UNIQUENESS OF THE MAXIMAL EXTENSION
OF A MONOTONE OPERATOR

by

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1. Introduction

R. Robert in [11] proved the following result:

Theorem [Robert] Let E be a Banach space and A a maximal monotone operator from E into its dual E^* and which satisfies: $\mathrm{Int}\,(\mathrm{conv\ dom}\ A) \neq \emptyset$ and $\mathrm{cl\ dom}\ A$ convex. Let A_0 be an operator from E into E^* which satisfies:

a) $\forall x \in E,\ A_0 x \subset Ax$

b) $\mathrm{dom}\ A_0$ is dense in $\mathrm{dom}\ A$

Then, for every x in $\mathrm{int\ dom}\ A$,

$$(1.1) \qquad Ax = \mathrm{cl}_{\sigma(E^*,E)}\ \mathrm{conv}\ \{x^*;\ (x,x^*) \in \mathrm{cl}_{E\times\sigma(E^*,E)}\ \mathrm{Gr}\ A_0\}$$

Using this result, P. Benilan gave the following theorem:

Theorem [Benilan] Under the above hypothesis, the graph of A is equal to

$$(1.2) \qquad \{(x,x^*) \in \mathrm{cl\ dom}\ A \times E^*;\ \forall y^* \in A_0 y,\ \langle x^*-y^*, x-y\rangle \geq 0\}$$

Benilan did not publish his result, but S. Menou guoted it with its proof in [10]. H. Attouch [9] also proved these two results for the case when A_0 is the minimal section of A, and applied them to the study of measurable dependence for a family of maximal monotone operators.

In Hilbert spaces, these results can be interpreted as saying that a monotone operator can be uniquely extended to a maximal monotone operator within the closure of the convex hull of its domain if the closure of the domain of the original operator is convex and the interior of the convex hull of its domain is nonempty. In Section 2, we give a direct proof of such a theorem without using Robert's theorem.

It is interesting to generalize (1.1) to all the domain of A. The difficulty is the unboundedness of a maximal monotone operator at its boundary points. In Section 3, we prove that a monotone operator is bounded on any open segment between an interior point and a boundary point belonging to its domain. As a corollary of this result, we also prove that a maximal monotone operator is bounded in a bounded, convex set P in the interior of its domain if and only if its minimal section is defined and bounded in $\mathrm{cl}\, P$. These results themselves extend the known result about the local boundedness of a monotone operator at an interior point of its domain [2] [4].

In Section 4, we give a formulation similar to (1.1) but in the whole domain. In Section 5, we use the above results to get a new formulation of convergence in the graph sense [9] for sequences of maximal monotone operators. We confine our discussion to Hilbert spaces in most places and to finite-dimensional spaces in some places.

We use H to denote a real Hilbert space. We use i and j to denote integers, t and s to denote real numbers, and $x, y, z, u, v, w, h, k, p$ and q with or without subscripts to denote vectors in H or R^n. We use $\mathrm{conv}\, D$, $\mathrm{cl}\, D$ and $\mathrm{int}\, D$ to denote the convex hull, the closure and the interior of a set D respectively, $\mathrm{dom}\, F$ to denote the domain of a function F, and ∂f to denote the subdifferential of a convex function f.

2. Uniqueness of the Maximal Extension

Our first theorem follows. It may be derived from Benilan's theorem, but we prove it directly.

Theorem 1. Suppose that F is a set-valued monotone operator in H, that $D = \text{int dom } F$ is convex and nonempty, and that $\text{cl } D = \text{cl dom } F$. Define

$$(2.1) \qquad F^*(x) := \{y \mid \langle y-v,\ x-u\rangle \geq 0, \quad \forall (u,v) \in \text{Graph } F\}$$

for all $x \in \text{cl } D$. Then F^* is the unique maximal monotone operator satisfying $\text{Graph } F \subset \text{Graph } F^*$ and $\text{dom } F^* \subset \text{cl } D$.

