ON THE SWIRLING FLOW BETWEEN ROTATING COAXIAL DISKS: EXISTENCE AND NONUNIQUENESS

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ABSTRACT

Consider solutions $\langle G(x,\varepsilon),H(x,\varepsilon)\rangle$ of the von Kármán equations for the swirling flow between two rotating coaxial disks

(1.1) $$\varepsilon H^{iv} + HH''' + GG' = 0$$

and

(1.2) $$\varepsilon G'' + HG' - H'G = 0$$

with boundary conditions

(1.3) $$H(0,\varepsilon) = H'(0,\varepsilon) = H(1,\varepsilon) = H'(1,\varepsilon) = 0$$

(1.4) $$G(0,\varepsilon) = s, \quad G(1,\varepsilon) = 1, \quad |s| < 1 .$$

In this work we establish the existence of solutions for ε small enough. In fact, if n is a given positive integer with sign $s = (-1)^n$ then there is - for $\varepsilon > 0$ sufficiently small - a solution with the additional property: $G(x,\varepsilon)$ has n interior zeros. If $n > 1$ there are at least two such solutions. If $s = 0$ there is at least one such solution for every positive integer n. The asymptotic "shape" of these solutions is described.

AMS (MOS) Subject Classifications: 34B15, 34E15, 35Q10

Key Words: Ordinary differential equations, Rotating fluids, Similarity solutions, Asymptotic behavior, Existence, Nonuniqueness

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ON THE SWIRLING FLOW BETWEEN ROTATING COAXIAL DISKS: EXISTENCE AND NONUNIQUENESS*

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Introduction

In 1921 T. von Kármán [5] developed the similarity equations for axi-symmetric, incompressible, steady flow - "swirling flow". Let (q_r, q_θ, q_x) be the coordinates of velocity in cylindrical coordinates, (r,θ,x). von Kármán assumed that there is a function $H(x,\varepsilon)$ such that

$$q_x = -H(x,\varepsilon) .$$

Then, as a direct consequence of the steady state Navier-Stokes equations one finds that (see [1], [5]) there is a function $G(x,\varepsilon)$ so that the velocity components are described by

$$q_r = \frac{r}{2} H'(x,\varepsilon), \qquad q_\theta = \frac{r}{2} G(x,\varepsilon) .$$

The functions $\langle G(x,\varepsilon), H(x,\varepsilon)\rangle$ satisfy the equations

$$\varepsilon H^{iv} + HH''' + GG' = 0 , \tag{1.1}$$

$$\varepsilon G'' + HG' - H'G = 0 . \tag{1.2}$$

The quantity $\varepsilon > 0$ is related to the bulk viscosity. Equation (1.1) can be integrated to yield

$$\varepsilon H''' + HH'' + \frac{1}{2} G^2 - \frac{1}{2} (H')^2 = \mu \tag{1.3}$$

where μ is a constant of integration.

In the case originally studied by von Kármán, the flow above a single disk, we have a problem on the infinite interval $[0,\infty]$ and the constant of integration is known, i.e.,

$$\mu = \frac{1}{2} \Omega_\infty^2$$

where $\Omega_\infty = G(\infty,\varepsilon)$.

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In [8] we considered the asymptotic behavior of families of solutions $\langle G,H \rangle$ which satisfy the bounds:

(1.5a) $$|H(x,\varepsilon)| + |H'(x,\varepsilon)| + |G(x,\varepsilon)| \leq B \ .$$

There is a point x_0, $0 < x_0 < 1$ and a constant δ, $0 < \delta \leq B$ such that

(1.5b) $$|H(x_0,\varepsilon)| \geq \delta \ .$$

Interestingly enough we discovered that there are no such families. Moreover, a careful analysis of our discussion makes it clear that the source of the difficulty is the condition that

$$\lim_{\varepsilon \to 0} \{|\Omega_0| + |\Omega_1|\} \neq 0 \ .$$

Furthermore, the physical boundary conditions (1.4a), (1.4b) together with the conditions (1.5a), (1.5b) imply that (in the limit as $\varepsilon \downarrow 0+$) the function $H(x,\varepsilon)$ may not have nodal zeros, i.e.

(1.6) $$\text{Lim } H(x,\varepsilon) \text{ may not change sign .}$$

Therefore we turned to the question of the existence of "pathological" solutions $\langle G,H \rangle$ which satisfy the differential equations (1.1), (1.2), the bounds (1.5a), (1.5b), the boundary condition (1.4a), (1.4b) and also satisfy (1.4c) with

(1.7a) $$|\Omega_0(\varepsilon)| + |\Omega_1(\varepsilon)| \neq 0, \qquad 0 < \varepsilon$$

(1.7b) $$\lim_{\varepsilon \to 0} \{|\Omega_0(\varepsilon)| + |\Omega_1(\varepsilon)|\} = 0 \ .$$

Moreover, (1.6) implies these solutions should be essentially "positive" in the sense that

$$\bar{H}(x) = \text{Lim } H(x,\varepsilon) \geq 0 \ .$$

Remark: The hypothesis (1.5a), (1.5b) implies that - after the extraction of a subsequence if necessary, the functions $H(x,\varepsilon)$ are convergent to a non-trivial limit function.

The results of [8] imply that $\bar{H}(x)$ must then have a special form. That is, there are n numbers, $\sigma_0, \sigma_1, \ldots, \sigma_n$, with

(1.8a) $$0 = \sigma_0 < \sigma_1 < \cdots < \sigma_n = 1 \ .$$

And, on the interval

$$[\sigma_j, \sigma_{j+1}], \qquad j = 0,1,\ldots,n-1$$

we have

(1.8b) $$\bar{H}(x) = \frac{H_{2,j}}{\tau_j^2}\,[1 - \cos \tau_j(x - \sigma_j)] ,$$

where

(1.8c) $$\tau_j = 2\pi(\sigma_{j+1} - \sigma_j)^{-1} .$$

In this paper we prove that if $\varepsilon > 0$ is small enough there are such pathological solutions. Moreover, such solutions exist for all n, $n = 1,2,\dots$.

The main result is

Theorem I: Let $n \geq 1$ be a given integer. let g_0, g_1 be given real numbers with

(1.9) $$g_1 \neq 0, \quad \text{sign } g_1 = (-1)^n .$$

Then there is an $\bar{\varepsilon} > 0$ and an $L = L(\varepsilon)$ such that, for all $\varepsilon \in (0,\bar{\varepsilon}]$ there is a pair of functions $\langle G(x,\varepsilon), H(x,\varepsilon)\rangle$ defined on the interval $[0,L(\varepsilon)]$ which satisfy the differential equations (1.1), (1.2) on that interval. In addition these functions satisfy the boundary conditions

(1.10a) $$H(0,\varepsilon) = H(L(\varepsilon),\varepsilon) = 0 ,$$

(1.10b) $$H'(0,\varepsilon) = H'(L(\varepsilon),\varepsilon) = 0 ,$$

(1.10c) $$G(0,\varepsilon) = g_0\varepsilon^{2/3}, \quad G(L(\varepsilon),\varepsilon) = g_1\varepsilon^{2/3} .$$

The functions $\langle G(x,\varepsilon), H(x,\varepsilon)\rangle$ also satisfy (1.5a), (1.5b). The function $H(x,\varepsilon)$ has n "humps". To be precise, there are exactly $n + 1$ numbers,

$$0 = \sigma_0 < \sigma_1(\varepsilon) < \cdots < \sigma_{n-1}(\varepsilon) < \sigma_n(\varepsilon) = L(\varepsilon)$$

at which $H(x,\varepsilon)$ has (relative) minima, that is

(1.11) $$H'(\sigma_j(\varepsilon),\varepsilon) = 0, \quad H''(\sigma_j(\varepsilon),\varepsilon) > 0 .$$

Moreover,

$$H(\sigma_j(\varepsilon),\varepsilon) \geq -k\varepsilon^{1/3}$$

and, between the $\sigma_j(\varepsilon)$, $H(x,\varepsilon)$ is essentially positive. In fact, $H(x,\varepsilon)$ is essentially given by (1.8b) while the function $G(x,\varepsilon)$ has the form

(1.12a) $$G(x,\varepsilon) \approx (-1)^j\tau_j H(x,\varepsilon), \quad \sigma_j(\varepsilon) < x < \sigma_{j+1}(\varepsilon)$$

where

(1.12b) $$\tau_j = \frac{2\pi}{\sigma_{j+1} - \sigma_j}, \qquad j = 0,1,\ldots,n-1 .$$

Finally, there is a constant $\tilde{\tau} < 0$ such that

(1.13) $$\lim_{\varepsilon \to 0} \tau_j(\varepsilon) = |\tilde{\tau}|^{j-1} .$$

Remark: The characterization of $G(x,\varepsilon)$ given by (1.2a), (1.2b) can be made more precise.

Case 1: $$g_0 > 0 .$$

In this case $G(x,\varepsilon)$ has exactly n interior zeros, say $\gamma_1,\gamma_2,\ldots,\gamma_n$, and

(1.14a) $$\gamma_j < \sigma_j, \qquad j = 1,2,\ldots,n$$

(1.14b) $$\sigma_j - \gamma_j = O(\varepsilon^{1/3}), \qquad j = 1,2,\ldots,n .$$

Case 2: $$g_0 < 0 .$$

In this case $G(x,\varepsilon)$ has exactly $(n+1)$ zeros, say $\gamma_0,\gamma_1,\gamma_2,\ldots,\gamma_n$ and (1.14a), (1.14b) hold. Moreover

(1.14c) $$\sigma_0 = 0 < \gamma_0, \quad \gamma_0 - \sigma_0 = O(\varepsilon^{1/3}) .$$

Once one has proven this result we obtain the pathological solutions on the interval $[0,1]$ by taking

(1.15a) $$\tilde{x} = x/L(\varepsilon), \qquad \varepsilon' = \varepsilon/L(\varepsilon) ,$$

(1.15b) $$\tilde{H}(\tilde{x},\varepsilon') = H(x,\varepsilon), \qquad \tilde{G}(\tilde{x},\varepsilon') = L(\varepsilon)G(x,\varepsilon) .$$

