can be quite useful in identifying B-derivatives and in establishing their properties. The next two corollaries illustrate some properties that can easily be proved using it.

COROLLARY A.3: Suppose f and g are Lipschitzian functions from an open set $\Omega \subset \mathbb{R}^m$ to $\mathbb{R}^k$. Let $x_0 \in \Omega$, and suppose f and g are B-differentiable at x_0. Then

a. If $\alpha \in \mathbb{R}$ then αf is B-differentiable at x_0 and $D(\alpha f)(x_0) = \alpha Df(x_0)$.

b. $f + g$ *is B-differentiable at* x_0 *with* $D(f+g)(x_0) = Df(x_0) + Dg(x_0)$.

PROOF: For (a), we just note that

$$f(x) = f(x_0) + Df(x_0)(x-x_0) + o(x-x_0) , \qquad (A.6)$$

and hence

$$(\alpha f)(x) = (\alpha f)(x_0) + \alpha Df(x_0)(x-x_0) + o(x-x_0) .$$

As the graph of $\alpha Df(x_0)$ is a cone, Part (b) of Theorem A.2 tells us that αf is B-differentiable at x_0 and $D(\alpha f)(x_0) = \alpha Df(x_0)$.

For (b), the proof is similar to that of (a), except that we write expressions like (A.6) for each of f and g . As $Df(x_0)$ and $Dg(x_0)$ are single-valued, $Df(x_0) + Dg(x_0)$ is a single-valued function on $\mathbb{R}^m$ whose graph is a cone. Again, Part (b) of Theorem A.2 gives us the result.

COROLLARY A.4: *Let* f *be a Lipschitzian function from an open set* $\Phi \subset \mathbb{R}^m$ *to* $\mathbb{R}^k$. Let $x_0 \in \Omega$, *and suppose* g *is a Lipschitzian function from an open set* $\Gamma \subset \mathbb{R}^k$, *with* $f(x_0) \in \Gamma$, *to* $\mathbb{R}^1$. *If* f *is B-differentiable at* x_0 *and* g *is B-differentiable at* $f(x_0)$, *then* $g \circ f$ *is B-differentiable at* x_0 *and*

$$D(g \circ f)(x_0) = Dg[f(x_0)] \circ Df(x_0) .$$

PROOF: First note that if (u,w) belongs to graph $Dg[f(x_0)] \circ Df(x_0)$, then with $v = Df(x_0)u$ we have $(u,v) \in$ graph $Df(x_0)$ and $(v,w) \in$ graph $Dg[f(x_0)]$. If $\alpha > 0$ then $\alpha v = Df(x_0)(\alpha u)$ and $\alpha w = Dg[f(x_0)](\alpha v)$. Hence $\alpha(u,w) \in$ graph $Dg[f(x_0)] \circ Df(x_0)$, so that this graph is a cone.

Now note that

$$f(x) = f(x_0) + Df(x_0)(x-x_0) + o(x-x_0)$$

so that

$$\begin{aligned}(g\circ f)(x) &= g[f(x_0)+Df(x_0)(x-x_0)+o(x-x_0)] \\ &= g[f(x_0)+Df(x_0)(x-x_0)]+o(x-x_0) \\ &= (g\circ f)(x_0)+Dg[f(x_0)](Df(x_0)(x-x_0))+o(x-x_0) \\ &= (g\circ f)(x_0)+[Dg[f(x_0)]\circ Df(x_0)](x-x_0)+o(x-x_0)\,, \qquad \text{(A.7)}\end{aligned}$$

where we have used the fact that g and $Df(x_0)$ are both Lipschitzian, as well as the approximation information given by Part (a) of Theorem A.2. Now we apply Part (b) of Theorem A.2 to (A.7) to prove the corollary.

Finally, we note that evidently any Fréchet derivative is also a B-derivative, since its graph is a cone and it has the approximation property treated in Theorem A.2.

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LOCAL STRUCTURE OF FEASIBLE SETS IN NONLINEAR PROGRAMMING, PART III: STABILITY AND SENSITIVITY

by

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LOCAL STRUCTURE OF FEASIBLE SETS IN NONLINEAR PROGRAMMING, PART III: STABILITY AND SENSITIVITY.

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ABSTRACT

This paper continues the local analysis of nonlinear programming problems begun in Parts I and II. In this part we exploit the tools developed in the earlier parts to obtain detailed information about local optimizers in the nondegenerate case. We show, for example, that these optimizers obey a weak type of differentiability and we compute their derivatives in this weak sense.

Key words: Nonlinear programming, stability, sensitivity analysis, nondegeneracy.

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1. Introduction. This paper continues the local analysis of nonlinear programming problems begun in [7] and [8]. There, we introduced a fundamental algebraic decomposition of the space around a feasible point of the basic problem

$$\begin{aligned} &\text{minimize}_x \quad f(x,p) \\ &\text{subject to} \quad h(x,p) = 0 \\ &\qquad\qquad\qquad x \in C, \end{aligned} \tag{1.1)(p}$$

where f and h are C^r functions $(r \geq 1)$ from $\Omega \times \Pi$ to $\mathbb{R}$ and $\mathbb{R}^m$ respectively, Ω and Π are open subsets of $\mathbb{R}^n$ and of a real Banach space P respectively, and C is a convex subset of $\mathbb{R}^n$. In [7] C was not assumed to be closed; in [8] it was assumed closed and (for most of the paper) polyhedral, and stronger results were thereby obtained. In this paper we assume throughout that C is polyhedral. The parameter p is used to study the behavior of the programming problem and its solutions under perturbations of the functions appearing in the problem.

In [8] we studied the idea of nondegeneracy, defined as follows: Suppose $p_0 \in \Pi$, and let x_0 be a feasible point for (1.1) (p_0). Denote the tangent cone to C at a point $x \in \mathbb{R}^n$ by $T_C(x)$, and the normal cone by $N_C(x)$. The feasible point x_0 is said to be <u>nondegenerate</u> if

$$h_x(x_0, p_0)\,[\text{lin}\ T_C(x_0)] = \mathbb{R}^m, \tag{1.2}$$

where $\text{lin}\ T_C(x_0)$ is the <u>lineality space</u> of the cone $T_C(x_0)$ (the largest subspace contained in it), and where h_x denotes the partial Fréchet derivative of h with respect to the x-variables. The property of nondegeneracy is stronger than that of <u>regularity</u>, studied in [7]: x_0 is said to be <u>regular</u> if

$$h_x(x_0,p_0)[T_C(x_0)] = \mathbb{R}^m. \tag{1.3}$$

In [7], we used regularity, together with the decomposition mentioned earlier, to derive optimality conditions and to examine the structure of the feasible set $\mathcal{F}(p_0)$ near x_0; here $\mathcal{F}(p)$ is defined to be $\{x \in C \mid h(x,p) = 0\} = C \cap h(\cdot,p)^{-1}(0)$. In [8], we showed that under the stronger hypothesis of nondegeneracy, considerably more could be done. Since we shall use the results of [8] in what follows, we summarize them here.

Given a point $x_0 \in \mathcal{F}(p_0)$, denote $h_x(x_0,p_0)$ by D. Let M be the subspace of $\mathbb{R}^n$ parallel to aff C, the affine hull of C. Let $K := M \cap \ker D$, and let L and J be subspaces complementary to K in M and in $\ker D$ respectively. The regularity hypothesis (1.3) implies that $\mathbb{R}^n = J \oplus K \oplus L$. The stronger nondegeneracy hypothesis (1.2) implies that L can be chosen to lie in $\operatorname{lin} T_C(x_0)$, and we shall assume that this has been done. Let P_J, P_K, and P_L be the projectors onto J, K, and L along, in each case, the other two spaces, and let $P_0 = P_J + P_K$, the projector onto $\ker D$ along L.

In [8, Th. 2.2] we showed that there were open neighborhoods U_* of the origin in $\mathbb{R}^n$, V_* of p_0 in P, and W_* of x_0, such that for each $p \in V_*$ the function

$$\theta_p := P_0[(\cdot) - x_0] \mid W_* \cap \mathcal{F}(p)$$

was a C^r diffeomorphism of $W_* \cap \mathcal{F}(p)$ onto $U_* \cap \Lambda$, where $\Lambda := (C - x_0) \cap K$. We also exhibited the inverse ψ_p of θ_p. This diffeomorphism property was a key result of [8] since it implied that we could replace (1.1)(p), for p near p_0 and x near x_0, by the problem

$$\text{minimize}_y \ \{\tilde{\phi}(y,p) \mid y \in \Lambda\}, \tag{1.4)(p}$$

where $\tilde{\phi}(y,p) := f[\theta_p^{-1}(y), p]$. In replacing (1.1)(p) by (1.4)(p) we have changed a problem whose feasible set is defined by nonlinear, parametrically dependent functions into one whose feasible set is a fixed, polyhedral convex set.

Note that the definition of $\tilde{\phi}$ just given makes sense only for arguments $y \in U_* \cap \Lambda$, since ψ_p is only defined there. This will be slightly inconvenient, so we shall extend $\tilde{\phi}$ to a function ϕ defined for all small $y \in \mathbb{R}^n$ and all p near p_0 in P. To do so, we recall from [7] and [8] the construction of θ_p and its inverse. We first define uniquely a particular generalized inverse D^- of D by the requirements

$$D D^- = I, \quad D^- D = P_L . \tag{1.5}$$

Next, we observe that the equation

$$0 = D^- h[x(y,p),p] + (I - D^- D)[x(y,p) - (x_0 + y)] \tag{1.6}$$

defines, for y near 0 and p near p_0, a C^r function $x(y,p)$. When $x(\cdot,p)$ is restricted to $U_* \cap \Lambda$ it becomes a diffeomorphism of $U_* \cap \Lambda$ onto $U_* \cap \mathcal{F}(p)$ whose inverse is θ_p. Details of this construction are in [7] and [8]. To obtain our desired function ϕ, we need only take the composition $f(\cdot,p) \circ x(\cdot,p) \circ P_0$.

