ITERATIVE METHODS FOR ELLIPTIC PROBLEMS

AND THE DISCOVERY OF "q"

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Seymour V. Parter

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Seymour V. Parter[1]

(1) Computer Sciences Department, University of Wisconsin, Madison, WI.

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ABSTRACT

Consider a direct iterative method for solving the linear system $AU = Y$ which arises from the discretization of a boundary value problem involving an elliptic partial differential operator L of order $2m$. Such iterative methods split A into a difference $A = M - N$. The critical question is the behavior of $\rho = \max\{|\lambda|, \lambda \text{ an eigenvalue of } M^{-1}N\}$. In this report we develop a theory for the determination of the asymptotic behavior of ρ as $h \to 0$. This theory depends on recognizing $(h^P N)$ as a weak approximation to a differential operator q of order $\leq m$. Then, under "appropriate conditions", $\rho \sim 1 - \Lambda_0 h^P$ where Λ_0 is the real part of the minimal eigenvalue of $Lu = \lambda qu$. The study of the appropriate conditions leads one to three interesting mathematical questions: (i) relevance, (ii) Spectral Approximation and (iii) general estimates.

1. Introduction

An important problem that arises often in applications is the solution of an elliptic boundary-value problem

(1.1) $$Lu = f \quad \text{in} \quad \Omega$$

(1.2) $$Bu = 0 \quad \text{on} \quad \partial\Omega$$

where Ω is a bounded (or unbounded) smooth domain in $\mathbb{R}^d$. The boundary operator B is in general a linear differential operator. However, the simplest case is (for second order problems) merely

(1.2') $$u = 0 \quad \text{on} \quad \partial\Omega .$$

Such elliptic boundary-value problems are frequently stated in their "weak form." Let $H_m(\Omega)$ be the functions defined on Ω which have m'th derivatives in $L_2(\Omega)$. Let $\tilde{H}_m \subset H_m(\Omega)$ be a subspace that incorporates (in an appropriate manner) the boundary conditions (1.2). Let $a(u,\phi)$ be a bilinear, continuous and coercive form defined on $\tilde{H}_m \otimes \tilde{H}_m$:

(1.3) $a(u,\phi)$ is bilinear, so that

(1.3a) $$a(\alpha u_1 + \beta u_2, \phi) = \alpha a(u_1,\phi) + \beta a(u_2,\phi) \quad \text{and}$$

(1.3b) $$a(u, \alpha\phi_1 + \beta\phi_2) = \bar{\alpha} a(u,\phi_1) + \bar{\beta} a(u,\phi_2) \; ;$$

(1.4) $a(u,\phi)$ is continuous, so

there is a constant $K > 0$ such that

(1.4a) $$|a(u,\phi)| \leq K\|u\|_m \cdot \|\phi\|_m, \quad \forall u, \phi \in \tilde{H}_m \; ;$$

(1.5) $a(u,\phi)$ is coercive, i.e.,

$a(u,u)$ is real and there is a constant $K_0 > 0$ such that

(1.5a) $$K_0 \|u\|_m^2 \leq a(u,u), \quad \forall u \in \tilde{H}_m .$$

Then, under certain appropriate conditions, the problem (1.1)-(1.2) takes the form:

<u>Find</u> $u \in \tilde{H}_m$ <u>so that</u>

(1.6) $$a(u,\phi) = F(\phi), \quad \forall \phi \in \tilde{H}_m$$

where F is a linear functional defined on $\tilde{H}_m$.

Condition (1.4)-(1.6) guarantees that there is a unique solution u^*. However, in general, one cannot explicitly exhibit u^*. Then, the desire for precise quantitative information about u^* leads to numerical approaches. The problem is "discretized," i.e., the problem is set in a finite-dimensional subspace, and we obtain a system of linear algebraic equations of the form

(1.7) $$AU = F .$$

Here A is an $n \times n$ matrix, n is large, and (somehow)

(1.8) $$A \sim a .$$

One approach to the solution of (1.7) is a direct iterative method based on a "splitting" of the matrix A. We write

(1.9) $$A = M - N ,$$

whence (1.7) takes the form

(1.10) $$MU = NU + F .$$

Given a first guess, say U^0, we then obtain iterates U^j from the scheme

(1.11) $$MU^{j+1} = NU^j + F .$$

We are now almost ready to discuss the problem of this report - the asymptotic behavior of the spectral radius of the iteration matrix

(1.12) $$K = M^{-1}N .$$

It is well known that the convergence of the iterative scheme (11) is determined by the spectral radius of K. Therefore we are concerned with the eigenvalue problem

(1.12a) $$\lambda MX = NX, \quad X \neq 0$$

and especially with

(1.12b) $$\rho := \max |\lambda| .$$

We suppose that this "splitting" has been arranged in some regular way and OUR PROBLEM is the study of the ASYMPTOTIC BEHAVIOR OF ρ as $n \to \infty$.

This is an old problem. There are results of Shortley and Weller - 1938 [18], a beautiful theory by Young - 1951 [21], which extended work of Frankel - 1950 [7], an intriguing paper by Garabedian - 1956 [8], a fine book by Varga - 1962 [19]; and right now there is an active school in Brussels, Belgium working with Beauwens [2], [3], [4] applying methods of analysis to these problems. Of course, many, many others have made, and are making contributions. I apologize to all whose names have been omitted.

The theory of Young is based on a combinatorial concept - Property A. The reader need not be expert in this theory. However, I do wish to mention one interesting and important consequence of this theory.

For (block) Jacobi schemes which have Property A, if λ is an eigenvalue of $M^{-1}N$, so is $(-\lambda)$!