Proof. It is already known that there exists a maximal monotone operator F' satisfying $\text{Graph } F \subset \text{Graph } F'$ and $\text{dom } F' \subset \text{cl } D$ [2]. By (2.1), any such maximal extension must be contained in F^*. Therefore, if we know that F^* is monotone, we can conclude that F^* is the unique maximal extension of F, satisfying $\text{dom } F^* \subset \text{cl } D$.

We now prove that F^* is monotone.

Suppose that $(x_1,y_1), (x_2,y_2) \in \text{Graph } F^*$. Write $u = (x_1+x_2)/2$; then $u \in \text{cl } D$. Choose any $u_0 \in \text{int dom } F$. Let $u_i = u + t_i(u_0-u)$, $0 < t_i < 1$, $t_i \to 0$, $i=1,2,3,\ldots$. Since $u_0 \in D$, $u \in \text{cl } D$ and $D = \text{int dom } F = \text{int } (\text{cl } D)$, we know that $u_i \in D \subset \text{dom } F$, $i=1,2,3,\ldots$ Therefore, we can choose $v_i \in F(u_i)$, $i=0,1,2,3,\ldots$, and we have

$$\langle y_1-y_2,\ x_1-x_2\rangle = \langle y_1-v_i,\ x_1-x_2\rangle + \langle y_2-v_i,\ x_2-x_1\rangle$$

$$= 2\langle y_1-v_i,\ x_1-u\rangle + 2\langle y_2-v_i,\ x_2-u\rangle$$

$$= 2\langle y_1-v_i,\ x_1-u_i\rangle + 2\langle y_2-v_i,\ x_2-u_i\rangle$$

$$+ 2\langle y_1-v_i,\ u_i-u\rangle + 2\langle y_2-v_i,\ u_i-u\rangle$$

$$= 2\langle y_1-v_i,\ x_1-u_i\rangle + 2\langle y_2-v_i,\ x_2-u_i\rangle$$

$$+ 4\langle v_0-v_i,\ u_i-u\rangle + 2\langle y_1+y_2-2v_0,\ u_i-u\rangle$$

$$= 2\langle y_1-v_i,\ x_1-u_i\rangle + 2\langle y_2-v_i,\ x_2-u_i\rangle$$

$$+ 4(t_i/(1-t_i))\langle v_0-v_i,\ u_0-u_i\rangle + 2t_i\langle y_1+y_2-2v_0,\ u_0-u\rangle$$

$$\geq 2t_i\langle y_1+y_2-2v_0,\ u_0-u\rangle\ , \qquad i=1,2,3,\ldots .$$

The last inequality is due to the monotonicity of F and to (2.1). Letting $i \to +\infty$, we have

$$\langle y_1-y_2,\ x_1-x_2\rangle \geq 0\,.$$

This completes the proof.

Corollary 1.1 If the interiors of the domains of two maximal monotone operators are identical and nonempty, and if they have a common single-valued selection in this interior, then these two maximal monotone operators are identical.

Corollary 1.2 If a maximal monotone operator has a cyclically monotone, single-valued selection over its domain, then this maximal monotone operator is also cyclically monotone.

The proof of Corollary 2 follows a line similar to the proof of Corollary 2.8, p. 39 of [2].

These two corollaries extend Corollary 2.2, p. 29 and Corollary 2.8, p. 39 of [2] to more general selections.

3. A Boundedness Property

It is known that a monotone operator in H is locally bounded at every interior point of its domain, and that a maximal monotone operator must be unbounded at every boundary point of its domain [4]. In the following theorem, we show that a monotone operator is bounded on an open segment between an interior point and a boundary point of its domain. This makes it easier to investigate some properties at a boundary point by approximating it from the interior.

Theorem 2. Suppose that F is a set-valued monotone operator in H, $z \in \text{int dom } F$, $x \in \text{dom } F$. Then

$$(3.1) \qquad \{y \mid y \in F(tx+(1-t)z),\ 0 < t < 1\}$$

is bounded.