It will be convenient for later use to record some of the first and (if $r \geqq 2$) second derivatives of $x(y,p)$ with respect to y and/or p.

Standard calculus applied to (1.6) yields, for any $r, s \in \mathbb{R}^n$ and $q \in P$,

$$x_y(y,p)(s) = A(y,p)^{-1} P_0 s$$

and

$$x_p(y,p)(q) = -A(y,p)^{-1} D^- h_p[x(y,p), p]q,$$

where

$$A(y,p) := P_0 + D^- h_x[x(y,p), p].$$

Note that since $P_0 = I - D^- D$, we have $A(0,p_0) = I$. We then obtain

$$x_{yy}(y,p)(r)(s) = -A(y,p)^{-1} D^- h_{xx}[x(y,p),p][x_y(y,p)r][x_y(y,p)s], \quad (1.7)$$

and

$$x_{yp}(y,p)(r)(q) = A(y,p)^{-1} D^- \{h_{xx}[x(y,p),p][x_p(y,p)q][x_y(y,p)r] - h_{xp}[x(y,p),p][x_y(y,p)r][q]\}. \quad (1.8)$$

These formulas become considerably simpler when evaluated at $(y,p) = (0,p_0)$.

The <u>reduced gradient</u> g_0 for (1.1)(p_0) at x_0 is the derivative of $\psi(\cdot,p_0)$ at 0: that is,

$$g_0 := \phi_y(0,p_0) = P_0^* f_x(x_0,p_0). \quad (1.9)$$

The first-order optimality criterion for (1.1)(p_0) is the inclusion $g_0 \in -N_C(x_0)$ [8, Prop. 3.1]. If we use (1.9) to write this as

$$f_x(x_0,p_0) - D^*[(D^-)^* f_x(x_0,p_0)] \in -N_C(x_0),$$

and if we write $\lambda(x_0,p_0) := (D^-)^* f_x(x_0,p_0)$ then we have

$$g_0 = f_x(x_0,p_0) - D^* \lambda(x_0,p_0) \in -N_C(x_0). \quad (1.10)$$

Note that the multipliers $\lambda(x_0,p_0)$ are reversed in sign from those in [8]. The purpose of the present sign convention is to facilitate the following simple geometric interpretation of g_0: if we recall that $(\ker D) \oplus L = \mathbb{R}^n$, we can see that also

$$(\ker D)^{\perp} \oplus L^{\perp} = \mathbb{R}^n. \tag{1.11}$$

The projectors on $(\ker D)^{\perp}$ and $L^{\perp}$, along the other subspace in each case, are P_L^* and P_0^* respectively, so since $P_L^{\perp} + P_0^* = I$ we have

$$P_L^* f_x(x_0,p_0) = f_x(x_0,p_0) - D^*\lambda(x_0,p_0) = g_0.$$

In other words, g_0 is the component of $f_x(x_0,p_0)$ in $L^{\perp}$ under the decomposition (1.11). If we take the rows of D as a basis for $(\ker D)^{\perp}$, then the multipliers $\lambda(x_0,p_0)$ are simply the coordinates, in that basis, of the complementary component of $f_x(x_0,p_0)$ in $(\ker D)^{\perp}$. When we remove this component from $f_x(x_0,p_0)$, we are left with the reduced gradient g_0.

The optimality condition (1.10) implies that, for each $c \in C$, $\langle g_0, c-c_0 \rangle \geqq 0$. The set C_0 consisting of those $c \in C_0$ for which this inequality holds as an equation $(\langle g_0, c-c_0 \rangle = 0)$ is a face of C. In fact, this face has special relevance to local optimization. If we define $\Lambda_0 := (C_0 - x_0) \cap K$, then we showed in [8, Th. 3.4] that for y near 0 and p near p_0, the local minimizers of the problem

$$\text{minimize}_y \ \{\phi(y,p) \mid y \in \Lambda\} \tag{1.12)(p}$$

are exactly the same as those of the more tightly constrained problem

$$\text{minimize}_y \ \{\phi(y,p) \mid y \in \Lambda_0\}. \tag{1.13)(p}$$

In fact, we can simplify (1.12)(p) and (1.13)(p) even more if we recall that C and hence C_0 are polyhedral. Therefore, near x_0 the sets $C - x_0$ and $C_0 - x_0$ coincide with their tangent cones $T_C(x_0)$ and $T_{C_0}(x_0)$, which we shall denote by T and T_0 respectively. It follows that Λ and Λ_0 coincide, near x_0, with $T \cap K$ and $T_0 \cap K$ respectively. The cone $T \cap K$ is actually $T_{\mathcal{F}(p_0)}(x_0)$, as shown in [7, Th. 3.1].

It then follows from our earlier diffeomorphism result that if we choose U_*, V_* and W_* to be small enough, $W_* \cap \mathcal{F}(p)$ will be diffeomorphic to $U_* \cap T \cap K$ via θ_p, and $W_* \cap \mathcal{F}(p) \cap C_0$ will be differmorphic to $U_* \cap T_0 \cap K$. Thus, we may replace (1.12)(p) and (1.13)(p) by

$$\min \quad \{\phi(y,p) \mid y \in T \cap K\} \qquad (1.14)(p)$$

and

$$\min \quad \{\phi(y,p) \mid y \in T_0 \cap K\} \qquad (1.15)(p)$$

respectively.

By this diffeomorphism technique we have replaced the problem of studying local minimizers of (1.1)(p) by that of studying local minimizers of the C^r function $\phi(\cdot,p)$ on the polyhedral convex cone $T_0 \cap K$ (which does not depend on p). We call $T_0 \cap K$ the <u>critical cone</u> for $(1.1)(p_0)$ at x_0. We shall see that this transformation enables us to use simple geometric properties of the critical cone to gain optimality and sensitivity information about (1.1)(p) in an easy and natural way. In the process we shall recover several criteria that have previously been proved for particular cases of (1.1)(p).

With the reduced gradient g_0 defined in (1.9) we can associate the cone $(g_0) := \{\lambda g_0 | \lambda \geq 0\}$. Using this notation it is easy to show that $T_0 = T \cap (g_0)^\circ$. As the inner product $\langle g_0, \cdot \rangle$ is non-negative on T because $(-g_0 \in N_C(x_0) = T^\circ)$, this also shows that T_0 is the face of T defined by $T_0 = \{t \in T | \langle g_0, t \rangle = 0\}$. We shall use these observations in Section 2.

The remainder of this paper is organized as follows: in Section 2, we study the critical cone $T_0 \cap K$. We show how to determine when this cone is actually a subspace, and how to compute its affine hull even when it is not a subspace. We also relate these results to particular criteria that have appeared in the literature for special cases of (1.1)(p_0). In Section 3, we show that strong convexity of $\phi(\cdot, p_0)$ on aff $(T_0 \cap K)$ ensures existence, local uniqueness, and Lipschitz continuity of a minimizer of (1.1)(p) for p near p_0. We show that this criterion generalizes earlier work of Kojima [2] and the author [4], and we provide a convenient test to determine when this strong convexity holds. Finally, we show that the minimizer will exhibit a weak kind of differentiability, which we call Bouligand differentiability. This concept is explained, and some of its properties are derived, in the Appendix. The final result of Section 3 shows how to compute the Bouligand derivative of the minimizer.

2. Properties of the critical cone. In this section we study various aspects of the critical cone $T_0 \cap K$ identified in Section 1. We show that the problem (1.1)(p), for p near p_0 and x near x_0, behaves essentially like an unconstrained minimization problem when $T_0 \cap K$ is a subspace, and we observe that in some familiar special cases of (1.1)(p) this will occur precisely when certain well-known conditions (strict complementary slackness or dual nondegeneracy) hold. Then we examine the more general situation when $T_0 \cap K$ is not a subspace. We show that in a particular case frequently seen in the literature, the critical cone $T_0 \cap K$ is the linear image of a certain cone occurring in the second-order optimality conditions. Finally, for this case we show how to compute the affine hull of $T_0 \cap K$, since that subspace will play an important part in the results of Section 3.

We have already observed that for all p near p_0 and for all y near 0, if y is to be a local minimizer of (1.12)(p), then y must lie in $T_0 \cap K$, and that therefore we can restrict our attention to the problem of minimizing ϕ on $T_0 \cap K$. If $T_0 \cap K$ is a subspace, let k be its dimension and let Q be an injective linear transformation from $\mathbb{R}^k$ onto $T_0 \cap K$. The problem of minimizing $\phi(y,p)$ in y on $T_0 \cap K$ is then evidently equivalent to that of minimizing $\phi(\cdot,p) \circ Q$ on $\mathbb{R}^k$, so that in this case our nonlinear programming problem has been reduced to a simple unconstrained minimization. It is therefore of interest to be able to determine whether in a particular problem $T_0 \cap K$ is in fact a subspace, and we show in Proposition 2.2 how to do this. In order to prove that proposition, we need the following lemma.

LEMMA 2.1: Let W be a closed convex cone in $\mathbb{R}^n$ and let $w \in \mathbb{R}^n$. Then $(w)^\circ \cap W$ is a subspace if and only if $-w \in \operatorname{ri} W^\circ$. In the latter case, we actually have $(w)^\circ \cap W = \operatorname{lin} W$.