Our approach to this problem can be outlined as follows. We rewrite (1.12a) as

$$\lambda(M-N)X = (1-\lambda)NX,$$

and if $\lambda \neq 0$ ($\lambda = 0$ is a particularly uninteresting case) then

(1.13) $$AX = \left[\frac{1-\lambda}{\lambda h^p}\right][h^p N]X .$$

That is,

(1.14a) $$AX = \mu \tilde{N} X ,$$

(1.14b) $$\mu = \frac{1-\lambda}{\lambda h^p} , \quad \tilde{N} = h^p N,$$

where p is as yet undetermined. Because A arises from L in some natural way we hope to find a related eigenvalue problem of the form

$$(1.15) \qquad Lu = \Lambda qu \; , \quad Bu = 0 \;\; \text{on} \;\; \partial\Omega \; ,$$

where q is a differential operator of degree lower than $2m$ ($\leq m$ in fact!). Thus in some sense

$$(1.16) \qquad h^p N \sim q \; .$$

Then, "if all goes right," there will be an eigenvalue λ of (1.12a) with

$$|\lambda| = \rho$$

and

$$\frac{1-\lambda}{\lambda h^p} = \mu = \Lambda_{min} + o(1),$$

where $\Lambda_{min} = \Lambda_0 + iT$ is a "minimal" eigenvalue of (1.15). Then

$$\lambda = \frac{1}{1+\mu h^p} = 1 - \Lambda_{min} h^p + o(h^p)$$

and

$$\rho = 1 - \Lambda_0 h^p + o(h^p) \; .$$

In his basic paper Garabedian [8] began with equation (1.11) as applied to the error $\varepsilon^{k+1} = u - u^{k+1}$ and wrote

$$M\left(\frac{\varepsilon^{j+1} - \varepsilon^j}{\Delta t}\right) = \frac{1}{\Delta t}(N-M)\varepsilon^j = \frac{-1}{\Delta t} A\varepsilon^j \; .$$

This expression led him to a time-dependent problem

$$\gamma \frac{\partial u}{\partial t} = \alpha u_{xt} + \beta u_{yt} + Lu \; ,$$

which in turn led to the eigenvalue problem

(1.18) $$L\phi = \lambda[\gamma\phi - \alpha\phi_x - \beta\phi_y] , \quad B\phi = 0$$

and the condition

$$\rho = e^{-\lambda_{min}\Delta t} = 1 - \lambda_0\Delta t + 0(\Delta t^2)$$

where λ_{min} is the "minimal" eigenvalue of (1.18).

As one develops this idea one finds many interesting mathematical questions that must be resolved before one can assert the validity of (1.19). Garabedian neither raised nor resolved these questions. Yet in the particular cases he discussed, we believe his results are correct. In one case (the five point star for the Laplace operator) these results follow independently from the theory of Young [21], [22]. There is a large amount of computational experience that indicates the validity of the results in the other case (the nine-point star for the Laplace operator).

In this report we develop the theory in a very general setting. The theory and results are useful and interesting in themselves. However, we shall also emphasize three interesting mathematical questions that this theory brings to the forefront. These are:

(i) Spectral approximation.

(ii) Relevance:= a study of properties of those eigenvalues λ with

$$|\lambda| = \rho .$$

(iii) Estimates:= determining the unimportant terms and providing a basis for proofs of Spectral approximation.

2. Examples

We start with a simple 1-dimensional example

$$-u'' = f\ , \quad 0 \le x \le 1 \tag{2.1}$$

$$u(0) = u(1) = 0\ . \tag{2.2}$$

Let us consider the simplest finite difference approach to this problem. Let

$$h = \frac{1}{n+1} \tag{2.3}$$

and let A be the $n \times n$ tridiagonal matrix

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & \bigcirc & \\ & -1 & 2 & -1 & & \\ & & \cdot & \cdot & \cdot & \\ & \bigcirc & & \cdot & \cdot & \cdot \\ & & & \cdot & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} = \frac{1}{h}\,[-1,2,-1]\ . \tag{2.4}$$

Most people would expect to find $(\frac{1}{h^2})$ rather than $\frac{1}{h}$. However, we have chosen this normalization both because it is appropriate for the presentation of our theory and because this normalization is appropriate from the finite element point of view.

To this end, with each vector

$$U = (u_1, u_2, \ldots u_n)^T$$

we associate a continuous piecewise linear function

$$u(x;h)$$

which is linear on each interval $kh \leq x \leq (k+1)h$ and satisfies

$$u(kh,h) = u_k \,, \quad u(0,h) = u(1,h) = 0 \,.$$

The space of all such functions is designated S_h. Then (see Figure 1) for every two such vectors U, V we have

$$\text{(2.5)} \qquad V^*AU = \int_0^1 \bar{v}'(x,h)u'(x,h)dx = a(u,v) \,.$$

Thus in this example we see exactly how A and $a(u,\phi)$ are associated - recall (1.8).

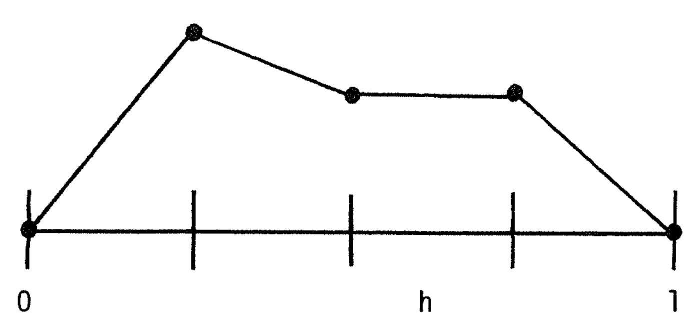

Figure 1

In any case, the boundary value problem (2.1) - (2.2) is replaced by the linear system

$$AU = F \tag{2.6}$$

where the vector F is related to the function $f(x)$. For example, in the simplest case

$$F = h(f(h), f(2h), \dots f(nh))^T .$$

The (point) Jacobi iterative scheme (which we certainly do not recommend, but discuss as a simple example) is given by

$$\frac{2}{h} I U^{k+1} = \frac{1}{h} \begin{bmatrix} 0 & 1 & 0 & & & 0 \\ 1 & 0 & 1 & 0 & & \\ 0 & 1 & 0 & 1 & 0 & \\ 0 & 0 & 1 & & \ddots & \ddots \\ & & & \ddots & & 1 \\ & & & & 1 & 0 \end{bmatrix} U^k + F . \tag{2.7}$$

Hence

$$A = M - N$$

with

$$M = \frac{2}{h} I \tag{2.8a}$$

and

(2.8b) $$N = \frac{1}{h}\begin{bmatrix} 0 & 1 & & \bigcirc & \\ 1 & 0 & 1 & & \\ 0 & 1 & 0 & 1 & \\ & \bigcirc & & & 1 \\ 0 & & & 1 & 0 \end{bmatrix} = \frac{1}{h}[1,0,1]\ .$$

It happens that the eigenvalues and eigenvectors of $M^{-1}N$ are known [19]. These are

