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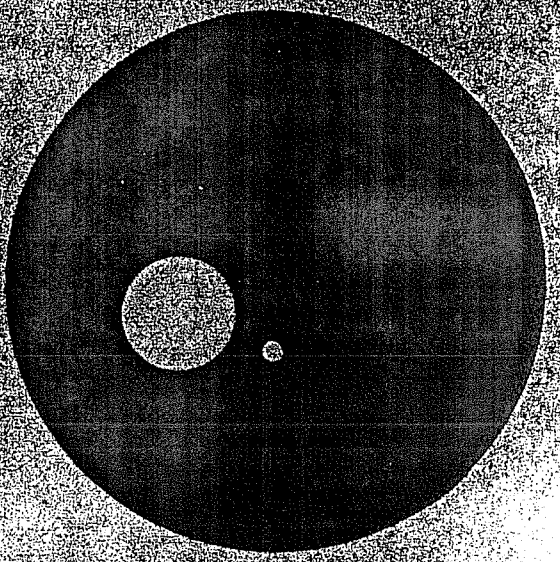

ALGORITHMS FOR THE GEOMETRY OF SEMI-ALGEBRAIC SETS

by

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ABSTRACT

ALGORITHMS FOR THE GEOMETRY OF SEMI-ALGEBRAIC SETS

Dennis Soulé Arnon

Under the supervision of Professor George E. Collins

Let A be a set of poiynomials in r variables with integer coefficients. An A-invariant cylindrical algebraic decomposition (cad) of r-dimensional Euclidean space (G. Collins, Lect. Notes Comp. Sci., 33, Springer-Verlag, 1975, pp 134-183) is a certain cellular decomposition of r-space, such that each cell is a semi-algebraic set, the polynomials of A are sign-invariant on each cell, and the cells are arranged into cylinders. The cad algorithm given by Collins provides, among other applications, the fastest known decision procedure for real closed fields, a cellular decomposition algorithm for semi-algebraic sets, and a method of solving nonlinear (polynomial) optimization problems exactly. The time-consuming calculations with real algebraic numbers required by the algorithm have been an obstacle to its implementation and use. The major contribution of this thesis is a new version of the cad algorithm for $r \leq 3$, in which one works with maximal connected A-invariant collections of cells, in such a way as to often avoid the most time-consuming algebraic number calculations. Essential to this new cad algorithm is an algorithm we present for determination of adjacencies among the cells of a cad. Computer programs for the cad and adjacency algorithms have been written, providing the first complete implementation of a cad algorithm. Empirical data obtained from application of these programs are presented and analyzed.

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CHAPTER 1

INTRODUCTION

In 1948, Tarski [TAR51] gave a quantifier elimination procedure, and hence a decision procedure, for the first order theory of real closed fields (let RCF denote the theory). Because difficult and important mathematical problems can be expressed in RCF ([TAR51] has examples; see also [KAH75] and [QUA71]), there was interest soon after the appearance of Tarski's paper in a computer implementation of the procedure (see e.g. [SEI54], [COL56], [COL57]). However, the computational infeasibility of the method became apparent. In the period 1954-73 two new quantifier elimination methods for RCF appeared ([SEI54], [COH69]) and improvements to Tarski's method were found ([HOL74]). But despite these improved algorithms and the enormous advances in hardware and software, an implementation suitable for nontrivial problems seemed beyond available capabilities in 1973. In a 1971 assessment of the impact of Tarski's classic paper, Abraham Robinson reaffirmed the possibility of applying such a computer program to substantial problems if it were available ([ROB71], p 140).

In 1973, G.E. Collins ([COL75],[COL76]) discovered a new and much more efficient quantifier elimination procedure for RCF. For any fixed number of variables, its computing time is a polynomial function of the length of the input formula, whereas the time for Tarski's and other previous algorithms is exponential in input formula length. Collins' algorithm offered renewed hope of a feasible implementation. The principal component of the new

method is an algorithm for constructing what Collins called a *cylindrical algebraic decomposition* (cad) of r-dimensional Euclidean space E^r. A cad algorithm takes as input a set $A = \{A_1, \ldots, A_n\}$ of polynomials in r variables with integer coefficients. Its output is a decomposition of E^r into finitely many disjoint connected subsets, called *cells* by Collins, with the following property. For each cell c and each A_i, either $A_i(x) < 0$ for all x in c, $A_i(x) = 0$ for all x in c, or $A_i(x) > 0$ for all x in c. (Thus each A_i is *sign-invariant* on c; we also say that each A_i is *invariant* on c, and that c is *A-invariant*, since we are not concerned with the value $A_i(x)$, but only its sign.) For each cell, Collins' original cad algorithm also constructs a particular point belonging to that cell. This sample point is determined exactly, in the sense that each of its coordinates is a real algebraic number which can be approximated to any desired precision.

A *semi-algebraic set* S (in E^r) is, roughly, the set of all points of E^r satisfying some given finite boolean combination φ of polynomial equations and inequalities in r variables (a precise definition is given in Section 2.1). φ is called a *defining formula* of S. In [COL75], Collins showed that each cell of a cad is a semi-algebraic set, by providing a subalgorithm of the cad algorithm which constructs defining formulas for the cells.

The subject of this thesis is certain algorithms which pertain to the geometric relationships between the cells of a cad. It is appropriate to mention existing literature on semi-algebraic geometry (e.g. [BRU79], [HIR74]), with which one may wish to compare our work. The fact that the cad algorithm can be used to decompose any semi-algebraic set into (semi-algebraic) cells ([ARN79]), is a further connection between cad's and semi-algebraic geometry.

Implementation of the cad algorithm began soon after its discovery. Between 1974 and 1977 significant work was done by Collins, A. Akritas of the University of Wisconsin, and R. Loos, M. Lauer, and E. Lauer of the University of Karlsruhe. These efforts built on the contributions of the many persons who had participated in the development of the SAC-1 and SAC-2 Computer Algebra systems.

F. Müller of the University of Heidelberg implemented portions of the cad algorithm between 1974 and 1977, as described in his dissertation [MUE77]. His objective was to obtain single solution points to (nonlinear) optimization problems involving polynomial functions. By using the cad algorithm, he was able to obtain exact algebraic number coordinates for such points. Since he sought only the first occurrence of a certain kind of cell in a cad, however, his algorithms discarded irrelevant cells, and in general simplified the cad algorithm wherever possible. For example, he did not need to construct defining formulas for cells, which is essential to the application of the cad algorithm to RCF quantifier elimination. An empirical observation he made about the algebraic number calculations performed by the cad algorithm was one of the starting points for the work reported in this thesis. We will say more about this observation below.

In 1978 several of the major subalgorithms (e.g. DEFINE, DECOMP, APROJ of [COL75]) of the cad algorithm remained unimplemented. In 1979-1980, Arnon carried out the first complete implementation of the cad and RCF quantifier elimination algorithms. It became apparent that the range of feasibility of the cad algorithm was limited. The bottleneck, as Loos had predicted and Müller had observed, was the time required for the exact calculations with real algebraic numbers, specifically the constructive version

of the primitive element theorem.

In a 1978 letter to Collins [KAH78], Peter J. Kahn of Cornell University had made the striking observation that a cad of E^r is what would be called in algebraic topology a *cellular decomposition* of E^r (see e.g. [MAS78], p 54). An *i-cell* in E^r, for any i, $0 \leq i \leq r$, is a subset of E^r which is homeomorphic to E^i. (E^0 is defined to be a single point). A cellular decomposition of E^r is a collection D of disjoint i-cells in E^r, for various i, $0 \leq i \leq r$, such that for each i, the union of the j-cells in D, $0 \leq j \leq i$, is a closed subset of E^r, and the union of D is E^r. Kahn went on to raise the possibility of using a cad to compute the homology groups of semi-algebraic sets, for which one would need to know the incidences among cells. For $i \geq 0$, an i-cell c^i is said to be *incident* on an (i+1)-cell c^{i+1} if c^i and c^{i+1} are disjoint, and if c^i is contained in $\bar{c}^{i+1}$, the closure of c^{i+1}. The cad algorithm as originally given, however, does not produce incidence information. In [ARN79], it was shown that incidence of cells of a cad is decidable, by expressing the assertion "c^i is incident on c^{i+1}" as a sentence in RCF. Applying a quantifier elimination algorithm for RCF, one would have an (inefficient) algorithm for incidence determination.

Following the implementation of the original cad algorithm, when it became clear that the time required for algebraic number calculations inhibited its use, it was seen that incidence determination might offer a means of alleviating the problem. Let $A_1, \ldots, A_n$ be the input polynomials to an invocation of the cad algorithm. Scott McCallum pointed out that, for cad's of E^2, the hardest algebraic number calculations, in the sense of requiring extension fields of highest degree, usually occur for the sample points of certain 0-dimensional cells of a cad with the following property: there exists

a 1-cell of the cad, on which the 0-cell is incident, such that each A_i has the same sign on both cells. The observation of F. Müller [MUE77] mentioned above complemented McCallum's. For cad's of E^2, the essence of Müller's observation is that, in most cases, the algebraic number calculations for all cells other than those that McCallum had distinguished could be done in extension fields of low degree. Thus arose the idea of "pasting", in some sense, the 0-cells McCallum had distinguished to the 1-cells with the same sign patterns on which they were incident, for the purpose of avoiding the algebraic number calculations for the 0-cells. Extending this idea, Arnon proposed, for cad's of E^r, the notion of an A-invariant *cluster* of cells, namely a collection C of cells of a cad such that, where R is the union of C, R is A-invariant. For $r \leq 3$, it has been found possible to modify the cad algorithm to do certain algebraic number calculations only for maximal A-invariant clusters, instead of individual cells, often yielding a substantial time saving. The price of this speedup is that one no longer obtains sample points for all cells of a cad, but this does not impede using the cad algorithm as a quantifier elimination algorithm for RCF.

