CONJUGATE CONE CHARACTERIZATION OF POSITIVE DEFINITE AND SEMIDEFINITE MATRICES

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ABSTRACT

Positive definite and semidefinite matrices are characterized in terms of positive definiteness and semidefiniteness on arbitrary closed convex cones in R^n. These results are obtained by generalizing Moreau's polar decomposition to a conjugate decomposition. Some typical results are: The matrix A is positive definite if and only if for some closed convex cone K, A is positive definite on K and $(A+A^T)^{-1}$ exists and is semidefinite on the polar cone $K°$. The matrix A is positive semidefinite if and only if for some convex polyhedral cone K or some general closed convex cone satisfying a certain condition, A is positive semidefinite on both K and the conjugate cone $K^A = \{s | x^T(A+A^T)s \leqq 0, \forall x \in K\}$, and $(A+A^T)x = 0$ for all x in K such that $x^TAx = 0$.

AMS (MOS) Subject Classifications: 15A63, 10C25, 90C20

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1. INTRODUCTION

In deriving local duality results for nonlinear programs in [5] the following characterization of symmetric positive definite matrices was established: An $n\times n$ real symmetric matrix A is positive definite if and only if A is positive definite on some arbitrary subspace of the n-dimensional Euclidean space R^n and A^{-1} exists and is positive semi-definite on the orthogonal complement of the subspace. It is the purpose of this paper to generalize this result by replacing the subspace by a closed convex cone and dropping the symmetry of A. In particular we will show in Theorem 3.6 that A is positive definite if and only if A is positive definite on some arbitrary closed convex cone in R^n and $(A+A^T)^{-1}$ exists and is positive semidefinite on the polar cone. The algebraic proof employed in [5] breaks down in attempting to replace the subspace by a closed convex cone and a completely different proof is given here based on the concept of a conjugate decomposition of a vector in R^n, which is an extension of the polar decomposition of Moreau [9], and which we define now.

1.1 Definition (Conjugate decomposition) Let K be a closed convex cone in R^n and let A be an $n\times n$ real matrix. A point a in R^n is said to have a conjugate decomposition with respect to K and A if there exists x and y such that

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$$(1.1) \qquad a = x + y,\ x \in K,\ y \in K^A := \{s \mid x^T(A+A^T)s \leqq 0,\ \forall x \in K\}$$
$$x^T(A+A^T)y = 0.$$

The closed convex cone K^A is called the conjugate cone to K with respect to A.

Note that for an arbitrary A and K it is in no way assured that a conjugate decomposition exists for each point a in R^n. If A is taken to be the $n \times n$ identity matrix then K^A degenerates to the polar cone

$$K^\circ := \{s \mid s^T x \leqq 0,\ \forall x \in K\}$$

and the polar decomposition of any vector a in R^n defined by

$$a = x + y, \text{ with } x \in K,\ y \in K^\circ,\ x^T y = 0$$

is assured by Moreau's theorem [9]. One of the principal results of this paper will be to establish in Theorem 2.3 the existence of a conjugate decomposition for any a in R^n when the matrix A is not necessarily positive definite nor even positive semidefinite. We shall do this by showing that the existence of a conjugate decomposition is equivalent to finding a stationary point of the following constrained optimization problem

$$(1.2) \qquad \underset{z}{\text{minimize}} \quad f(z) := (z-a)^T A(z-a) \quad \text{subject to} \quad z \in K.$$

We define a stationary point x of (1.2) as any x satisfying the following minimum principle necessary optimality condition [7, Theorem 9.3.3]

$$x \in K,\ (z-x)^T \nabla f(x) \geqq 0,\quad \forall z \in K$$

that is

$$x \in K, \ (z-x)^T(A+A^T)(x-a) \geqq 0, \ \forall z \in K.$$

By taking $z = 0$ and $z = 2x$, which are points in the cone K, these conditions are equivalent to

$$x \in K, \ x^T(A+A^T)(x-a) = 0, \ z^T(A+A^T)(x-a) \geqq 0, \ \forall z \in K$$

which in turn are equivalent to

(1.3) $$x \in K, \ a - x \in K^A, \ x^T(A+A^T)(x-a) = 0.$$

Upon setting $y := a - x$ we get $a = x + y$ and see that (1.3) is equivalent to the conjugate decomposition (1.1). Hence we have the following preliminary result. A similar result for subspaces rather than cones is contained in [1, Theorem 8.4].