(2.9) $$\lambda_k = \cos\pi kh\ , \quad X^k = (\sin\pi kjh)_{j=1}^{n}\ .$$

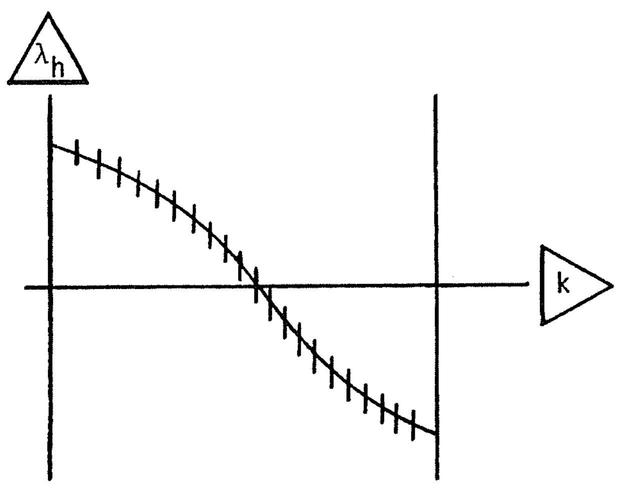

Figure 2

<u>Remark 1</u>: This particular scheme does indeed possess property A - and it is indeed true that the eigenvalues appear in pairs

$$(\lambda_k,\ \lambda_{n-k+1} = -\lambda_k)\ .$$

Remark 2: Consider the eigenvalues λ with

$$|\lambda| = \rho .$$

There are two, λ_1 and λ_n. In this case

(2.10) $$\rho = \lambda_1 = \cos \pi h = 1 - \frac{1}{2} \pi^2 h^2 + o(h^2) .$$

Let us write

(2.11) $$A = \frac{2}{h} I - W - W^T = D - W - W^T$$

where W is a strictly lower triangular matrix. Then the Jacobi scheme just discussed is given by

(2.12) $$\frac{2}{h} I \, U^{k+1} = (W+W^T)U^k + F$$

and the corresponding eigenvalue problem

$$\lambda MX = NX , \quad X \neq 0$$

is a "nice" standard problem for the eigenvalues of a real symmetric matrix N relative to a positive definite matrix M.

The associated point Gauss-Seidel iteration scheme is given by

(2.13) $$(\frac{2}{h} I - W)U^{k+1} = W^T U^k + F$$

and its associated eigenvalue problem is not a regular problem: N is not symmetric and M is not positive definite. But there are some saving features. The matrix

$$K = M^{-1} N$$

is a nonnegative irreducible matrix. Hence the Perron-Frobenius theory [19] tells us that there is exactly one eigenvalue λ with

$$|\lambda| = \rho$$

and, in fact, that λ satisfies

$$\lambda = \rho \ !$$

Indeed, the theory of Young [21], [22] tells us that

$$\rho = (\cos \pi h)^2 = 1 - \pi^2 h^2 + O(h^4) \ . \tag{2.14}$$

Another important scheme based on the representation (2.11) is the SOR-ω (successive overrelaxation) scheme. Let ω be chosen with

$$0 < \omega < 2 \ .$$

Consider the scheme

$$\frac{1}{\omega}(D-\omega W)U^{k+1} = \frac{1}{\omega}[\omega W^T + (1-\omega)D]U^k + F \ . \tag{2.15}$$

This method is known to be convergent [19] and if $\mu \neq 0$ is an eigenvalue of this SOR-ω scheme then there is a k, $1 \leq k \leq n$, for which μ satisfies the equation

$$(\mu + \omega - 1)^2 = \omega^2 (\cos \pi k h)^2 \mu \ . \tag{2.16}$$

For our purposes we need only observe that (see page 111 of [19]): for each ω, $0 < \omega < 2$, there is an eigenvalue μ with

$$|\mu| = \rho$$

and

(2.17) $$\text{Re}\ \mu > 0 .$$

Remark: In all three cases we have discussed the "relevance" question - as defined in section 1. In a few pages we shall see why this is indeed relevant.

Let us now return to our general problem

(*) $$\lambda MU = NU .$$

Then

$$\lambda(M-N)U = \lambda AU = (1-\lambda)NU .$$

Hence if $\lambda \neq 0$ we obtain

$$AU = \left[\frac{1-\lambda}{\lambda h^p}\right] [h^p N]U .$$

That is

(2.18a) $$AU = \mu\tilde{N}U$$

where

(2.18b) $$\mu = \frac{1-\lambda}{\lambda h^p} , \quad \tilde{N} = h^p N .$$

The exponent p has not yet been chosen. Consider the Jacobi iterative scheme (2.7).

Let us write (2.18a) in the "weak form": Find U so that

$$V^*AU = \mu V^*\tilde{N}U \,, \quad \forall V \in C^n \,. \tag{2.18c}$$

Using the basic (finite element) fact (2.5) we have

$$\int_0^1 \bar{v}'(x,h)u'(x,h)dx = \mu h^{p-1} \sum_{k=1}^{n} \bar{v}_k[u_{k+1}+u_{k-1}] \,. \tag{2.19}$$

Since the left-hand-side of (2.19) is an integral and the right-hand-side is "almost" a Riemann sum, we choose

$$p = 2 \,. \tag{2.20}$$

We write

$$[u_{k+1}+u_{k-1}] = 2u_k + h\{\frac{u_{k+1}-2u_k+u_{k-1}}{h}\}$$

and (2.19) becomes

$$(1 + h^2\mu)\; V^*AU = 2\mu\; h\sum \bar{v}_k u_k \,,$$

or

$$(1 + h^2\mu) \int_0^1 \bar{v}'(x,h)u'(x,h)dx \sim 2\mu \int_0^1 \bar{v}(x,h)u(x,h)dx \,.$$

If we ignore the term $h^2\mu$ we see that we have convinced ourselves that the eigenvalue μ of (2.18) - with $p = 2$ - is related to eigenvalues of the weak problem

(2.21) $$a(u,\phi) = \Lambda \int_0^1 2u(x)\phi(x)dx \ .$$

Our first question is: does (2.21) have eigenvalues? Because (2.21) is the weak form of

(2.22a) $$-u'' = \Lambda q(x)u \ , \quad 0 \leq x \leq 1$$

(2.22b) $$u(0) = u(1) = 0$$

with

(2.22c) $$q(x) = 2$$

the answer is yes. In particular, we come to the first part of our discussion of spectral problems. The problem (2.21) (or equivalently (2.22)) possesses a minimal eigenvalue $\Lambda_0 > 0$. In fact

(2.23) $$\Lambda_0 = \frac{1}{2}\pi^2 .$$

Our next question is: Do there exist eigenvalues of (2.18) near Λ_0? Let us be more precise: Let us define

Spectral Approximation (a): Let $\delta >$ be given. Then there is an $h_0 > 0$ such that for each h, $0 < h \leq h_0$, there is an eigenvalue $\mu(h)$ of (2.18) satisfying

$$|\Lambda_0 - \mu(h)| < \delta \ .$$

Now our question is: Is Spectral Approximation (a) true? In this case the answer is yes!