Proof (only if): Suppose $(w)^\circ \cap W$ is a subspace. If $-w \notin \operatorname{ri} W^\circ$, then by the proper separation theorem [9, Th. 11.3] there is some $v \in \mathbb{R}^n$ with $\langle v,-w \rangle \geqq 0$ and $\langle v,y \rangle \leqq 0$ for each $y \in W^\circ$ (so that $v \in W^{\circ\circ} = W$). Further, either $\langle v,-w \rangle > 0$ or $\langle v,y_0 \rangle < 0$ for some $y_0 \in W^\circ$.

As $v \in W$ with $\langle v,w \rangle \leqq 0$, it follows that $v \in (w)^\circ \cap W$. But the latter set is a subspace by hypothesis, so $-v \in (W^\circ) \cap W$. Thus $\langle -v,w \rangle \leqq 0$ so in fact $\langle v,-w \rangle = 0$, which implies $\langle v,y_0 \rangle < 0$ for some $y_0 \in W^\circ$. This contradicts the fact that $-v \in W$ and hence $\langle -v,y_0 \rangle \leqq 0$. Thus $-w \in \operatorname{ri} W^\circ$.

(if): Designate by B_0 the intersection of the unit ball B with $\operatorname{aff} W^\circ$. Since $-w$ ri W, there is some $\varepsilon > 0$ with $-w + \varepsilon B_0 \subset W^\circ$. Thus

$$\varepsilon B_0 = w + (-w + \varepsilon B_0) \subset (w) + W^\circ$$

The right-hand side is a cone, so we actually have $\operatorname{aff} W^\circ \subset (w) + W^\circ$. However, $(w) \subset -W^\circ$ by hypothesis, so $(w) + W^\circ \subset \operatorname{aff} W^\circ$, so in fact $(w) + W^\circ = \operatorname{aff} W^\circ$. It follows that

$$(w)^\circ \cap W = [(w)+W^\circ]^\circ = (\operatorname{aff} W^\circ)^\circ = \operatorname{lin} W,$$

as required. This completes the proof.

Using Lemma 2.1 we can now develop a convenient criterion for $T_0 \cap K$ to be a subspace. In keeping with our convention for $T_C(x_0)$ we write N for $N_C(x_0)$.

PROPOSITION 2.2: $T_0 \cap K$ is a subspace if and only if $-g_0 \in \operatorname{ri} N$, and in that case one has $T_0 \cap K = \operatorname{lin}(T \cap K)$.

PROOF: We apply Lemma 2.1 with $w = g_0$ and $W = T \cap K$. As $T_0 = T \cap (g_0)^\circ$, we see that the statement that $T_0 \cap K$ is a subspace is equivalent to saying $(w)^\circ \cap W$ is a subspace. By Lemma 1, this is equivalent to $-w \in \text{ri } W^\circ$: that is, to $-g_0 \in \text{ri } (T \cap K)^\circ = (\text{ri } N) + K^\perp$. Lemma 2.1 also tells us that then $T_0 \cap K = \text{lin } (T \cap K)$. Hence the proof of the proposition amounts to proving that $-g_0 \in (\text{ri } N) + K^\perp$ if and only if $-g_0 \in \text{ri } N$. The "if" part is obvious. For the "only if" part, suppose $-g_0 = r + v$, $r \in \text{ri } N$, $v \in K^\perp$. We know $L \subset \text{lin } T$, so $\text{aff } N \subset L^\perp$: hence $r \in L^\perp$. As $g_0 = P_0^* f_x(x_0, p_0)$, $g_0 \in \text{im } P_0^* = L^\perp$. It follows that both $-g_0$ and r belong to $L^\perp$: hence $v \in K^\perp \cap L^\perp = (K+L)^\perp = M^\perp = \text{lin } N$. But $N = N + \text{lin } N$, so $\text{ri } N = \text{ri } N + \text{lin } N$. As $r \in \text{ri } N$ and $v \in \text{lin } N$, we have $-g_0 \in \text{ri } N$, which proves the proposition.

Proposition 2.2, and the discussion preceding it, makes precise the idea that a nonlinear optimization problem may be "locally essentially unconstrained," and it provides a test for determining just when this property holds. In the rest of this section we show that in two familiar special cases of (1.1)(p), this test reduces to properties already familiar in the literature.

Example I: Standard linear programming. Here we are concerned with the problem

$$\min \quad \{\langle c,x \rangle \mid Ax = b,\ x \geqq 0\},$$

so we can set $f(x) = \langle c,x \rangle$, $h(x) = Ax - b$, and $C = \mathbb{R}^n_+$. We suppress the perturbation parameter p since it plays no role in this example. If we suppose that x_0 is a basic feasible point corresponding to the basis B

and the partition [B N] of A, then we pointed out in [8] that nondegeneracy in the sense used here corresponds to the requirement that $x_B > 0$ (that is, to primal nondegeneracy in the usual linear programming sense). In this case we take $L = \text{lin}\, T_C(x_0) = \mathbb{R}^B \times \{0\}^N$, where the superscripts indicate that individual factors are to be taken to be $\mathbb{R}$ or $\{0\}$ according as the particular index is in B or N. We then have

$$D^- = \begin{bmatrix} B^{-1} \\ \\ 0 \end{bmatrix}, \quad P_L = \begin{bmatrix} I & B^{-1}N \\ \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad P_0 = \begin{bmatrix} 0 & -B^{-1}N \\ \\ 0 & I \end{bmatrix}.$$ The multipliers

are $\lambda = (D^-)^* f_x(x_0) = [(B^{-1})^* 0]c = c_B B^{-1}$, and the reduced gradient is

$$g_0 = P_0^* f_x(x_0) = \begin{bmatrix} 0 & 0 \\ \\ -(B^{-1}N)^* & I \end{bmatrix} c = [0_B , c_N - c_B B^{-1} N] ,$$

where we have abused notation slightly in order to write the multipliers and the reduced gradient in familiar forms.

In this case we have $T = \mathbb{R}^B \times (\mathbb{R}_+)^N$, so $N = \{0\}^B \times (\mathbb{R}_-)^N$. Hence, $-g_0$ will belong to N whenever $c_N - c_B B^{-1}N \geqq 0$, the familiar linear programming optimality criterion. The problem will be "essentially unconstrained" near x_0 whenever $-g_0 \in \text{ri}\, N$: that is, when $c_N - c_B B^{-1}N > 0$. This is the criterion usually referred to in the linear programming literature as "dual nondegeneracy."

It is of interest here to compute $T_0 \cap K$ to see what kind of subspace we are dealing with. In this case $\text{aff}\, C = \mathbb{R}^n$, so J has dimension zero, and thus $K = J + K = \text{im}\, P_0$. Using the expressions for P_0 and T given above, we see that the cone $T \cap K$ is given by

$$T \cap K = \left\{ \begin{bmatrix} -B^{-1}Ns \\ \\ s \end{bmatrix} \,\middle|\, s \geqq 0 \right\} .$$

The cone $T_0 \cap K$ consists of those elements v of $T \cap K$ for which $\langle g_0 , v \rangle \leqq 0$; that is, for which $\langle C_N - C_B B^{-1} N , s \rangle \leqq 0$. However, if $T_0 \cap K$ is to be a subspace then as already pointed out we have $C_N - C_B B^{-1} N > 0$, and as $s \geqq 0$ this implies $s = 0$. Hence in the case of standard linear programming, $T_0 \cap K$ is a subspace if and only if it is just the origin. This should not come as any particular surprise, since we know from linear programming theory that dual nondegeneracy implies a unique "corner solution," and the solution of an unconstrained minimization problem with a linear objective function will be unique if and only if the dimension of the space over which the minimization is done is actually zero.

Example II: Nonlinear programming with inequality and equality constraints. We consider next the nonlinear optimization problem

$$\begin{aligned} \text{minimize} \quad & d(z) \\ \text{subject to} \quad & c_E(z) = 0 \\ & c_A(z) \leqq 0 \\ & c_I(z) \leqq 0 , \end{aligned} \tag{2.1}$$

where $z \in \mathbb{R}^k$ and d, c_E, c_A, and c_I are C^r functions $(r \geqq 1)$ from an open set $\Omega \subset \mathbb{R}^k$ into $\mathbb{R}$, $\mathbb{R}^e$, $\mathbb{R}^a$, and $\mathbb{R}^i$ respectively. Let $z_0 \in \Omega$ and suppose that $c_E(z_0) = 0$, $c_A(z_0) = 0$ and $c_I(z_0) < 0$: thus c_A and c_I identify the inequality constraints that are respectively active and inactive at z_0. We shall write d_z for $d_z(z_0)$.

To convert (2.1) to the form (1.1)(p), we introduce slack variables s_A and s_I. We let $x := (z, s_A, s_I)$ and $f(x) := d(z)$,

$$h(x) := \begin{bmatrix} c_E(z) \\ c_A(z) + s_A \\ c_I(z) + s_I \end{bmatrix},$$

and $C := \mathbb{R}^k \times \mathbb{R}^a_+ \times \mathbb{R}^i_+$. Then an equivalent formulation of (2.1) is

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && h(x) = 0 \\ &&& x \in C. \end{aligned} \tag{2.2}$$

Note that we have suppressed the perturbation parameter in (2.1) and (2.2); in the analysis that we shall do here it would simply remain constant, so there is no point in writing it out.

Now suppose that $x_0 = (z_0, -c_A(z_0), -c_E(z_0))$ is a nondegenerate feasible point of (2.2). Write G_E for $c_E'(z_0)$, and define G_A and G_I similarly. We shall determine the various elements of the reduced problem at z_0 in terms of these matrices. In particular, we shall compute $T_0 \cap K$ and its affine hull.