1.2 <u>Theorem</u> Let A be an $n \times n$ real matrix and let K be a closed convex cone in R^n. A point a in R^n has a conjugate decomposition (1.1) $a = x + y$ if and only if x is a stationary point of (1.2), that is x satisfies (1.3), and in which case $y = a - x$.

It is convenient to introduce now the following.

1.3 <u>Definition</u> Let $K \subset R^n$ and let A be an $n \times n$ real matrix. Then:

(i) A is positive semidefinite on $K \Leftrightarrow \left\langle x \in K \Rightarrow x^TAx \geqq 0\right.$

(ii) A is positive definite on $K \Leftrightarrow \left\langle 0 \neq x \in K \Rightarrow x^TAx > 0\right.$

(iii) A is positive semidefinite plus on $K \Leftrightarrow \left\langle \begin{array}{l} x \in K \Rightarrow x^TAx \geqq 0 \\ x^TAx = 0, \ x \in K \Rightarrow (A+A^T)x = 0 \end{array}\right.$

Note that if $K = R^n_+ := \{x \mid x \geqq 0, \ x \in R^n\}$, the above three classes of matrices in Definition 1.3 become respectively the classes of copositive, strictly copositive and copositive plus matrices [2,6]. Note that (ii) does not in general imply the strict convexity of x^TAx on K unless K is a subspace.

With the above preliminaries at hand we can outline the principal thrust of the paper. In Section 2 we shall establish by means of the equivalence between (1.1) and (1.3) the existence of a conjugate decomposition of arbitrary points in R^n for special types of cones and matrices in R^n. In Theorem 2.3 we show that if K is a convex polyhedral cone, or K is a general closed convex cone satisfying a certain condition, and A is positive semidefinite plus on K then each point in R^n has a conjugate decomposition with respect to K and A. In Corollary 2.2 we show that if K is any general closed convex cone in R^n, and if A is positive definite on K then each point in R^n has a conjugate decomposition with respect to K and A. Theorem 2.9 establishes the uniqueness of this conjugate decomposition under the added assumption that A is positive definite on the affine hull of K. In Section 3 we utilize the conjugate decomposition results of Section 2 to characterize positive definite and semidefinite matrices. In Theorem 3.1 we show that for any convex polyhedral cone or for a special closed convex general cone, the matrix A is positive semidefinite if and only if A is positive semidefinite plus on K and positive semidefinite on K^A. In Corollaries 3.3 and 3.4 we characterize positive semidefinite matrices in terms of copositive and copositive plus matrices. In Theorem 3.5 we characterize a positive definite matrix A by being positive definite on K and K^A, or by being positive definite on K and $(A+A^T)^{-1}$ being positive semidefinite on $K°$. Finally Corollary 3.9 characterizes positive definite matrices in terms of copositive and strictly copositive matrices.

A brief word about notation. We shall denote the 2-norm and ∞-norm of a vector x in R^n by $\|x\|_2$ and $\|x\|_\infty$ respectively. For an $n\times n$ matrix A, $\ker A := \{x|Ax=0\}$. For a subspace S of R^n, $S^\perp$ will denote the orthogonal complement $\{y|x^Ty=0, \forall x\in S\}$. For a set S in R^n, cl (S), will denote the closure of S. For $f\colon R^n \to R$, ∇f will denote $n\times 1$ gradient vector. R^n_+ will denote $\{x|x\geqq 0, x\in R^n\}$ while R^n_- will denote $\{x|x\leqq 0, x\in R^n\}$. For a point x in R^n the projection (or equivalently the orthogonal projection) on a closed subset S of R^n is that unique point $P(x)$ in S which satisfies

$$\|x - P(x)\|_2 = \min_{P\in S} \|x - P\|_2 .$$

2. CONJUGATE DECOMPOSITION

We shall establish in this section a number of results which guarantee the existence of a conjugate decomposition of any vector in R^n. We begin with a simple existence result.

2.1 Lemma Let K be a general closed convex cone in R^n and let A be an $n\times n$ real matrix. If A is positive definite on K, then (1.2) has a solution.