Because the answer is yes, there is an eigenvalue

$$\lambda = \lambda(h) = \frac{1}{1+h^2\mu(h)}$$

of (*) and

$$\lambda = 1 - \frac{1}{2}\pi^2 h^2 + o(h^2) .$$

Hence, by the definition of ρ

$$\rho \geq 1 - \frac{1}{2}\pi^2 h^2 + o(h^2) . \qquad (2.24)$$

In order to complete the discussion and reobtain (2.10) we must know that the eigenvalue problems (2.18) and (2.21) are relevant to the eigenvalue problem (*). For example, suppose we did not know the eigenvalues of (2.18) and (as is <u>not</u> the case) the eigenvalues λ with $|\lambda| = \rho$ all satisfied

$$\lambda < 0 ;$$

then the corresponding μ satisfy

$$|\mu(h)| = \left|\frac{1-\lambda}{\lambda h^2}\right| \nearrow +\infty .$$

Thus, these eigenvalues of (2.18) are not relevant to the limiting eigenvalue problem (2.21). Let us define

<u>Relevance</u>: There are constants $C_0 > 0$, $h_0 > 0$, and for $0 < h \leq h_0$ there is an eigenvalue λ with $|\lambda| = \rho$ and

$$\left|\frac{1-\lambda}{h^p}\right| \leq C_0 . \qquad (2.25)$$

In addition to the question of relevance we must discuss

Spectral Approximation (b): Let $\mu(h_n)$ be a sequence of eigenvalues of (2.18) with $h_n \to 0$. Suppose

$$\mu(h_n) \to \mu_\infty .$$

Then μ_∞ is an eigenvalue of the limiting equation (2.21).

In this case, because of the variational characterization of eigenvalues of (2.18) and the variational characterization of the eigenvalues of (2.21), it is not difficult to prove that Spectral Approximation (b) is true and we can complete the argument as follows. Let $\lambda(h)$ be a relevant eigenvalue with

$$|\lambda(h)| = \rho(h) .$$

Because we know (from very general considerations) that the Jacobi scheme is convergent, we deduce from (2.24) that

$$1 - \frac{1}{2}\pi^2 h^2 + o(h^2) \le \lambda(h) \le 1 ,$$

and hence the relevance condition is satisfied with

$$C_0 = \pi^2 .$$

Therefore the associated eigenvalues $\mu(h)$ of (2.18) are uniformly bounded. Hence a subsequence, say $\mu(h_n)$, converges to an eigenvalue of (2.21), say μ_∞. Then

$$\frac{1-\lambda}{h^2} = \mu_\infty + o(1)$$

and

$$\lambda = 1 - \mu_\infty h^2 + o(h^2) \ .$$

Hence, because μ_∞ is of the form

$$\mu_\infty = \frac{1}{2} k^2\pi^2 \ ,$$

we get

(2.26) $$\lambda = \rho \leq 1 - \frac{k^2}{2}\pi^2 h^2 + o(h^2) \ .$$

Combining (2.26) with (2.24) yields (2.10).

This approach was used by Parter [11], [12] to establish a general theorem - in the self-adjoint case. More recently Parter and Steuerwalt [14] have extended that argument to finite difference equations for the general second order elliptic operator, and to related parabolic problems.

Now let us consider the Gauss-Seidel iterative scheme. Following our recipe we obtain (2.18) with

(2.27) $$N = \frac{1}{h}\,[0,0,1]$$

and

(2.28) $$V^*\tilde{N}U = h^{p-1} \sum_{k=1}^{n-1} \bar{v}_k u_{k+1} \ .$$

Hence, arguing as before we have

(2.29a) $$p = 2$$

and

(2.29b) $$V^*\tilde{N}U = h\sum_{k=1}^{n} \bar{v}_k u_k + h[h\sum_{k=1}^{n} \bar{v}_k(\frac{u_{k+1}-u_k}{h})] .$$

Because

$$|h\sum\bar{v}_k(\frac{u_{k+1}-u_k}{h})| \leq [h\sum\bar{v}_k^2]^{\frac{1}{2}}\left[h\sum(\frac{u_{k+1}-u_k}{h})^2\right]^{\frac{1}{2}} .$$

our equation (2.19) becomes

(2.30) $$\int_0^1 \bar{v}'(x)u'(x)dx = \mu h\sum\bar{v}_k u_k + \mu h\varepsilon(v,u)$$

where (using standard inequalities)

(2.30b) $$|\varepsilon(v,u)| \leq C_0 ||v||_{L_2} \cdot ||u'||_{L_2} .$$

In this case (2.30) leads to the related problem

(2.31) $$a(u,\phi) = \Lambda\int_0^1 u(x)\bar{\phi}(x)dx , \quad \forall\phi \in S_h .$$

This is the weak form of the problem

(2.32a) $$-u'' = \Lambda q(x)u , \quad 0 \leq x \leq 1$$

(2.32b) $$u(0) = u(1) = 0$$

with

(2.32c) $$q(x) = 1 .$$

In this case

$$\Lambda_0 = \pi^2$$

and we are led to conjecture that

$$\rho = 1 - \pi^2 h^2 + o(h^2) \quad (2.33)$$

As we know - see (2.14) - this is correct!! Let us see what we must prove in order to apply our general arguments.

Once more, general considerations imply that the method is convergent. And, as we have indicated earlier, the Perron-Frobenius theory implies that ρ is itself an eigenvalue. However, as we have said, in this instance the eigenvalue problem (1.12a) is not a self-adjoint problem while the eigenvalue problem (2.32a) is a self-adjoint problem. Hence in this case the verification of the spectral approximation property is not so simple.

Now let us consider the point SOR iterative scheme (2.15). With our knowledge of the nature of the final results based on the SOR theory of Young [21], [22] let us set

$$\omega = 2 - Ch , \quad (2.34a)$$

$$\tilde{N} = hN , \quad \mu = \frac{1-\lambda}{\lambda h} , \quad (2.34b)$$

i.e.,

$$p = 1 .$$

A direct computation shows that

$$V^*\tilde{N}U = h\sum[\tfrac{1}{2}C]\bar{v}_k u_k + h\sum\bar{v}_k[\frac{u_{k+1}-u_k}{h}] \tag{2.35}$$
$$+ O(h^2\sum\bar{v}_k u_k) .$$

Thus we are led to compare the basic discrete eigenvalue problem (2.18a) with the eigenvalue problem

$$-u'' = \Lambda[\tfrac{1}{2}Cu + u'] , \quad 0 \leq x \leq 1 \tag{2.36a}$$

$$u(0) = u(1) = 0 . \tag{2.36b}$$

Thus, in this case

$$q = \frac{1}{2}C + \frac{d}{dx} . \tag{2.37}$$

Once more we ask our basic questions:

(i) Does (2.36) have eigenvalues?