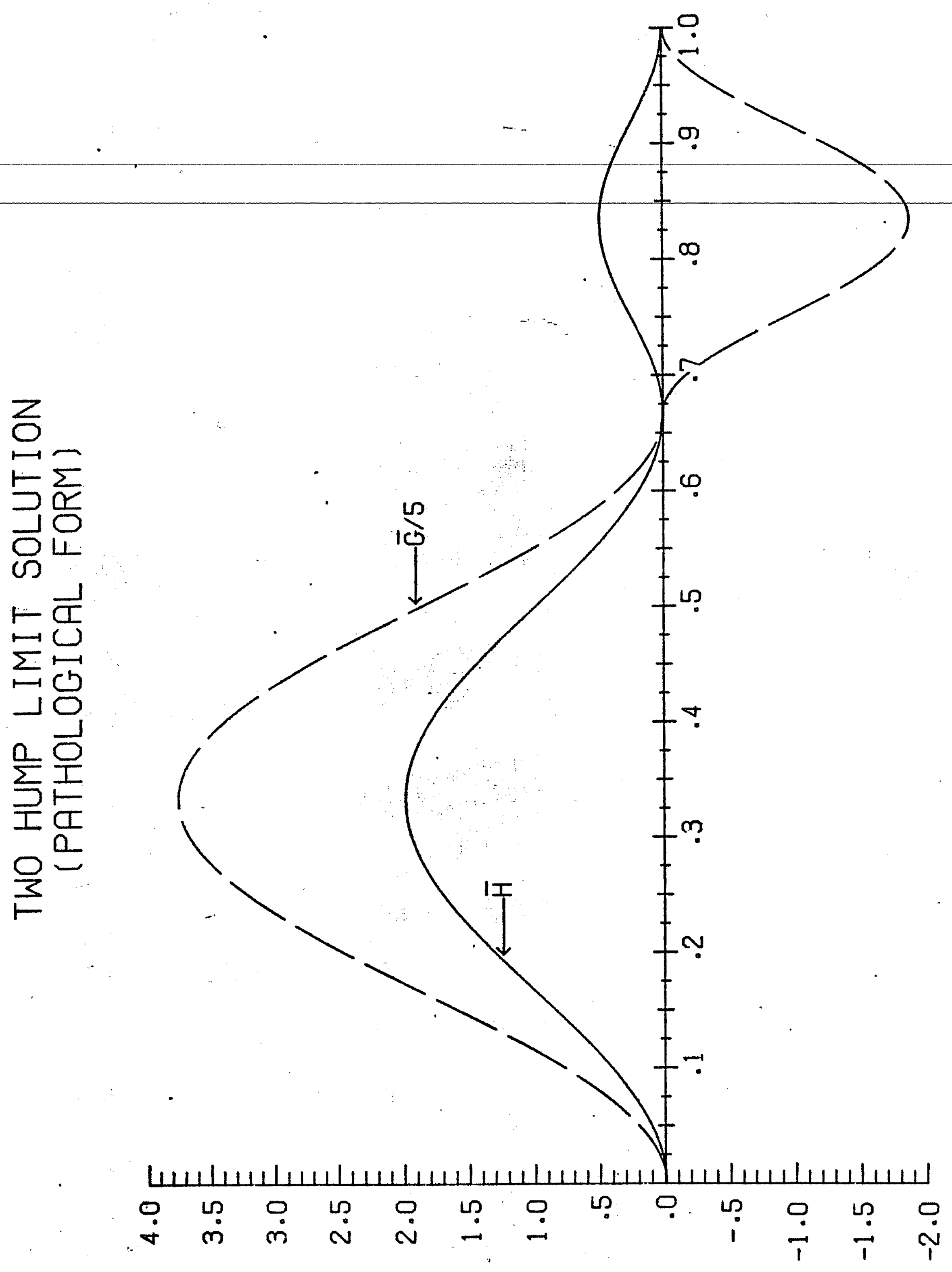
The functions $\langle \tilde{G},\tilde{H} \rangle$ satisfy (1.1), (1.2) - with x replaced by $\tilde{x}$ and ε replaced by ε'. In addition these functions satisfy the boundary conditions (1.4a), (1.4b), (1.4c) with

$$\Omega_0 = \Omega_0(\varepsilon') = g_0 L^{5/3}(\varepsilon')^{2/3} ,$$
$$\Omega_1 = \Omega_1(\varepsilon') = g_1 L^{5/3}(\varepsilon')^{2/3} .$$

Finally, if we set

(1.16a) $$\tilde{\varepsilon} = \frac{\varepsilon'}{|\Omega_1(\varepsilon')|} = \frac{\varepsilon^{1/3}}{|g_1 L^{5/3}|} ,$$

(1.16b) $$\bar{H}(\tilde{x},\tilde{\varepsilon}) = \frac{1}{|\Omega_1(\varepsilon')|}\tilde{H}(\tilde{x},\varepsilon') = O(\tilde{\varepsilon}^{-2}) ,$$


TWO HUMP LIMIT SOLUTION
(PATHOLOGICAL FORM)
Ḡ/5
H̄
4.0
3.5
3.0
2.5
2.0
1.5
1.0
.5
.0
-.5
-1.0
-1.5
-2.0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0

$$\tilde{G}(\tilde{x},\tilde{\varepsilon}) = \frac{1}{\Omega_1(\varepsilon')}\tilde{G}(\tilde{x},\varepsilon') = O(\tilde{\varepsilon}^{-2}) \tag{1.16c}$$

we obtain a solution of (1.1), (1.2), (1.4a), (1.4b), (1.4c) with ε replaced by $\tilde{\varepsilon}$, x replaced by $\tilde{x}$, with Ω_0 replaced by $\tilde{\Omega}_0$, and Ω_1 replaced by $\tilde{\Omega}_1$ where

$$\tilde{\Omega}_0 = g_0/g_1, \quad \tilde{\Omega}_1 = 1 .$$

For completeness sake we formulate this last result as

<u>Theorem II</u>: Let $n > 1$ be an integer. Let s be a given real number. Then for ε small enough there is a solution $\langle G,H\rangle$ of (1.1), (1.2), (1.4a), (1.4b), (1.4c) with

$$\Omega_0 = s, \quad \Omega_1 = 1 . \tag{1.17}$$

This solution may be described in a manner similar to the description given in Theorem I. There are exactly $(n + 1)$ numbers

$$0 = \sigma_0(\varepsilon) < \sigma_1(\varepsilon) < \cdots < \sigma_{n-1}(\varepsilon) < \sigma_n(\varepsilon) = 1$$

at which $H(x,\varepsilon)$ has its relative minima, i.e.,

$$H'(\sigma_j(\varepsilon),\varepsilon) = 0, \quad H''(\sigma_j(\varepsilon),\varepsilon) > 0 . \tag{1.18a}$$

Moreover, between the $\sigma_j(\varepsilon)$ the function $H(x,\varepsilon)$ is essentially positive. That is, for any given $\delta > 0$, $2\delta < \sigma_{j+1} - \sigma_j$ we have, for small ε,

$$H(x,\varepsilon) > 0, \quad \sigma_j(\varepsilon) + \delta < x < \sigma_{j+1}(\varepsilon) - \delta . \tag{1.18b}$$

Furthermore

$$\mathrm{Lim}\left(\frac{2\pi}{\sigma_{j+1}(\varepsilon) - \sigma_j(\varepsilon)}\right) = \xi_j = \xi_0|\tilde{\tau}|^j . \tag{1.19}$$

The function $G(x,\varepsilon)$ has at least n nodal zeros; $0 < \gamma_1(\varepsilon) < \gamma_2(\varepsilon) < \cdots < \gamma_n(\varepsilon) < 1$.

Moreover

$$\gamma_j(\varepsilon) < \sigma_j(\varepsilon), \quad \sigma_j - \gamma_j = O(\varepsilon), \quad j = 1,2,\ldots,n . \tag{1.20a}$$

If $s \neq 0$ and

$$\mathrm{sign}\ s = (-1)^{n+1}$$

then $G(x,\varepsilon)$ has $(n + 1)$ zeros. The additional zero, $\gamma_0(\varepsilon)$ satisfies

$$0 < \gamma_0(\varepsilon) = O(\varepsilon) . \tag{1.20b}$$

Furthermore

$$\|H\| \approx (\varepsilon^{-2}), \quad \|G\| \approx (\varepsilon^{-2}) . \tag{1.21}$$

Remark: One can choose to characterize the solution $\langle G,H\rangle$ by the number of "humps" or by the number of interior (nodal) zeros of $G(x,\varepsilon)$. Suppose we choose to discuss the number of humps. If $s \neq 0$ let $\langle \tilde{G}(x,\varepsilon),\tilde{H}(x,\varepsilon)\rangle$ be the solution described in Theorem II with n humps and

$$\bar{s} = \frac{1}{s}, \quad \bar{\varepsilon} = |s|\varepsilon .$$

Then

$$\tilde{H}(x,\varepsilon) = -\frac{1}{|s|} H(1 - x,\bar{\varepsilon}) ,$$
$$\tilde{G}(x,\varepsilon) = \frac{1}{s} G(1 - x,\bar{\varepsilon})$$

is another solution of (1.1), (1.2), (1.4a), (1.4b) and (1.17). On the other hand, if one chooses to look at the number of interior zeros of $G(x,\varepsilon)$ we have the following situation

Case 1: $s > 0$. For every even $\bar{n} > 2$ there are at least two solutions $\langle G,H\rangle$, $\langle \tilde{G},\tilde{H}\rangle$ of (1.1), (1.2), (1.4a), (1.4b) and (1.17) with $G(x,\varepsilon),\tilde{G}(x,\varepsilon)$ having exactly $\bar{n}$ interior zeros and which also satisfy

$$H(x,\varepsilon) > 0, \quad \tilde{H}(x,\varepsilon) > 0, \qquad \text{(essentially)}$$

Let $n = \bar{n}$ and $\langle G(x,\varepsilon),H(x,\varepsilon)\rangle$ be the solution described in Theorem II. From (1.9) we see that

$$g_1 > 0, \quad g_0 > 0 .$$

Hence $G(x,\varepsilon)$ has exactly $n = \bar{n}$ interior zeros. In addition, let $n = \bar{n} - 1$ and $\langle \tilde{G}(x,\varepsilon),\tilde{H}(x,\varepsilon)\rangle$ be the solution described in Theorem II. Then

$$g_1 < 0, \quad g_0 < 0 \qquad \text{(essentially)}$$

and $\tilde{G}(x,\varepsilon)$ has exactly $n + 1 = \bar{n}$ interior zeros.

Case 2: $s < 0$. For every odd $\bar{n} > 3$ there are at least two solutions $\langle G,H\rangle,\langle \tilde{G},\tilde{H}\rangle$ of (1.1), (1.2), (1.4a), (1.4b) and (1.17) with $G(x,\varepsilon),\tilde{G}(x,\varepsilon)$ having exactly $\bar{n}$ interior zeros and also satisfy

$$H(x,\varepsilon) > 0, \quad \tilde{H}(x,\varepsilon) > 0, \qquad \text{(essentially)} .$$

If $\bar{n} = 1$ there is at least one solution $\langle G,H\rangle$ of (1.1), (1.2), (1.4c), (1.4b), and (1.7) with $G(x,\varepsilon)$ having exactly $\bar{n} = 1$ interior zeros while

$$H(x,\varepsilon) > 0 \qquad \text{(essentially)}.$$

Let $n = \bar{n}$ and $\langle G(x,\varepsilon),H(x,\varepsilon)\rangle$ be the solution described in Theorem II. From (1.9) we see that

$$g_1 < 0, \quad g_0 > 0 .$$

Hence $G(x,\varepsilon)$ has exactly $n = \bar{n}$ interior zeros. If $\bar{n} > 1$ let $n = \bar{n} - 1$ and let $\langle \tilde{G}(x,\varepsilon), \tilde{H}(x,\varepsilon) \rangle$ be the solution described in Theorem II. Then

$$g_1 > 0, \quad g_0 < 0$$

and $\tilde{G}(x,\varepsilon)$ has exactly $n + 1 = \bar{n}$ interior zeros.