Proof By assumption, there exists $\gamma > 0$ such that

$$x^TAx \geqq \gamma||x||_2^2 \qquad \forall x \in K$$

Define

$$S := \{x \mid ||x||_2 \leqq \frac{||(A+A^T)a||_2}{\gamma},\ x\in K\}$$

Then, for any x in K but not in S we have that

$$\begin{aligned} f(x) = (x-a)^TA(x-a) &\geqq \gamma||x||_2^2 - x^T(A+A^T)a + f(0) \\ &\geqq ||x||_2(\gamma||x||_2 - ||(A+A^T)a||_2) + f(0) \\ &\geqq f(0) \end{aligned}$$

Since 0 is in S it follows that

$$\inf_{x\in K} f(x) = \inf_{x\in S} f(x)$$

Therefore the existence of a solution to (1.2) follows from the compactness of S. $\square$

Combining Lemma 2.1 and Theorem 1.2 gives the following.

2.2 Corollary Let K be a general closed convex cone in R^n and let A be an $n\times n$ real matrix which is positive definite on K. Then each vector in R^n has a conjugate decomposition with respect to K and A.

We next give a useful sufficient condition for conjugate decomposition in terms of positive semidefinite plus matrices.

2.3 Theorem Let A be an $n\times n$ real matrix and let K be a general closed convex cone in R^n satisfying one of the three equivalent conditions

(2.1a) $(A+A^T)(K)$ is closed

(2.1b) $K + \ker(A+A^T)$ is closed

(2.1c) $P(K)$, the projection of K on $(\ker(A+A^T))^\perp$, is closed

or let K be a convex polyhedral cone in R^n. If A is positive semidefinite plus on K then each vector in R^n has a conjugate decomposition with respect to K and A.

Proof That conditions (2.1a), (2.1b) and (2.1c) are equivalent follows from Lemma A.1 of the Appendix. By Theorem 1.2 it is sufficient to show that (1.2) has a solution and hence a stationary point. Let $L := \ker(A+A^T)$ and let $P(x)$ denote the projection on the subspace $L^\perp$ using the 2-norm. For any x in R^n let $x = y + z$ with $y \in L^\perp$ and $z \in L$. Then

$$\begin{aligned} f(x) &= (x-a)^T A(x-a) \\ &= (y+z)^T A(y+z) - a^T(A+A^T)(y+z) + a^T Aa \\ &= y^T Ay + y^T(A+A^T)z + z^T Az - a^T(A+A^T)(y+z) + a^T Aa \\ &= y^T Ay - a^T(A+A^T)y + a^T Aa \qquad (\text{Since } z \in L) \\ &= f(y). \end{aligned}$$

Therefore

$$\inf\{f(x)\,|\,x \in K\} = \inf\{f(y)\,|\,y \in P(K)\}$$

If $\bar{y}$ solves the problem

$$(2.2) \qquad \underset{y}{\text{minimize}} \quad (y-a)^T A(y-a) \qquad \text{subject to} \quad y \in P(K)$$

then any $\bar{x}$ in K with $P(\bar{x}) = \bar{y}$ is a solution of (1.2). Hence we need only show that (2.2) is solvable for any a.

Clearly since K is a convex cone and $P(\cdot)$ is a linear operator, then $P(K)$ is also a convex cone. We want to show that $P(K)$ is also closed. When K is polyhedral, $P(K)$ is closed because for any point of closure c of $P(K)$ the linear program $\inf\{\|x-c\|_\infty \,|\, x \in P(K)\} = 0$ has a solution [3,8] x in $P(K)$ and hence $c = x \in P(K)$. When K is a general closed convex cone then $P(K)$ is closed by assumption (2.1c).

Let $0 \neq y \in P(K)$ and let x by any point in K such that $P(x) = y$. It follows from $y \neq 0$ that $x \notin \ker(A+A^T)$. Consequently since A is positive semidefinite plus on K, $y^T Ay = x^T Ax > 0$. By Lemma 2.1, (2.2) has a solution, which in turn implies that (1.2) has a solution. $\square$

Note that a sufficient condition for (2.1c) is that

$$K \cap \ker(A+A^T) \subset -K.$$

To see this note that this condition and the fact that $\ker(P) = \ker(A+A^T)$ imply that $K \cap \ker P \subset (-K) \cap K$ and hence by Theorem 9.1 of Rockafellar [10] $P(K)$ is closed.