As provided out in [8], nondegeneracy of x_0 means that the matrix $\begin{bmatrix} G_E \\ G_A \end{bmatrix}$ has full row rank. Thus, we can write

$$\begin{bmatrix} G_E \\ G_A \end{bmatrix} = [0 \ R] \, Q^T,$$

where Q is a $k \times k$ orthogonal matrix and R is a nonsingular, upper triangular $(e+a)\times(e+a)$ matrix. If we now write $Q=[Q_1\ Q_2]$, where Q_1 is $k\times(k-e-a)$ and Q_2 is $k\times(e+a)$, then

$$\begin{bmatrix} G_E \\ G_A \end{bmatrix} = RQ_2^T$$

Referring to the definition of C in (2.2), we can see that $T_C(x_0) = \mathbb{R}^k \times \mathbb{R}^a \times \mathbb{R}^i$, and thus $\operatorname{lin} T_C(x_0) = \mathbb{R}^k \times \{0\}^a \times \mathbb{R}^i$. Choose $L = (\operatorname{im} Q_2) \times \{0\}^a \times \mathbb{R}^i$, so that $L \subset \operatorname{lin} T_C(x_0)$. In this case $\operatorname{aff} C = \mathbb{R}^{k+a+i}$, so J is the zero subspace and $K = \ker D$, where

$$D = \begin{bmatrix} G_E & 0 & 0 \\ G_A & I & 0 \\ G_I & 0 & I \end{bmatrix}.$$

To show that the subspaces K and L satisfy $K \oplus L = \mathbb{R}^{k+a+i}$, suppose first that $y \in K \cap L$. Partition y as (y_1, y_2, y_3), with $y_1 \in \mathbb{R}^k$, $y_2 \in \mathbb{R}^a$, and $y_3 \in \mathbb{R}^i$. As $y \in \ker D$, we have

$$\begin{aligned} G_E y_1 \qquad\qquad &= 0 \\ G_A y_1 + y_2 \qquad &= 0 \\ G_I y_1 \qquad + y_3 &= 0, \end{aligned} \tag{2.3}$$

but as $y \in L$ we also have $y_1 \in \operatorname{im} Q_2$ (so that, say, $y_1 = Q_2 w$) and $y_2 = 0$. Using $y_2 = 0$ in (2.3) we find that

$$0 = \begin{bmatrix} G_E \\ G_A \end{bmatrix} y_1 = (RQ_2^T)(Q_2 w) = Rw,$$

since $Q_2^T Q_2 = I$. As R is nonsingular we must have $w = 0$ and thus $y_1 = 0$. Using this in (2.3) we find that $y_3 = 0$. Hence K and L are independent.

As $\text{im}\, Q_2$ has dimension $e+a$, we see that $\dim L = e+a+i$. But D has full row rank, so its kernel K has dimension $(k+a+i) - (e+a+i) = k-e$. As K and L are independent we have $\dim(K+L) = k+a+i$, and hence $\mathbb{R}^{k+a+i} = K \oplus L$.

To compute D^- we follow the procedure suggested in [7], defining a $(k+a+i) \times (e+a+i)$ matrix E by

$$E = \begin{bmatrix} Q_2 & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix},$$

so that

$$DE = \begin{bmatrix} R & 0 \\ G_I Q_2 & I \end{bmatrix}, \quad (DE)^{-1} = \begin{bmatrix} R^{-1} & 0 \\ -G_I Q_2 R^{-1} & I \end{bmatrix}.$$

Thus

$$D^- = E(DE)^{-1} = \begin{bmatrix} Q_2 R^{-1} & 0 \\ 0 & 0 \\ -G_I Q_2 R^{-1} & I \end{bmatrix}.$$

It follows that the multipliers at x_0 are

$$f_x(x_0) D^- = [d_z Q_2 R^{-1}, 0]\,. \tag{2.4}$$

For future reference we partition R^{-1} into $[Z_E\ Z_A]$, where Z_E is $(e+a)\times e$ and Z_A is $(e+a)\times a$. With this partitioning we can display the multipliers for each type of constraint by rewriting (2.4) as

$$f_x(x_0)D^- = [d_zQ_2Z_E, d_zQ_2Z_A, 0] =: (\lambda_E, \lambda_A, \lambda_I) .$$

To find the projectors on L and on $\ker D$, we compute

$$P_L = D^-D = \begin{bmatrix} Q_2Q_2^T & Q_2Z_A & 0 \\ 0 & 0 & 0 \\ G_IQ_1Q_1^T & -G_IQ_2Z_A & I \end{bmatrix} ,$$

where we have used $Q_1Q_1^T = I - Q_2Q_2^T$ and $R^{-1} = [Z_E\ Z_A]$.

Then

$$P_0 = I - P_L = \begin{bmatrix} Q_1Q_1^T & -Q_2Z_A & 0 \\ 0 & I & 0 \\ -G_IQ_1Q_1^T & G_IQ_2Z_A & 0 \end{bmatrix} ,$$

and the reduced gradient at x_0 is

$$f_x(x_0)P_0 = [d_zQ_1Q_1^T , -d_zQ_2Z_A , 0] . \tag{2.5}$$

Note that in the second position of the reduced gradient we have the negatives of the multipliers for the active inequality constraints.

We can now compute the cones $T\cap K$ and $T_0\cap K$. For $T\cap K$, recall that $T_C(x_0) = \mathbb{R}^k \times \mathbb{R}_+^a \times \mathbb{R}^i$ and that (2.3) describes the vectors (y_1, y_2, y_3) in $\ker D$. Putting these together we see that

$$T\cap K = \{(y_1, -G_Ay_1, -G_Iy_1) \mid G_Ey_1=0, G_Ay_1 \leqq 0\} . \tag{2.6}$$

Up to now we have assumed only that x_0 was a nondegenerate feasible point. To find $T_0 \cap K$ we assume that x_0 is a stationary point, so that the reduced gradient belongs to $-N_C(x_0)$. Recalling that $N_C(x_0) = \{0\}^k \times \mathbb{R}^a_- \times \{0\}^i$ and using (2.5), we see that stationarity implies

$$d_z Q_1 = 0 , \quad d_z Q_2 Z_A \leqq 0 . \tag{2.7}$$

The points y of $T_0 \cap K$ are those in $T \cap K$ that make a non-positive inner product with the reduced gradient. Using (2.5) and (2.6) we see that this implies $(-d_z Q_2 Z_A)(-G_A y_1) \leqq 0$. However, as $d_z Q_2 Z_A = \lambda_A \leqq 0$ by (2.7) and $G_A y_1 \leqq 0$ since $y \in T \cap K$, we see that λ_A and $G_A y_1$ must have complementary supports. Hence we have

$$T_0 \cap K = \begin{bmatrix} I \\ -G_A \\ -G_I \end{bmatrix} (U_0) , \tag{2.8}$$

where we have defined the polyhedral convex cone U_0 by

$$U_0 := \{y_1 \in \mathbb{R}^k \mid G_E y_1 = 0,\ G_A y_1 \leqq 0 \text{ and } (G_A y_1)j = 0 \text{ if } (\lambda_A)j \neq 0\} . \tag{2.9}$$

This cone U_0 is the familiar cone appearing in the second-order optimality conditions (see, e.g., [3, §10.3]).

From Proposition 2.2 we know that $T_0 \cap K$ will be a subspace if and only if the reduced gradient belongs to $-\text{ri } N_C(x_0)$: in this case, this means precisely that every component of λ_A is negative. This property is usually referred to as "strict complementary slackness" in the nonlinear programming literature, so we have shown that the problem (2.1) reduces locally to an essentially unconstrained minimization if at the stationary point x_0 one has (1) strict complementary slackness and (2) linear independence of the gradients of the active (binding) constraints.

As a final step in the analysis of (2.1) we compute the affine hull of $T_0 \cap K$, since we shall need this information in the next section. For this purpose, let us suppose that the rows of G_A have been ordered so that all of those corresponding to negative components of λ_A are placed before all corresponding to zero components of λ_A. Then we can partition G_A accordingly, as $\begin{bmatrix} G_- \\ G_0 \end{bmatrix}$, and we can partition the component matrix Z_A of R^{-1} as $[Z_- \ Z_0]$. Now recall that

$$I = RQ_2^T Q_2 R = \begin{bmatrix} G_E \\ G_- \\ G_0 \end{bmatrix} Q_2 \ [Z_E \ Z_- \ Z_0] .$$

It follows that $\begin{bmatrix} G_E \\ G_- \end{bmatrix} (Q_2 Z_0) = 0$ and $G_0(Q_2 Z_0) = I$. Referring to (2.8) we see that each column of $-Q_2 Z_0$ belongs to the cone U_0 defined earlier. However, these same columns form a basis for $\ker \begin{bmatrix} G_E \\ G_- \end{bmatrix}$, so $\ker \begin{bmatrix} G_E \\ G_- \end{bmatrix} \subset$ aff U_0. However, it follows from (2.9) that $U_0 \subset \ker \begin{bmatrix} G_E \\ G_- \end{bmatrix}$, so actually aff $U_0 = \ker \begin{bmatrix} G_E \\ G_- \end{bmatrix} = \operatorname{im}(Q_2 Z_0)$. We then have

$$\text{aff } T_0 \cap K = \begin{bmatrix} I \\ -G_A \\ -G_I \end{bmatrix} \left(\ker \begin{bmatrix} G_E \\ G_- \end{bmatrix}\right) . \qquad (2.10)$$

3. Stability and sensitivity analysis of an optimizer. In Section 2 we studied the critical cone $T_0 \cap K$ associated with a local minimizer x_0. The results of that section made no use of the perturbation parameter p, which remained fixed at p_0.