It is essential to this clustering version of the cad algorithm to have an efficient incidence algorithm. For technical reasons, it became convenient to speak of adjacency of cells rather than incidence. Two connected subsets X_1 and X_2 of E^r are said to be *adjacent* if their union is a connected set. (It will be shown in Chapter 3 that, under certain hypotheses, if c^i and c^{i+1} are respectively an i-cell and an (i+1)-cell of a cad of $E^r, r \leq 3$, then c^i is incident on c^{i+1} if and only if c^i and c^{i+1} are adjacent.) In 1980 Arnon found a new and more efficient adjacency (incidence) algorithm. The idea of the new algorithm was to evaluate a defining formula for $\bar{c}^{i+1}$ at a sample point

of c^i, and thereby decide whether c^i is contained in $\bar{c}^{i+1}$. This algorithm may thus be called the "formula closure adjacency algorithm" . The implementation of a clustering cad algorithm was completed soon afterwards. The formula closure adjacency algorithm, and hence this first clustering cad algorithm, suffered from the limitation that the primitive parts of the r-variate input polynomials, and of certain of their partial derivatives, could not vanish identically on the vertical line over any point of E^{r-1} (this can only happen if $r \geq 3$).

In a 1979 M.Sc. thesis [MCC79] on constructive triangulation of (algebraic) curves and surfaces, McCallum gave algorithms for computing a simplicial decomposition of a single curve in E^2 or a single surface in E^3, and also gave algorithms for incidence determination in such decompositions. His incidence algorithms are based on the idea of drawing a sufficiently small box about a point of a curve (or surface), and counting the intersections of the curve (or surface) with the sides of the box. It was observed that the McCallum incidence algorithms could be extended to provide adjacency algorithms for cad's of E^2 and E^3, and would have the advantage over the formula closure algorithm of not requiring the construction of defining formulas. However, the McCallum E^3 adjacency algorithm would have a restriction similar to that on the formula closure algorithm, namely that the trivariate input polynomials not vanish identically on any vertical line over a point in E^2. In 1980, using the McCallum E^2 adjacency algorithm as a subalgorithm, a new cad adjacency algorithm for E^3 was found jointly by Collins, Arnon, and McCallum, which does not place the "non-vanishing" limitation on the input polynomials (and which does not require the construction of defining formulas). The new E^3 adjacency algorithm proceeds by reducing

the given E^3 adjacency problem to a number of E^2 adjacency problems. The McCallum E^2 adjacency algorithm and the new E^3 algorithm have been made a part of a second version of the clustering cad algorithm. In the remainder of the thesis these two adjacency algorthms will be referred to as the "section boundary adjacency algorithm for 2-space". and the "section boundary adjacency algorithm for 3-space" respectively.

The major contribution of this thesis is the clustering cad algorithm. In Chapter 2 we review the original cad algorithm. Chapter 3 contains results on the nature of cell boundaries which are essential to all that follows. Chapter 4 develops the clustering cad algorithm. Chapter 5 develops the section boundary adjacency algorithms. In Chapter 6 we describe various experiments which we have performed with the computer programs for our algorithms. Chapter 7 summarizes the thesis.

The investigations reported in this thesis are related to the problem of "solving systems of polynomial equations" in computer algebra. Programs for this task, utilizing some of the classical elimination theory of algebraic geometry, have been available in several computer algebra systems, as described in [WIL62], [MOS66], and [YUN73].

CHAPTER 2

THE ORIGINAL CYLINDRICAL ALGEBRAIC DECOMPOSITION ALGORITHM

Section 2.1 contains definitions and a first sketch of the algorithm. Section 2.2 is a new and complete exposition of the underlying mathematical results. Sections 2.3 - 2.5 discuss certain important aspects of the algorithm, and Section 2.6 presents an abstract version of it.

2.1 A synopsis.

Our first concern in this section is to precisely define the notion of an A-invariant cylindrical algebraic decomposition. We then summarize the "original" cad algorithm, which terminology shall refer as in Chapter 1 to the cad algorithm in [COL75]. For any $r \geq 0$, let I_r denote $Z[x_1, \ldots, x_r]$, the ring of integral polynomials in r variables. We view I_r, $r \geq 1$, as $I_{r-1}[x_r]$, and with respect to this view, write $ldcf(F)$ for the leading coefficient of any F in I_r, and $deg(F)$ for the degree of any F in I_r. (The degree and leading coefficient of the zero polynomial are both zero.)

Throughout this thesis, by the term "formula" we shall mean "well-formed formula of RCF", as defined in [TAR51]. We will also adhere to the convention that $\varphi(x_1, \ldots, x_r)$ denotes a formula φ in which all occurrences of $x_1, \ldots, x_r$ are free, each x_i may or may not occur in φ, and no variables besides $x_1, \ldots, x_r$ occur free in φ.

Definition. A *standard atomic formula* is a formula of one of the six forms $A = 0$, $A > 0$, $A < 0$, $A \neq 0$, $A \geq 0$, and $A \leq 0$, where $A \in I_r$ for some r. A *standard formula* is any formula which can be constructed from standard atomic formulas using propositional connectives and quantifiers. A *standard prenex formula* is a standard formula of the form

$$\Psi(x_1, \ldots, x_k) = (Q_{k+1}x_{k+1})\ldots(Q_r x_r)\varphi(x_1, \ldots, x_r)$$

where $\varphi(x_1, \ldots, x_r)$ is a quantifier-free standard formula, $0 \leq k \leq r$, and each $(Q_i x_i)$ is either an existential quantifier (Ex_i) or a universal quantifier (Ax_i).

Clearly for any formula, there is an equivalent standard prenex formula.

Definition. A *definable set in* E^k, $k \geq 1$, is a subset S of E^k such that for some formula $\Psi(x_1, \ldots, x_k)$, S is the set of points in E^k which satisfy Ψ. Ψ is called a *defining formula* for S. Given any formula Ψ, we write $S(\Psi)$ to denote the definable set determined by it.

Definition. A definable set S is *semi-algebraic* if S has a defining formula which is quantifier-free.

It follows immediately from the definition that the semi-algebraic sets in E^r are a boolean class of subsets of E^r. That is, they are closed under finite unions, finite intersections, and complementation. Because of the existence of a quantifier elimination procedure for RCF, a subset of E^r is definable if and only if it is semi-algebraic. [HIR74] gives some other equivalent definitions of semi-algebraic set.

Definition. A *region* in E^r, $r \geq 1$, is a nonempty connected subset of E^r.

We note that a "region" is usually required to also be open. Our usage, while unconventional, will be a great convenience in this thesis.

Definition. For any $r \geq 1$ and any subset X of E^r, a *decomposition* of X is a finite collection of disjoint regions in E^r whose union is X. A decomposition is *algebraic* if each of its regions is a semi-algebraic set.

Remark. Since an i-cell is a region, a cellular decomposition of E^r as defined in Chapter 1 is a decomposition of E^r according to the definition just given.

Definition. For any region c in E^{r-1}, $r \geq 1$, the *cylinder over c*, written Z(c), is the Cartesian product $c \times E \subset E^r$. c is called the *base* of Z(c). A *cylinder* in E^r is a cylinder over c, for some region c in E^{r-1}.

Definition. Given a region c in E^{r-1}, $r \geq 1$, a *section* of Z(c) is a subset s of Z(c) such that for some continuous function $f : c \to E$, $s = graph(f) = \{ < \alpha, f(\alpha) > \mid \alpha \in c \}$. s is said to be the *f-section* of Z(c). In the case r=1, one must have $c = E^0$, and we identify any function f defined on c with its unique value $\alpha \in E$.

Definition. For any region c in E^{r-1}, $r \geq 1$, a *sector* of Z(c) is a subset s of Z(c) such that one of the following four conditions holds.

(1) There is a continuous function $f : c \to E$ such that $s = \{ < \alpha, b > \in Z(c) \mid b < f(\alpha) \}$. (s is the $(-\infty, f)$*-sector* of Z(c)).

(2) There are continuous functions $f_1 < f_2 : c \to E$ such that $s = \{ < \alpha, b > \in Z(c) \mid f_1(\alpha) < b < f_2(\alpha) \}$. (s is the (f_1, f_2)*-sector* of Z(c)).

(3) There is a continuous function $f:c \to E$ such that $s = \{ < a,b > \in Z(c) \mid b > f(a)\}$. (s is the (f,∞)-*sector* of Z(c)).

(4) $s = Z(c)$. (s is the $(-\infty,\infty)$-*sector* of Z(c)).

Observations.

(1) Sections and sectors of cylinders are regions, i.e. they are connected.

(2) Where $\pi:E^r \to E^{r-1}$ is the standard projection, if c is a region in E^r with $r \geq 1$, and if s is a section or sector of Z(c), then $\pi(s) = c$.

Definition. Let c be a region in E^{r-1}, $r \geq 1$, and suppose we are given continuous functions $f_1 < f_2 < \cdots < f_k:c \to E$, $k \geq 0$. The $(f_1, \ldots, f_k)$-*induced stack over c*, written $S(f_1, \ldots, f_k, c)$, is defined as follows. If k = 0, then $S(f_1, \ldots, f_k, c)$ is $\{Z(c)\}$. If $k \geq 1$, then $S(f_1, \ldots, f_k, c)$ is the collection of the following subsets of Z(c):

$(-\infty, f_1)$-sector

f_1-section

(f_1, f_2)-sector

f_2-section

.
.
.

(f_{k-1}, f_k)-sector

f_k -section

(f_k, ∞)-sector .

Observation.. $S(f_1, \ldots, f_k, c)$ is a decomposition of Z(c).

Definition. Let c be a region in E^{r-1}, $r \geq 1$. A *stack over c*, written S(c), is

$S(f_1,\ldots,f_k,c)$ for some continuous functions $f_1 < f_2 < \cdots < f_k : c \to E$, $k \geq 0$. A *stack* in E^r is a stack over c for some region c in E^{r-1}.

Definition. Let S and T be stacks in E^r, $r \geq 1$. T is a *refinement* of S if every element of S is a union of elements of T.