(ii) Is Spectral Approximation (a) true?

(iii) Is Spectral Approximation (b) true?

(iv) What about the relevance of the eigenvalues of the discrete problem (2.18) to the eigenvalue problem (2.36)?

In this simple case all the answers are known, and the theory applies.

We hope these examples have given some notion of how one proceeds with these ideas. However, in all the examples so far, (because we have been dealing with "point" iterative schemes) it would seem that the "weak form" of eigenvalue problem is unnecessary. We can readily recognize (2.18a) as a consistent approximation to the eigenvalue problem

(2.38) $$-u'' = \Lambda q u \ , \quad u(0) = u(1) = 0 \ .$$

Hence we close this section with a simple - but complex - example: the "4-point Jacobi Scheme." However, let us once more remind the reader that this procedure is not being suggested as a means of solving this simple problem. It is an interesting example of the ideas involved in this theory. Let

(2.39) $$n = 4\sigma$$

and let A_4 be the 4×4 matrix

(2.40a) $$A_4 = \frac{1}{h} [-1,2,-1] \ ,$$

and give M and N the induced block tridiagonal structures

(2.40b) $$M = \begin{bmatrix} A_4 & & \bigcirc \\ & A_4 & \\ \bigcirc & & A_4 \end{bmatrix} ,$$

(2.40c) $$N = \frac{1}{h}\begin{bmatrix} 0 & R & & & & \\ R^T & 0 & R & & \bigcirc & \\ & R^T & 0 & R & & \\ & & R^T & \ddots & & \\ & \bigcirc & & & \ddots & R \\ & & & & R^T & 0 \end{bmatrix} .$$

where $$R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

As usual, our iterative scheme is given by (1.11). We take

(2.41a) $$\tilde{N} = h^2 N , \quad p = 2 ,$$

(2.41b) $$\mu = \frac{1-\lambda}{\lambda h^2} .$$

Then

(2.42) $$V^*\tilde{N}U = h \sum_{s=1}^{\sigma-1} [\bar{v}_{4s+1}u_{4s} + \bar{v}_{4s}u_{4s+1}] .$$

Clearly, if

$$v_{4s+1} \approx v_{4s} , \quad u_{4s+1} \approx u_{4s}$$

then (2.42) suggests that $V^*\tilde{N}U$ is a Riemann sum based on a Δx of $4h$. But the multiplier of (2.42) is h . So we rewrite (2.42) as

(2.43a) $$V^*\tilde{N}U \approx 4h \sum \bar{v}_{4s}u_{4s}q$$

where

(2.43b) $$q = \frac{1}{2} = \frac{2}{4} .$$

Thus we are led to the eigenvalue problem

(2.44a) $$-u'' = \Lambda qu , \quad u(0) = u(1) = 0 ,$$

(2.44b) $$q = \frac{1}{2} .$$

In this way, after answering our mathematical questions of Spectral Approximation and "relevance," we have

(2.45) $$\rho_4 = 1 - 2\pi^2 h^2 + o(h^2) .$$

Similarly, if we consider the k-point block Jacobi iterative scheme we obtain

(2.46) $$\rho_k = 1 - \frac{k}{2}\pi^2 h^2 + o(h^2) .$$

While the results (2.45) and (2.46) are interesting, we invite the reader to see $\tilde{N}$ - as given by (2.40c) - as a consistent approximation to the function $q(x) = 1/2$. In fact, the operators $\tilde{N}$ that arise in block schemes with large blocks all behave as does (2.40c). They are rather nasty operators. They are the "homogenization" operators of Babuška [1] and Lions [5].

3. A More Formal Outline

Let us reconsider what we have done. Beginning with a matrix A that is related to the differential operator L or, equivalently, to the bilinear form $a(u,\phi)$, we write

$$A = M - N\,, \quad (M^{-1} \text{ exists})$$

and consider the eigenvalue problem

$$\lambda MU = NU\,. \tag{3.1}$$

Our concern is with the spectral radius

$$\rho = \max|\lambda|\,. \tag{3.2}$$

We rewrite (3.1) as

$$V^*AU = \mu V^*\tilde{N}U\,, \quad \forall V \in C^n\,, \tag{3.3a}$$

where

$$\mu = \frac{1-\lambda}{\lambda h^p}\,, \quad \tilde{N} = h^p N\,. \tag{3.3b}$$

Having done this we seek to discover an operator q such that

$$q\colon H_m \to L_2$$

and, in some way, we may recognize (3.3) as an approximation to the weak eigenvalue problem

$$a(u,\phi) = \Lambda \int\int \bar{\phi} q[u]dxdy\,, \quad \forall \phi \in \tilde{H}_m\,. \tag{3.4}$$

Then we assume

A.1) <u>The eigenvalue problem (3.4) possesses a "minimal" eigenvalue</u>

(3.5) $$\Lambda_{min} = \Lambda_0 + iT .$$

with $\Lambda_0 > 0$. By "minimal" we mean that for any eigenvalue Λ we have

$$0 < \Lambda_0 \leq \text{Re } \Lambda ,$$

and if $\text{Re } \Lambda = \Lambda_0$ then

$$|\Lambda_{min}| \leq |\Lambda| .$$

A.2) <u>Spectral Approximation</u> (a), (b): for h small enough there is an eigenvalue $\mu(h)$ near Λ_{min}, and the limit of every convergent sequence of eigenvalues $\{\mu(h_n)\}$ of (3.3a) is an eigenvalue Λ of (3.4).

A.2) <u>Relevance</u>: There are positive constants C_0, h_0 so that for every $h \leq h_0$ there is an eigenvalue λ of the basic eigenvalue problem (3.1) that satisfies

$$|\lambda| = \rho$$

and

$$\left|\frac{1-\lambda}{h^p}\right| \leq C_0 .$$

Under these circumstances it is an easy matter to show that

(3.6a) $$\rho = 1 - \Lambda_0 h^p + o(h^p) .$$

When one can prove the truth of Spectral Approximation (a), (b) but cannot discuss the relevance question one obtains

(3.6b) $$\rho \geq 1 - \Lambda_0 h^p + o(h^p) .$$

This result involves two of the three mathematical topics discussed in the introduction. We have not yet discussed "estimates."