We note here that in the polyhedral case, Theorem 2.3 can also be established by using Eaves' existence results for quadratic programming [4, Corollary 4].

It is important to note that condition (2.1) is essential when K is not polyhedral as shown by the following example.

2.4 Example Let $K = \{(x_1,x_2,x_3)|2x_1x_3 \geqq x_2^2,\ x_1 \geqq 0,\ x_3 \geqq 0\}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad a = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Note first that $(A+A^T)(K)$ is not closed because $(0,1,0)$ is not in $(A+A^T)(K)$ but $(\varepsilon,1+\varepsilon,0)$ is for any $\varepsilon > 0$. Now since a is not in K, and since for any $\varepsilon > 0$ the point $z = (\varepsilon,1+\varepsilon,\frac{(1+\varepsilon)^2}{2\varepsilon})$ is in K and $(z-a)A(z-a) = 2\varepsilon^2$, it follows that problem (1.2) has no solution. If $a = x + y$ is a conjugate decomposition of a with respect to K and A, then it follows from the semidefiniteness of A and Theorem 1.2 that x is a minimum solution of (1.2), which is a contradiction. Hence such a decomposition cannot exist even though A is positive semidefinite plus on K.

Under certain circumstances the roles of K and K^A may be interchanged. This is a consequence of the following.

2.5 Lemma Let A be an $n\times n$ real matrix and let K be a general closed convex cone in R^n satisfying (2.1) or let K be a convex polyhedral cone in R^n. Then

$$K^{AA} = K + \ker(A+A^T)$$

Proof Let $\bar{A} := A + A^T$ and for any set S in R^n define

$$\bar{A}^{-1}(S) := \{x|\bar{A}x\in S\}$$

Note that $\bar{A}^{-1}(S)$ is well defined even if $\bar{A}$ is not invertible. Since

$$K^A = \{y | y\bar{A}x \leqq 0,\ \forall x \in K\} = \{y | \bar{A}y \in K^\circ\} = \bar{A}^{-1}(K^\circ)$$

it follows that

$$(K^A)^\circ = (\bar{A}^{-1}(K^\circ))^\circ = \mathrm{cl}(\bar{A}(K^{\circ\circ}))$$

where the last equality follows from Rockafellar's Corollary 16.3.2 [10]. Hence

$$(K^A)^\circ = \mathrm{cl}\ (\bar{A}(K)) = \bar{A}(K)$$

where the last equality obtains from either the polyhedral assumption on K or from assumption (2.1a). We now have

$$K^{AA} = \bar{A}^{-1}((K^A)^\circ) = \bar{A}^{-1}(\bar{A}(K)) = \{y | \bar{A}y \in \bar{A}(K)\}$$

Consequently

$$\begin{aligned} y \in K^{AA} &\Leftrightarrow \bar{A}y = \bar{A}x \quad \text{for some} \quad x \in K \\ &\Leftrightarrow y - x \in \ker(\bar{A}) \quad \text{for some} \quad x \in K \\ &\Leftrightarrow y \in K + \ker(\bar{A}) . \qquad \square \end{aligned}$$

Lemma 2.5 can now be used to replace K by K^A in Theorem 2.3.

2.6 <u>Theorem</u> Let A be an $n \times n$ real matrix and let K be a general closed convex cone in R^n satisfying (2.1) or let K be a convex polyhedral cone in R^n. If A is positive semidefinite plus on K^A then each vector in R^n has a conjugate decomposition with respect to K and A.

Proof It is evident that K^A is a closed convex cone. Furthermore, $\ker(A+A^T) \subset -K^A \cap K^A$. Hence $K^A \cap \ker(A+A^T) \subset -K^A$. By applying Theorem 2.3 to the cone K^A instead of K we have that for any vector a in R^n, there exist $\hat{y} \in K^A$ and $\hat{x} \in K^{AA}$ such that $a = \hat{x} + \hat{y}$ and $\hat{y}^T(A+A^T)\hat{x} = 0$. By Lemma 2.5 there exist x in K and z in $\ker(A+A^T)$ such that $\hat{x} = x + z$. Let $y = \hat{y} + z$, then $a = x + y$, $x \in K$, $y \in K^A$ and $x^T(A+A^T)y = (\hat{x}-z)^T(A+A^T)(\hat{y}+z) = \hat{x}^T(A+A^T)\hat{y} = 0$. □

2.7 Corollary Let K be any closed convex cone in R^n. If A is positive definite on K^A then $(A+A^T)^{-1}$ exists and each vector in R^n has a conjugate decomposition with respect to K and A.