Definition. A stack is *algebraic* if each of its elements is a semi-algebraic set.

Definition. A decomposition D of E^r, $r \geq 1$, is *cylindrical* if

(1) r=1 and D is a stack in E^1, or

(2) r>1 and there is a cylindrical decomposition D' of E^{r-1} such that for every c in D', if S(c) is the set of all regions of D which meet Z(c), then S(c) is a stack over c, and

$$D = \bigcup_{c \in D'} S(c).$$

It is clear that D' is unique, and thus associated with any cylindrical decomposition D of E^r, for $r \geq 2$, are *induced* cylindrical decompositions D_i of E^i for i=r-1,r-2,...,1.

A *cylindrical algebraic decomposition* of E^r, $r \geq 1$, is a decomposition of E^r which is both cylindrical and algebraic. Associated with any cad D of E^r, $r \geq 2$, are induced cad's D_i of E^i for i=r-1,r-2,...,1.

As mentioned in Chapter 1, Kahn in his 1978 letter [KAH78] made the important observation that each region of a cad is an i-cell for some i. It is clear how the dimensions of the cells of a cad D of E^r are inductively determined. If r = 1, then each point α_k is a 0-cell, and each open interval is a 1-

cell. If r > 1, then assume the dimensions of the cells of the induced cad D' of E^{r-1} are known. Each cell d of D is an element of a stack S over a cell c of D'. If d is a section of S, then $dim(d) = dim(c)$, and if d is a sector of s, then $dim(d) = dim(c) + 1$.

Definition. Let D, D^* be cad's of $E^r, r \geq 1$. D^* is a ***refinement*** of D if for every cell c of D, there is a collection of cells of D^* whose union is c.

Definition. Let F be any element of I_r, $r \geq 1$. Let X be a subset of E^r. F is said to be *invariant* on X (and X is said to be *F-invariant)*, if one of the following three conditions holds:

(1) $F(\alpha) > 0$ for all α in X. ("F has positive sign on X").

(2) $F(\alpha) = 0$ for all α in X. ("F has zero sign on X").

(3) $F(\alpha) < 0$ for all α in X. ("F has negative sign on X").

Thus F is invariant on X if and only if the sign of F is invariant on X. Let $A = \{A_1, \ldots, A_n\}$, $n \geq 1$, be any set of elements of I_r. X is said to be *A-invariant* if each A_i is invariant on X. A stack in E^r or a decomposition of E^r is A-invariant if each of its regions is.

Note that a constant polynomial, e.g. the zero polynomial, is invariant on any subset of E^r.

Observation. Suppose $A \subset I_r, r \geq 1$ and let S be a stack in E^r. If S is A-invariant, then any refinement of S is A-invariant. If D is an A-invariant cad of E^r, then any refinement of D is A-invariant.

The original cad algorithm for determining an A-invariant cad of E^r can be summarized as follows. If the input polynomials A = $\{A_1, \ldots, A_n\}$ are

univariate, we take the real roots of $\prod_{i=1}^{n} A_i$ as real numbers $\alpha_1, \ldots, \alpha_k$ determining a cylindrical decomposition D of E^1 (we may assume without loss of generality that each A_i is nonzero). Since each α_i is algebraic, one easily shows that D is algebraic, hence D is a cad of E^1. Also, D is clearly A-invariant. If A is a subset of I_r, r>1, suppose that we have a means of constructing a finite subset P(A) of I_{r-1} such that if c is any cell of a P(A)-invariant cad of E^{r-1}, then there is an A-invariant algebraic stack over c. We call the cad algorithm recursively with P(A) as input to obtain a P(A)-invariant cad D' of E^{r-1}. We then determine the sections and sectors of each A-invariant algebraic stack over a cell of D' to obtain an A-invariant cad D of E^r. Section 2.2 presents the original cad algorithm in more detail, via a proof of the theorem (Thereom 2.2.1) that there exists an A-invariant cad for any $A \subset I_r$. In the course of proving Theorem 2.2.1, we define an appropriate P(A) and establish that it has the required property.

2.2 The existence of A-invariant cylindrical algebraic decompositions.

Definition. An *algebraic point* of $E^r, r \geq 1$, is a point of E^r each of whose coordinates is a real algebraic number.

Theorem 2.2.1. Given any finite set $A = \{A_1, \ldots, A_n\}$ of polynomials in I_r, $r \geq 1$, there exists an A-invariant cylindrical algebraic decomposition of E^r, every cell of which contains an algebraic point.

Proof. We may assume each A_i is nonzero, since if A' is the set of nonzero elements of A, any A'-invariant cad of E^r is A-invariant. Furthermore, we may assume $n \geq 1$, since if n = 0 then $\{E^r\}$ is an A-invariant cad of E^r. We

proceed by induction on r as follows. The case r=1 is straightforward, for, as noted above, the real roots of $\prod_{i=1}^{n} A_i$ determine an A-invariant cad D of E^1, and clearly each cell of D contains an algebraic point. For r>1, Theorems 2.2.17 and 2.2.18 below establish the existence of a map P which associates with any finite subset A of I_r a finite subset P(A) of I_{r-1}, such that for any P(A)-invariant cad of E^{r-1}, there is an A-invariant algebraic stack S over each cell c of this cad. Theorem 2.2.2 below establishes that if c contains an algebraic point, then so does every cell of S. By the inductive hypothesis a P(A)-invariant cad D' of E^{r-1} exists, such that every cell of D' contains an algebraic point. From this we immediately obtain the existence of an A-invariant cad of E^r such that each cell contains an algebraic point, namely the union of the stacks over the cells of D'. ▪

The remainder of this section is devoted to proving the theorems which serve as lemmas for Theorem 2.2.1 (Theorems 2.2.2, 2.2.17, and 2.2.18).

Notation. Let F be an element of I_r, $r \geq 2$, let G be an element of I_{r-1}, let c be a region in E^{r-1}, and let α be a point of c. We write $F_\alpha(x_r)$ or F_α to denote $F(\alpha, x_r)$, and G_α to denote $G(\alpha)$.

Definition. A complex number is ***strictly complex*** if it is non-real.

Definition. Let F be a nonzero element of I_r, $r \geq 2$. Let c be a region in E^{r-1}. Let $V^*(F)$ denote the variety of F, i.e. $V^*(F)$ is the set of all $<a_1, \ldots, a_r>$, with each a_i complex, such that $F(a_1, \ldots, a_r) = 0$. Let V(F) denote the real variety of F, i.e. V(F) is the set of all $<a_1, \ldots, a_r>$ in $V^*(F)$ with each a_i real. We say $V^*(F)$ is *delineable* on c if the following two conditions are satisfied:

(1) $V(F) \cap Z(c)$ consists of finitely many disjoint sections $s_1, \ldots, s_m$ of Z(c), $m \geq 0$, such that for each i, $1 \leq i \leq m$, the following condition holds: there is a positive integer e_i such that if s_i is an f_i*-section*, then $f_i(\alpha)$ is a root of $F_\alpha(x_r)$ of multiplicity e_i for all α in c. (e_i is the *multiplicity* of s_i.) If $m \geq 1$, then $s_1, \ldots, s_m$ are called the *F-sections* of Z(c). Without loss of generality we may assume $f_1 < f_2 < \cdots < f_m$; $S(f_1, \ldots, f_m, c)$ is called the *F-induced stack over* c, written S(F,c). When convenient, we write $s_1 < s_2 < \cdots < s_m$. The sectors of S(F,c) are called the *F-sectors* of Z(c).

(2) The multiset of the multiplicities of the distinct strictly complex roots of $F(\alpha, x_r)$ is invariant over all α in c. (See [KNU69], p. 411, for a definition of multiset).

We will usually abbreviate "$V^*(F)$ is delineable" to "F is delineable".

Theorem 2.2.2. Let $F \in I_r$, $r \geq 2$, be delineable on a region c in E^{r-1} which contains an algebraic point. Then every region of S(F,c) contains an algebraic point.

Proof. Let $\alpha \in c$ be an algebraic point. If S(F,c) has $m \geq 0$ sections, then $F(\alpha, x_r)$ has m real roots $b_1, \ldots, b_m$, each of which is a real algebraic number. Thus $< \alpha, b_1 >, \ldots, < \alpha, b_m >$ are algebraic points each of which lies on a distinct section of S(F,c). Clearly for every sector s of S(F,c), there exists a rational b such that $< \alpha, b > \in s$. ∎

Lemma 2.2.3. Let $F \in I_r$, $r \geq 1$, and let R be a region in E^r. If $F(x) \neq 0$ for all $x \in R$, then either $F(x) > 0$ for all $x \in R$, or $F(x) < 0$ for all $x \in R$.

Proof. Let $U_1 = \{x \mid x \in E^r \ \& \ F(x) > 0\}$, and let $U_2 = \{x \mid x \in E^r \ \& \ F(x) < 0\}$. Since F is a continuous function $E^r \rightarrow E$, every point of U_1 is an interior point,

hence U_1 is open. Similarly, U_2 is open. Let $V_1 = U_1 \cap R$ and $V_2 = U_2 \cap R$. V_1 and V_2 are both open in the subspace topology on R, and $R = V_1 \cup V_2$. Since R is connected, there do not exist nonempty disjoint open subsets of R whose union is R. Hence either $V_1 = \emptyset$ or $V_2 = \emptyset$. ▪

Lemma 2.2.4. Let $F \in I_r$, $r \geq 2$, be delineable on a region c in E^{r-1}. Then S(F,c) is F-invariant.

Proof. S(F,c) could only fail to be F-invariant if some sector of S(F,c) is not F-invariant. By definition of delineability, F does not vanish at any point of a sector of S(F,c). Hence by Lemma 2.2.3, every sector of S(F,c) is F-invariant, hence S(F,c) is F-invariant. ▪

Definition. Let c be a region in E^r, $r \geq 1$, and T a function defined on c. Let the topology on c be the induced topology from E^r. T is ***locally invariant*** on c if for any $\alpha \in c$, there is an open neighborhood M of α in c such that $T(\beta) = T(\alpha)$ for all $\beta \in M$.