In the discussion of the point Jacobi iterative scheme (2.7) we were led to (2.19). We then wrote

(3.7a) $$h\sum \bar{v}_k(u_{k+1}+u_{k-1}) = h\sum(2)\bar{v}_k u_k + \varepsilon(v,u)$$

and

(3.7b) $$h\sum \bar{v}_k(u_{k+1}+u_{k-1}) = \int_0^1 2\bar{v}(x)u(x)dx + E(v,u) .$$

In dealing with the 4-point Jacobi scheme we wrote - see (2.42) -

(3.8a) $$h\sum_{s=1}^{\sigma-1}(\bar{v}_{4s+1}u_{4s} + \bar{v}_{4s}u_{4s+1}) = h\sum 2\bar{v}_{4s}u_{4s} + \tilde{\varepsilon}(v,u)$$

and

(3.8b) $$h\sum_{s=1}^{\sigma-1}(\bar{v}_{4s+1}u_{4s} + \bar{v}_{4s}u_{4s+1}) = \int_0^1 (\tfrac{1}{2})\bar{v}u dx + \tilde{E}(v,u) .$$

In these formulae, the terms $\varepsilon(v,u)$, $\tilde{\varepsilon}(v,u)$, $E(v,u)$, $\tilde{E}(v,u)$ represent "errors" that can be estimated. The question is, is it sufficient merely to recognize these errors as being small provided that $u(x)$, $v(x)$ are "nice, smooth" functions, or are more precise estimates required? In our work to date we have met two situations. In the first case - and by far the most common one -- one can easily prove: if (3.3a) holds and $v(x)$ is smooth then

(3.9a) $$V^*AU = a(u,v) + o(||v||_s + ||u||_{H_m})$$

(3.9b) $$V^*\tilde{N}U = \int\int \bar{v}q[u]dxdy + o(||v||_s + ||u||_{H_m})$$

where $||v||_s$ denotes an appropriate norm of v. In this case the proof of the validity of the Spectral approximation hypothesis is relatively straightforward. In the finite element case this result follows from the (by now) standard theories - e.g. see [10]. For the finite difference case we carried out an appropriate modification of the standard argument for a special problem in the non self-adjoint case [13]. The ideas in that work generalize. Of course, when both (3.3a) and (3.4) are self-adjoint the Spectral Approximation problem is much simpler. One can use the variational principles that characterize eigenvalues to obtain the desired results. Such proofs of Spectral Approximation based on variational arguments are quite old - see [20].

In our general work with Steuerwalt on finite element equations we used very precise estimates, which we will discuss in Section 5, to obtain a new convergence theorem. Moreover, as we will see in Section 5, those estimates are extremely useful in the discovery of "q."

4. Finite Difference Methods - 2nd Order Problems

We now consider a special case that arises often in practice: 2nd order linear elliptic operators in 2 dimensional regions, i.e.

(4.1)
$$Lu := -[(au_x)_x + (bu_x)_y + (bu_y)_x + (cu_y)_y] + d_1 u_x + d_2 u_y + d_0 u \ .$$

Actually, the fact that we are considering only 2-dimensional problems is merely a notational convenience. The ideas of this section generalize to elliptic problems in any number of dimensions.

In many finite difference equations for such problems we may write the unknown as u_{kj}, where we imagine

$$u_{kj} \cong u(k\Delta x, j\Delta y) \ .$$

That is, the underlying space of functions are continuous piecewise linear functions and the unknowns $\{u_{kj}\}$ are merely the function values at selected points in the region Ω. In this setting it is convenient to use double indices to describe the vector $\{u_{kj}\}$ of unknowns. Correspondingly, the matrix A of our basic problem (7) is described by pairs of double indices. We write (7) as

(4.2)
$$\sum_{\sigma\mu} a_{kj,\sigma\mu} u_{\sigma\mu} = f_{kj} \ .$$

Let's assume that

(4.3a)
$$V^*AU = a(u,v) + \varepsilon_a(u,v)$$

where

(4.3b) $$\varepsilon_a(u,v) = o(a(u,u) + a(v,v)) .$$

In many iterative schemes we have a relatively simple splitting (1.9) that satisfies

<u>Condition S</u>:

(1) If $a_{kj,\sigma\mu} = 0$ then $n_{kj,\sigma\mu} = m_{kj,\sigma\mu} = 0$.

(2) If $n_{kj,\sigma\mu} \neq 0$ then $n_{kj,\sigma\mu} = -a_{kj,\sigma\mu}$.

If Condition S is satisfied then we take

(4.4) $$\tilde{N} = h^2 N \ , \quad \mu = \frac{1-\lambda}{\lambda h^2} .$$

<u>Remark</u>: Both the Jacobi iterative scheme and the Gauss-Seidel scheme satisfy condition S. The SOR Scheme ($\omega \neq 1$) does not.

<u>Remark</u>: The fact that $p = 2$ and we use h^2 is related to the fact that this is a second order equation, <u>not</u> to the fact that this problem is set in 2 dimensions. In d dimensions we have (we hope the notation is clear)

$$a_{k_1 k_2 \dots k_d, \sigma_1 \sigma_2 \dots \sigma_d} = O(h^{d-2}) .$$

Let us compute $V^*\tilde{N}U$. We have

(4.5) $$V^*\tilde{N}U = h^2 \sum_{kj} \sum_{\sigma\mu} n_{kj,\sigma\mu} \bar{v}_{kj} u_{\sigma\mu} .$$

Thus

(4.6a) $$V^*\tilde{N}U = h^2 \sum_{kj} [\sum_{\sigma\mu} n_{kj,\sigma\mu}] \bar{v}_{kj} u_{kj} + \hat{\varepsilon}(u,v)$$

where

$$\hat{\varepsilon}(u,v) = h^2 \sum_{kj} \bar{v}_{kj}[\sum_{\sigma\mu} n_{kj,\sigma\mu}(u_{\sigma\mu}-u_{kj})] \ . \tag{4.6b}$$

In finite difference methods there is usually a fixed integer $r > 0$ such that

$$n_{kj,\sigma\mu} = 0$$

unless

$$|k-\sigma| < r \quad \text{and} \quad |j-\mu| < r \ .$$

Hence for "smooth" functions u we expect $\hat{\varepsilon}(u,v)$ to be small. Specifically, we anticipate

$$|\hat{\varepsilon}(u,v)| \leq Kh\{h^2\sum|v_{kj}|^2\}^{\frac{1}{2}} a(u,u)^{\frac{1}{2}} \ . \tag{4.6c}$$

Thus, at first glance, our candidate for q is given by

$$\sum_{\sigma\mu} n_{kj,\sigma\mu} \sim \hat{q}_{kj} \ .$$

However, as we have seen in the case of the 4-point block Jacobi scheme, it may well be that $\hat{q}_{kj} = 0$ for most (k,j). Indeed, if our "blocks" are large, this will always be the case. Hence, we must take this fact into account.