Proof. Suppose that $u \in F(z)$, $v \in F(x)$. Write $h = x - z$. For $0 < t < 1$ and $y \in F(tx+(1-t)z)$, we have

$$\langle v-y, h\rangle = (1/(1-t))\langle v-y, x-(tx+(1-t)z)\rangle \geq 0,$$

$$(3.2) \qquad \langle y-u, h\rangle = (1/t)\langle y-u, (tx+(1-t)z)-z\rangle \geq 0,$$

$$\langle u, h\rangle \leq \langle y, h\rangle \leq \langle v, h\rangle .$$

Given any $w \in H$, let $k = \lambda w$, $\lambda > 0$, such that

$$z + k \in \text{dom } F, \; z - k \in \text{dom } F$$

Choose

$$p \in F(z+k), \; q \in F(z-k).$$

For $0 < t < 1$ and $y \in F(tx+(1-t)z)$, we have

$$
(3.3) \quad \begin{aligned}
0 &\le \langle p-y, z+k-(tx+(1-t)z)\rangle = \langle p-y, k-th\rangle , \\
\langle y, k\rangle &\le \langle p, k\rangle - t\langle p, h\rangle + t\langle y, h\rangle \\
&\le \langle p, k\rangle - t\langle p, h\rangle + t\langle v, h\rangle \\
&\le \langle p, k\rangle + |\langle p, h\rangle| + |\langle v, h\rangle| ;
\end{aligned}
$$

$$
(3.4) \quad \begin{aligned}
0 &\le \langle q-y, z-k-(tx+(1-t)z)\rangle = \langle q-y, -k-th\rangle , \\
\langle y, k\rangle &\ge \langle q, k\rangle + t\langle q, h\rangle - t\langle y, h\rangle \\
&\ge \langle q, k\rangle + t\langle q, h\rangle - t\langle v, h\rangle \\
&\ge \langle q, k\rangle - |\langle q, h\rangle| - |\langle v, h\rangle| .
\end{aligned}
$$

From (3.3) and (3.4), we know that for all $y \in F(tx+(1-t)z)$, $0 < t < 1$, $\langle y, k\rangle$, i.e., $\langle y, w\rangle$ is bounded, i.e., (3.1) is weakly bounded. This means that (3.1) is bounded [7][8]. The theorem is proved.

Corollary 2.1 Suppose that F is a set-valued monotone operator in H, and that P is a bounded convex set satisfying $P \subset \text{int dom } F$. If F has a single-valued selection F_0 which is defined and bounded at all the boundary points of P, then F is bounded on P.

Proof. Fix $z \in P \subset \text{int dom } F$. Take x as an arbitrary boundary point of P, $v = F_0(x) \in F(x)$. Using the same argument as the proof of Theorem 2 and noticing that v and h are bounded, we get this ~~corollary.~~

Corollary 2.2 Suppose that F is a maximal monotone operator in H, and that P satisfies the same hypotheses as above. A necessary and sufficient condition that F be bounded on P is that the minimal section of F be defined and bounded in $\text{cl } P$.

Proof. The sufficiency follows from Corollary 2.1. We prove necessity. Every point x in $\text{cl } P$ can be approached by an open segment in $P \subset \text{int dom } F$. Since F is bounded on this open segment, there is a sequence $\{(x_n, v_n) \in \text{Graph } F, n=0,1,2,\ldots\}$, such that x_n converges to x and v_n weakly converges to some point v. According to Theorem 1 and the maximality of F, $(x,v) \in \text{Graph } F$ and $\|v\| = \lim_{n\to\infty} \|v_n\|$. Therefore, the points v defined in this way are bounded for all $x \in \text{cl } P$ by the bound of F on P. Hence, the minimal section of F is defined and bounded on $\text{cl } P$. The corollary is proved.

4. Another Form of the Maximal Extension

In this section, we confine our discussion to R^n. To give another form of the maximal extension of a monotone operator, we use Theorem 2 and the fact that the recession cone of the function value set of a maximal monotone operator at each point of its domain is exactly the normal cone of the closure of its domain at that point [6].