Case 3: $s = 0$. For every $\bar{n} \geq 1$ (even or odd) there is at least one solution $\langle G(x,\varepsilon), H(x,\varepsilon) \rangle$ with $G(x,\varepsilon)$ having exactly $\bar{n}$ interior zeros and

$$H(x,\varepsilon) > 0 \quad \text{(essentially)} .$$

Let $n = \bar{n}$ and let $\langle G(x,\varepsilon), H(x,\varepsilon) \rangle$ be the solution described in Theorem II.

The basic Theorem I is proven via a "shooting" argument. The basic estimates follow from the following analysis. When $H(x,\varepsilon)$ is small, i.e.

$$H(x,\varepsilon) = O(\varepsilon^{2/3})$$

then one studies the "stretched" problem: let

(1.22a) $$\xi = \frac{x - x_0}{\varepsilon^{1/3}} ,$$

(1.22b) $$h(\xi,\varepsilon) = \varepsilon^{-2/3} H(x,\varepsilon), \quad g(\xi,\varepsilon) = \varepsilon^{-2/3} G(x,\varepsilon) .$$

The functions $\langle g,h \rangle$ satisfy the equations

(1.23a) $$h''' + hh'' + \frac{1}{2} \varepsilon^{2/3} g^2 - \frac{1}{2} (h')^2 = \mu/\varepsilon^{2/3} = \bar{\mu}\varepsilon^{1/3} ,$$

(1.23b) $$g'' + hg' - h'g = 0 .$$

With $\mu = O(\varepsilon)$, i.e., $\bar{\mu} = O(1)$. We find that

(1.24a) $$h(\xi,\varepsilon) \to \bar{h}(\xi) ; \quad \text{a quadratic function}$$

of the form

(1.24b) $$\bar{h}(\xi) = \frac{\delta}{2} (\xi - \xi_0)^2 .$$

Furthermore

$$g(\xi,\varepsilon) \to \bar{g}(\xi)$$

where $\bar{g}(\xi)$ satisfies

(1.25) $$\bar{g}'' + \bar{h}\bar{g}' - \bar{h}'\bar{g} = 0 .$$

The solutions of this problem are discussed in the Appendix. On the other hand, when $H(x,\varepsilon)$ is "large", then the development in [8] shows that $H, H', H'', H''', G, G', G''$ can all be estimated in terms of

$$\left|\frac{G(x,\varepsilon)}{H(x,\varepsilon)}\right| .$$

Fortunately, we do not require that $H(x,\varepsilon)$ be too large. In fact,

$$H(x,\varepsilon) > k\varepsilon^{2/3}$$

is sufficient. Hence the requirements of "small" $H(x,\varepsilon)$ and "large" $H(x,\varepsilon)$ overlap and we are able to give a complete analysis.

Realizing these facts one proceeds as follows.

Starting Procedure (See Theorem 3.2.)

For every choice of $\bar{\mu}, h_2, \alpha, g_0$ there is a solution $\langle g,h \rangle$ of (1.23a), (1.23b) on the interval $[0,\alpha]$ which also satisfies the boundary conditions

$$h(0,\varepsilon) = h'(0,\varepsilon) = 0, \quad h''(0,\varepsilon) = h_2 > 0 , \tag{1.26a}$$

$$g(0,\varepsilon) = g_0, \quad g(\alpha,\varepsilon) = h(\alpha,\varepsilon) . \tag{1.26b}$$

The results of Section 3 show that this solution $\langle G,H \rangle$ of (1.1), (1.2) - originally defined only on the interval $[0,\alpha\varepsilon^{1/3}]$ may be continued to entire interval $[0,2\pi - \delta]$ and, on this interval

$$H(x,\varepsilon) \approx \frac{h_2}{\tau^2}[1 - \cos \tau x], \quad G(x,\varepsilon) \approx \tau H(x,\varepsilon) \tag{1.27}$$

where $0 < \tau < 1$ and $\tau \to 1$ as $\alpha \to \infty$. The results of Section 4 show that this solution may in fact be continued to the larger interval $2\pi\left[1 + \frac{1}{|\tilde{\tau}|}\right] - \delta$ where $\tilde{\tau}$ is a negative number described in the Appendix. Furthermore, for small ε we have

$$H(x,\varepsilon) \approx \frac{h_2}{\tau_1^2}(1 - \cos \tau_1(x - 2\pi)], \quad 2\pi + \delta \leq x \leq 2\pi\left[1 + \frac{1}{|\tilde{\tau}|}\right] - \delta , \tag{1.28a}$$

$$G(x,\varepsilon) \approx -\tau_1 H(x,\varepsilon), \quad 2\pi + \delta \leq x \leq 2\pi\left[1 + \frac{1}{|\tilde{\tau}|}\right] - \delta , \tag{1.28b}$$

$$\tau_1 \approx \tau|\tilde{\tau}| . \tag{1.28c}$$

Thus we have exhibited 2 "humps". Proceeding in this way we construct a solution with n humps. Let $x_n = x_n(\varepsilon)$ be the n'th relative minimum of $H(x,\varepsilon)$. In Section 5 we employ an elementary degree theory argument to show that one may choose $\bar{\mu}, h_2$ so that

$$H(x_n,\varepsilon) = 0 \ ,$$
$$G(x_n,\varepsilon) = g_1$$

provided that

$$\text{sign } g_1 = (-1)^{n+1} \ .$$

In this way we prove Theorem I.

2. Existence of Solutions Away from Turning Points

In this section we are concerned with solutions of the equations (1.1), (1.2)

$$\varepsilon H^{iv} + HH''' + GG' = 0 \ , \tag{2.1}$$

$$\varepsilon G'' + HG' - H'G = 0 \ , \tag{2.2}$$

with initial data

$$d^{\nu}H(x_0)/dx^{\nu} = H_{\nu}, \ \nu = 0,1,2,3, \quad d^{\nu}G(x_0)/dx^{\nu} = G_{\nu}, \quad \nu = 0,1 \ , \tag{2.3a}$$

where

$$H_0 > 0 \ . \tag{2.3b}$$

In dealing with this case we use a basic estimate of Kreiss (lemma 2.1 of [7]) which we include for the sake of completeness.

Lemma 2.1: Consider the differential equation

$$\varepsilon \frac{dy}{dx} + a(x)y = F(x), \qquad \alpha < x < \beta \ , \tag{2.4}$$

where a, F are continuous functions with

$$a(x) > 0$$

and $\varepsilon > 0$ is a positive constant. The solutions of (2.4) satisfy the estimate

$$|y(x)| \le \|F/a\|_{\alpha,x} + \sigma(x,\alpha)|y(\alpha)|, \qquad \alpha \le x \ . \tag{2.5}$$

Here

$$\|f\|_{\alpha,x} \equiv \max_{\alpha \le t \le x} |f(t)| \tag{2.6a}$$

$$\sigma(x,\alpha) = \exp\{-\frac{1}{\varepsilon}\int_{\alpha}^{x} a(t)dt\} \ . \tag{2.6b}$$

Proof: The solutions of (2.4) are given by

$$y(x) = y(\alpha)\sigma(x,\alpha) + \int_{\alpha}^{x} \exp\{\frac{1}{\varepsilon}\int_{x}^{t} a(s)ds\} \cdot \frac{F(t)}{\varepsilon}\, dt \ .$$

We rewrite this as

$$y(x) = y(\alpha)\sigma(x,\alpha) + \int_{\alpha}^{x} \frac{F(t)}{a(t)}\, d\left(\exp\{\frac{1}{\varepsilon}\int_{x}^{t} a(s)ds\}\right) \ .$$

For fixed $x > \alpha$ the function

$$\exp\left\{\frac{1}{\varepsilon}\int_x^t a(s)ds\right\}$$

is monotone increasing as t increases from α to x. The estimate (2.5) follows from the mean value theorem.

<u>Lemma 2.2</u>: Let $\langle G(x,\varepsilon),H(x,\varepsilon)\rangle$ be a solution of (2.1), (2.2), (2.3a), (2.3b). Let

$$s(x,x_0) = \exp\left\{-\frac{1}{\varepsilon}\int_{x_0}^{x} H(t,\varepsilon)dt\right\} .$$

Let

$$G''(x_0,\varepsilon) = G_2 .$$

Then, for $x \geq x_0$ we have

(2.7a) $$|G'(x,\varepsilon)| \leq \|G/H\|_{x_0,x} \cdot \|H'\|_{x_0,x} + s(x,x_0)|G_1| ,$$

(2.7b) $$|G''(x,\varepsilon)| \leq \|G/H\|_{x_0,x} \cdot \|H''\|_{x_0,x} + s(x,x_0)|G_2| ,$$

(2.7c) $$|H'''(x,\varepsilon)| \leq \|G/H\|_{x_0,x} \cdot \|G'\|_{x_0,x} + s(x,x_0)|H_3| .$$

<u>Proof</u>: The estimates (2.7a), (2.7c) follow from Lemma 2.1 and equations (2.2), (2.1) respectively. Differentiating (2.2) we have

(2.8) $$\varepsilon G''' + HG'' = H''G .$$

The estimate (2.7b) follows from (2.8) and Lemma 2.1.

<u>Lemma 2.3</u>: Let $\langle G,H\rangle$ be a solution of the above problem in some interval $x_0 \leq x \leq x_1$ with the following properties

(2.9) $$H > 0,\ \|G/H\|_{x_0,x_1} \leq M,\ x_1 - x_0 \leq \min(1/M,1) .$$

Then there are constants K_{ij} which depend only on H_ν, $\nu = 0,1,2,3;$ G_ν, $\nu = 0,1$ and M and not on ε such that

$$\|d^jG/dx^j\|_{x_0,x_1} \le K_{1j}, \quad j = 0,1 ,$$

(2.10)

$$\|d^jH/dx^j\|_{x_0,x_1} \le K_{2j}, \quad j = 0,1,2,3 .$$

Also

(2.11) $$\|G''\|_{x_0,x_1} \le MK_{22} + |G_2|, \quad G_2 = G''(x_0,\varepsilon) .$$

Proof: For any solution of the above equations we may apply Lemma 2.2 and obtain the estimates (2.7a), (2.7b), (2.7c). Also, the Taylor expansion

$$H'(x) = H_1 + (x - x_0)H_2 + \int_{x_0}^{x} (\xi - x_0)H'''(\xi)d\xi$$

gives us

(2.12) $$\|H'\|_{x_0,x} \le |H_1| + |x - x_0|H_2 + \frac{1}{2}(x - x_0)^2\|H'''\|_{x_0,x} .$$

Therefore by (2.9) and (2.7)

$$\|H'''\|_{x_0,x_1} \le M^2\|H'\|_{x_0,x_1} + M|G_1| + |H_3| \le \frac{1}{2}(x_1 - x_0)^2M^2\|H'''\|_{x_0,x_1} +$$

$$+ M^2\{|H_1| + (x_1 - x_0)|H_2|\} + M|G_1| + |H_3| .$$

By (2.9) $(x_1 - x_0)^2M^2 \le 1$ and $x_1 - x_0 \le 1$. Therefore

$$\|H'''\|_{x_0,x_1} \le 2M^2(|H_1| + |H_2|) + 2M|G_1| + 2|H_3| .$$

Thus we have proved the estimate for H'''. By Taylor expansion (see (2.12)) we obtain the estimates for H,H',H'', and by (2.7a) and (2.7b) they follow also for G,G',G''.