Here we allow p to vary near p_0, and we study the questions of existence, uniqueness, and stability of a local minimizer near x_0 when the problem is perturbed. We show that, under appropriate conditions, not only does a local minimizer exist near x_0 for each p near p_0, but when regarded as a function of p this optimizer is Lipschitzian. In fact, it is almost differentiable in the sense that it can be well approximated by a simpler function that can, in principle, be computed using information available at the solution. The "almost" results from the fact that this approximating function is not affine (unless $T_0 \cap K$ is a subspace): in general, the portion of its graph near p_0 is a cone rather than an affine set.

Recall that we have transformed the problem (1.1)(p) into the considerably simpler problem (1.15)(p), whose feasible set is the polyhedral convex cone $T_0 \cap K$. Of course, without making some assumptions about the functions involved in the problem we cannot expect to find any interesting results about stability. Thus, we consider next what kinds of assumptions we should make about (1.15)(p) in order to ensure that its minimizer is unique and well behaved for p near p_0. Then, of course, any information we gain about that minimizer can be immediately translated into corresponding information about a minimizer of (1.1)(p) near x_0.

We shall use the concept of strong convexity in connection with (1.15) (p). This idea is defined as follows:

DEFINITION 3.1: Let γ be a function from $\mathbb{R}^n$ to $(-\infty, +\infty]$. γ is <u>strongly convex with modulus</u> $\rho > 0$ if for each $x_0, x_1 \in \mathbb{R}^n$ and each $\lambda \in (0,1)$,

$$\gamma[(1-\lambda)x_0 + \lambda x_1] \leq (1-\lambda)\gamma(x_0) + \lambda\gamma(x_1) - \tfrac{1}{2}\rho\lambda(1-\lambda)\|x_0 - x_1\|^2 .$$

We say γ is strongly convex on a subset U of $\mathbb{R}^n$ if the function equal to γ on U and to $+\infty$ off U is strongly convex.

The following lemma characterizes strong convexity when the function in question is restricted to an affine set and when enough differentiability is present.

LEMMA 3.2: <u>Let</u> γ <u>be a</u> C^2 <u>function from a neighborhood</u> N <u>of</u> x_0 <u>in</u> $\mathbb{R}^n$ <u>to</u> $\mathbb{R}$. <u>Let</u> A <u>be an affine set containing</u> x_0 <u>and let</u> S <u>be the subspace parallel to</u> A. <u>Then</u> γ <u>is strongly convex on a neighborhood of</u> x_0 <u>in</u> A, <u>if and only if</u> $\gamma''(x_0)$ <u>is positive definite on the subspace</u> S.

PROOF (only if): Suppose γ is strongly convex with modulus ρ on a neighborhood of x_0 in A. We can take this neighborhood to be $M \cap A$, where $M \subset N$ is a neighborhood of x_0 in $\mathbb{R}^n$. Let $s \in S$; we can assume with no loss of generality that s is small enough so that $x_0 + s$ and $x_0 - s$ belong to M. As $s \in S$, these points also lie in A, hence in $M \cap A$. Thus we have

$$\tfrac{1}{2}\rho\|s\|^2 = \tfrac{1}{2}\rho(\tfrac{1}{2})(\tfrac{1}{2})\|2s\|^2 \leq \tfrac{1}{2}\gamma(x_0+s) + \tfrac{1}{2}\gamma(x_0-s) - \gamma(x_0) .$$

But

$$\gamma(x_0+s) = \gamma(x_0) + \gamma'(x_0)s + \tfrac{1}{2}\langle s,\gamma''(x_0)s\rangle + o(\|s\|^2) ,$$

$$\gamma(x_0-s) = \gamma(x_0) - \gamma'(x_0)s + \tfrac{1}{2}\langle s,\gamma''(x_0)s\rangle + o(\|s\|^2) ,$$

so

$$\tfrac{1}{2}\rho\|s\|^2 \leqq \tfrac{1}{2}\langle s,\gamma''(x_0)s\rangle + o(\|s\|^2) .$$

Since this inequality holds for all small $s \in S$, it follows that for all $s \in S$, $\langle s,\gamma''(x_0)s\rangle \geqq \rho\|s\|^2$.

(if): Suppose $\gamma''(x_0)$ is positive definite on S, and let $s \in S$. Since γ is C^2, there is an open convex neighborhood M of x_0 in $\mathbb{R}^n$ such that if $X \in M$ then $\gamma''(x)$ is positive definite on s with modulus ρ. Define an auxiliary function θ from the neighborhood $\{t | w + ts \in M \cap A\}$ of the origin in $\mathbb{R}$ to $\mathbb{R}$ by

$$\theta(t) : \gamma(w+ts) - \gamma'(w)(w+ts) .$$

Then we have

$$\gamma(w+ts) - \gamma(w) - \gamma'(w)ts = \theta(t) - \theta(0) = \int_0^t \theta'(v)dv$$

$$= \int_0^t [\gamma'(w+vs) - \gamma'(w)]\, s\, dv = \int_0^t \int_0^v \langle s,\gamma''(w+ys)s\rangle \, dy\, dv$$

$$\geqq \rho\|s\|^2 \int_0^t \int_0^v dy\, dv = \tfrac{1}{2}\rho t^2\|s\|^2 . \tag{3.1}$$

Now let x_1 and x_2 be any two points in $M \cap A$, and let $\lambda \in (0,1)$. Apply (3.1) with $s = x_1 - x_2$ and with $w = (1-\lambda)x_1 + \lambda x_2$, taking the indicated choices for t to obtain the inequalities shown:

$$t = \lambda : \gamma(x_1) - \gamma(w) - \lambda\gamma'(w)(x_1-x_2) \geqq \tfrac{1}{2}\rho\lambda^2\|x_1-x_2\|^2 ; \tag{3.2}$$

$$t = \lambda-1 : \gamma(x_2) - \gamma(w) + (1-\lambda)\gamma'(w)(x_1-x_2) \geqq \tfrac{1}{2}\rho(1-\lambda)^2\|x_1-x_2\|^2 . \tag{3.3}$$

Multiplying (3.2) by $(1-\lambda)$ and (3.2) by λ and adding, we obtain

$$(1-\lambda)\gamma(x_1) + \lambda\gamma(x_2) - \gamma[(1-\lambda)x_1+\lambda x_2] \geqq \tfrac{1}{2}\rho\lambda(1-\lambda)\|x_1-x_2\|^2 ,$$

which completes the proof.

Our approach to stability analysis of (1.15)(p) will be the following: we consider a local minimizer x_0, and we make the assumption that $\phi(\cdot,p_0)$ is strongly convex on aff $(T_0 \cap K)$. Since aff $(T_0 \cap K)$ is itself a subspace, we see from Lemma 3.2 that our assumption is equivalent to the assumption that $\phi_{yy}(x_0,p_0)$ is positive definite on aff $(T_0 \cap K)$. As we shall see later, this latter assumption is well known in the special case of nonlinear programming with inequality and equality constraints, being exactly the "strong second order sufficient condition" used by Kojima [2] and the author [4].

With this assumption, we consider the nonlinear generalized equation

$$0 \in \phi_y(y,p) + N_{T_0 \cap K}(y) , \tag{3.4)(p}$$

which expresses the condition for a stationary point of (1.15)(p) to exist. We can use (3.4)(p) since our assumption will guarantee that not only $\phi(\cdot,p_0)$, but also $\phi(\cdot,p)$ for p near p_0, is strongly convex. Hence the stationary-point condition (3.4)(p) is equivalent to the minimization condition in (1.15)(p).

Applying to (3.4)(p) the analytical machinery devised in [4], we can prove that for each p near p_0 there is a unique minimizer $Y(p)$ of (1.15)(p) near 0, and that the function Y is Lipschitzian. Of course, this immediately implies the existence of a Lipschitzian function $X(p)$ yielding the unique local minimizer of (1.1)(p) near x_0.

Finally, we shall use some additional results about generalized equations to show that the functions Y and X are almost differentiable, in the sense that they can be well approximated near p_0 by relatively simple functions that are in principle computable. However, the graphs of these simple functions are not subspaces (as would be the case with derivatives) but rather cones.

To begin the detailed analysis we make the assumption, for the remainder of this section, that x_0 is a nondegenerate local minimizer of (1.1)(p_0) and that f and h are C^2 on a neighborhood of (x_0,p_0). Of course, this then implies that 0 is a local minimizer of (1.15)(p_0), and that ϕ is C^2 on a neighborhood of $(0,p_0)$. We also assume that $\phi(\cdot,p_0)$ is strongly convex on a neighborhood of 0 in aff $(T_0 \cap K)$. By Lemma 3.2, this is equivalent to assuming that $\phi_{yy}(0,p_0)$ is positive definite on aff $(T_0 \cap K)$. Since ϕ is C^2, there are neighborhoods U_0 of 0 in $\mathbb{R}^n$ (open and convex) and V_0 of p_0 in π, such that if $(y,p) \in U_0 x V_0$, than $\phi_{yy}(y,p)$ is positive definite on aff $(T_0 \cap K)$ with, say, constant $\rho > 0$: i.e., for each $s \in$ aff $(T_0 \cap K)$, $\langle s,\phi_{yy}(y,p)s\rangle \geq \rho\|s\|^2$. Now Lemma 3.2 (in particular, the proof of the "if" direction) implies that for each $p \in V_0$, $\phi(\cdot,p)$ is strongly convex on U_0. This tells us that $\phi(\cdot,p)$ has a local minimizer $Y(p)$ on $U_0 \cap (T_0 \cap K)$ if and only if the stationary point condition (3.4)(p) holds for $y = Y(p)$; further, if such a minimizer exists it is the unique global minimizer of $\phi(\cdot,p)$ on $U_0 \cap (T_0 \cap K)$. Thus we have now to investigate the solvability of (3.4)(p).

