A SUFFICIENT CONDITION FOR THE EXISTENCE OF OPTIMAL SOLUTIONS TO INTEGER AND MIXED-INTEGER PROGRAMS

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Abstract

A sufficient condition for the existence of optimal solutions to feasible integer or mixed-integer programs with bounded objective functions is shown to be the finiteness of the set of extreme points. This condition is more general than those previously given. A further question is presented.

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1. Introduction

Unlike a linear programming problem (LP), an integer programming problem (IP) or a mixed-integer programming problem (MIP) may have no optimal solution even if it is feasible and its objective function is bounded on its feasible set (when the latter property holds the problem is said to be bounded). Consider the following IP (from Meyer [1]):

$$(P)\qquad \begin{aligned} &\text{maximize} && -\alpha x_1 + x_2, \\ &\text{subject to} && -\alpha x_1 + x_2 \le 0, \\ &&& x_1 \ge 1, \\ &&& x_2 \ge 0, \\ &&& x_1,\ x_2 \ \text{integer}, \end{aligned}$$

where α is a positive irrational number. This problem is feasible since $(x_1, x_2) = (1,0)$ is a feasible point. Its objective function is bounded above by 0. In fact, the supremum of the objective function on the feasible set is 0, but no feasible point can attain this value because of the irrationality of α. Therefore, this IP has no optimal solution at all.

Meyer [1] gave some sufficient conditions for the existence of optimal solutions to feasible and bounded IP's and MIP's. Let (M) denote the problem

$$(M)\qquad \begin{aligned} &\text{minimize} && c_1 x + c_2 y, \\ &\text{subject to} && A_1 x + A_2 y = b, \\ &&& x \ge 0, \\ &&& y \ge 0, \\ &&& x \ \text{integer}, \end{aligned}$$

where $x \in R^{n_1}$ and $y \in R^{n_2}$, and the problem data c_1, c_2, A_1, A_2, b are vectors and matrices of sizes

$$(1 \times n_1),\ (1 \times n_2),\ (m \times n_1),\ (m \times n_2),\ (m \times 1),$$

respectively. If either

(i) $n_2 = 0$,

or (ii) A_1 and A_2 are comprised of rational data,

or (iii) x is bounded on the feasible set

and, if (M) is bounded and feasible, then (M) has an optimal solution.

Meyer also proved that S, the feasible set of (M), has only a finite number of extreme points under any of the above conditions. Later, Meyer and Wage [2] proved further that the convex hull of S is polyhedral under any of those conditions.

If S has a polyhedral convex hull, then it is clear that (M) is either unbounded, infeasible, or has an optimal solution. However, if we know only that S has a finite number of extreme points, the convex hull is not necessarily polyhedral, and it may not even be closed. In Section 2, we give an example to show this.

However, if S has at most a finite number of extreme points, then (M) has an optimal solution if (M) is bounded and feasible. In Section 3, we prove this main result. In this way, we generalize the results of [1]. This also shows that non-existence of optimal solutions is caused by an infinite number of extreme points.

Is the inverse proposition correct? It seems so, but has not yet been proven. In Section 4, we consider this question.

2. An Example

We now give a three-dimensional example. The feasible set of this MIP has only one extreme point, but its convex hull is not polyhedral and not even closed. The set is

$$S = \{(x_1,x_2,x_3) \mid -\alpha x_1+x_2+x_3=0,\ x_1,x_2,x_3 \geq 0,\ x_1,x_2 \text{ integer}\}$$

where α is a positive irrational number.

<u>Example</u>. The set S given above only has one extreme point $(0,0,0)$, but

$$\text{conv}(S) = \{(x_1,x_2,x_3) \mid -\alpha x_1+x_2+x_3=0,\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 > 0\} \cup \{(0,0,0)\}, \qquad (2.1)$$

which is not polyhedral and not closed.

It is clear that $(0,0,0)$ is an extreme point of S. For any point $x = (x_1,x_2,x_3)$ in S, it is obvious that $2x = (2x_1,2x_2,2x_3)$ is also in S. Since x is a convex combination of $(0,0,0)$ and $2x$, there is no other extreme point of S.

Clearly,

$$\text{conv}(S) \subset \{(x_1,x_2,x_3) \mid -\alpha x_1+x_2+x_3=0,\ x_1,x_2,x_3 \geq 0\}. \qquad (2.2)$$

From [1] or [4, p. 194], there exists a sequence of points $\{(x_1^i,\ x_2^i,\ x_3^i,)\} \subset S$, such that

$$x_1^i \to +\infty \qquad (2.3)$$

and

$$0 \leq \alpha - x_2^i/x_1^i \leq (x_1^i)^{-2}$$

i.e.

$$0 \le x_3^i \le (x_1^i)^{-1} . \tag{2.4}$$

Now we suppose $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ satisfies

$$-\alpha\bar{x}_1 + \bar{x}_2 + \bar{x}_3 = 0,\ \bar{x}_1,\ \bar{x}_2 \ge 0,\ \bar{x}_3 > 0 .$$

From (2.4) and (2.5), there is a point $(x_1^k, x_2^k, x_3^k) \in S$, such that

$$x_1^k > \bar{x}_1 \quad \text{and} \quad x_3^k < \bar{x}_3 .$$

Letting $\bar{x}_1/x_1^k = \lambda$, note that

$$x_2^k = \alpha x_1^k - x_3^k > \frac{\alpha}{\lambda}\bar{x}_1 - \bar{x}_3 > \frac{1}{\lambda}(\alpha\bar{x}_1 - \bar{x}_3) = \frac{1}{\lambda}\bar{x}_2 > \bar{x}_2 .$$

We also have

$$(\bar{x}_1,\bar{x}_2,\bar{x}_3) = \frac{\bar{x}_1}{x_1^k}\left[\frac{x_1^k\bar{x}_2}{\bar{x}_1 x_2^k}(x_1^k,x_2^k,x_3^k) + (1-\frac{x_1^k\bar{x}_2}{\bar{x}_1 x_2^k})(x_1^k,0,\alpha x_1^k)\right] + (1-\frac{\bar{x}_1}{x_1^k})(0,0,0),$$

where equality holds for the third component because it holds for the first two components and because the points on both sides have the property that $x_3 = \alpha x_1 - x_2$. It is easily seen that this representation of $(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ is a convex combination of points of S.