Theorem 3. Suppose that F is a set valued monotone operator in R^n, that $D = \text{int dom } F$ is nonempty and convex, that $\text{cl } D = \text{cl dom } F$ and that F^* is the unique maximal extension defined by (2.1). Then for all $x \in \text{cl } D$,

$$F^*(x) = \text{cl } (\text{conv } S(x)) + N(x) , \tag{4.1}$$

where $N(x)$ is the normal cone to $\text{cl } D$ at x, and $S(x)$ is the set of all limits of sequences

$$\{y_i | y_i \in F(t_i x+(1-t_i)z),\ 0 < t_i < 1,\ t_i \to 1\}$$

for every $z \in D$.

Proof. Since F^* is closed, we know that $S(x) \subset F^*(x)$. Since $F^*(x)$ is closed and convex, we know that

$$F^*(x) \supset \text{cl } (\text{conv } S(x)) .$$

From [6], we know that $N(x)$ is the recession cone of $F^*(x)$. Therefore,

$$F^*(x) \supset \text{cl } (\text{conv } S(x)) + N(x) . \tag{4.2}$$

The opposite inclusion must now be proved. If $F^*(x)$ is empty, that is trivial. Suppose $F^*(x)$ is nonempty. Just as in the proof of Theorem 25.6 of [3], it follows that

$$F^*(x) \subset \text{cl}\ (\text{conv}\ E) + N(x)\ ,$$

where E is the set of all exposed points of $F^*(x)$. To prove

$$F^*(x) \subset \text{cl}\ (\text{conv}\ S(x)) + N(x)\ , \tag{4.3}$$

it suffices to prove

$$E \subset S(x)\ . \tag{4.4}$$

Given any exposed point w of $F^*(x)$, there exists by definition a supporting hyperplane to $F^*(x)$, which meets $F^*(x)$ only at w. Thus there exists a vector h with $\|h\| = 1$ such that h is normal to $F^*(x)$ at w but not normal to $F^*(x)$ at any other points, i.e.,

$$\langle h, w\rangle > \langle h, v\rangle\ , \qquad \forall v \in F^*(x),\ v \neq w\ . \tag{4.5}$$

Since $N(x)$ is the recession cone of $F^*(x)$, the latter condition on v implies in particular that

$$\langle h, k\rangle < 0\ , \qquad \forall k \in N(x),\ k \neq 0\ .$$

Hence there does not exist a vector $k \neq 0$ such that

$$\langle u, k\rangle \leq \langle x, k\rangle \leq \langle x+th, k\rangle$$

for every $u \in \text{dom}\ F^*$ and every nonnegative number $t \geq 0$. In other words, the half line $\{x+th|t\geq 0\}$ cannot be separated from dom F^*. It follows from Theorem 11.3 of [3] that this half line must meet the interior of dom F^*. There is a positive number $s > 0$ such that

$$z = x + sh \in \text{int dom } F^* = \text{int dom } F .$$

From Theorem 2,

$$\{y|y\in F(tx+(1-t)z),\ 0<t<1\}$$

is bounded. There is thus a convergent subsequence

$$\{y_i|y_i\in F(t_ix+(1-t_i)z),\ 0<t_i<1,\ t_i\to 1\}$$

$$y_i \to y_* .$$

According to the closedness of F^*,

$$y_* \in F^*(x) . \tag{4.6}$$

From the monotonicity of F^*,

$$<y_i-w,\ h> = (1/s)<y_i-w,\ z-x>$$
$$= (1/(s(1-t_i)))<y_i-w,\ (t_ix+(1-t_i)z)-x> \geq 0 .$$

Letting $i \to \infty$, $t_i \to 1$, we have

$$<y_*-w,\ h> \geq 0 ,$$

i.e.,

$$<w,\ h> \leq <y_*,\ h> . \tag{4.7}$$

Comparing (4.7) with (4.5) and (4.6), we know that

$$y_* = w .$$

~~i.e., $w \in S(x)$, so (4.4) holds. This completes the proof.~~

Corollary 3.1 Theorem 3 is still true if $S(x)$ is replaced by the set $\bar{S}(x):=$