Fortunately, quite a lot is known about inclusions like (3.4)(p); these <u>generalized equations</u> exhibit solvability and regularity properties analogous to those of conventional nonlinear equations. A survey of this area is given in [6], and many details and proofs are in [4].

From results in [4] and [6] we can see that the key to analyzing the behavior of (3.4)(p) is the linearization given by

$$0 \in \phi(0,p_0) + \phi_{yy}(0,p_0)y + N_{T_0 \cap K}(y) . \tag{3.5}$$

For example, [6, Th. 4.4] or [4, Th. 2.1, Cor. 2.2] show that if the inverse of the operator on the right in (3.5) is (locally) single-valued and Lipschitzian near 0, then for any p near p_0 (3.4)(p) will have a locally unique solution $Y(p)$ that is Lipschitzian in p. In fact, under our assumptions such a solution will even be globally unique.

To investigate the inverse of the operator in (3.5), note that y solves

$$w \in \phi_y(0,p_0) + \phi_{yy}(0,p_0)y + N_{T_0 \cap K}(y) \tag{3.6}$$

if and only if y solves the convex quadratic programming problem

$$\text{minimize} \quad \{\langle \phi_y(0,p_0)-w,y\rangle + \tfrac{1}{2}\langle y,\phi_{yy}(0,p_0)y\rangle \mid y \in T_0 \cap K\} . \tag{3.7}$$

Since $\phi_{yy}(0,p_0)$ is positive definite on aff $(T_0 \cap K)$, we can see that for each $w \in \mathbb{R}^n$ (3.7) has a unique global optimizer $y(w)$, and such an optimizer is then also the unique solution of (3.6). Further, if w_1 and

w_2 are two points in $\mathbb{R}^n$, then

$$w_i - \phi_y(0,p_0) - \phi_{yy}(0,p_0)y(w_i) \in N_{T_0 \cap K}(y(w_i)),\ i = 1,2 .$$

The definition of normal cone then yields

$$\langle w_1 - \phi_y(0,p_0) - \phi_{yy}(0,p_0)y(w_1),\ y(w_1) - y(w_2) \rangle \geqq 0 ,$$

$$\langle w_2 - \phi_y(0,p_0) - \phi_{yy}(0,p_0)y(w_2),\ y(w_1) - y(w_2) \rangle \leqq 0 .$$

Subtracting the second inequality from the first, we find that

$$\langle w_1 - w_2,\ y(w_1) - y(w_2) \rangle \geqq \langle \phi_{yy}(0,p_0)[y(w_1)-y(w_2)],\ y(w_1) - (w_2) \rangle$$
$$\geqq \rho \| y(w_1) - y(w_2) \|^2 ,$$

so that we have

$$\| y(w_1) - y(w_2) \| \leqq \rho^{-1} \| w_1 - w_2 \| ,$$

and therefore the inverse operator we are considering is in fact Lipschitzian. (Another way to reach this conclusion would have been to show that the operator in (3.6) is strongly monotone and then to use general results about such operators. In this case the direct argument seemed simpler.)

We are now able to prove the following theorem about minimizers of (1.1)(p) and (1.15)(p):

THEOREM 3.3: <u>Suppose that</u> x_0 <u>is a nondegenerate local minimizer of</u> (1.1)(p_0), <u>that</u> f <u>and</u> h <u>are</u> C^2 <u>on a neighborhood of</u> (x_0,p_0), <u>and that</u> $\phi_{yy}(0,p_0)$ <u>is positive definite on</u> aff $(T_0 \cap K)$.

<u>Then there exist neighborhoods</u> U <u>of</u> 0 <u>in</u> $\mathbb{R}^n$, V <u>of</u> p_0 <u>in</u> P, <u>and</u> W <u>of</u> x_0 <u>in</u> $\mathbb{R}^n$, <u>and Lipschitzian functions</u> $Y : V \to U$ <u>and</u> $X : V \to W$, <u>such that for each</u> $p \in V$, $Y(p)$ <u>is the unique minimizer of</u> (1.15)(p) <u>in</u> U, <u>and</u> $X(p)$ <u>is the unique minimizer of</u> (1.1)(p) <u>in</u> W.

PROOF: Our analysis just prior to the statement of the theorem showed that the generalized equation (3.4)(p_0) was regular at the origin in the sense of [6] (i.e., its linearization has a Lipschitzian inverse). Applying [6, Th. 4.4], we see that there are neighborhoods U of 0 in $\mathbb{R}^n$ and V of p_0 in P (which we can take to be contained in $U_0 \cap U_*$ and $V_0 \cap V_*$ respectively), and a Lipschitzian function $Y : V \to U$ such that, for each $p \in V$, $Y(p)$ is the unique solution of (3.4)(p) in U. However, our previous remarks show that this is equivalent to saying that $Y(p)$ is the unique minimizer of (1.15)(p) on U. Finally, we set $X = \psi_p \circ Y$; since $Y(p)$ remains in $U_* \cap \Lambda$ we know from our earlier diffeomorphism results that $X(p)$ will then be the unique minimizer of (1.1)(p) on $W := \psi_p(U)$. As W is a neighborhood of x_0, we have completed the proof.

At this point two remarks are in order. First, the review in Section 1 showed that the multipliers λ were C^r functions of x: thus, as $X(p)$ varies with p the corresponding multipliers will be Lipschitzian in p, and possibly smoother: in fact they will share whatever smoothness properties X may have (up to C^r). Second, there is indeed a situation in which X will be at least C^2: it is precisely the case that we studied in Section 2, in which $T_0 \cap K$ is a subspace. We saw that this case would arise whenever $-g_0$ belonged to the relative interior of $N_C(x_0)$.

To see why X is in fact C^r in this case, we return to the generalized equation (3.4)(p). If $T_0 \cap K$ is a subspace, say of dimension k, we can write (3.4)(p) equivalently as

$$\begin{aligned} &\phi_y(y,p) \in (T_0 \cap K)^{\perp} , \\ &y \in T_0 \cap K . \end{aligned} \tag{3.8}$$

If we let Q be an injective linear transformation from $\mathbb{R}^k$ to $\mathbb{R}^n$ with $T_0 \cap K = \text{im } Q$, then we can reformulate (3.8) as

$$Q^T \phi_y(Qz,p) = 0 , \tag{3.9}$$

where we have replaced y by Qz. The expression (3.9) is a system of nonlinear equations, and we can analyze its solution using the implicit-function theorem. To do so, we examine its first derivative in z at $z = 0$ (hence $y = 0$), which is $Q^T \phi_{yy}(0,p_0)Q$. This is a positive definite (hence nonsingular) linear transformation from $\mathbb{R}^k$ to $\mathbb{R}^k$, since we have assumed that $\phi_{yy}(0,p_0)$ is positive definite on aff $(T_0 \cap K) = \text{im } Q$. The implicit-function theorem then tells us that there is a C^r solution $z(p)$ of (3.9) for p near p_0, which is unique in some neighborhood of the origin in $\mathbb{R}^k$. Putting $Y(p) = Qz(p)$, we obtain a C^r solution of the minimization problem (1.15)(p) and thence a C^r solution $X(p)$ of (1.1)(p). Derivatives of X and Y can then be computed, using Q and the derivatives of ϕ.

Returning to the general case, we investigate the positive definiteness condition on $\phi_{yy}(0,p_0)$ in the special case of nonlinear inequality and equality constraints. This case was dealt with in detail in Section 2. There, we found that

$$\text{aff } (T_0 \cap K) = \begin{bmatrix} I \\ -G_A \\ -G_I \end{bmatrix} \left(\ker \begin{bmatrix} G_E \\ G_- \end{bmatrix}\right) . \tag{2.10}$$

Since we are interested only in $\phi_{yy}(0,p_0)$, we shall suppress the perturbation parameter p in writing the problem functions. Using the notation of Section 2 we have

$$f_{xx}(x_0)rs = d_{zz}(z_0)r_z s_z \; ; \; h_{xx}(x_0)rs = \begin{bmatrix} (c_E)_{zz}(z_0)r_z s_z \\ (c_A)_{zz}(z_0)r_z s_z \\ (c_I)_{zz}(z_0)r_z s_z \end{bmatrix} \tag{3.10}$$

where r_z and s_z denote the portions of r and s corresponding to z (the first k components). Incidentally, the form of (3.10) shows that the introduction of slack variables to convert (2.1) to (2.2) made no essential difference in the derivatives of the problem functions; this point is of interest when considering the computational effectiveness of such a procedure.

Using the derivative formulas from Section 1, one finds

$$\phi_{yy}(0)rs = f_{xx}(x_0)(P_0r)(P_0s) - f_x(x_0)D^- h_{xx}(x_0)(P_0r)(P_0s) \; . \tag{3.11}$$

Recalling that $f_x(x_0)D^- = (\lambda_E,\lambda_A,\lambda_I)$, and using (3.10), one obtains from (3.11) an expression in the original problem functions from (2.1) , namely

$$\begin{aligned} \phi_{yy}(0)rs = d_{zz}(z_0)(P_0r)_z(P_0s)_z - \langle \lambda_E,(c_E)_{zz}(z_0)(P_0r)_z(P_0s)_z \rangle \\ - \langle \lambda_A,(c_A)_{zz}(z_0)(P_0r)_z(P_0s)_z \rangle \; , \end{aligned} \tag{3.12}$$

since we know $\lambda_I = 0$.