Proof Note that $\ker(A+A^T) \subset K^A$ and for any y in $\ker(A+A^T)$, $y^TAy = 0$. Since A is positive definite on K^A it follows that $\ker(A+A^T) = \{0\}$ and consequently $(A+A^T)^{-1}$ exists. Clearly then all the assumptions of Theorem 2.6 hold and any vector in R^n has a conjugate decomposition with respect to K and A. □

The following example shows that the conjugate decomposition of a vector need not be unique.

2.8 Example Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $K = R^2_+$. Clearly A is positive definite on K. Because the problem (1.2) with $a = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is here equivalent to

$$\text{minimize} \quad (x_1+x_2-1)^2 \quad \text{subject to} \quad x_1 \geqq 0,\ x_2 \geqq 0$$

it follows that the point $x = \begin{pmatrix} \lambda \\ 1-\lambda \end{pmatrix}$ with $\lambda \in [0,1]$ is a solution of (1.2).

Hence for any $\lambda \in [0,1]$, $x := \begin{bmatrix} \lambda \\ 1-\lambda \end{bmatrix} \in K$, $y := \begin{bmatrix} -1-\lambda \\ 1+\lambda \end{bmatrix} \in K^A$, $x^T(A+A^T)y = 0$, and $a = x + y$.

A sufficient condition for the uniqueness of a conjugate decomposition is given by the following.

2.9 Theorem Let K be a general closed convex cone in R^n and let the $n \times n$ real matrix A be positive definite on the affine hull $\mathrm{aff}(K)$ of K or the affine hull $\mathrm{aff}(K^A)$ of K^A. Then each vector in R^n has a unique conjugate decomposition with respect to K and A.

Proof The existence of a conjugate decomposition follows immediately from Corollary 2.2 or Corollary 2.7. Suppose now that

$$a = x + y = \bar{x} + \bar{y}$$

are conjugate decompositions of a point a in R^n. Then $x - \bar{x} = \bar{y} - y$ and

$$\begin{aligned} (x-\bar{x})^T(A+A^T)(x-\bar{x}) &= (x-\bar{x})^T(A+A^T)(\bar{y}-y) \\ &= x^T(A+A^T)\bar{y} + \bar{x}^T(A+A^T)y \\ &\leqq 0 \end{aligned}$$

This can hold only if $x = \bar{x}$ since A is positive definite on $\mathrm{aff}(K)$. The proof is similar for the case when A is positive definite on $\mathrm{aff}(K^A)$. $\square$

3. CHARACTERIZATION OF POSITIVE DEFINITE AND SEMIDEFINITE MATRICES

In this section we utilize the conjugate decomposition results established in Section 2 to characterize positive definite and semidefinite matrices and we begin with the latter.

3.1 Theorem Let A be an $n \times n$ real matrix and let K be a general closed convex cone in R^n satisfying (2.1) or let K be a convex polyhedral cone in R^n. A is positive semidefinite if and only if A is positive semidefinite plus on K and positive semidefinite on K^A.

Proof (Necessity) If A is positive semidefinite then it is obviously positive semidefinite on both K and K^A. Since $x^TAx = 0$ is a global minimum of x^TAx it follows that $\nabla(x^TAx) = (A+A^T)x = 0$, and hence A is positive semidefinite plus on K.

(Sufficiency) If A is positive semidefinite on K and positive semidefinite plus on K^A then it follows from Theorem 2.3 that for each a in R^n we have the conjugate decomposition

$$a = x + y \quad \text{with} \quad x \in K,\ y \in K^A,\ x^T(A+A^T)y = 0$$

Hence

$$a^TAa = x^TAx + x^T(A+A^T)y + y^TAy = x^TAx + y^TAy \geqq 0 \qquad \square$$

The following example shows that A merely being positive semidefinite on K and K^A, without being semidefinite plus on K^A, is not enough to ensure that A is positive semidefinite.

3.2 Example Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $K = R^2_+$. Then $K^A = R^2_-$. Clearly A is positive semidefinite on both K and K^A, but A is not positive semidefinite.