Lemma 2.2.5. Let c be a region in E^{r-1}, $r \geq 2$, and T a function defined on c. If T is locally invariant on c, then T is invariant on c.

Proof. Choose any γ in c. Let S_1 be the union of the T-invariant neighborhoods $M(\alpha)$, for all those α in c such that $T(\alpha) = T(\gamma)$. Let S_2 be the union of the T-invariant neighborhoods $M(\alpha)$, for all those α in c such that $T(\alpha) \neq T(\gamma)$. $c = S_1 \cup S_2$, S_1 is nonempty, and if S_2 is also nonempty, we have a partition of c into disjoint nonempty open subsets, contradicting its connectivity. Hence S_2 is empty, hence $S_1 = c$, i.e. T is invariant on c. ▪

Definition. Let J be a unique factorization domain. Let F,G be nonzero elements of J[x]. If deg(F) > deg(G), let $F_1 = F$ and $F_2 = G$, else let $F_1 = G$ and

$F_2 = F$. Let $F_1, F_2, \ldots, F_k$, $k \geq 2$, be a polynomial remainder sequence as defined in [BRT71]. Let $n_i = deg(F_i)$, $1 \leq i \leq k$. Then $n_1, n_2, \ldots, n_k$ is the *degree sequence* of F and G.

Definition. Let F and G be nonzero elements of J[x], J a unique factorization domain. Let k = min(deg(F),deg(G)). For $0 \leq j < k$, we write $S_j(F,G)$ to denote the j^{th} subresultant of F and G [BRT71]. For $0 \leq j < k$, the j^{th} *principal subresultant coefficient of F and G*, written $psc_j(F,G)$, is the coefficient of x_j^j in $S_j(F,G)$. We define $psc_k(F,G)$ to be $1 \in J$. The *psc set of F and G*, written PSC(F,G), is

$$\{psc_j(F,G) \mid 0 \leq j \leq k \ \ \& \ psc_j(F,G) \neq 0\}$$

Theorem 2.2.6. Let F and G be nonzero elements of J[x], J a unique factorization domain. Let $n_1, n_2, \ldots, n_k$, $k \geq 2$, be the degree sequence of F and G. Then

(1) $n_k = deg(gcd(F,G))$, and

(2) For any j, $0 \leq j \leq n_2$, $psc_j(F,G) \neq 0$ if and only if $j = n_i$ for some i, $2 \leq i \leq k$.

Proof. Let $F_1, F_2, \ldots, F_k$ be a polynomial remainder sequence whose first two terms are F and G; thus $n_i = deg(F_i)$, $2 \leq i \leq k$. As pointed out on p. 506 of [BRT71], $F_k \sim gcd(F,G)$, where $\sim$ denotes similarity. Hence $n_k = deg(gcd(F,G))$. Suppose for some j, $0 \leq j \leq n_2$, that $psc_j(F,G) \neq 0$. If $j = n_2$ we are done, so suppose $j < n_2$. Then by the fundamental thoerem of polynomial remainder sequences [BRT71], either $j = n_i$ for some i, $3 \leq i \leq k$, or $j = n_{m-1} - 1$, for some m, $3 \leq m \leq k$. If $j = n_i$ we are done, so suppose $j = n_{m-1} - 1$. Then $n_{m-1} - 1 \geq n_m$. Suppose $n_{m-1} - 1 > n_m$. By the

fundamental theorem of p.r.s., $S_{n_{m-1}-1}(F,G) \sim S_{n_m}(F,G)$, so

$$deg(S_{n_m-1}(F,G)) = deg(S_{n_m}(F,G)) = n_m < n_{m-1} - 1 = j.$$

Hence $psc_j(F,G) = 0$, a contradiction. So $j = n_{m-1} - 1 = n_m$, and we are done. Suppose conversely that $j = n_i$ for some i, $2 \le i \le k$. If i = 2, then $psc_{n_2}(F,G) = 1 \neq 0$. If $i \ge 3$, then by the fundamental theorem of p.r.s., $S_{n_i}(F,G) \sim F_i$, hence $deg(S_{n_i}(F,G)) = deg(F_i) = n_i$, hence $psc_{n_i}(F,G) \neq 0$. ∎

Corollary 2.2.7. Let F and G be nonzero elements of J[x], J a unique factorization domain. Then $deg(gcd(F,G))$ = k if and only if k is the least j such that $psc_j(F,G) \neq 0$.

Proof. Follows immediately from Theorem 2.2.6. ∎

Definition For $F \in I_r, r \ge 1$, the *derivative of F*, written $der(F)$ or F', is the partial derivative of F with respect to x_r.

Theorem 2.2.8. Let F be a nonzero element of $I_r, r \ge 2$. Let c be a region in E^{r-1} such that $(ldcf(F))_\alpha \neq 0$ for all α in c. Then the following are equivalent:

(1) F is delineable on c.

(2) The number of distinct (complex) roots of $F_\alpha(x_r)$ is invariant over all α in c.

(3) If $deg(F) \ge 2$, then the least k such that $psc_k(F_\alpha, F'_\alpha) \neq 0$ is invariant over all α in c.

Proof. We show (1) iff (2), and (2) iff (3).

(1) => (2): An immediate consequence of the definition of delineability.

(2) => (1): If F_α has no roots for all α in c, the assertion is immediate, so suppose that F_α has $p \ge 1$ distinct complex roots for all $\alpha \in c$.

Choose some $\alpha \in c$, and let $z_1, \ldots, z_p$ be the distinct complex roots of F_α. There exists an s satisfying $0 \leq s \leq p$, such that $z_1 < z_2 < \cdots < z_s$ are real, and $z_{s+1}, \ldots, z_p$ are strictly complex. Let e_i be the multiplicity of z_i for $1 \leq i \leq p$. Let δ be a positive real number less than $1/2\, sep(F_\alpha)$, where $sep(F_\alpha)$ is the minimum distance in the complex plane between any two distinct roots of F_α. For $1 \leq i \leq p$, let C_i be the circle in the complex plane of radius δ centered at z_i. By Theorem (1,4) of [MAR66], there is a neighborhood M in c of α such that for all $\beta \in M$, and for $1 \leq i \leq p$, F_β has e_i roots, multiplicities counted, in C_i. Since F_β and F_α each have p distinct roots, and since the interiors of the C_i's are disjoint, F_β has one root of multiplicity e_i in C_i for $1 \leq i \leq p$. The root y_i of F_β in C_i cannot be strictly complex if $i \leq s$, for then the complex conjugate of y_i would be a second distinct root of F_β in C_i, a contradiction. The root of F_β in C_i cannot be real if $i > s$, for by the choice of δ, C_i contains no real points when $i > s$, since complex roots occur in conjugate pairs.

Thus where $\sigma(\beta)$ is defined to be the number of real roots of F_β for any $\beta \in c$, σ is locally invariant on c, hence by Lemma 2.2.5, σ is invariant on c with the value $s = \sigma(\alpha)$. For any $\beta \in c$, let $z_1(\beta) < z_2(\beta) < \cdots < z_s(\beta)$ be the real roots of F_β. For $1 \leq i \leq s$, we define a function $f_i : c \to e$ by setting $f_i(\beta) = z_i(\beta)$ for any $\beta \in c$. From our argument above, one sees that each f_i is continuous. For $1 \leq i \leq s$ and for any $\beta \in c$, define $\varepsilon_i(\beta)$ to be the multiplicity of $f_i(\beta)$ as a root of F_β. From our argument above one also sees that for $1 \leq i \leq s$, ε_i is a locally invariant function on c, hence by Lemma 2.2.5, ε_i is invariant on c. Hence F satisfies condition (1) of the definition of delineability.

For any $\beta \in c$, define $m(\beta)$ to be the multiset of the multiplicities of the distinct strictly complex roots of F_β. Again from our argument above, one sees that m is locally invariant on c, hence m is invariant on c. Thus F satisfies condition (2) of the definition of delineability, and so is delineable on c.

(2) => (3): Since $(ldcf(F))_\alpha$ is nonzero for all α in c, $n_\alpha = deg(F_\alpha)$ is invariant over all α in c. If $n_\alpha = 0$ or 1, then the assertion is trivial, so assume $n_\alpha \geq 2$ for all α in c. For any α in c, let k_α be the degree of $C_\alpha(x_r) = gcd(F_\alpha(x_r), F'_\alpha(x_r))$. Where p_α is the number of distinct roots of F_α, one easily sees that $p_\alpha = n_\alpha - k_\alpha$. Since p_α and n_α are invariant over all $\alpha \in c$, so is k_α. Hence by Theorem 2.2.7, the least k such that $psc_k(F_\alpha, F'_\alpha) \neq 0$ is invariant over all $\alpha \in c$.

(3) => (2): If $deg(F)$ is 0 or 1 then the number of distinct roots of $F_\alpha(x_r)$ is clearly invariant over all α in c, so assume $deg(F) \geq 2$. Let $\bar{k}$ be the nonnegative integer such that for all α in c, $\bar{k}$ is the least k such that $psc_k(F_\alpha, F'_\alpha)$ is nonzero. Then by Theorem 2.2.7, $deg(gcd(F_\alpha, F'_\alpha)) = \bar{k}$ is invariant over all α in c. Since $n_\alpha = deg(F_\alpha)$ is invariant over all α in c, we have that p_α, the number of distinct complex roots of F_α, is invariant over all α in c, since $p_\alpha = n_\alpha - \bar{k}$. ∎

Theorem 2.2.9. Let A and B be elements of I_r, $r \geq 2$, both of which are delineable on a region c in E^{r-1}. Assume also that $(ldcf(A))_\alpha \neq 0$ and $(ldcf(B))_\alpha \neq 0$ for every α in c. Then the following are equivalent:

(1) AB is delineable on c.