Now if r and s lie in aff $(T_0 \cap K)$ they surely lie in ker $h_x(x_0)$, so the multiplication by P_0 will be superfluous. Thus we can see from (3.12) that if $r , s \in$ aff $(T_0 \cap K)$ then

$$\phi_{yy}(0)rs = d_{zz}(z_0)r_z s_z - \langle \lambda_E,(c_E)_{zz}(z_0)r_z s_z \rangle - \langle \lambda_A,(c_A)_{zz}(z_0)r_z s_z \rangle . \tag{3.13}$$

The expression on the right of (3.13) is simply the quadratic form defined by the second derivative of the standard Lagrangian of (2.1) at z_0, evaluated at the pair (r_z, s_z). Referring to (2.10), we see that r and s belong to aff $(T_0 \cap K)$ if and only if r_z and s_z belong to

$$\ker \begin{bmatrix} G_E \\ G_- \end{bmatrix}$$; that is, to the subspace orthogonal to the gradients of (i) the equality constraints, and (ii) those inequality constraints having nonzero multipliers. Hence, our requirement that $\phi_{yy}(0)$ be positive definite on aff $(T_0 \cap K)$ reduces, in this special case, to the requirement that the second derivative of the standard Lagrangian be positive definite on the subspace just described. This is, as we noted above, precisely the "strong second-order sufficient condition" used by Kojima [2] and the author [4] in their analyses of this particular case.

Although we have shown that the functions X and Y are Lipschitzian, it is possible to gain some additional information about them by showing that they are in fact differentiable in a certain sense, weaker than that of Fréchet differentiability. We call this weak form of differentiability Bouligand differentiability (B-differentiability). It is defined, and some of its properties are derived, in the Appendix. In the remainder of this section we shall use these results without further comment, and we shall also assume that the space P is finite-dimensional: say, $\mathbf{R}^k$.

We are going to prove that the functions X and Y of Theorem 3.3 are B-differentiable, and to exhibit their B-derivatives. In order to do this, we shall require a lemma in which we use the idea of a polyhedral function or multifunction (multivalued function). This simply means a function or multifunction whose graph is the union of finitely many polyhedral convex sets. Such multifunctions are treated in some detail in [5].

LEMMA 3.4: Let y be a single-valued polyhedral function from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $y(0) = 0$. Then there is a neighborhood N of the origin such that for any $z \in N$ and any $\lambda \in [0,1]$, $y(\lambda z) = \lambda y(z)$.

PROOF: By [5, Lemma 1] there is some convex neighborhood N of the origin such that if any component of graph y intersects $N \times \mathbb{R}^m$, then that component actually contains $(0,0)$. Now let $z \in N$ and $\lambda \in [0,1]$. The pair $(z,y(z))$ belongs to some component of graph y, and as $z \in N$ that component also contains $(0,0)$. Thus it must also contain $(\lambda z, \lambda y(z)) = (1-\lambda)(0,0) + \lambda(z,y(z))$. Hence $(\lambda z, \lambda y(z))$ belongs to graph y, and by single-valuedness we then have $y(\lambda z) = \lambda y(z)$, which proves Lemma 3.4.

The main result about B-differentiability of X and Y is the following.

THEOREM 3.5: Assume the hypotheses of Theorem 3.3, and assume further that $P = \mathbb{R}^k$. Then the functions X and Y of Theorem 3.3 are B-differentiable at p_0, and one has for small $q \in \mathbb{R}^k$

$$DY(p_0)(q) = y \circ [-\phi_{yp}(0,p_0)](q) , \tag{3.14}$$

and

$$DX(p_0)(q) = DY(p_0)(q) - D^{-}h_p(x_0,p_0)(q) , \tag{3.15}$$

where $y(w)$ is the single-valued Lipschitzian function defined by (3.6) or (3.7).

PROOF: We first obtain the expression for $DY(p_0)$. Recall that

$$\phi_y(0,p_0+q) = \phi_y(0,p_0) + \phi_{yp}(0,p_0)(q) + o(q) .$$

We also know from [6, Th. 4.5] or [4, Th. 2.3] that since $Y(p)$ solves (3.4)(p),

$$Y(p_0+q) = y[\phi_y(0,p_0) - \phi_y(0,p_0+q)] + o(q) ,$$

where y is defined by (3.6) or (3.7). As y is shown (in the discussion following (3.7)) to be Lipschitzian, we have

$$\begin{aligned} Y(p_0+q) &= y[-\phi_{yp}(0,p_0)(q) + o(q)] \\ &= y\circ[-\phi_{yp}(0,p_0)](q) + o(q) . \end{aligned}$$

Lemma 3.4 shows that there is a function v defined from $\mathbb{R}^n$ to $\mathbb{R}^n$, whose graph is a cone, and a neighborhood N of the origin such that $v(w) = y(w)$ whenever $w \in N$. It follows that for small q, $v\circ[-\phi_{yp}(0,p_0)](q) = y\circ[-\phi_{yp}(0,p_0)](q)$. Thus, for such q we have (since $y(p_0) = 0$)

$$Y(p_0+q) = Y(p_0) + v\circ[-\phi_{yp}(0,p_0)](q) + o(q) .$$

Since the graphs of v and $-\phi_{yp}(0,p_0)$ are cones, so is that of their composition. Applying Theorem A.2 to the Lipschitzian function Y, we conclude that Y is B-differentiable at p_0, and that

$$DY(p_0) = v\circ[-\phi_{yp}(0,p_0)] .$$

However, for small q, $-\phi_{yp}(0,p_0)(q)$ belongs to the region near 0 where v agrees with y; hence we have (3.14).

To establish (3.15) we note that for p near p_0 $X(p) = x[y(p),p]$, where the function x is as defined in Section 1. Hence, using the chain rule for B-derivatives (Corollary A.4), we find that X is B-differentiable at p_0 with

$$DX(p_0)(q) = x_y(0,p_0) \circ DY(p_0)(q) + x_p(0,p_0)(q)$$
$$= DY(p_0)(q) - D^{-}h_p(x_0,p_0)(q) ,$$

which is (3.15). This completes the proof.

We note in closing that Aubin [1] derived very general results about solvability and sensitivity of convex optimization problems. He used contingent derivatives, which are multivalued generalizations of the Bonligand derivatives employed here. We have chosen to use a direct approach in analyzing (1.15)(p), rather than to attempt to apply Aubin's results, because with the direct approach we can apply special information that we have about (1.15)(p) (e.g., strong convexity) to prove sharper results than would be true for general convex optimization.

APPENDIX: BOULIGAND DERIVATIVES

This appendix presents results about Bouligand derivatives that are needed in the last part of Section 3. Only those results that are needed here will be presented.

In [1], Aubin defined contingent derivatives and analyzed some of their properties. These contingent derivatives are obtained by considering the contingent cone (originally introduced by Bouligand) to the graph by a multivalued function, at a point in that graph, to be the graph of a certain operator. This operator is the contingent derivative of the multivalued function at that point of its graph. Of course, in general the contingent derivative will itself be multivalued.

We shall be concerned here with a special case of the contingent derivative, which we call the Bouligand derivative. This special case arises when the function involved is single-valued and Lipschitzian on a neighborhood of the point in question and the contingent derivative at that point is also single-valued. In this situation the Bouligand derivative has some strong properties not shared by contingent derivatives in general.

Throughout this appendix we assume that f is a function from an open set $\Omega \subset \mathbb{R}^m$ to $\mathbb{R}^k$, which is Lipschitzian on Ω with modulus λ. If $x_0 \in \Omega$ we consider the contingent derivative $Df(x_0)$, defined by letting graph $Df(x_0)$ be the contingent cone to graph f at $(x_0, f(x_0))$: that is, the cone K defined by

$(v,w) \in K$ if and only if there exist $\rho_n > 0$ and (v_n, w_n) with $w_n = f(x_0+v_n) - f(x_0)$, $v_n \to 0$, and $\rho_n(v_n, w_n) \to (v,w)$.

We observe that $Df(x_0)\, v$ is nonempty for any $v \in \mathbb{R}^m$. Indeed, if we consider the set $\{\tau^{-1}[f(x_0+\tau v) - f(x_0)] \mid \tau > 0\}$ we see that for small τ, no element of this set has norm greater than $\lambda\|v\|$ because f is Lipschitzian. Hence there is a sequence $\tau_n \downarrow 0$ with $\tau_n^{-1}[f(x_0+\tau_n v) - f(x_0)]$ converging to some w. Taking $v_n = \tau_n v$ and $\rho_n = \tau_n^{-1}$ in the above definition, we see that $w \in Df(x_0)v$.

Of course, in general $Df(x_0)$ contains more than one point. Our next definition deals with the special case of single-valued $Df(x_0)$.

DEFINITION A.1: If $Df(x_0)$ is single-valued (i.e., a function) we call it the <u>Bouligand derivative</u> (<u>B-derivative</u>) of f at x_0.

One of the most useful properties of the Fréchet derivative is that of approximation. The following theorem shows that the B-derivative retains the approximation property of the Fréchet derivative, that it inherits the Lipschitz modulus of f, and that it provides the best approximation to f near x_0 among all functions whose graphs are cones.

THEOREM A.2: <u>Let</u> f <u>be Lipschitzian from the open set</u> $\Omega \subset \mathbb{R}^m$ to $\mathbb{R}^k$, <u>with modulus</u> λ, <u>and let</u> $x_0 \in \Omega$.

<u>a</u>. <u>If</u> f <u>is B-differentiable at</u> x_0, <u>then</u> $Df(x_0)$ <u>is Lipschitzian on</u> $\mathbb{R}^m$ <u>with modulus</u> λ, <u>and one has</u> $f(x) = f(x_0) + Df(x_0)(x-x_0) + o(x-x_0)$.

<u>b</u>. <u>If</u> d <u>is any function on</u> $\mathbb{R}^m$ <u>such that</u> graph d <u>is a cone and</u> $f(x) = f(x_0) + d(x-x_0) + o(x-x_0)$, <u>then</u> f <u>is B-differentiable at</u> x_0 <u>and</u> $Df(x_0) = d$.