We shall now use these estimates to derive existence theorems.

Theorem 2.1: Consider the initial value problem (2.1)-(2.3) and assume that

(2.13) $$H_0 \ge \delta > 0 .$$

Then there is an interval $x_0 \le x \le x_1$, $x_1 - x_0 > 0$, independent of ε, in which the above problem has for all ε with $0 < \varepsilon \le 1$ a unique solution. Moreover the estimates

(2.10) of Lemma 2.3 are valid. The constants K_{ij} depend only on $H_\nu, \nu = 0,1,2,3;$ $G_\nu, \nu = 0,1,2$ and δ.

Proof: Let $\varepsilon > 0$ be fixed. From the general existence theory for ordinary differential equations it follows that there is an interval $x_0 < x < x_1$ where the conditions of Lemma 2.3 are satisfied with $M = 2(|G_0/H_0| + 1)$. We want to estimate x_1. Taylor expansion gives us

$$|H(x) - H_0| < (x - x_0)K_{21}, \; |G(x) - G_0| < (x - x_0)K_{11} \; .$$

Therefore $H(x) > \frac{1}{2} H_0$, $|G(x)| < |G_0| + |H_0|$ and $|H(x)/G(x)| < M$ for

$$0 < x - x_0 < \min(1/K_{11}, H_0/(2K_{21})) = x_1 - x_0 \; .$$

Thus the solution exists in this interval and the theorem is proven.

Now consider the limit process $\varepsilon \to 0$. We want to prove

Theorem 2.2: Assume that G_ν, H_ν are functions of ε with

(2.14a) $$\lim_{\varepsilon \to 0} H_\nu = \bar{H}_\nu, \nu = 0,1,2, \quad \lim_{\varepsilon \to 0} G_0 = \bar{G}_0, \quad \bar{H}_0 > \delta > 0 \; .$$

Assume also that

(2.14b) $$H_3, G_1, G_2 \text{ are uniformly bounded} \; .$$

Let x_0, x_1 be as Theorem 2.1. Then the solutions of the initial value problem (2.1)-(2.3) converge on any interval $x_0 + \delta < x < x_1$, $\delta > 0$ to the solution of the reduced problem

(2.15a) $$\bar{H}\bar{H}''' + \bar{G}\bar{G}' = 0, \quad d^\nu \bar{H}/dx^\nu|_{x=x_0} = \bar{H}_\nu, \quad \nu = 0,1,2 \; ,$$

(2.15b) $$\bar{H}\bar{G}' - \bar{H}'\bar{G} = 0, \quad \bar{G}(x_0) = G_0 \; .$$

Proof: (2.14) and (2.10) show that

$$\varepsilon G'' = O(\varepsilon) \to 0 \; .$$

Differentiating (2.1) we obtain for $y = H^{iv}$ the equation

$$\varepsilon y' + Hy + H'H''' + (G')^2 + GG'' = 0 \; .$$

Therefore by Lemma 2.1 and (2.14)

(2.16) $$|y(x)| < \left\| \frac{H'H''' + (G')^2 + GG''}{H} \right\|_{x_0,x} + |y(x_0)| s(x,x_0) \; .$$

By (2.14) $\varepsilon H'''(x_0)$ is bounded. Therefore $y(x_0)s(x,x_0)$ is bounded for $x > x_0 + \delta$.

Therefore $\varepsilon y \to 0$ and the theorem follows by standard compactness arguments. This proves the theorem.

It is easy to see that the solution of (2.15) has the form

$$\bar{H}(x) = \bar{H}_0 + \frac{\bar{H}_1}{\tau}\sin\tau(x - x_0) + \frac{\bar{H}_2}{\tau^2}(1 - \cos\tau(x - x_0))$$

(2.17)

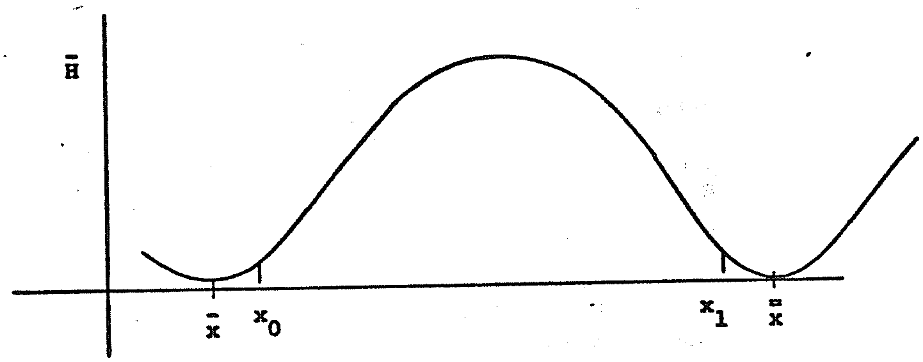
$$\bar{G}(x) = \tau\bar{H}(x), \quad \tau = \bar{G}_0/\bar{H}_0 .$$

Up till now we have only proved the convergence to G,H in the interval $x_0 \leq x \leq x_1$. However, we obtain immediately uniform convergence in any interval $x_0 \leq x \leq \bar{x}_1$ where $\bar{H}(x) \geq \sigma$, σ any constant > 0. This we can also express in another way. Let $\bar{x} < x_0 < \bar{\bar{x}}$ be the first points to the left and right of x_0 with $\bar{H}(\bar{x}) = \bar{H}(\bar{\bar{x}}) = 0$. Then we can prove existence of solutions $\langle G,H \rangle$ of (2.1), (2.2) and uniform convergence to $\langle \bar{G},\bar{H} \rangle$ in any interval $\bar{x} < x_0 \leq x \leq x_1 < \bar{\bar{x}}$. (Of course, if we move x_0 then we have to change the initial conditions to obtain the same $\langle \bar{G},\bar{H} \rangle$).

In [8] we proved that a necessary condition to obtain an order one solution of the rotating disc problem is that $\bar{H}'(\bar{x}) = \bar{H}'(\bar{\bar{x}}) = 0$ i.e. we can write $\langle \bar{G},\bar{H} \rangle$ in the form

(2.18)
$$\bar{H} = \frac{\bar{H}_2}{\tau^2}(1 - \cos\tau(x - \bar{x})), \quad \bar{G} = \tau\bar{H}, \quad \bar{x} \leq x \leq \bar{x} + \frac{2\pi}{|\tau|} .$$

Hence, we shall seek such solutions.



For later purposes we write the equation (2.1) in another form. We integrate (2.1) and get

$$\varepsilon H''' + HH'' + \frac{1}{2}(G^2 - H'^2) = \tilde{\mu} \; . \tag{2.19}$$

To obtain a limit solution of the form (2.18) it is necessary and sufficient that $\lim_{\varepsilon \to 0} \tilde{\mu} = 0$ because a simple calculation shows that $\langle \bar{G}, \bar{H} \rangle$ satisfies the equation

$$\bar{H}\bar{H}'' + \frac{1}{2}(\bar{G}^2 - (\bar{H}')^2) = 0 \; .$$

For our purposes the right choice is

$$\tilde{\mu} = \varepsilon \bar{\mu} \; . \tag{2.20}$$

Instead of H_3 we can give $\bar{\mu}$ as initial condition and compute H_3 from (2.19). In particular H_3 is bounded if

$$H_0 H_2 + \frac{1}{2}(G_0^2 - H_1^2) = O(\varepsilon), \quad \bar{\mu} \text{ bounded} \; . \tag{2.21}$$

Finally we collect a number of formulas which we will need later.

<u>Lemma 2.4:</u> We can write the equations (2.1), (2.2) in the form

$$H'''(x) = H'''(x_0) e^{-\frac{1}{\varepsilon}\int_{x_0}^{x} H(\eta)d\eta} - \frac{1}{\varepsilon}\int_{x_0}^{x} e^{-\frac{1}{\varepsilon}\int_{\xi}^{x} H(\eta)d\eta} G(\xi)G'(\xi)d\xi \; , \tag{2.22}$$

$$G'(x) = G'(x_0) e^{-\frac{1}{\varepsilon}\int_{x_0}^{x} H(\eta)d\eta} + \frac{1}{\varepsilon}\int_{x_0}^{x} e^{-\frac{1}{\varepsilon}\int_{\xi}^{x} H(\eta)d\eta} H'(\xi)G(\xi)d\xi \; , \tag{2.23}$$

$$G'(x) = (G(x)/H(x))H'(x) - \varepsilon G''(x)/H(x) \; , \tag{2.24}$$

$$\frac{d}{dx}(G/H) = (HG' - GH')/H^2 = -\varepsilon G''(x)/H^2(x) \; . \tag{2.25}$$

Also

$$G''(x) = G''(x_0) e^{-\frac{1}{\varepsilon}\int_{x_0}^{x} H(\eta)d\eta} + \frac{1}{\varepsilon}\int_{x_0}^{x} e^{-\frac{1}{\varepsilon}\int_{\xi}^{x} H(\eta)d\eta} H''(\xi)G(\xi)d\xi \; . \tag{2.26}$$

<u>Proof:</u> These equations follow directly from (2.1), (2.2) and (2.8).