Thus,

$$\{(x_1,x_2,x_3) | -\alpha x_1 + x_2 + x_3 = 0,\ x_1 \ge 0,\ x_2 \ge 0,\ x_3 > 0\} \subset \mathrm{conv}(S)$$

and since

$$\{(x_1,x_2,0)|x_1>0 \text{ or } x_2>0\} \cap S = \emptyset ,$$

(2.1) holds. □

However, even if the convex hull of an IP or MIP is not closed, the problem will have an optimal solution if it is bounded and feasible and has a finite number of extreme points. We prove this in the next section.

3. A Sufficient Condition

We describe the theorem conversely: a necessary condition for the nonexistence of an optimal solution is an infinite number of extreme points.

Theorem. Consider the problem described in Section 1,

$$\text{(M)} \qquad \begin{array}{ll} \text{minimize} & c_1x + c_2y \\ \text{subject to} & A_1x + A_2y = b, \\ & \quad x \geq 0 \\ & \quad y \geq 0 \\ & \quad x \ \text{integer} \end{array}$$

where $x \in R^{n_1}$, $y \in R^{n_2}$, $c_1 \in R^{n_1}$, $c_2 \in R^{n_2}$, $b \in R^m$, A_1 and A_2 are matrices of appropriate sizes. Suppose S, the feasible set of (M) is not empty and $c_1x + c_2y$ is bounded from below on S. If (M) has no optimal solution then S has infinite number of extreme points.

Proof. According to the condition, we have

$$\begin{aligned} r &= \inf\{c_1x+c_2y \mid (x,y) \in S\} > -\infty \\ c_1x + c_2y &> r, \ \forall (x,y) \in S \end{aligned} \tag{3.1}$$

Denote the set of extreme points of S as E. Suppose E is finite. Then we have

$$r_1 = \min\{c_1x+c_2y \mid (x,y) \in E\} > r \tag{3.2}$$

Choose $(\hat{x},\hat{y}) \in S$ such that $c_1\hat{x} + c_2\hat{y} < \frac{r+r_1}{2}$.

Choose $c_3 \in R^{n_1}$, $c_4 \in R^{n_2}$, $c_3 > 0$, $c_4 > 0$ such that

$$c_3\hat{x} + c_4\hat{y} < \frac{r_1 - r}{2}$$

Then

$$(c_1+c_3)\hat{x} + (c_2+c_4)\hat{y} < r_1 \tag{3.3}$$

Consider the problem

$$\text{(MM)} \qquad \begin{array}{ll} \text{minimize} & (c_1+c_3)x + (c_2+c_4)y, \\ \text{subject to} & (x,y) \in S \end{array}$$

From (3.3), (MM) is equivalent to the problem

$$\text{(MN)} \qquad \begin{array}{ll} \text{minimize} & (c_1+c_3)x + (c_2+c_4)y, \\ \text{subject to} & (x,y) \in S, \\ & (c_1+c_3)x + (c_2+c_4)y \leq r_1. \end{array}$$

For any (x,y) in the feasible set of (MN), from (3.1),

$$c_3x + c_4y \leq r_1 - c_1x - c_2y < r_1 - r.$$

Therefore, (MM) is equivalent to problem

$$\text{(MP)} \qquad \begin{array}{ll} \text{minimize} & (c_1+c_3)x + (c_2+c_4)y, \\ \text{subject to} & (x,y) \in S, \\ & c_3x + c_4y \leq r_1 - r. \end{array}$$

Since $c_3 > 0$ and $c_4 > 0$, the feasible set of (MP) is bounded and x can only assume a finite number of values. The feasible set of (MP) is thus a union of a finite number of polyhedral sets. Moreover, the objective function of (MP) is bounded below, since

$$(c_1+c_3)x + (c_2+c_4)y \geq c_1x + c_2y > r.$$

Thus, there is an optimal solution of (MP).

Since (MP) and (MM) are equivalent, each optimal solution of (MP) is an optimal solution of (MM), and vice versa. Therefore, there is an optimal solution of (MM). It is easily shown from $x \geq 0$ and $y \geq 0$ that there is an optimal extreme point (x^*,y^*) of (MM), so that

$$\begin{aligned} &(c_1+c_3)x^* + (c_2+c_4)y^* \\ &\leq (c_1+c_3)\hat{x} + (c_2+c_4)\hat{y} < r_1. \end{aligned} \tag{3.4}$$

On the other hand, for any $(x,y) \in E$, by (3.3), we have

$$(c_1+c_3)x + (c_2+c_4)y \geq c_1x + c_2y \geq r_1,$$

contradicting (3.4). This proves the theorem.

4. An Open Question

From Theorem 2, an inverse question arises:

If

$$S = \{(x,y) | A_1x+A_2y=b,\ x,y\geq 0,\ x \text{ integer}\}$$

has an infinite number of extreme points, does there exist a $(c_1,c_2) \in R^{n_1} \times R^{n_2}$ such that problem (M) is bounded but does not have an optimal solution?

It seems that the answer should be "yes" but this is an open question.

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