(4.8) $\{x \mid (x_i, y_i) \to (x, y),\ (x_i y_i) \in \text{Graph } F,\ i=1,2,3,\ldots\}.$

Proof. Since F^* is closed, we know that $\bar{S}(x) \subset F^*(x)$. Since $F^*(x)$ is closed and convex, we know that

$$F^*(x) \supset \text{cl conv } \bar{S}(x).$$

Again, since $N(x)$ is the recession cone of $F^*(x)$,

$$F^*(x) \supset \text{cl conv } \bar{S}(x) + N(x).$$

Since $S(x) \subset \bar{S}(x)$, we get the opposite direction from (4.1).

Notice that Corollary 3.1 is the same as Robert's theorem applied to R^n if $x \in \text{int dom } F$.

5. Sequences of Maximal Monotone Operators

The following results were suggested by the referee. The definition about convergence in the graph sense (or, equivalently, in the resolvent sense) may be seen in [9].

Theorem 4. Let A be a maximal monotone operator in H, $\operatorname{int}(\operatorname{conv} \operatorname{dom} A) \neq \emptyset$ and $\operatorname{cl} \operatorname{dom} A$ be convex. Let $\{A_i,\ i=0,1,2,\ldots\}$ be a sequence of maximal monotone operators in H, with $A_i \to A$ in the graph sense of [9]. Denote their minimal sections by $\overset{\circ}{A}_i$ and $\overset{\circ}{A}$ respectively. Then for every x in $\operatorname{int} \operatorname{dom} A$,

$$Ax = \operatorname{cl} \operatorname{conv} T(x). \tag{5.1}$$

$$T(x) = \{\text{weak-limit } \overset{\circ}{A}_i(y_i) \,|\, y_i \to x,\ y_i \in \operatorname{dom} A_i\}, \tag{5.2}$$

where the closure is also in the weak sense.

Proof. According to Robert's theorem,

$$Ax = \operatorname{cl} \operatorname{conv} S(x),$$

where the closure is in the weak sense (we will not mention this further in the proof) and

$$S(x) = \{\text{weak-limit } \overset{\circ}{A}(x_j) \,|\, x_j \to x\}.$$

From Theorem 1.1 of [9], we know that $\overset{\circ}{A}_i \to \overset{\circ}{A}$ and therefore there exists a double sequence

$$\{\overset{\circ}{A}_i(x_j) \,|\, \overset{\circ}{A}_i(x_j) \to \overset{\circ}{A}(x_j) \ \text{ as } \ i \to \infty\}.$$

According to Lemma 1.6 of [9], we know that there exists $\{y_i \,|\, y_i \to x,\ y_i \in \operatorname{dom} A_i\}$ such that

$$\text{weak-limit } \overset{\circ}{A}_i(y_i) = \text{weak-limit } \overset{\circ}{A}(x_j).$$

Thus, we have proved

$$Ax \subset \text{cl conv } T(x).$$

However, from (5.2) and the fact that $A_i \to A$, $\overset{\circ}{A}_i(y_i) \in A_i(y_i)$, we see that $T(x) \subset Ax$. Since Ax is convex and closed in the weak sense, we know that

$$Ax \supset \text{cl conv } T(x).$$

Thus, we have proved the theorem.

Theorem 5. If $H = R^n$ in Theorem 4, then for every x in dom A,

$$(5.3) \qquad Ax = \text{cl conv } T(x) + N(x),$$

where $T(x)$ is defined by (5.2) and $N(x)$ is the normal cone of cl dom A at x.

Proof. This time we use Corollary 3.1 instead of Robert's theorem and follow the same argument. Notice that the conditions of that corollary can be obtained from the hypothesis of Theorem 4 by using Theorem 0.3 of [10].

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