3. Existence of Solutions when H_0 is Small

We consider again the initial value problem (2.1)-(2.3) and write (2.2) in the integrated form (2.13) with $\tilde{\mu}$ of the form (2.14). Consider initial data of the form

(3.1) $\quad H_0 = \varepsilon^{2/3}h_0, \quad H_1 = \varepsilon^{1/3}h_1, \quad H_2 = h_2, \quad \tilde{\mu} = \varepsilon\bar{\mu}, \quad h_\nu > 0, \quad \nu = 0,1,2 ,$

(3.2) $\quad G_0 = \varepsilon^{2/3}g_0, \quad G_1 = \varepsilon^{1/3}g_1, \quad g_\nu > 0, \quad \nu = 0,1 .$

$\bar{\mu}, h_\nu, g_\nu$ can be functions of ε but we assume that

$$\bar{\mu}, g_0, g_1, h_0, h_1, h_2 \quad \text{and} \quad H_3 = -\frac{h_0h_2 - \frac{1}{2}h_1^2}{\varepsilon^{1/3}} + \bar{\mu} - \frac{1}{2}\varepsilon^{1/3}g_0^2$$

are bounded independently of ε. We assume also that

(3.3) $$G_2 = h_1g_0 - h_0g_1 > 0 ,$$

and is bounded independently of ε. We want to prove

Theorem 3.1: There is an interval $x_0 \le x \le x_1$, $x_1 - x_0 > 0$ independent of ε, in which the above problem has a unique solution for all ε with $0 < \varepsilon \le 1$. Moreover

(3.4) $$0 < \frac{1}{2}H_2 \le H''(x) \le \frac{3}{2}H_2, \quad x_0 \le x \le x_1 ,$$

and the estimates (2.5) of Lemma 2.1 are valid. $x_1 - x_0$ and the constants K_{ij} depend only on $H_\nu, \nu = 0,1,2,3; g_0/h_0$ and $G_\nu, \nu = 0,1,2$. Also

(3.5) $$\varepsilon G''(x) = O(\varepsilon), \quad H(x) \ge H_0 + \frac{1}{4}H_2(x - x_0)^2 .$$

Proof: Let $x_0 \le x \le x_1$ be the largest interval satisfying (3.4). Then (2.20), (3.2), (3.3) and (3.4) imply that $G''(x) \ge 0$. Therefore by (2.19)

$$\|G/H\|_{x_0,x_1} \le G_0/H_0 = g_0/h_0 = M .$$

Thus Lemma 2.1 shows that $d^\nu H/dx^\nu$, $\nu = 0,1,2,3$; $d^\nu G/dx^\nu$, $\nu = 0,1,2$ are bounded if (3.4) holds and $x_1 - x_0 \le \min(1/M,1)$. By Taylor expansion

$$|H''(x) - H''(x_0)| \le (x - x_0)\|H'''\|_{x,x_0} .$$

Therefore, we can find x_1, independent of ε, such that (3.4) is valid. Then (3.5) follows from Lemma 2.1 and the assumption that $h_0 > 0$, $h_1 > 0$ and that G_2 is bounded independently of ε. This proves the theorem.

Now we consider the limiting process $\varepsilon \to 0$. Assume that

$$\varepsilon^{2/3}h_0 \to 0,\ \varepsilon^{1/3}h_1 \to 0,\ h_2 \to \bar{h}_2 > 0,\ \varepsilon^{2/3}g_0 \to 0,\ \varepsilon^{1/3}g_1 \to 0,\ g_0/h_0 \to \tau_1 > 0 .$$

g_0, h_0, g_1, h_1 can be large. However, we assume that $\bar{\mu}, \varepsilon^{1/3}g_0^2$ are uniformly bounded and that

$$g_1 = (g_0/h_0)h_1 - \ell/h_0, \quad h_1^2 = 2h_0h_2 + 2\varepsilon^{1/3}p \tag{3.6}$$

where $\ell > 0, p$ are also uniformly bounded. This assumption guarantees that we have bounds for

$$G_2 = \ell > 0 \text{ and } H_3 = p + \bar{\mu} - \frac{1}{2}\varepsilon^{1/3}g_0^2 .$$

By Section 2 and Theorem 3.1 $\langle G,H \rangle$ converge in a neighbourhood of x_1 to a solution $\langle \bar{G},\bar{H} \rangle$ of the reduced equation. $\langle \bar{G},\bar{H} \rangle$ is of the form (2.12) with $\bar{x} = x_0$. We want to derive a relation between τ_1 and τ. By (2.19) and (3.5)

$$\frac{G(x_1)}{H(x_1)} = \frac{G(x_0)}{H(x_0)} - \varepsilon \int_{x_0}^{x} \frac{G''(x)}{H^2(x)}\, dx$$

with

$$\varepsilon \int_{x_0}^{x_1} \frac{G''}{H^2}\, dx < \text{const.}\ \varepsilon \int_{x_0}^{x} \frac{dx}{(H_0 + \frac{1}{4}H_2(x - x_0)^2)^2}$$

$$= \text{const.}\ \frac{\varepsilon}{H_0^2} \int_{x_0}^{\infty} \frac{dx}{\left(1 + \frac{1}{4}\left(\frac{x - x_0}{H_0^{1/2}}\right)^2\right)^2} < \text{const.}\ \frac{\varepsilon}{H_0^{3/2}} = \frac{\text{const.}}{h_0^{3/2}}$$

i.e.

$$\frac{G(x_1)}{H(x_1)} = \frac{G(x_0)}{H(x_0)} + O\left(\frac{1}{h_0^{3/2}}\right) = \frac{g_0}{h_0} + O\left(\frac{1}{h_0^{3/2}}\right) . \tag{3.7}$$

Now let $\varepsilon \to 0$ then $G(x_1)/H(x_1) \to \tau$ and therefore

$$\tau = g_0/h_0 + O(1/h_0^{3/2}) = \tau_1 + O(1/h_0^{3/2}) . \tag{3.8}$$

We consider now a two-point boundary value problem for the equations (2.1), (2.13) in an interval

$$0 \le x \le x_0 = \alpha\varepsilon^{1/3}, \qquad \alpha = \text{const.} > 0 .$$

The boundary conditions are

(3.9) $\quad H(0) = H'(0) = 0, \quad H''(0) = h_2 > 0, \quad G(0) = \varepsilon^{2/3} g_0, \quad G(x_0) = \tau_1 H(x_0), \; \tau_1 > 0 .$

We want to prove

<u>Theorem 3.2:</u> The boundary value problem (2.2), (2.13), (3.9) has a solution with the following properties. $G, G', G'', H, H', H'', H'''$ can be estimated by h_2, τ_1, g_0 and $\varepsilon^{1/3}\bar{\mu}$. Also, $G'(x_0) > 0$, $G''(x_0) > 0$ provided α is big enough.

<u>Proof:</u> Introduce new variables by

$$x = \varepsilon^{1/3}\tilde{x}, \quad G = \varepsilon^{2/3} g, \quad H = \varepsilon^{2/3} h .$$

Then the above equations become

(3.10)
$$\dddot{h} + h\ddot{h} + \frac{1}{2}\left(\varepsilon^{2/3} g^2 - \dot{h}^2\right) = \varepsilon^{1/3}\bar{\mu}$$
$$\ddot{g} + h\dot{g} - g\dot{h} = 0 ,$$

(3.11) $\quad h(0) = \dot{h}(0) = 0, \; \ddot{h}(0) = h_2, \; g(0) = g_0, \; g(\alpha) = \tau_1 h(\alpha) .$

We can solve the reduced equations

(3.12) $\quad \dddot{\bar{h}} + \bar{h}\ddot{\bar{h}} - \frac{1}{2}\dot{\bar{h}}^2 = 0, \quad h(0) = \dot{h}(0) = 0, \quad \ddot{h}(0) = h_2 ,$

(3.13) $\quad \ddot{\bar{g}} + \bar{h}\dot{\bar{g}} - \bar{g}\dot{\bar{h}} = 0, \quad \bar{g}(0) = g_0, \quad \bar{g}(\alpha) = \tau_1 \bar{h}(\alpha) .$

The solution of (3.10) is

(3.14)
$$\bar{h} = \frac{1}{2} h_2 \tilde{x}^2 .$$

Introducing this expression into (3.13) and the boundary conditions gives us

(3.15) $\quad \ddot{\bar{g}} + \frac{h_2}{2}\tilde{x}^2\dot{\bar{g}} - h_2\tilde{x}\bar{g} = 0, \quad \bar{g}(0) = g_0, \quad \bar{g}(\alpha) = \tau_1\bar{h}(\alpha) .$

By the Appendix the general solution of the differential equation (3.15) is of the form

(3.16) $\quad \bar{g} = \lambda_1\varphi_1\left(\left(\frac{1}{2}h_2\right)^{1/3}\tilde{x}\right) + \lambda_2\left[\frac{1}{2}h_2\right]^{1/3}\varphi_2\left(\left(\frac{1}{2}h_2\right)^{1/3}\tilde{x}\right) ,$

where $\varphi_1(t)$ decays exponentially for $t \to \infty$, $\varphi_1(0) \ne 0$ and

$$\lim_{t\to\infty}\frac{\varphi_2(t)}{t^2} = 1, \qquad \lim_{t\to\infty}\frac{\varphi_2'(t)}{2t} = 1, \qquad \lim_{\tilde{x}\to\infty}\frac{\varphi_2''(t)}{2} = 1 .$$

Therefore, if α is sufficiently large, then

$$(3.17)\qquad \lambda_2 \sim \tau_1, \quad \lambda_1 \sim \frac{g_0 - \tau_1[\frac{1}{2} h_2]^{1/3}\varphi_2(0)}{\varphi_1(0)} \; .$$

and

$$(3.18)\qquad \dot{\bar{g}} > 0, \; \ddot{\bar{g}} > 0, \qquad \bar{g}/\bar{h} \sim \tau_1 \quad \text{for all sufficiently large } x \; .$$

By a standard perturbation argument it follows that the full system (3.8), (3.9) has a solution with

$$(3.19)\qquad \| d^\nu g/d\tilde{x}^\nu - d^\nu \bar{g}/d\tilde{x}^\nu \|_{0,\alpha} + \| d^\nu h/d\tilde{x}^\nu - d^\nu \bar{h}/d\tilde{x}^\nu \|_{0,\alpha} < C_\nu \varepsilon^{1/3}, \quad \nu = 0,1,2 \; .$$

Also $\dddot{\bar{h}} \equiv 0$ implies

$$(3.20)\qquad \| \dddot{h} \|_{0,\alpha} < C_3 \varepsilon^{1/3} \; .$$

We return now to the original variables. By (3.17)

$$(3.21)\qquad \begin{gathered} H(x) = \frac{h_2}{2} x^2 + O(\varepsilon), \; H'(x) = h_2 x + O(\varepsilon^{2/3}), \; H''(x) = h_2 + O(\varepsilon^{1/3}), \; H'''(x) = \dddot{h}/\varepsilon^{1/3} \; . \\ G = \varepsilon^{2/3}\bar{g} + O(\varepsilon), \quad G' = \varepsilon^{1/3}\dot{\bar{g}} + O(\varepsilon^{2/3}), \quad G'' = \ddot{\bar{g}} + O(\varepsilon^{1/3}) \; . \end{gathered}$$

This proves the theorem.