(2) Any A-section s_A and any B-section s_B of Z(c) are either disjoint or identical, and the number of distinct common strictly complex roots of A_α and B_α is invariant over all α in c.

(3) If A and B both have positive degree, then the least k such that $psc_k(A_\alpha(x_r), B_\alpha(x_r)) \neq 0$ is invariant over all α in c.

Proof. For the duration of this proof we define D = AB, $C_\alpha = gcd(A_\alpha, B_\alpha)$, for any $\alpha \in c$, and $p(A_\alpha)$, $p(B_\alpha)$, $p(D_\alpha)$, and $p(C_\alpha)$ to be the number of distinct complex roots of A_α, B_α, D_α, and C_α respectively, for any $\alpha \in c$. We note that

$$p(D_\alpha) = p(A_\alpha) + p(B_\alpha) - p(C_\alpha)$$

for any α in c.

(1) => (2) Let s_A be an f_A*-section* of A on c and s_B be an f_B*-section* of B on c. Since AB is delineable on c, s_A and s_B are contained in sections s'_A and s'_B of AB on c respectively. Clearly $s_A = s'_A$ and $s_B = s'_B$. By definition of delineability of AB, s_A and s_B must either be disjoint or identical.

By the argument just concluded, we have that the number of distinct common real roots of A_α and B_α is invariant over all $\alpha \in c$, i.e. the number of distinct real roots of C_α is invariant over all $\alpha \in c$. Since D, A, and B are delineable on c, $p(D_\alpha)$, $p(A_\alpha)$, and $p(B_\alpha)$ are invariant over all $\alpha \in c$. Hence since

$$p(C_\alpha) = p(A_\alpha) + p(B_\alpha) - p(D_\alpha),$$

$p(C_\alpha)$ is invariant over all $\alpha \in c$, hence the number of distinct common strictly complex roots of A_α and B_α is invariant over all $\alpha \in c$.

(2) => (3) If either $A_\alpha(x_r)$ or $B_\alpha(x_r)$ has degree zero for all α in c the assertion is trivial, so assume both have positive degree. Since any A-section and

any B-section are either disjoint or identical, and since the number of distinct common strictly complex roots of A_α and B_α is invariant over all α in c, the number of distinct roots of C_α is invariant over all α in c. But then the multiplicities of the distinct roots of C_α are also invariant over all α in c, since the multiplicity of any root of C_α is the minimum of its multiplicities as a root of A_α and as a root of B_α, which by delineability of A and B on c are both invariant over all α in c. Thus the total number of roots of C_α, multiplicities counted, is invariant over all α in c, i.e. the degree of C_α is invariant over c. Then by Theorem 2.2.7, the least k such that $psc_k(A_\alpha,B_\alpha)$ is nonzero is invariant over all α in c.

(3) => (1) We first note that since $(ldcf(A))_\alpha \neq 0$ and $(ldcf(B))_\alpha \neq 0$ for all α in c, $(ldcf(D))_\alpha \neq 0$ for all α in c. If either A or B has degree 0, then D is obviously delineable, so assume that both A and B have positive degree. By Theorem 2.2.8, if $p(D_\alpha)$ is invariant over all α in c, then D is delineable on c. Since A and B are delineable on c, $p(A_\alpha)$ and $p(B_\alpha)$ are invariant over all α in c by Theorem 2.2.8, so it suffices to show $p(C_\alpha)$ invariant over all α in c. We establish this by showing that for every α in c, there is a neighborhood M in c of α such that for every β in M, $p(C_\beta) = p(C_\alpha)$, then applying Lemma 2.2.5.

By assumption, the least k such that $psc_k(A_\alpha,B_\alpha)$ is nonzero is invariant over all α in c. Then by Theorem 2.2.7, $k_\alpha = deg(C_\alpha)$ is invariant over all α in c. Choose any α in c. Since $p(A_\beta)$ and $p(B_\beta)$ are invariant over all β in c, since the roots of A_α and the roots of B_α together comprise the roots of D_α, and since $sep(D_\alpha) \leq sep(A_\alpha)$ and $sep(D_\alpha) \leq sep(B_\alpha)$, by two applications of Theorem (1,4) of [MAR66] and an argument like that used in the proof of Theorem 2.2.8, we can find a neighborhood M in c of α and disjoint circles

centered at the roots of D_α such that for each circle C_i, the following conditions hold. Either A_β has one root of invariant multiplicity in C_i for all β in M, or A_β has no roots in C_i for all β in M, and either B_β has one root of invariant multiplicity in C_i for all β in M, or B_β has no roots in C_i for all β in M. Furthermore, for any β in M, each root of A_β is in some C_i and each root of B_β is in some C_i. Thus every root of $C_\beta(x_r)$, for any $\beta \in M$, is in one of those circles C_i which contains a root of both A_α and B_α, i.e. a root of C_α. Thus $p(C_\beta) \le p(C_\alpha)$. Let $C_1, \ldots, C_h$ be the circles containing a root of C_α; thus $h = p(C_\alpha)$. For $1 \le i \le h$, for any $\beta \in M$, let

$e_i(\beta)$ = the number of roots of C_β in C_i, multiplicities counted.

$e_i'(\beta)$ = the multiplicity of the unique root $z_i'(\beta)$ of A_β in C_i.

$e_i''(\beta)$ = the multiplicity of the unique root $z_i''(\beta)$ of B_β in C_i.

Since for all $\beta \in M$, and for $1 \le i \le h$, we have $e_i'(\beta) = e_i'(\alpha)$ and $e_i''(\beta) = e_i''(\alpha)$, we have

$$e_i(\beta) = \begin{cases} \min(e_i'(\alpha), e_i''(\alpha)) = e_i(\alpha) & , z_i'(\beta) = z_i''(\beta) \\ 0 & , z_i'(\beta) \neq z_i''(\beta) \end{cases}$$

for any $\beta \in M$, and $e_i(\alpha) \ge 1$. Furthermore, for all $\beta \in M$,

$$k_\beta = \sum_{i=1}^{h} e_i(\beta).$$

Suppose for some $\beta \in M$ and for some i, $1 \le i \le h$, that C_i does not contain a root of C_β. Then $z_i'(\beta) \neq z_i''(\beta)$, hence $e_i(\beta) = 0$, hence $k_\beta < k_\alpha$, a contradiction. Thus for every $\beta \in M$, and for each i, $1 \le i \le h$, C_i contains a (unique) root of C_β. Hence for all $\beta \in M$, $p(C_\beta) = p(C_\alpha)$. ∎

Theorem 2.2.10. Let A,B be elements of $I_r, r \ge 2$. Let c be a region in E^{r-1}. Assume $(ldcf(A))_\alpha \neq 0$ and $(ldcf(B))_\alpha \neq 0$, for every $\alpha \in c$. If AB is delineable on c, then A is delineable on c.

Proof. By Theorem 2.2.8 and Lemma 2.2.5, it suffices to show that the number of distinct roots of A is locally invariant on c. Let $\alpha \in c$. Let D = AB, and let $z_1, \ldots, z_p$ be the distinct roots of D at α. by Theorem (1,4) of [MAR66], there are disjoint circles $C_1, \ldots, C_p$, with z_i the center of C_i, and a neighborhood M in c of α such that if e_i is the multiplicity of z_i as a root of AB at α, then for every $\beta \in M$, D_β has e_i roots in C_i, multiplicities counted. By another application of the same theorem, we may choose M small enough that if e_i' is the multiplicity of z_i as a root of A at α (zero in case z_i is not a root of A at α), then A_β has e_i' roots in C_i, multiplicities counted, for every $\beta \in M$. By a third application of the theorem, we may assume that the same is true with B in place of A. Since D is delineable on c, it follows from Theorem 2.2.8 that D_β has exactly p distinct roots for every $\beta \in M$. Hence since each C_i contains at least one root of D_β, it must contain exactly one, for every $\beta \in M$. Let z_i be a root of A_α. Let $\beta \in M$ and let $\bar{z}_i$ be the unique root of D_β in C_i. If $\bar{z}_i$ is not a root of A_β then A_β has no roots in C_i. But if e_i' is the multiplicity of z_i as a root of A_α then $e_i' > 0$ and C_i contains e_i' roots of A_β, multiplicities counted, a contradiction. So $\bar{z}_i$ is a root of A_β, indeed the unique root of A_β in C_i. Further, if z_i is not a root of A_α, then $e_i' = 0$, and hence if $\beta \in M$, then A_β has no roots in C_i. So A_β and A_α have the same number of distinct roots for every $\beta \in M$. ▪

Corollary 2.2.11. Let $A_1, \ldots, A_n$, $n \geq 1$, be elements of I_r, $r \geq 2$, such that $(ldcf(A_i))_\alpha \neq 0$ for $1 \leq i \leq n$ and every $\alpha \in c$. If $\prod_{i=1}^{n} A_i$ is delineable on c, then each A_i is delineable on c.

Notation. For an element A_i of I_r, we write $A_{i,\alpha}$ for $(A_i)_\alpha$.

Theorem 2.2.12. Let $A_1, A_2, \ldots, A_n, n \geq 1$ be elements of $I_r, r \geq 2$, each of which is delineable on a region c in E^{r-1}. Assume also that $(ldcf(A_i))_\alpha \neq 0$ for $1 \leq i \leq n$ and every α in c. Then $\prod_{i=1}^{n} A_i$ is delineable on c if and only if for all i and j, $1 \leq i < j \leq n$, $A_i A_j$ is delineable on c.