Assuming that the nonzero values of $\hat{q}_{kj}$ occur with some regular pattern throughout the mesh we may try

$$q(x_k,y_j) \sim \alpha\, \hat{q}_{k,j}$$

where α is a parameter that describes the density of points (k,j) at which $\hat{q}_{kj}$ is (structurally) nonzero.

Now one has a candidate for q and must proceed to consider the problems of Spectral approximation and Relevance. As mentioned earlier, the work of Parter and Steuerwalt [14] and Parter [12] discuss this problem in great detail.

5. Finite Element Equations

In dealing with finite element equations one can be overwhelmed with the complexity of the search for "q." In a recent paper [15] Parter and Steuerwalt presented a general theory for the linear systems that arise in the finite element approximation of elliptic problems.

In addition to presenting this general approach in the finite element setting, [15] deals with a specific group of problems. These problems illustrate the complexity as well as the value and significance of appropriate "estimates." Let

$$\Omega := \{(x,y),\ 0 < x,y < 1\} \tag{5.1}$$

and let L be given by (4.1). We assume this problem is coercive. The finite element space S_h is the space of tensor products of Hermite cubic splines based on

$$\Delta x = \frac{1}{p_x+1}\ ,\quad \Delta y = \frac{1}{p_y+1}\ .$$

Thus to each point $(k\Delta x, j\Delta y)$ we associate a 4-vector

$$U_{kj} = [u_{kj}, \Delta x(u_x)_{kj}, (\Delta y(u_y)_{kj}, \Delta x\Delta y(u_{xy})_{kj}]\ .$$

Therefore in general at each point there are 144 coefficients and V*NU involves $v, v_x, v_y, v_{xy}, u, u_x, u_y, u_{xy}$. In Figure 3 we exhibit the "9-point block star" associated with the simplest problem

$$-\Delta u = f\ ,\quad \Delta x = \Delta y = h\ .$$

Since the finite element theory usually gives estimates on the H_1 norm

-54/175	-1/20	1/20	13/2100	-102/175	0	0	0	-54/175	1/20	1/20	-13/2100
1/20	3/700	-13/2100	1/3150	0	2/175	0	-2/315	-1/20	3/700	13/2100	1/3150
-1/20	-13/2100	3/700	-1/3150	0	0	-22/525	0	-1/20	13/2100	3/700	1/3150
13/2100	-1/3150	1/3150	1/2100	0	2/315	0	-4/1575	-13/2100	-1/3150	-1/3150	1/2100
-102/175	0	0	0	624/175	0	0	0	-102/175	0	0	0
0	-22/525	0	0	0	128/525	0	0	0	-22/525	0	0
0	0	2/175	2/315	0	0	128/525	0	0	0	2/175	-2/315
0	0	-2/315	-4/1575	0	0	0	16/1575	0	0	2/315	-4/1575
-54/175	-1/20	-1/20	-13/2100	-102/175	0	0	0	-54/175	1/20	-1/20	13/2100
1/20	3/700	13/2100	-1/3150	0	2/175	0	2/315	-1/20	3/700	-13/2100	-1/3150
1/20	13/2100	3/700	-1/3150	0	0	-22/525	0	1/20	-13/2100	3/700	1/3150
-13/2100	1/3150	1/3150	1/2100	0	-2/315	0	-4/1575	13/2100	1/3150	-1/3150	1/2100

Figure 3. The nine-point decomposition of A.

and does not estimate the point values of u_x, u_y and u_{xy} it would seem difficult to discover the needle "q" in that messy haystack. However, the finite element spaces generally have some regularity and estimates. In this case we obtain

Theorem 5.1: Let $h = \sqrt{\Delta x \Delta y}$. Then there is a constant $K > 0$, depending on $r = \frac{\Delta y}{\Delta x}$, such that for every $u, v \in S_h$ and every $\phi \in C^1(\bar{\Omega})$ we have

(5.2a) $$h^2 \sum_{ij} |v_{ij}|^2 \leq K||v||^2_{L_2} ,$$

(5.2b) $$h^2 \sum_{ij} u_{ij} v_{ij} \phi_{ij} = \int\int uv\phi dxdy + \delta(u,v,\phi) ,$$

(5.2c) $$h^2 \sum_{ij} [|v_{ij}-v_{i+1,j}|^2 + |v_{ij}-v_{i,j+1}|^2] \leq Kh^2 || \nabla v||^2_{L_2} ,$$

(5.2d) $$h^2 \sum_{ij} \sum_{|\alpha|=1}^{6} (\Delta x)^{\alpha_1} (\Delta y)^{\alpha_2} |(D^{\alpha} u)_{ij}|^2 \leq Kh^2 || \nabla u||^2_{L_2} ,$$

(5.2e) $$h^2 \sum_{ij} \left\{ \sum_{|\alpha|+|\beta| \geq 1} (\Delta x)^{\alpha_1+\alpha_2} (\Delta y)^{\beta_1+\beta_2} |(D^{\alpha} u)_{ij} (D^{\beta} v)_{ij}| \right\} \leq K\eta(u,v,h) ,$$

where

(5.3a) $$|\delta(u,v,\phi)| \leq K[1 + || \nabla \phi||_{\infty}]\eta(u+v,u-v,h)$$

and

(5.3b) $$\eta(u,v,h) = h[||u||_{L_2} ||\nabla v||_{L_2} + ||v||_{L_2} || \nabla u||_{L_2} + h|| \nabla u||_{L_2} || \nabla u||_{L_2}] .$$

Let us see what these estimates mean. The estimate (5.2b) asserts that

$$h^2 \sum \bar{v}_{ij} u_{ij} \sim \int\int \bar{v} u dxdy .$$

The estimate (5.2c) justifies the sort of index shifting that we did in (4.5), (4.6). Generally speaking, (5.2c), (5.2d), (5.2e) tell us that we may ignore many terms - both in the discovery of q and in verification of Spectral approximation. This is particularly true if the splitting satisfies Condition S of section 4.