Assume now that α, h_2, g_0, τ_1 and $\bar{\mu}$ are fixed and let $\varepsilon \to 0$. At $x_0 = \alpha\varepsilon^{1/3}$ the conditions of Theorem 3.1 are satisfied. Thus $\langle G,H \rangle$ can be continued and converges to a solution $\langle \bar{G},\bar{H} \rangle$ of the reduced equation (2.12). Here

$$(3.22)\qquad h_2 = H''(0) = \bar{H}_2$$

and τ,τ_1 satisfy the relation (3.8) with $h_0 = \frac{1}{2}\alpha^2 + O(\varepsilon)$ i.e.

$$(3.23)\qquad \tau = \tau_1 + O(1/\alpha^3) + O(\varepsilon) \; .$$

We summarize the result in

<u>Theorem 3.3:</u> Consider the two-point boundary value problem (2.1), (2.13), (3.9). Assume that h_2, g_0, τ_1 and $\bar{\mu}$ are fixed and α sufficiently large let $\varepsilon \to 0$. Then $\langle G,H \rangle$ converges uniformly in any interval $0 < x < x_1 < \bar{\bar{x}}$ to a solution $\langle \bar{G},\bar{H} \rangle$ of the reduced equation (2.12) with τ and H_2 satisfying the relations (3.22) and (3.23) respectively.

4. Existence of Solutions Through a Turning Point

In the previous sections we have shown that we can construct a solution of (2.1), (2.2) for $0 = \bar{x} < x < x_1 < \bar{\bar{x}} = \frac{2\pi}{|\tau|}$ which for $\varepsilon \to 0$ converges to a solution of the reduced equations (2.12). For simplicity we assume that $\tau > 0$ otherwise we change G to $-G$. Also, x_1 can be arbitrarily close to $\bar{\bar{x}}$. We shall now show that the solutions of (2.1), (2.2) can be continued through the turning point $\bar{\bar{x}}$. Let $x_1 < \bar{\bar{x}}$ be a point near $\bar{\bar{x}}$ where

$$H(x_1) > 0,\ H'(x_1) < 0,\ H''(x_1) \sim \bar{H}_2 > 0,\ G(x_1) > 0,\ G'(x_1) < 0,\ G''(x_1) > 0$$

for all sufficiently small ε. We want to show

<u>Lemma 4.1.</u> Consider (2.1) in the form (2.13) and assume that $\tilde{\mu}$ is given by (2.14). For sufficiently small ε there is a point $x_2 > x_1$ with

$$\text{(4.1)} \qquad \begin{aligned} &H(x_2) = \varepsilon^{2/3}h_0 > 0,\ H'(x_2) = \varepsilon^{1/3}h_1 < 0,\ \tfrac{1}{2}\bar{H}_2 < H''(x_2) = h_2 < \tfrac{3}{2}\bar{H}_2\,, \\ &G(x_2) = \varepsilon^{2/3}g_0 > 0,\ G'(x_2) = \varepsilon^{1/3}g_1 < 0\,. \end{aligned}$$

Here $h_0 > 0$ is a sufficiently large constant and h_1, g_0, g_1 are constants which depend only on h_0 and not on ε. Also

$$\text{(4.2)} \qquad H(x) > 0,\quad H'(x) < 0,\quad G(x) > 0,\quad G'(x) < 0,\quad G''(x) > 0$$

and $H'''(x)$ is uniformly bounded and can be estimated independently of μ in the whole interval $x_1 \le x \le x_2$.

<u>Proof:</u> There are two possibilities.

<u>1)</u> $H(x) > \varepsilon^{2/3}h_0$ <u>for all</u> $x \geq x_1$. We want to show that there must be a point x_3 where $H(x)$ has a minimum. Let $x_1 \le x \le x_4$ be an interval where $G(x) \geq 0$, $H''(x) \geq 0$. Then by (2.20) also $G''(x) > 0$ and (2.19) gives us

$$\text{(4.3)} \qquad 0 < \frac{G(x)}{H(x)} = \frac{G(x_1)}{H(x_1)} - \varepsilon \int_{x_1}^{x} \frac{G''}{H^2}\,dx \le \frac{G(x_1)}{H(x_1)}, \qquad x_1 \le x \le x_4\,.$$

Thus by Lemma 2.1 $H'''(x), G''(x)$ are uniformly bounded. Therefore, choosing x_1 sufficiently close to $\bar{\bar{x}}$ and $x_4 - x_1$ sufficiently small guarantees

$$\text{(4.4)} \qquad \tfrac{1}{2}\bar{H}_2 \le H''(x) \le \tfrac{3}{2}\bar{H}_2, \qquad x_1 \le x \le x_4\,.$$

Also, in the same way as in Section 3

$$\varepsilon \int_{x_1}^{x} \frac{G''}{H^2}\,dx < \text{const.}\ \varepsilon \int_{x_1}^{x} \frac{dx}{(H(x_4) + \frac{1}{2} H_2(x - x_4))^2} < \text{const.}\ h_0^{-3/2}$$

i.e., for sufficiently large h_0

$$\frac{1}{2}\frac{G(x_1)}{H(x_1)} < \frac{G(x)}{H(x)} < \frac{G(x_1)}{H(x_1)}, \quad \text{i.e.} \quad G(x) > 0\ . \tag{4.5}$$

Therefore we can find always an interval $x_1 < x < x_4$ with the above properties whose length $x_4 - x_1$ does not depend on ε and x_1. Choosing x_1 sufficiently near to $\bar{\bar{x}}$ makes $H'(x_1)$ as small as we like because H, H' converge to $\bar{H}, \bar{H}'$. Therefore (4.4) implies that there must be a point $x_3 \in (x_1, x_4)$ with $H'(x_3) = 0$. At this minimum (2.13) gives us

$$HH'' + \frac{1}{2} G^2 = \varepsilon\bar{\mu} - \varepsilon H'''\ . \tag{4.6}$$

For sufficiently small ε and $\bar{\bar{x}} - x_1$ (4.5) and (2.6c) show that H''' is bounded. Therefore $H(x_3) = O(\varepsilon)$ which is a contradiction.

<u>2). There is a point</u> x_2 <u>with</u> $H(x_2) = \varepsilon^{2/3} h_0$ <u>and</u> $H'(x) < 0$ <u>for</u> $x_1 < x < x_2$. Using (4.3) we find again that $G''(x) > 0$ and that (4.4) and (4.5) hold for $x_1 < x < x_2$. In particular $G(x_2) = \varepsilon^{2/3} g_0$. Also, by (2.13),

$$|H'(x_2)| = \sqrt{HH'' + O(\varepsilon)} = O(\varepsilon^{1/3})\ .$$

Therefore by (2.18)

$$0 < -G'(x_2) = O(\varepsilon^{1/3})\ .$$

By Lemma 2.1 the bound on H''' depends only on the bound for G/H which is independent of $\bar{\mu}$. Therefore H''' can be estimated independently of $\bar{\mu}$. This proves the lemma.

Now we can proceed as in Section 3. For $x > x_2$ we introduce new variables

$$h(x) = H(x)/\varepsilon^{2/3}, \quad g(x) = G(x)/\varepsilon^{2/3}, \quad x - x_2 = \varepsilon^{1/3}\tilde{x}, \quad \tilde{x} > 0\ .$$

Then we obtain the equations (3.8)

$$\begin{aligned} &\dddot{h} + h\ddot{h} + \frac{1}{2}(\varepsilon^{2/3} g^2 - \dot{h}^2) = \varepsilon^{1/3}\bar{\mu} \\ &\ddot{g} + h\dot{g} - g\dot{h} = 0 \end{aligned} \tag{4.7}$$

with boundary conditions

(4.8) $$h(0) = h_0,\ \dot{h}(0) = h_1,\ \ddot{h}(0) = h_2,\ g(0) = g_0,\ \dot{g}(0) = g_1 .$$

Also

(4.9) $\dddot{h}(0) = \varepsilon^{1/3}H'''(x_2) = O(\varepsilon^{1/3})$, $H'''(x_2)$ is bounded independently of $\bar{\mu}$,

and

(4.10) $$g_0/h_0 = \tau_1 = G(x_1)/H(x_1) + O(1/h_0^{3/2}) = \tau + O(\varepsilon + h_0^{-3/2})$$
$$g_1 = \tau_1 h_1 + O(1/h_0) = \tau h_1 + O(\varepsilon + 1/h_0) .$$

(4.9) and (4.7) show that

(4.11) $$2h_0h_2 - h_1^2 = O(\varepsilon^{1/3}), \quad \text{i.e.} \quad h_1 = -\sqrt{h_0h_2} + O(\varepsilon^{1/3}) ,$$

and the uniform boundedness of $H'''(x)$ for $x_1 \leq x \leq x_2$ shows that

(4.12) $$\lim_{\varepsilon \to 0} h_2 = \bar{H}_2 .$$

We can solve the reduced problem

(4.13) $$\dddot{\bar{h}} + \bar{h}\ddot{\bar{h}} - \frac{1}{2}\dot{\bar{h}}^2 = 0, \quad h(0) = h_0, \quad \dot{h}(0) = h_1, \quad \ddot{h}(0) = h_2$$

(4.14) $$\ddot{\bar{g}} + \bar{h}\dot{\bar{g}} - \bar{g}\dot{\bar{h}} = 0, \quad \bar{g}(0) = g_0, \quad \dot{\bar{g}}(0) = g_1 .$$

By (4.11) the solution of (4.13) is in any finite interval $0 \leq \tilde{x} \leq \tilde{x}_4$ of the form

(4.15) $$\bar{h} = \frac{1}{2}h_2(\tilde{x} - \tilde{x}_3)^2 + O(\varepsilon^{1/3})$$

where $\tilde{x}_3$ is determined by

$$h_0 = \frac{1}{2}h_2\tilde{x}_3^2 + O(\varepsilon^{1/3}), \quad \text{i.e.} \quad \tilde{x}_3 = \sqrt{\frac{2h_0}{h_2}} + O(\varepsilon^{1/3}) .$$

Replacing $\bar{h}$ in (4.14) by $\frac{1}{2}h_2(\tilde{x} - \tilde{x}_3)^2$ and introducing a new variable
$$\xi = \left(\frac{1}{2}h_2\right)^{1/3}(\tilde{x} - \tilde{x}_3)$$

gives us

(4.16) $$d^2\bar{\bar{g}}/d\xi^2 + \xi^2 d\bar{\bar{g}}/d\xi - 2\xi\bar{\bar{g}} = 0, \ \xi \geq \xi_0 = -\left(\frac{1}{2}h_2\right)^{1/3}\tilde{x}_3 = -\left[\frac{h_0}{\left(\frac{1}{2}h_2\right)^{1/3}}\right]^{1/2} + O(\varepsilon^{1/3})$$

with boundary conditions

(4.17) $$\bar{\bar{g}}(\xi_0) = g_0, \quad d\bar{\bar{g}}(\xi_0)/d\xi = \left(\frac{1}{2}h_2\right)^{-1/3} g_1 .$$