PROOF: There is no loss of generality in assuming for the proof that $x_0 = 0$ and $f(x_0) = 0$.

Suppose f is B-differentiable at x_0. We show first that if $v \in \mathbb{R}^m$, then

$$Df(0)v = \lim_{\rho \downarrow 0} \rho^{-1} f(\rho v) . \qquad \text{(A.1)}$$

Note that for small ρ, the quantity $\rho^{-1} f(\rho v)$ is bounded in norm by $\lambda \|v\|$. Therefore it has one or more cluster points; we shall show that there is only one, namely $Df(0)v$, and this will establish (A.1).

Suppose that for some sequence $\{\rho_i\}$ converging to 0, we have $\rho_i^{-1} f(\rho_i v) \to y$, where y is some element of $\mathbb{R}^k$. Then $(\rho_i v, f(\rho_i v))$ belongs to graph f for each i, and $\rho_i^{-1}(\rho_i v, f(\rho_i v)) \to (v,y)$. It follows from B-differentiability that $y = Df(0)v$, and this proves (A.1).

To show that $Df(0)$ is Lipschitzian, choose any points x_1 and x_2 in $\mathbb{R}^m$, and let $\varepsilon > 0$. Choose ρ to be a positive number so small that ρx_1 and ρx_2 belong to Ω and (by (A.1))

$$\|\rho^{-1} f(\rho x_i) - Df(0)x_i\| < \tfrac{1}{2}\varepsilon \quad (i = 1,2) .$$

Then we have

$$\begin{aligned} \|Df(0)x_1 - Df(0)x_2\| &\leqq \|Df(0)x_1 - \rho^{-1} f(\rho x_1)\| \\ &\quad + \rho^{-1}\|f(\rho x_1) - f(\rho x_2)\| + \|\rho^{-1} f(\rho x_2) - Df(0)x_2\| \\ &< \lambda\|x_1 - x_2\| + \varepsilon . \end{aligned}$$

As ε was arbitrary, we see that $Df(0)$ is Lipschitzian on $\mathbb{R}^m$ with modulus λ.

To show that $Df(0)$ has the approximation property we want, it suffices to prove that for any sequence $\{x_n\}$ converging to 0, with $x_n \neq 0$ for each n, there is a subsequence, say $\{x_k\}$ for k belonging to some index set I, such that

$$\|x_k\|^{-1}\|f(x_k) - Df(0)x_k\| \to 0 .$$

Given $\{x_n\}$ we select I so that $v_k/\|x_k\|$ converges to some $v \in \mathbb{R}^m$. Choose $\varepsilon > 0$ and let K be so large that if $k \geqq K$ then (by (A.1))

$$\| \|x_k\|^{-1} f(\|x_k\| v) - Df(0)v\| < \tfrac{1}{2}\varepsilon$$

and

$$\|v - \|x_k\|^{-1} x_k\| < (4\lambda)^{-1}\varepsilon .$$

Write ρ_k for $\|x_k\|$; then for $k \geqq K$ we have

$$\rho_k^{-1}\|f(x_k) - Df(0)x_k\| \leqq \rho_k^{-1}\|f(x_k) - f(\rho_k v)\|$$

$$+ \|\rho_k^{-1} f(\rho_k v) - Df(0)v\| + \|Df(0)v - Df(0)(\rho_k^{-1}x_k)\|$$

$$\leqq \lambda\rho_k^{-1}\|x_k - \rho_k v\| + \tfrac{1}{2}\varepsilon + \lambda\|v - \rho_k^{-1}x_k\| < \varepsilon ,$$

where the bounds on the first and third terms come from the Lipschitzian properties of f and $Df(0)$ respectively. It follows that $\|x_k\|^{-1}\|f(x_k) - Df(0)x_k\| \to 0$, and this completes the proof of (a).

To prove (b), let d be a function with the properties described in (b). Let $(v,w) \in$ graph d. For small $\rho > 0$, $\rho v \in \Omega$, and therefore by (b)

$$f(\rho v) = d(\rho v) + o(\rho) .$$

However, $d(\rho v) = \rho d(v) = \rho w$. Thus

$$(v,w) = \rho^{-1}(\rho v, f(\rho v)) + o(\rho) ,$$

and it follows that (v,w) belongs to the contingent cone of graph f at $(0,0)$. Hence graph $d \subset$ graph $Df(0)$ (the contingent derivative).

Now suppose $(v,w) \in$ graph $Df(0)$. By definition, there are sequences $\{\rho_n\} \subset \mathbb{R}$ and $\{x_n\} \subset \Omega$ with $\rho_n \downarrow 0$, $x_n \to 0$, and such that

$$(v,w) = \lim_{n\to\infty} \rho_n^{-1}(x_n , f(x_n)) .$$

Hence

$$x_n = \rho_n v + o(\rho_n) \tag{A.2}$$

and

$$f(x_n) = \rho_n w + o(\rho n) . \tag{A.3}$$

As f is Lipschitzian, we derive from (A.2) and (A.3) the fact that

$$f(\rho_n v) = \rho_n w + o(\rho_n) . \tag{A.4}$$

However, by hypothesis

$$f(\rho_n v) = d(\rho_n v) + o(\rho_n) . \tag{A.5}$$

From (A.4), (A.5), and the fact that $d(\rho_n v) = \rho_n d(v)$ (because graph d is a cone), we obtain

$$w = d(v) + \rho_n^{-1} o(\rho_n) .$$

and letting $n \to \infty$ we find that $w = d(v)$, so that graph $Df(0) \subset$ graph d. But d is single-valued, and therefore since we have shown that $d = Df(0)$, we conclude that f is B-differentiable at 0 with B-derivative equal to d. This proves Theorem A.2.

This theorem can be quite useful in identifying B-derivatives and in establishing their properties. The next two corollaries illustrate some properties that can easily be proved using it.

COROLLARY A.3: _Suppose_ f _and_ g _are Lipschitzian functions from an open set_ $\Omega \subset \mathbb{R}^m$ _to_ $\mathbb{R}^k$. _Let_ $x_0 \in \Omega$, _and suppose_ f _and_ g _are B-differentiable at_ x_0. _Then_

_a_. _If_ $\alpha \in \mathbb{R}$ _then_ αf _is B-differentiable at_ x_0 _and_ $D(\alpha f)(x_0) = \alpha Df(x_0)$.

b. $f + g$ *is B-differentiable at* x_0 *with* $D(f+g)(x_0) = Df(x_0) + Dg(x_0)$.

PROOF: For (a), we just note that

$$f(x) = f(x_0) + Df(x_0)(x-x_0) + o(x-x_0) , \tag{A.6}$$

and hence

$$(\alpha f)(x) = (\alpha f)(x_0) + \alpha Df(x_0)(x-x_0) + o(x-x_0) .$$

As the graph of $\alpha Df(x_0)$ is a cone, Part (b) of Theorem A.2 tells us that αf is B-differentiable at x_0 and $D(\alpha f)(x_0) = \alpha Df(x_0)$.

For (b), the proof is similar to that of (a), except that we write expressions like (A.6) for each of f and g . As $Df(x_0)$ and $Dg(x_0)$ are single-valued, $Df(x_0) + Dg(x_0)$ is a single-valued function on $\mathbb{R}^m$ whose graph is a cone. Again, Part (b) of Theorem A.2 gives us the result.

COROLLARY A.4: *Let* f *be a Lipschitzian function from an open set* $\Phi \subset \mathbb{R}^m$ *to* $\mathbb{R}^k$. Let $x_0 \in \Omega$, *and suppose* g *is a Lipschitzian function from an open set* $\Gamma \subset \mathbb{R}^k$, *with* $f(x_0) \in \Gamma$, *to* $\mathbb{R}^1$. *If* f *is B-differentiable at* x_0 *and* g *is B-differentiable at* $f(x_0)$, *then* $g \circ f$ *is B-differentiable at* x_0 *and*

$$D(g \circ f)(x_0) = Dg[f(x_0)] \circ Df(x_0) .$$

PROOF: First note that if (u,w) belongs to graph $Dg[f(x_0)] \circ Df(x_0)$, then with $v = Df(x_0)u$ we have $(u,v) \in$ graph $Df(x_0)$ and $(v,w) \in$ graph $Dg[f(x_0)]$. If $\alpha > 0$ then $\alpha v = Df(x_0)(\alpha u)$ and $\alpha w = Dg[f(x_0)](\alpha v)$. Hence $\alpha(u,w) \in$ graph $Dg[f(x_0)] \circ Df(x_0)$, so that this graph is a cone.

Now note that

$$f(x) = f(x_0) + Df(x_0)(x-x_0) + o(x-x_0)$$

so that

$$
\begin{aligned}
(g\circ f)(x) &= g[f(x_0) + Df(x_0)(x-x_0) + o(x-x_0)] \\
&= g[f(x_0) + Df(x_0)(x-x_0)] + o(x-x_0) \\
&= (g\circ f)(x_0) + Dg[f(x_0)](Df(x_0)(x-x_0)) + o(x-x_0) \\
&= (g\circ f)(x_0) + [Dg[f(x_0)]\circ Df(x_0)](x-x_0) + o(x-x_0) , \qquad \text{(A.7)}
\end{aligned}
$$

where we have used the fact that g and $Df(x_0)$ are both Lipschitzian, as well as the approximation information given by Part (a) of Theorem A.2. Now we apply Part (b) of Theorem A.2 to (A.7) to prove the corollary.

Finally, we note that evidently any Fréchet derivative is also a B-derivative, since its graph is a cone and it has the approximation property treated in Theorem A.2.

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