Moreover, with the estimates of Theorem 5.1 one can prove: There is a relevant eigenvalue λ with $|\lambda| = \rho$ that satisfies (A.3) provided that there is such an eigenvalue with eigenvector U satisfying

$$\text{Re } U^* N U \geq 0 .$$

See Theorem 5.1 of [15]. Notice that block Jacobi schemes that possess Property A always possess this property. This result yields new convergence proofs that do not depend on having self-adjoint problems or equations of "positive type." The main results are:

<u>Theorem 5.2</u>: Consider the k-line block Jacobi method (horizontal lines). Then

(5.4) $$q(x,y) = \frac{12}{5} \frac{1}{kr} c(x,y) , \quad r = \frac{\Delta y}{\Delta x} ,$$

and the spectral radius $\rho_J(k)$ satisfies

(5.5a) $$\rho_J(k) = 1 - \frac{5k}{12}\Gamma_0(\Delta y)^2 + o(h^2)$$

where Γ_0 is the minimal eigenvalue of the eigenvalue problem

(5.5b) $$L\psi = \Lambda\, c(x,y)\psi \ .$$

The spectral radius $\rho_{GS}(k)$ for the k-line Gauss-Seidel scheme satisfies

(5.6a) $$\rho_{GS}(k) = 1 - \frac{5}{6}k\Gamma_0(\Delta y)^2 + o(h^2) \ .$$

Similarly, the spectral radius $\rho_b(k)$ of the SOR scheme with optimal choice of ω satisfies

(5.6b) $$\rho_b(k) = 1 - 2\left(\frac{5k}{6}\Gamma_0\right)^{\frac{1}{2}}\Delta y + o(h) \ .$$

Theorem 5.3. Let the unknowns be ordered lexicographically and consider the point Gauss-Seidel iterative scheme. Then

$$q(x,y) = \frac{156}{175}\left[ra(x,y) + \frac{1}{r}c(x,y)\right] ,$$

and

$$\rho \geq 1 - \Lambda_0 h^2 + o(h^2) \ .$$

Notice: In this case we cannot claim asymptotic equality. In the self-adjoint case we know [19, Theorem 3.6] that the method is convergent. After all, Gauss-Seidel is always convergent for positive definite problems. Unfortunately, it appears that none of the proofs of this fact gives any qualitative information about the eigenvalues λ with $|\lambda| = \rho$.

6. Finite Elements and Point SOR

The results of the previous section do not yield results for the "point" SOR scheme applied to the finite element equations. Nevertheless there are several reasons for seeking such results. For one thing, experimental results by Fix and Larsen [6] and Rice [17] imply that for regular grids and

$$h < \frac{1}{10}$$

the SOR iterative schemes are preferable to direct solution methods.

Part of the strength of the finite element method is the fact that one need not have a nice regular array of mesh points. One puts points where they are needed and effects the necessary triangulation. Thus we would not always expect to be able to use nice regular blocks for iterative methods. Hence in these situations if one chooses to use iterative methods rather than a direct solve, we would expect to be using point SOR rather than k-line schemes or other regular block schemes. Finally, because this point SOR scheme does not fall within the Young theory (i.e., we do not have property A) there does not exist an efficient adaptive algorithm (see [9]) to determine the optimal ω.

For these reasons Parter and Steuerwalt undertook a study of point SOR for the model problem

$$-\Delta u = f$$

where the coefficients of A are given in Figure 3. We set

$$\tilde{N} = hN , \quad \omega = 2 - Ch$$

and find that

$$V^*\tilde{N}U = h^2 C \frac{156}{175} \sum \bar{v}_{kj} u_{kj} +$$

$$(6.1) \quad h^2 \frac{210}{175} \sum \bar{v}_{kj} \left[\frac{u_{k,j+1} - u_{k,j}}{h}\right] + h^2 \frac{102}{175} \sum \bar{v}_{kj} \left[\frac{u_{k+1,j} - u_{k,j}}{h}\right]$$

$$+ \frac{h^2}{10} \sum [(\bar{v}_y)_{kj} u_{kj} - \bar{v}_{kj} (u_y)_{kj}] + J(u,v)$$

where the "junk term" $J(u,v)$ is "small" for nice enough (u,v). Thus we have a candidate for q, namely

$$(6.2) \qquad qu := C \frac{156}{175} u + \frac{102}{175} u_x + u_y .$$

This leads us to consider the eigenvalue problem

$$(6.3) \qquad \begin{aligned} -\Delta u &= -(u_{xx} + u_{yy}) = \Lambda qu \quad \text{in} \quad \Omega , \\ u &= 0 \quad \text{on} \quad \partial\Omega . \end{aligned}$$

As usual we must ask about Spectral approximation and Relevance. The truth of Spectral approximation does <u>not</u> follow from the standard theories, and we cannot say anything about the Relevance question.

In this case, because the operator L is the Laplacian and the operator q also has constant coefficients we may apply the Garabedian change of variables

$$(6.4) \qquad u = v \exp\{- \frac{\Lambda}{2} (\frac{102}{175} x + y)\}$$

and complete the analysis of our (lower bound) candidate

$$\tilde{\rho}(\omega) = 1 - \Lambda_0(C)h \ .$$

Let

(6.5a) $$c_b = \frac{\sqrt{2}}{156}\pi[41029]^{\frac{1}{2}}$$

then

Case 1: $0 < c < c_b$

(6.5b) $$\Lambda_0(c) = \frac{150\pi}{[41029]^{\frac{1}{2}}}\frac{c}{c_b}$$

Case 2: $c_b < c < 2/h$

(6.5c) $$\Lambda_0(c) = \frac{150\pi}{[41029]^{\frac{1}{2}}}\frac{c}{c_b} - \frac{1}{[41029]}\{(156c)^2 - 2\pi^2(41029)\}^{\frac{1}{2}} \ .$$

We observe that

$$\frac{d}{dc}\Lambda_0(c)\Big|_{c=c_{b+}} = -\infty$$

and $\Lambda_0(c)$ is linear in c for $0 < c < c_b$.

While we cannot assert that we have found $\rho(\omega)$, we have found a candidate for a lower bound $\rho(\tilde{\omega})$ that exhibits the characteristic features of $\rho(\omega)$ in the classical case where property A holds. It is our hope that the nature of $\tilde{\rho}(\omega)$ will be useful in the construction of adaptive algorithms for the determination of an optimal ω.

7. Concluding Remarks

In the previous sections we have attempted to elucidate a theory for the estimation of the asymptotic behavior of the spectral radii of iterative schemes for elliptic problems. This theory has two main interesting mathematical points.

(1) The theory does not lose sight of the elliptic operator L.

(2) The theory leads one to study other mathematical questions that are interesting.

Finally, while we have not attempted to describe the usefulness of the theory, it is very useful - even though the basic constant Λ_0 is generally not accessible.

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