By (4.10) and (4.11)

$$\frac{\bar{\bar{g}}(\xi_0)}{\xi_0^2} = \frac{g_0}{\xi_0^2} = \left(\frac{1}{2}h_2\right)^{1/3}\frac{g_0}{h_0} = \left(\frac{1}{2}h_2\right)^{1/3}\tau + O(h_0^{-3/2} + \varepsilon) ,$$

(4.18)

$$\frac{d\bar{\bar{g}}(\xi_0)/d\xi}{2\xi_0} = \frac{\left(\frac{1}{2}h_2\right)^{-1/3} g_1}{-2\sqrt{\frac{h_0}{\left(\frac{1}{2}h_2\right)^{1/3}}}} + O(\varepsilon^{1/3}) = -\frac{\left(\frac{1}{2}h_2\right)^{1/3} g_1}{\sqrt{2h_0h_2}} + O(\varepsilon^{1/3})$$

$$= \left(\frac{1}{2}h_2\right)^{1/3}\tau + O(h_0^{-3/2} + \varepsilon^{1/3}) .$$

Thus by the Appendix

$$\bar{\bar{g}}(\xi) = \left(\frac{1}{2}h_2\right)^{1/3}\tau g_1(\xi) + O(h_0^{-3/2} + \varepsilon^{1/3})$$

where $g_1(\xi)$ is monotone decaying with

$$\lim_{\xi\to-\infty}\frac{g_1(\xi)}{\xi^2} = 1, \quad g_1(0) < 0, \quad \lim_{\xi\to+\infty}\frac{g_1(\xi)}{\xi^2} = -|\tilde{\tau}| = \tilde{\tau} .$$

For the original equations (4.7) a standard perturbation analysis gives us in any finite interval $0 < \tilde{x} < \tilde{x}_4$,

$$h(\tilde{x}) = \frac{1}{2}h_2(\tilde{x} - \tilde{x}_3)^2 + O(\varepsilon^{1/3}) ,$$

(4.19)

$$g(\tilde{x}) = \left(\frac{1}{2}h_2\right)^{1/3}\tau g_1\left(\left[\frac{1}{2}h_2\right]^{1/3}(\tilde{x} - \tilde{x}_3)\right) + O(h_0^{-3/2} + \varepsilon^{1/3}) .$$

Thus we can shoot through the turning point. In particular, we can choose $\tilde{x}_4$ so large that the conditions of Theorem 3.1 are satisfied. Furthermore

$$g(\tilde{x}_4)/h(\tilde{x}_4) = \tau\tilde{\tau} + O(h_0^{-3/2} + \varepsilon^{1/3}) .$$

Thus we can continue the solution of our problem to the next turning point $\bar{\bar{\bar{x}}} \cong 2\pi\left(\frac{1}{\tau} + \frac{1}{\tau|\tilde{\tau}|}\right)$ where we can repeat the process.

In the next section we need

<u>Lemma 4.2:</u> For ε sufficiently small $H'''(x)$ can be estimated independently of $\bar{\mu}$.

Proof: Away from the turning points this is clear because by Lemma 2.1 a bound for $H'''(x)$ depends only on a bound for $G/H \sim \tau$. To estimate H''' for $0 \leq \tilde{x} \leq \tilde{x}_4$ we can neglect the term $\varepsilon^{2/3}g^2$ in (4.7). Differentiating gives us

$$\overset{\cdot\cdot\cdot\cdot}{h} + h\,\overset{\cdot\cdot\cdot}{h} = 0, \qquad \overset{\cdot\cdot\cdot}{h}(\tilde{x}) = e^{-\int_0^{\tilde{x}} h(\xi)d\xi}\;\overset{\cdot\cdot\cdot}{h}(0) .$$

We know that $\overset{\cdot\cdot\cdot}{h}(0)/\varepsilon^{1/3}$ is bounded independently of $\bar{\mu}$ and therefore the same is true for $\overset{\cdot\cdot\cdot}{h}(\tilde{x})$ because $h(\xi) \geq -O(\varepsilon^{1/3})$. This proves the lemma.

5. Existence of Solutions

Let $n \geq 1, g_0, g_1$ be given with

$$\text{sign } g_1 = (-1)^n .$$

Let

(5.1) $$\tau = |\tilde{\tau}|^{n-1}$$

where $\tilde{\tau}$ is the constant given in (A.6). Let

(5.2a) $$\underline{h}_2 = |(g_1/[100\tau g_1(0)])^3|$$

where $g_1(x)$ is the function described by (A.1) with $\delta = 1$ and (A.2a). Let

(5.2b) $$\bar{h}_2 = |(200g_1/[\tau g_1(0)])^3| + 1 .$$

Let Q be the bound on $|H'''(x)|$ determined in Lemma 4.2. Let

(5.3) $$\beta = Q + 1 .$$

For the remainder of this section we require that

(5.4a) $$|\bar{\mu}| \leq \beta, \quad \text{i.e.} \quad |\tilde{\mu}| \leq \beta\varepsilon, \qquad \tau_1 = 1 ,$$

(5.4b) $$\underline{h}_2 \leq h_2 \leq \bar{h}_2 .$$

Let $\langle G(x,\varepsilon;\bar{\mu},h_2), H(x,\varepsilon;\bar{\mu},h_2)\rangle$ be the solution of (2.1), (2.2) which arise from the pair $\langle g(\xi,\varepsilon;\bar{\mu},h_2), h(\xi,\varepsilon;\bar{\mu},h_2)\rangle$ which satisfy the boundary value problem (3.8), (3.9). The constant α is fixed with $\alpha >> 1$.

By the arguments developed in the previous sections we obtain the following results.

Theorem 5.1: Let $n \geq 1$ be a fixed integer. If α is chosen large enough and ε small then $\langle G(x,\varepsilon;\bar{\mu},h_2), H(x,\varepsilon;\bar{\mu},h_2)\rangle$ exist on an interval $[0,B]$ whose length B is of order 1. The function $G(x,\varepsilon;\bar{\mu},h_2)$ has at least n zeros

(5.5a) $$0 < x_1 < x_2 < \cdots x_n < B .$$

The function $H(x;\varepsilon;\bar{\mu},h_2)$ has at least n relative minima

(5.5b) $$0 < y_1 < y_2 < \cdots < y_n < B .$$

If $g_0 < 0$ then $G(x,\varepsilon;\mu,h_2)$ has another zero x_0 with $0 < x_0 < x_1$. These numbers satisfy

(5.6a) $$0 < x_j - y_j \leq o(1) \quad \text{as} \quad \varepsilon \to 0, \qquad j = 1,2,\ldots,n ,$$

(5.6b) $$x_1 = 2\pi + o(1) ,$$

(5.6c) $$x_{j+1} - x_j = (2\pi)/|\tilde{\tau}|^{j-1} + o(1), \qquad j = 1,2,\ldots,n .$$

Also

(5.6d) $$H''(x_n,\varepsilon,\bar{\mu},h_2) = h_2 + o(1) .$$

On the interior of the interval (x_{j-1},x_j), $j = 1,2,\ldots,n$ we have

(5.7a) $$H(x,\varepsilon;\bar{\mu},h_2) = \frac{h_2}{|\tilde{\tau}|^{2(j-1)}}[1 - \cos|\tilde{\tau}|^{j-1}(x - x_{j-1})] + o(1)$$

while

(5.7b) $$G(x,\varepsilon;\bar{\mu},h_2) = (\tilde{\tau})^{j-1}H(x,\varepsilon;\bar{\mu},h_2) + o(1) .$$

If $\tilde{\mu} = \beta\varepsilon$ then

(5.8a) $$H(x_n,\varepsilon;\beta,h_2) > 0 ,$$

and, if $\mu = -\beta\varepsilon$ then

(5.8b) $$H(x_n,\varepsilon;-\beta,h_2) < 0 .$$

If $h_2 = \bar{h}_2$, then

(5.9a) $$G(x_n,\varepsilon,\mu,\bar{h}_2) = [(\tfrac{1}{2}\bar{h}_2)^{1/3}|\tilde{\tau}|^{n-1}g_1(0)]\varepsilon^{2/3} + o(\varepsilon^{2/3}) > g_1\varepsilon^{2/3} .$$

Similarly, if $h_2 = \underline{h}_2$, then

(5.9b) $$G(x_n,\varepsilon,\mu,\underline{h}_2) = [(\tfrac{1}{2}\underline{h}_2)^{1/3}|\tilde{\tau}|^{n-1}g_1(0)]\varepsilon^{2/3} + o(\varepsilon^{2/3}) < \bar{g}_1\varepsilon^{2/3} .$$

<u>Corollary</u>: There is a choice of $\bar{\mu},h_2$ which satisfy (5.4a), (5.4b) and

(5.10a) $$H(x_n,\varepsilon;\bar{\mu},h_2) = 0$$
$$G(x_n,\varepsilon;\bar{\mu},h_2) = g_1\varepsilon^{2/3} .$$

<u>Proof</u>: Let $\varepsilon > 0$ be so small that Theorem 5.1 holds. Consider the mapping

(5.11a) $$(\bar{\mu},h_2) \to (G(x_n,\varepsilon;\bar{\mu},h_2),H(x_n,\varepsilon;\bar{\mu},h_2)) .$$

With

(5.11b) $$|\bar{\mu}| \leq \beta, \quad \underline{h}_2 \leq h_2 \leq \bar{h}_2 .$$

Since $H''(x_n,\varepsilon,\bar{\mu};h_2) \approx h_2 > 0$ the implicit function theorem shows that x_n defined by the n'th min:

$$H'(x_n,\varepsilon;\bar{\mu},h_2) = 0$$

is a continuous function of $(\bar{\mu},h_2)$. Thus, the mapping (5.11a), (5.11b) is a continuous mapping.

The properties (5.5a), (5.5b), (5.6a), (5.6b) together with an elementary "degree" theory argument see [20] shows that there is a solution of (5.7a), (5.7b). To see this we consider the homotropy

$$G_t(x_n,\varepsilon;\bar{\mu},h_2) = tg_1\left[2 - \frac{3}{2}\left(\frac{\bar{h}_2 - h_2}{\bar{h}_2 - \underline{h}_2}\right)\right]\varepsilon^{2/3} + (1 - t)G(x_n,\varepsilon;\bar{\mu},h_2), \quad 0 \leq t \leq 1 ,$$

$$H_t(x_n,\varepsilon;\bar{\mu},h_2) = t\bar{\mu} + (1 - t)H(x_n,\varepsilon;\bar{\mu},h_2), \qquad 0 \leq t \leq 1 .$$

As t varies from 0 to 1 the inequalities (5.5a), (5.5b), (5.6a), (5.6b) continue to hold. Thus, throughout the homotropy, there is no solution on the boundary of the region described by (5.11b). For $t = 1$ the equations read

$$\left[2 - \frac{3}{2}\left(\frac{\bar{h}_2 - h_2}{\bar{h}_2 - \underline{h}_2}\right)\right]g_1 = g_1 , \tag{5.12a}$$

$$\bar{\mu} = 0 . \tag{5.12b}$$

There is a unique solution,

$$h_2 = \frac{1}{3}\bar{h}_2 + \frac{2}{3}\underline{h}_2, \quad \bar{\mu} = 0 .$$

Thus, there is a solution $h_2(t),\bar{\mu}(t)$ for every $t \in [0,1]$. In particular, there is a solution for $t = 1$ and our problem has a solution (see [20]).