A useful characterization of positive semidefinite matrices obtains if we set $K = R^n_+$ in Theorem 3.1.

3.3 Corollary Let A be an $n\times n$ real matrix. Then

$$A \text{ is positive semidefinite} \Leftrightarrow \begin{cases} \text{(a)} & x \geqq 0 \Rightarrow x^TAx \geqq 0, \\ \text{(b)} & x^TAx = 0,\ x \geqq 0 \Rightarrow (A+A^T)x = 0, \text{ and} \\ \text{(c)} & (A+A^T)x \geqq 0 \Rightarrow x^TAx \geqq 0 \end{cases}$$

Proof Set $K = R^n_+$ in Theorem 3.1 and note that

$$K^A = \{y | y^T(A+A^T)x \leqq 0,\ \forall x \geqq 0\} = \{y | (A+A^T)y \leqq 0\}$$

Hence $y^TAy = (-y^T)A(-y) \geqq 0$ for $y \in K^A$ is equivalent to condition (c) above. The Corollary then follows from Theorem 3.1. □

Note that since condition (a) in Corollary 3.3 characterizes copositive matrices, while conditions (a) and (b) characterize copositive plus matrices we have the following consequence to Corollary 3.3.

3.4 Corollary Let A be an $n\times n$ real matrix. A is positive semidefinite if and only if:

(a) A is copositive and satisfies conditions (b) and (c) of Corollary 3.3,

or

(b) A is copositive plus and satisfies condition (c) of Corollary 3.3.

Just as we established Theorem 3.1 from Theorem 2.3, we can similarly use Theorem 2.6 to obtain the following result where the roles of K and K^A have been interchanged.

3.5 Theorem Let A be an $n\times n$ real matrix and let K be a general closed convex cone in R^n satisfying (2.1) or let K be a convex polyhedral cone in R^n. A is positive semidefinite if and only if A is positive semidefinite on K and positive semidefinite plus on K^A.

We observe that if A is positive definite on K then condition (2.1) is automatically satisfied because $K \cap \ker(A+A^T) = \{0\}$. Hence we have the following important characterization of positive definite matrices.

3.6 Theorem Let A be an $n\times n$ real matrix and let K be any general closed convex cone in R^n. The following statements are equivalent:

(a) A is positive definite

(b) A is positive definite on both K and K^A

(c) A is positive definite on K, $(A+A^T)^{-1}$ exists and is positive semidefinite on $K° = \{y | x^Ty \leqq 0, \forall x \in K\}$.

Proof (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c): Trivial.

(b) $\Rightarrow$ (a): By Corollary 2.2, any nonzero vector a in R^n has a conjugate decomposition $a = x + y$ with respect to K and A, with both x and y not being zero simultaneously. Hence

$$a^TAa = (x+y)^TA(x+y) = x^TAx + y^TAy > 0$$

(c) $\Rightarrow$ (a): It follows from the existence of $(A+A^T)^{-1}$ that $y \in K^A$ if and only if $y = (A+A^T)^{-1}z$ and $z \in K°$. Hence if $(A+A^T)^{-1}$ is positive semidefinite on $K°$ and $y \in K^A$ then $y^TAy = \frac{1}{2}z^T(A+A^T)^{-1}z \geqq 0$. Since A is positive definite on the general closed cone K, then $K \cap \ker(A+A^T) \subset -K$. Hence it follows from Theorem 3.1 that A is positive semidefinite and so is $A + A^T$. Since $A + A^T$ is nonsingular, it must be positive definite and so is A. □

By taking $K = R^n_+$ in the last theorem we obtain the following interesting characterizations of positive definite matrices in terms of copositive, copositive plus and strictly copositive matrices.

3.7 Corollary Let A be an $n \times n$ real matrix. Then

$$A \text{ is positive definite} \Longleftrightarrow \begin{cases} 0 \neq x \in R^n_+ \Rightarrow x^TAx > 0 \\ x \in R^n_+ \Rightarrow x^T(A+A^T)^{-1}x \geqq 0 \end{cases}$$

Interchanging the roles of A and $(A+A^T)^{-1}$ in Corollary 3.7 gives the following.