Proof. Let $A = \prod_{i=1}^{n} A_i$. Since for each i, $(ldcf(A_i))_\alpha$ is nonvanishing for all α in c, we have $(ldcf(A))_\alpha \neq 0$ for all α in c. Suppose that for all i and j, $i < j$, $A_i A_j$ is delineable on c. Then by Theorem 2.2.9, for all i and j, every A_i-section and every A_j-section of Z(c) are either disjoint or identical, and the number of common strictly complex roots of $A_{i,\alpha}$ and $A_{j,\alpha}$ is invariant over all α in c. From this one easily sees that the number of distinct roots of A_α is invariant over all α in c. Hence by Theorem 2.2.8, A is delineable on c. Suppose conversely that A is delineable on c. Then since for all i and j, $1 \leq i < j \leq n$, $A_i A_j$ divides A, and $(ldcf(A_i A_j))_\alpha \neq 0$ for all $\alpha \in c$, by Corollary 2.2.11, $A_i A_j$ is delineable on c. ▪

Corollary 2.2.13 Let $A_1, A_2, \ldots, A_n, n \geq 1$ be elements of $I_r, r \geq 2$, each of which is delineable on a region c in E^{r-1}. Assume also for $1 \leq i \leq n$ that $(ldcf(A_i))_\alpha \neq 0$ for every α in c. Then the following are equivalent:

(1) $\prod_{i=1}^{n} A_i$ is delineable on c.

(2) For any i and j, $1 \leq i < j \leq n$, each A_i-section s_i and each A_j-section s_j of Z(c) are either disjoint or identical, and the number of common strictly complex roots of $A_{i,\alpha}$ and $A_{j,\alpha}$ is invariant over all α in c.

(3) For any i and j, $1 \leq i < j \leq n$, such that A_i and A_j both have positive degree, the least k such that $psc_k(A_{i,\alpha}(x_r), A_{j,\alpha}(x_r)) \neq 0$ is invariant over

all α in c.

Corollary 2.2.14. Let F be a primitive nonzero element of $I_r, r \geq 2$. Let c be a region in E^{r-1} such that $(ldcf(F))_\alpha \neq 0$ for all $\alpha \in c$. Then F is delineable on c if and only if its greatest squarefree divisor is delineable on c. (See [COL73] for a definition of greatest squarefree divisor).

Proof. Let F^* denote the greatest squarefree divisor of F. Since $(ldcf(F))_\alpha \neq 0$ for all $\alpha \in c$, we have $(ldcf(H))_\alpha \neq 0$ for any factor H of F and for all $\alpha \in c$. If F is delineable on c, then by Theorem 2.2.10, F^* is delineable on c. Suppose F^* is delineable on c. From Corollary 2.2.13 it is easily seen that $(F^*)^k$ is delineable on c for any $k \geq 1$. There exists a $k \geq 1$ such that F divides $(F^*)^k$. Where $K = (F^*)^k / F$, clearly $(ldcf(K))_\alpha \neq 0$ for all $\alpha \in c$. Hence by Theorem 2.2.10, F is delineable on c. ∎

Definition. Let F be an element of I_r, $r \geq 2$. Let c be a region in E^{r-1}. F is *cylindrical on* c if $F(a_1, \ldots, a_r) = 0$ for every $<a_1, \ldots, a_r>$ in Z(c). If F is cylindrical on c, the *F-induced stack over* c, written S(F,c), is $\{Z(c)\}$. If c consists of a single point α, we say F is *cylindrical at* α. If for a point $\beta \in E^{r-1}$ F is not cylindrical at β, we say F is *noncylindrical at* β.

Definition. Let F be an element of $I_r, r \geq 2$. Let c be a region in E^{r-1}. F is *regular* on c if F is either delineable or cylindrical on c.

Theorem 2.2.15. If $F \in I_r$ is regular on a semi-algebraic region c in E^{r-1}, $r \geq 2$, then S(F,c) is algebraic.

Proof. Let φ be a defining formula for c. If F is cylindrical on c or there are no F-sections of Z(c), then S(F,c) = { Z(c) }, and since φ also defines Z(c), S(F,c) is algebraic. Suppose F is delineable on c with F-sections

$s_1 < s_2 < \cdots < s_m, m \geq 1$, and let s_i be an f_i-section. It suffices to show that each region of S(F,c) is definable. Let x denote an (r-1)-tuple $< x_1, \ldots, x_{r-1} >$, and let y stand for x_r. Then for $2 \leq k \leq m-1$, s_k is the set of all points $<x,y>$ satisfying the following formula:

$$\varphi(x) \;\&\; (Ey_1)(Ey_2)\cdots(Ey_{k-1})[\, y_1 < y_2 < \cdots < y_{k-1} < y$$
$$\&\; F(x,y_1) = 0 \;\&\; F(x,y_2) = 0 \;\&\; \cdots \;\&\; F(x,y_{k-1}) = 0 \;\&\; F(x,y) = 0$$
$$\&\; (Ay_{k+1})\{\, (\, y_{k+1} \neq y_1 \;\&\; y_{k+1} \neq y_2 \;\&\; \cdots \;\&\; y_{k+1} \neq y_{k-1} \;\&$$
$$y_{k+1} \neq y \;\&\; F(x,y_{k+1}) = 0\,) \Rightarrow y_{k+1} > y \,\}\,].$$

Defining formulas for s_1 and s_m can be obtained by obvious modifications to the above formula. For $1 \leq k \leq m$, let φ_k denote the defining formula for s_k. For $2 \leq k \leq m$, the (f_{k-1}, f_k)-sector of S(F,c) is the set of all points <x,y> satisfying the formula:

$$\varphi(x) \;\&\; (Ey_{k-1})(Ey_k)[\, y_{k-1} < y < y_k \;\&\; \varphi_{k-1}(x,y_{k-1}) \;\&\; \varphi_k(x,y_k)\,].$$

Clearly defining formulas for the $(-\infty, f_1)$-sector and the (f_m, ∞)-sector of S(F,c) can be obtained by straightforward modifications to the formula just given. ▪

Theorem 2.2.16. Let c be a region in E^{r-1}, $r \geq 2$, and suppose that $F \in I_r$ is delineable on c. Then $deg(F_a)$ is invariant over all $\alpha \in c$.

Proof. From the definition of delineability one sees that there exists $p \geq 0$ such that F_a has p distinct complex roots for all $\alpha \in c$. One also sees that where $e_1(\alpha), \ldots, e_p(\alpha)$ are the multiplicities of the distinct roots of F_a, the multiset $\{e_1(\alpha), \ldots, e_p(\alpha)\}$ is invariant over all $\alpha \in c$. Since $deg(F_a) = \sum_{i=1}^{p} e_i(\alpha)$ for any $\alpha \in c$, $deg(F_a)$ is invariant over all $\alpha \in c$. ▪

Definition. Suppose $F \in I_r, r \geq 2$ is regular on a region c in E^{r-1}. The *degree of F on* c, or the *c-degree of F*, written $deg_c(F)$, is defined as follows. If F is cylindrical on c, then $deg_c(F) = 0$. If F is delineable on c, then $deg_c(F) = deg(F_\alpha)$ for any α in c.

Definition. Let A be a set of polynomials in I_r, $r \geq 2$, which is regular on some region c in E^{r-1}. The *noncylindrical product of elements of A on* c, written A_c , is the product of all the elements of A which are delineable on c. If there are no such elements, then A_c is the constant polynomial 1 in I_r.

Definition. Let c be a region in E^{r-1}. Let $A = \{A_1, \ldots, A_n\}$ be a subset of I_r. A is *regular* on c if (1) each A_i is regular on c, and (2) A_c is delineable on c.

Definition For any $F \in I_r$, the *leading term* of F, written $ldt(F)$, is

$$ldcf(F) \cdot x_r^{deg(F)}.$$

The *reductum* of F, written $red(F)$, is $F - ldt(F)$. For any $k \geq 0$, the *kth reductum of F*, written $red^k(F)$, is defined by induction on k:

$$red^0(F) = F.$$

$$red^{k+1}(F) = red(red^k(F)).$$

Theorem 2.2.17. Let $A = \{A_1, \ldots, A_n\}$, $n \geq 1$ be a subset of I_r, $r \geq 2$. If A is regular on a semi-algebraic region c in E^{r-1}, then $S(A_c, c)$ is A-invariant and algebraic.

Proof. By Theorem 2.2.15, $S(A_c, c)$ is algebraic. For any $A_i \in A$ which is delineable on c, there is a unique reductum G_i of A_i such that $A_i = G_i$ on c and $ldcf(G_i)$ is nonvanishing on c. There is a unique reductum G_c of A_c such that $A_c = G_c$ on c and $ldcf(G_c)$ is nonvanishing on c. We have that $G_c = \prod\{G_i \mid A_i \in A$ delineable on c$\}$, and that G_c is delineable on c since A_c is.

If $A_j, A_k \in A$ are delineable on c with $j \neq k$, then applying Corollary 2.2.13 to G_c, G_j, and G_k, we have that each G_j-section is either disjoint from or identical to any G_k-section, hence each A_j-section is either disjoint from or identical to any A_k-section. Thus each A_j that is delineable on c is invariant on each section of $S(A_c, c)$. Since for each A_j that is delineable on c, every sector of $S(A_c, c)$ is a subset of some sector of $S(A_j, c)$, A_j is invariant on each sector of $S(A_c, c)$. The cylindrical elements of A are clearly invariant on each region of $S(A_c, c)$, so $S(A_c, c)$ is A-invariant. ▪

Definition. For any $F \in I_r$, the *reducta set of F*, written RED(F), is

$$\{red^k(F) \mid 0 \leq k \leq deg(F) \ \& \ red^k(F) \neq 0\}$$

Definition. Let $A = \{A_1, \ldots, A_n\}$, $n \geq 1$, be a set of polynomials in I_r, $r \geq 2$. The *projection* of A, written PROJ(A), is a set of polynomials in I_{r-1} defined as follows. For each i, $1 \leq i \leq n$, let $R_i = RED(A_i)$. Let

$$P_1(A) = \bigcup_{i=1}^{n} \bigcup_{G_i \in R_i} (\{ldcf(G_i)\} \cup PSC(G_i, G'_i))$$

$$P_2(A) = \bigcup_{1 \leq i < j \leq n} \bigcup_{G_i \in R_i \& G_j \in R_j} PSC(G_i, G_j)$$

Then PROJ(A) is the union of $P_1(A)$ and $P_2(A)$.