This corollary implies the truth of Theorem I. Theorem II and Theorem III follow as indicated in the Introduction.

Appendix

In this Appendix we are concerned with the equation

$$g'' + \delta x^2 g' - 2\delta x g = 0, \quad \delta > 0 . \tag{A.1}$$

In fact, we need only consider the case $\delta = 1$. For, if $g(x;1)$ is a solution of (A.1) - with $\delta = 1$ - then, for any $\delta > 0$, a direct calculation shows that the function

$$Y(x;\delta) = g(\delta^{1/3}x,1)$$

is a solution of (A.1) with this value of δ.

Our first concern is with the asymptotic behavior of solutions $g(\xi)(= g(\xi,1))$ as $\xi \to \pm\infty$.

A simple calculation using the Liouville-Green (or WKBJ) approximation (see chapter 6 of [14]) leads to the following results.

Case 1: As $x \to -\infty$ there are two linearly independent solutions $g_1(x), g_2(x)$ and

$$g_1(x) \sim x^2, \quad x \to -\infty , \tag{A.2a}$$

$$g_2(x) \sim x^{-4}\exp[-\frac{x^3}{3}], \quad x \to -\infty . \tag{A.2b}$$

Thus, there is a unique function, $g_1(\xi)$, which satisfies (A.1) with $\delta = 1$ and

$$g_1(x)/x^2 \to 1, \quad x \to -\infty . \tag{A.3}$$

Furthermore, a more careful asymptotic expansion of $g_1(x)$, e.g., using the methods described in [25, pp. 52-61] yields,

$$g_1(x) \sim x^2(1 + 2/3x^2), \quad x \to -\infty , \tag{A.4a}$$

$$g_1'(x) \sim 2x(1 + 2/3x^2)(1 - 1/x^2), \quad x \to -\infty , \tag{A.4b}$$

and

$$g_1''(x) \sim 2, \quad x \to -\infty . \tag{A.4c}$$

Case 2: The same calculations show that: as $x \to +\infty$ there are two linearly independent functions $\varphi_1(x), \varphi_2(x)$ which satisfy (A.1), with $\delta = 1$, and

$$\varphi_1(x) \sim x^{-4}\exp[-\frac{x^3}{3}], \quad x \to +\infty , \tag{A.5a}$$

$$\varphi_2(x) \sim x^2, \quad x \to +\infty . \tag{A.5b}$$

Since the function $g_1(x)$ which is characterized by (A.3) can be written as a linear combination of $\varphi_1(x)$ and $\varphi_2(x)$ we see that there is a unique constant, call it $\tilde{\tau}$, such that

(A.6) $$\lim_{x\to+\infty} g_1(x)/x^2 = \tilde{\tau} .$$

<u>Lemma A.1:</u> Let $\varphi_1(x)$ be the solution of (A.1) - with $\delta = 1$ - described by (A.5a). Then

(A.7) $$\varphi_1(0) \neq 0 .$$

<u>Proof:</u> Suppose (A.7) is false. Then

$$A = \varphi_1'(0) \neq 0 .$$

Suppose $A > 0$. An easy argument based on the maximum principle - or based on the identity

$$\frac{d}{dx}\left\{\varphi_1'\exp[\tfrac{x^3}{3}]\right\} = 2x\varphi_1\exp[\tfrac{x^3}{3}]$$

shows that

(A.8a) $$\varphi_1'(x) > 0, \qquad 0 \leq x < \infty .$$

(A.8b) $$\varphi_1(x) > 0, \qquad 0 < x < \infty .$$

The identity

(A.9) $$\frac{d}{dx}\left\{\varphi_1''\exp[\tfrac{x^3}{3}]\right\} = 2\varphi_1\exp[\tfrac{x^3}{3}]$$

and the fact

$$\varphi_1''(0) = 0$$

implies that

$$\varphi_1''(x) \geq 0, \qquad 0 \leq x < \infty .$$

Thus

$$\varphi_1(x) \geq Ax$$

which contradicts (A.5a). If $A < 0$ we apply the above argument to $-\varphi_1(x)$.

We now turn to a more detailed discussion of the function $g_1(x)$.

<u>Theorem A:</u> Let $g_1(x)$ be the function which satisfies (A.1) - with $\delta = 1$ - and (A.3). Then

(A.10a) $$g_1'(x) < 0, \qquad -\infty < x < \infty ,$$

(A.10b) $$xg_1'' \leq 0, \qquad -\infty < x < \infty .$$

Let $\bar{g}$ denote the unique point at which $g_1(x)$ vanishes, i.e., (A.11a) $$g_1(\bar{g}) = 0 .$$

Then

(A.11b) $$-(2/3)^{1/3} < \bar{g} < 0 .$$

Finally, the number $\tilde{\tau}$ of (A.6) satisfies

(A.12) $$\tilde{\tau} < 0 .$$

Proof: For negative values $x << -R < 0$ we have

$$g_1(x) > 0, \quad g_1'(x) < 0, \quad x << -R < 0 .$$

Since $g_1(x)$ satisfies (A.1), $g_1(x)$ cannot have a positive relative minimum on the interval $(-\infty,0)$. Thus, either

(A.13) $$g_1(x) > 0, \quad -\infty < x < 0 ,$$

or there is a first point $\bar{g} < 0$ at which (A.11a) holds. Suppose (A.13) holds. Then (A.9) shows that

$$g_1''(x) > 0, \quad -\infty < x < 0 .$$

Let $x_1 << -1$ and $x_1 < x < 0$. Then

(A.14) $$0 < g_1(x)/x^2 = g_1(x_1)/x_1^2 - \int_{x_1}^{x} [g_1''(t)]/t^4 dt < g_1(x_1)/x_1^2 .$$

Let $x_1 \to -\infty$. Then

$$0 < g_1(x)/x^2 < \lim_{x_1 \to -\infty} g_1(x_1)/x_1^2 = 1 .$$

Thus, if (A.13) holds,

$$0 < g_1(x) \leq x^2, \quad -\infty < x < 0 .$$

But then

$$g_1(0) = g_1'(0) = 0$$

and

$$g_1(x) \equiv 0 .$$

Since this is impossible, there is a first point $\bar{g} < 0$ at which (A.11a) holds. Moreover

(A.15a) $$g_1'(x) < 0, \quad -\infty < x \leq \bar{g} < 0 ,$$

(A.15b) $$g_1''(x) > 0, \quad -\infty < x \leq \bar{g} < 0 .$$

We now estimate $\bar{g}$ from below. We have

$$g_1'''(\bar{g}) = 2g_1(\bar{g}) - \bar{g}^2 g_1''(\bar{g}) < 0 .$$

Let $x_1 < \bar{g}$ be any point such that

(A.16) $$g_1'''(x) < 0, \qquad x_1 < x < \bar{g} .$$

Then

$$0 < g_1''(\bar{g}) < g_1''(x) < g_1''(x_1), \qquad x_1 < x < \bar{g} < 0 .$$

Returning to (A.14) we have

$$g_1(\bar{g})/\bar{g}^2 = 0 = g_1(x_1)/x_1^2 - \int_{x_1}^{\bar{g}} [g''(t)/t^4]dt .$$

That is

$$0 < g_1(x_1)/x_1^2 = \int_{x_1}^{\bar{g}} [g_1''(t)/t^4]dt < g_1''(x_1) \int_{x_1}^{\bar{g}} [1/t^4]dt .$$

Thus

(A.17) $$0 < g_1(x_1)/x_1^2 < [g_1''(x_1)/3][(1/|\bar{g}|^3) - (1/|x_1|^3)] < g_1''(x_1)/(3|\bar{g}|^3) .$$

Let $x_1 \to -\infty$, under the condition that (A.16) hold. Either $x_1 \to \bar{x}_1$ a finite point at which

$$g_1'''(\bar{x}_1) = 0$$

or $x_1 \to -\infty$. In either case

$$2g_1(x_1)/x_1^2 \to g_1''(\bar{x}_1) .$$

Thus, (A.17) yields

$$0 < |\bar{g}|^3 < 2/3 .$$

Thus we have proven that (A.11b) holds for the "first" zero of $g_1(x)$.

Our next task is to extend the range of the inequalities (A.15a), (A.15b) to the larger interval $(-\infty,0)$.

Let $\delta = 1$ and let $Y_1(x), Y_2(x)$ be the special solutions of (A.1) which also satisfy

(A.18a) $$Y_1(0) = 0, \qquad Y_1'(0) = -1 ,$$

(A.18b) $$Y_2(0) = -1, \qquad Y_2'(0) = 0 .$$

Let r_1 and r_2 be the smallest (in absolute value) negative zeros of $Y_1(x)$ and $Y_2(x)$ respectively. It is an easy matter to obtain infinite series solutions for

$Y_1(x), Y_2(x)$ and see that

$$|r_1| > 1, \qquad |r_2| > 1 .$$

These estimates, together with (A.11b) and the oscillation theorems (see [16, page 42]) show that

$$g_1(x) < 0, \qquad \bar{g} < x \leq 0 .$$

Thus

$$g_1(0) < 0 . \tag{A.19}$$

Let

$$g_1(x) = d_1Y_1(x) + d_2Y_2(x) .$$

Since $g_1(\bar{g}) = 0$ and $Y_1(\bar{g}) > 0$, $Y_2(\bar{g}) < 0$ we see that d_1 and d_2 are of the same sign. But (A.19) gives

$$d_2(-1) = g_1(0) < 0 .$$

Thus,

$$d_1 > 0, \qquad d_2 > 0 ,$$

and

$$g_1'(0) = -d_1 < 0 .$$

But, (A.1) implies that $g_1(x)$ cannot have a negative relative maximum in the interval $(\bar{g},0)$. Since $g_1'(\bar{g}) < 0$ and $g_1'(0) < 0$ we see that

$$g_1'(x) < 0, \qquad \bar{g} \leq x \leq 0 .$$

On this interval we have

$$g_1'' = 2xg_1 - x^2g_1' > 0, \qquad \bar{g} \leq x < 0 .$$

Hence, we have the inequalities (A.10a) and (A.10b) on the interval $(-\infty,0]$. The completion of the proof now follows from the initial conditions $g_1(0), g_1'(0)$, the maximum principle and the identity (A.9).

<u>Remark:</u> Since the theorem holds the quantities $\bar{g}$ and $\tilde{\tau}$ can be determined - to any desired accuracy - by numerical computations. Results of Jerry Browning of NCAR indicate that

$$\tilde{\tau} \sim -2 ,$$

$$\bar{g} \sim -.91 .$$

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