3.8 Corollary Let A be an $n \times n$ real matrix. Then

$$A \text{ is positive definite} \Longleftrightarrow \begin{cases} x \in R^n_+ \Rightarrow x^TAx \geqq 0 \\ 0 \neq x \in R^n_+ \Rightarrow x^T(A+A^T)^{-1}x > 0 \end{cases}$$

3.9 Corollary A necessary and sufficient condition that a copositive (strictly copositive) matrix A be positive definite is that $(A+A^T)^{-1}$ exists and is strictly copositive (copositive).

The following characterization of positive definite matrices which was obtained by entirely different arguments in [5] is a simple consequence of Theorem 3.6 where K is taken to be a subspace of R^n.

3.10 Corollary [5] Let S be any subspace in R^n, let $S^\perp$ be its orthogonal complement and let A be an $n \times n$ symmetric matrix. A is positive definite if and only if A is positive definite on S and A^{-1} exists and is positive semidefinite on $S^\perp$.

Appendix

A.1 Lemma Let M be an $m \times n$ real matrix and let K be any set in R^n. The following are equivalent

(a) $M(K)$ is closed

(b) $K + \ker(M)$ is closed

(c) $P(K)$, the projection of K on $(\ker(M))^\perp$, is closed.

Proof (b) $\Rightarrow$ (c): Since $P(x) \in (\ker(M))^\perp$ and $P(x) - x \in \ker(M)$, it follows that $P(x) \in (\ker(M))^\perp \cap (x+\ker(M))$ and consequently $P(K) = (\ker(M))^\perp \cap (K+\ker(M))$. Since the subspace $(\ker(M))^\perp$ is closed, it follows that $P(K)$ is closed if $K + \ker(M)$ is closed.

(a) $\Rightarrow$ (b): Let $\{y^k+w^k\} \subset K + \ker(M)$ and let $y^k + w^k \to \bar{x}$. We want to show that $\bar{x} \in K + \ker(M)$ when $M(K)$ is closed. Let $z^k = M(y^k+w^k) = My^k \in M(K)$. It follows from $\|z^k\| \leqq \|M\| \, \|y^k+w^k\|$ and the closedness of $M(K)$ that there exists a subsequence $\{z^{k_i}\}$ such that $z^{k_i} \to \bar{z}$ and $\bar{z} \in M(K)$. Let $\bar{z} = M\bar{y}$, $\bar{y} \in K$. Then

$$M\bar{x} = \lim M(y^{k_i}+w^{k_i}) = \lim z^{k_i} = \bar{z} = M\bar{y}$$

Let $\bar{w} = \bar{x} - \bar{y}$, then $M\bar{w} = 0$ and $\bar{w} \in \ker(M)$. Hence $\bar{x} = \bar{y} + \bar{w} \in K + \ker(M)$.

(a) $\Leftarrow$ (c): Since $P(x) - x \in \ker(M)$ for $x \in K$ it follows that $M(P(K) - K) = 0$ or that $M(P(K)) = M(K)$. Hence we need to show that $M(P(K))$ is closed when $P(K)$ is closed. When M is a matrix of zeros this is trivial. So suppose M is not a matrix of zeros. Define

$$\rho := \min \{\|Mu\| \;\big|\; \|u\| = 1,\ u \in (\ker(M))^{\perp}\} > 0 .$$

Let $\{z^k\} \subset M(P(K))$ and $z^k \to \bar{z}$. We want to show that $\bar{z} \in M(P(K))$. Let $z^k = MP(x^k)$ with $x^k \in K$. Hence

$$\|z^k\| = \|MP(x^k)\| \geqq \rho\|P(x^k)\|$$

where the last inequality follows from the definition of ρ and $P(x^k) \in (\ker(M))^{\perp}$. Consequently, the sequence $\{P(x^k)\}$ is bounded (since $\{z^k\}$ is bounded), and since it is contained in the closed set $P(K)$, it must have a subsequence $\{P(x^{k_i})\}$ converging to a $\bar{u} \in P(K)$. Let $\bar{u} = P(\bar{x})$ with $\bar{x} \in K$. Since $z^{k_i} = MP(x^{k_i})$, $z^{k_i} \to \bar{z}$ and $P(x^{k_i}) \to \bar{u} = P(\bar{x})$, it follows that $\bar{z} = M\bar{u} = MP(\bar{x})$, $\bar{x} \in K$. Hence $\bar{z} \in M(P(K))$. $\square$

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