Theorem 2.2.18. For $A \subset I_r$, $r \geq 2$, if c is a PROJ(A)-invariant region in E^{r-1}, then A is regular on c.

Proof. Consider any A_i in A. By definition, PROJ(A) includes every nonzero coefficient of A_i, so each coefficient of A_i either vanishes everywhere or nowhere on c. If every coefficient of A_i vanishes on c, then A_i is cylindrical on c. Suppose some coefficient of A_i is nonvanishing on c. Then there is a unique reductum G_i of A_i such that A_i is equal to G_i on c and $ldcf(G_i)$ is

nonvanishing on c. Then since PROJ(A) includes $ldcf(G_i)$ and every element of $PSC(G_i, G'_i)$, the least k such that $psc_k(G_{i,\alpha}, G'_{i,\alpha}) \neq 0$ is invariant over all α in c. Hence by Theorem 2.2.8, G_i is delineable on c, hence A_i is delineable on c. Thus every element of A is regular on c. Consider now any two elements A_i and A_j of A which are delineable on c. Let G_i and G_j be the unique reducta of A_i and A_j respectively such that $A_i = G_i$ on c and $A_j = G_j$ on c. PROJ(A) includes $PSC(G_i, G_j)$, hence the least k such that $psc_k(G_{i,\alpha}(x_r), G_{j,\alpha}(x_r)) \neq 0$ is invariant over all α in c. Hence by Theorem 2.2.12, $G_i G_j$ is delineable on c. Thus where G_c is the unique reductum of A_c such that $A_c = G_c$ on c and $ldcf(G_c)$ is nonvanishing on c, by Corollary 2.2.13, G_c is delineable on c. Hence A_c is delineable on c. Hence A is regular on c. ∎

This completes the proof of the theorems needed for the proof of Theorem 2.2.1.

2.3 Bases

Definition. By an *algebraic polynomial* we mean a polynomial over the algebraic number field $Q(\gamma)$, for some algebraic number γ.

Definition. An element of I_0 is positive if it is a positive integer. An element of I_r, $r \geq 1$, is *positive* if its leading coefficient, as an element of I_{r-1}, is positive.

Definition. Let J be a unique factorization domain. A *basis* is a set $B = \{B_1, \ldots, B_s\}$, $s \geq 0$, of pairwise relatively prime elements of J[x], such that each B_i is primitive, ample, and of positive degree. A basis B is *square-*

free if each B_i is squarefree. A basis B is *irreducible* if each B_i is irreducible.

The definition of an *ample* element of J[x] is given in [COL75] and [COL73]. For the bases B of interest in this thesis, we will always have either $B \subset I_r$, $r \geq 1$, or $B \subset Q(\gamma)[x]$, for some real algebraic number γ. An element of I_r is ample if and only if it is positive, and an element of $Q(\gamma)[x]$ is ample if and only if it is monic.

Definition. Let J be a unique factorization domain, and let A be a subset of J[x] such that each element of A is primitive and of positive degree. A basis $B \subset$ J[x] is a *basis for A* if

(1) For each $A_i \in A$, there exist (not necessarily distinct) $B_1, \ldots, B_k \in B$, $k \geq 0$, such that

$$A_i = c \prod_{j=1}^{k} B_j$$

for some unit c of J, and

(2) Each $B_j \in B$ divides some $A_i \in A$.

Definition. For a unique factorization domain J, let A be a set of primitive polynomials in J[x] of positive degree. Given bases B and B' for A, we say that B' is a *refinement* of B if every element of B' is a divisor of some element of B. Clearly the set B^* of ample irreducible divisors of elements of A is a basis for A. B^* is the finest basis for A in the sense that it is a refinement of every other basis for A. Since B^* is also a squarefree basis, we refer to it as the *finest squarefree basis* for A. As shown in [COL75] (p. 146), A also has a *coarsest squarefree basis*, that is, a squarefree basis such that every other squarefree basis for A is a refinement of it.

Definition. Let F be an element of $I_r = I_{r-1}[x_r]$. The *content* of F, written $cont(F)$, is the greatest common divisor of the coefficients of F. The *primitive part* of F, written $pp(F)$, is $F \,/\, cont(F)$. The content of the zero polynomial is defined to be zero, and the primitive part of the zero polynomial is defined to be zero. Let A be a subset of $I_r, r \geq 2$. We define CONT(A) to be the set of contents of elements of A, and PP(A) to be the set of the primitive parts of positive degree of elements of A.

We will use bases to obtain a new version of the projection operator that is computationally more efficient than that defined in Section 2.2, chiefly because we apply it to polynomials of lower degree. Where $A \subset I_r$, $r \geq 2$, is the set of input polynomials to the cad algorithm, and B is a basis for PP(A), our new projection of A will be $CONT(A) \cup PROJ(B)$. This new version of PROJ is justified by the following theorem.

Theorem 2.3.1 Let A be a subset of $I_r, r \geq 2$, and let B be a basis for PP(A). Let c be a region in E^{r-1} which is both PROJ(B)-invariant and CONT(A)-invariant. Then A is regular on c, and $S(B_c, c)$ is A-invariant.

Proof. Since PROJ(B') $\subset$ PROJ(B) for any subset B' of B, by Theorem 2.2.17 any subset of B is regular on c. Hence if B' is a subset of B each of whose elements is delineable on c, then the product of B' is delineable on c. Then by Corollary 2.2.14, any power product of elements of B' is delineable on c.

For any $A_i \in A$, if $cont(A_i)$ vanishes on c, then A_i is cylindrical on c. Suppose $cont(A_i)$ is nonvanishing on c. If $pp(A_i)$ is the power product of elements of B of which at least one is cylindrical on c, then A_i is cylindrical on c. If $pp(A_i)$ is the power product of noncylindrical elements of B, then by our argument above, $pp(A_i)$ is delineable on c, hence A_i is delineable on c.

Since $pp(A_c)$ is the power product of noncylindrical elements of B, $pp(A_c)$ is delineable on c. $cont(A_c)$ is nonvanishing on c, since A_c is the product of polynomials whose contents are nonvanishing on c, hence A_c is delineable on c. Hence A is regular on c. We have $V(A_c) = V(pp(A_c)) = V(gsfd(pp(A_c)))$, where gsfd denotes greatest squarefree divisor. $gsfd(pp(A_c))$ is a factor of B_c, hence $V(gsfd(pp(A_c))) \subset V(B_c)$, hence $S(B_c,c)$ is a refinement of $S(A_c,c)$, hence since $S(A_c,c)$ is A-invariant, so is $S(B_c,c)$. ∎

The use of bases will be important to nearly all the results of this thesis.

Definition Let D be a cad of $E^r, r \geq 2$, and c a cell of D'. The *D-stack over c*, written $S_D(c)$, is the stack over c consisting of the cells of D contained in Z(c).

Notation. For any cad D of $E^r, r \geq 2$, we will write D' to denote the cad of E^{r-1} induced by D. If $r \geq 3$, we write D'' to denote the cad of E^{r-2} induced by D'.

Definition. Let D be a cad of $E^r, r \geq 1$. A *section* of D is defined as follows. If r=1, then any 0-cell of D is a section of D. If r>1, then D is the union of the stacks $S_D(c)$, for each $c \in D'$, and a section of D is a section of any $S_D(c)$. A *sector* of D is similarly defined.

Definition. A cad D of $E^r, r \geq 1$, is $(B^1, \ldots, B^r)$ *basis-determined* if $B^r \subset I_r$ is a basis, and

1. D is B^r-invariant.
2. every section of D is contained in V(B) for some $B \in B^r$.
3. if r>1, then D' is $(B^1, \ldots, B^{r-1})$ basis-determined.

Theorem 2.3.2. Let D be a cad of $E^r, r \geq 1$, and let B^i be a basis contained in I_i for $1 \leq i \leq r$. D is $(B^1, \ldots, B^r)$ basis-determined if and only if the following three conditions are satisfied:

(1) For each cell c of D', and for each $B \in B^r$, either $V(B) \cap Z(c)$ is the union of finitely many sections of $S_D(c)$, or $Z(c) \subset V(B)$.

(2) Every section of D is contained in V(B) for some $B \in B^r$.

(3) if r>1, then D' is $(B^1, \ldots, B^{r-1})$ basis-determined.

Proof. By induction on r. For r=1 the assertion is clear. Suppose that r>1, that D is $(B^1, \ldots, B^r)$ basis-determined, let $c \in D'$, and let $B \in B^r$. If V(B) meets a sector s of $S_D(c)$, then by the B-invariance of D, B vanishes on s, hence $Z(c) \subset V(B)$. If V(B) meets a section t of $S_D(c)$, then by the B-invariance of D, B vanishes on t. Hence if $Z(c) \not\subset V(B)$, then $V(B) \cap Z(c)$ is the union of finitely many sections of $S_D(c)$.

Assume now that conditions (1)-(3) are satisfied. Consider any $c \in D'$ and any $B \in B^r$. If $Z(c) \subset V(B)$, then clearly $S_D(c)$ is B-invariant. Assume $V(B) \cap Z(c)$ is the union of finitely many sections of $S_D(c)$. If V(B) meets a section t of $S_D(c)$, then $t \subset V(B)$, and hence B is invariant on t. If R is a region of $S_D(c)$ which V(B) does not meet, then by an argument similar to that used in the proof of Lemma 2.2.4, B is invariant on R. Hence $S_D(c)$ is B-invariant. Hence D is B^r-invariant. ∎

2.4 Sample point construction.

Sample point construction plays an essential part in the extension of a cad of E^{r-1} to a cad of E^r by the cad algorithm. If $A \subset I_r, r \geq 2$, is the set of input polynomials to the cad algorithm, if B is a basis for A, if c is any cell in

the induced cad of E^{r-1}, and if α is a sample point for c, then the number of real roots of $B^* = \{B_i(\alpha,x_r) \mid B_i \in B \ \& \ deg(B_i(\alpha,x_r)) > 0\}$ is the number of sections of $S(B_c,c)$. As suggested by Theorem 2.2.2, the section sample points we will construct for $S(B_c,c)$ are the points $< \alpha,b >$, where b is a real root of B^*. The sector sample points we will construct have the form $< \alpha,b >$ with b rational.

Our algorithms for sample point construction utilize two distinct representations for real algebraic numbers. The first is the representation of a real algebraic number as an element of the field of all real algebraic numbers. The second is the representation of an algebraic number as an element of an algebraic number field $Q(\gamma)$, for some real algebraic number γ.

Let γ be an element of the field of all real algebraic numbers. There is a unique monic irreducible polynomial M(x) in Q[x] such that $M(\gamma)=0$. M(x) is the *rational minimal polynomial* of γ. The unique primitive, positive integral polynomial $\overline{M}(x)$ similar to M(x) is the *integral minimal polynomial* of γ. An *acceptable isolating interval* for γ is an interval with rational number endpoints containing γ but no other root of M(x), and satisfying certain other conditions. These conditions are relevant to the root isolation algorithms, bu are irrelevant to all the matters discussed in this thesis. A representation for γ is the pair consisting of its integral minimal polynomial and an acceptable isolating interval for it.

Consider now the algebraic number field $Q(\gamma)$. Let n be the degree of M(x). Let Ψ denote the familiar isomorphism from $Q(\gamma)$ to Q[x]/(M(x)). For any a in $Q(\gamma)$, $\Psi(a)$ can be viewed as a polynomial A(x) in Q[x] of degree less

than n, such that a=A(γ). Then we take the representation for a to be A(x).

For any algebraic point $\alpha = < \alpha_1, \ldots, \alpha_r >$ in E^r, there exists a real algebraic number γ such that $Q(\gamma) = Q(\alpha_1, \ldots, \alpha_r)$. We call γ a ***primitive element*** for α. α is represented as a triple $(\overline{M}, I, \overline{\alpha})$, where $\overline{M}$ is the integral minimal polynomial of γ, I is an acceptable isolating interval for γ, and $\overline{\alpha} = < \overline{\alpha}_1, \ldots, \overline{\alpha}_r >$, where $\overline{\alpha}_i$ is the representation for α_i as an element of $Q(\gamma)$.

We first discuss the construction of sample points for cad's of E^1. For a set A of univariate input polynomials, the cad algorithm calculates the finest squarefree basis B for $PP(A)$, and constructs (acceptable) isolating intervals for the real roots of B. Thus any 0-cell in the cad of E^1 will be the unique root γ of some $\overline{M} \in B$ lying in some acceptable isolating interval I which we have constructed. If γ is rational, then our representation for this 0-cell is $(\overline{M}, I, < \gamma >)$. If γ is irrational, then since our representation for γ as an element of $Q(\gamma)$ is just the polynomial "x", the sample point representation for this 0-cell is the triple $(\overline{M}, I, (x))$. For a 1-cell in a cad of E^1, we pick some rational number r contained in it as its sample point, trivially compute the integral minimal polynomial $\overline{M}$ of r and some acceptable isolating interval I for r, and then form the triple $(\overline{M}, I, <r>)$ as the representation of r.

Now let A be a subset of I_r, $r \geq 2$, and let D be an A-invariant cad of E^r. Let B be the finest squarefree basis for PP(A). Given a cell c of D', we explain how we construct sample points for $S(B_c, c)$. We assume that an algebraic sample point α for c has been constructed; let γ denote the primitive element for α. α is represented as a triple $(\overline{M}, I, \overline{\alpha})$, where $\overline{M}$ is the integral minimal polynomial of γ, I is an acceptable isolating interval for γ, and $\overline{\alpha}$ is an (r-1)-tuple $< \overline{\alpha}_1, \ldots, \overline{\alpha}_{r-1} >$, such that each $\overline{\alpha}_i$ is an element of $Q(\gamma)$.

Sector sample points are straightforward, for since any such point has the form $<\alpha,b>$ with b a rational number, its representation is $(\overline{M},I,<\overline{\alpha}_1,\ldots,\overline{\alpha}_{r-1},b>)$. To construct section sample points, we proceed as follows. Let $B^* = \{B_i(\alpha,x_r) \mid B_i \in B \ \& \ deg(B_i(\alpha,x_r)) > 0\}$. We compute a coarsest squarefree basis $\overline{B}$ for B^*. The set of real roots of elements of this basis is the same as the set of rth coordinates of points of sections of $S(B_c,c)$ in $Z(\alpha)$, so we isolate these roots using an algorithm such as that described in [RUM76]. Let β be a real root of an element $\overline{B}_j$ of $\overline{B}$. We apply the NORMAL algorithm of [LOO73] to obtain a univariate integral polynomial N(x) which has β among its roots, and an isolating interval J for β as a root of N(x). We then find the unique positive irreducible factor $\overline{N}(x)$ of N(x) which has a root in J. $\overline{N}(x)$ is the integral minimal polynomial of β. We then apply the SIMPLE algorithm of [LOO73] to $\overline{M}(x)$, $\overline{N}(x)$, I, and J to obtain an integral polynomial U(x) and an interval K. K is an isolating interval for a real root ϑ of U(x) having the property that $Q(\vartheta) = Q(\gamma,b)$. We find the unique positive irreducible factor $\overline{U}(x)$ of U(x) which has a root in K. $\overline{U}(x)$ is the integral minimal polynomial of ϑ. SIMPLE also produces rational polynomials $m(x)$ and $n(x)$, which are respectively the representation of γ as an element of $Q(\vartheta)$ and the representation of β as an element of $Q(\vartheta)$. Let U^* be the rational minimal polynomial of ϑ. For $1 \le i \le r-1$, let $\alpha_i^*(x) = \overline{\alpha}_i(m(x)) \ mod \ U^*(x)$. Since $\alpha_i = \overline{\alpha}_i(\gamma)$, $\gamma = m(\vartheta)$, and $U^*(\vartheta) = 0$, we have $\alpha_i^*(\vartheta) = \overline{\alpha}_i(m(\vartheta)) = \overline{\alpha}_i(\gamma) = \alpha_i$. Hence if $\alpha^* = <\alpha_1^*,\ldots,\alpha_{r-1}^*,n>$, then the representation for the section sample point $<\alpha,\beta>$ is $(\overline{U}, K, \alpha^*)$.

2.5 Defining formula construction

Definition. For $F \in I_r$, $r \ge 1$, and for any $k \ge 0$, the *kth derivative of F*, written

$der^k(F)$, is defined by induction on k:

$$der^0(F) = F,$$

$$der^{k+1}(F) = der(der^k(F)).$$

Definition. For any nonzero F in I_r, $r \geq 1$, the *derivative set of F*, written DER(F), is

$$\{pp(\, der^k(F)\,) \mid 0 \leq k \leq deg\,(F)\}.$$

If F is the zero polynomial, then DER(F) is the empty set.

Definition. Let F be an element of I_r, $r \geq 2$. Let c be a region in E^{r-1}. F is *derivative-regular* on c if

(1) F is regular on c, and

(2) every element of DER(F) is regular on c.

Lemma 2.5.1. Let $F \in I_r$, $r \geq 2$, be regular on a region R in E^{r-1}. If $deg_R(F) \geq 1$, then $cont(F')$ is nonvanishing on R.

Proof. Let $F = \sum_{i=0}^{n} F_i x_r{}^i$, and let $c = cont(F')$. For any $\alpha \in R$ such that c vanishes at α, $F_n, F_{n-1}, \ldots, F_1$ all vanish at α. Hence since $k \geq 1$ and F_k does not vanish on R, c does not vanish at any $\alpha \in R$. ▪

Lemma 2.5.2. Let $F \in I_r$, $r \geq 2$, be regular on a region R in E^{r-1}. Suppose $k = deg_R(F) \geq 1$, and let $G = pp(F')$. Then $deg_R(G) = k-1$.

Proof. Let $F = \sum_{i=0}^{n} F_i x_r{}^i$, and let $c = cont(F')$. By Lemma 2.5.1, c does not vanish at any $\alpha \in R$. Let $G = \sum_{i=0}^{n-1} G_i x_r{}^i$. Since $F_i = i \cdot c \cdot G_i$, for $1 \leq i \leq n$, since $F_n, F_{n-1}, \ldots, F_{k+1}$ all vanish on R, and since F_k is nonvanishing on R, we have that $G_{n-1}, \ldots, G_k$ all vanish on R, and G_{k-1} is nonvanishing on R. Hence

$deg_R(G) = k-1$. ∎

The next theorem has the same conclusion as Theorem 2.2.15 but stronger hypotheses. Its proof implicitly exhibits the structure of the quantifier-free defining formulas constructed by the cad algorithm for the cells of S(F,c), whereas Theorem 2.2.15's proof was nonconstructive.

Theorem 2.5.3. If $F \in I_r$ is derivative-regular on a semi-algebraic region c in $E^{r-1}, r \geq 2$, then S(F,c) is algebraic.

Proof. Let φ_c be a quantifier-free defining formula for c. We proceed by induction on $k = deg_c(F)$. If $k = 0$, then $S(F,c) = \{Z(c)\}$ and since φ_c also defines Z(c), S(F,c) is algebraic. If $deg_c(F)=1$, then F is delineable on c with one section of odd multiplicity, hence S(F,c) consists of one section and two sectors, and F has opposite signs on the two sectors. Hence $(\varphi_c \ \& \ F>0)$, $(\varphi_c \ \& \ F=0)$, and $(\varphi_c \ \& \ F<0)$ are defining formulas for the regions of S(F,c) (which of the first and third formulas applies to which sector depends on the sign of F on the sectors).

Suppose now that $k \geq 2$, and suppose that $S(K,c)$ is algebraic for any $K \in I_r$ with K derivative-regular on c and $deg_c(K) < k$. Let G = $pp(F')$. By Lemma 2.5.2, $deg_c(G) = deg_c(F) - 1$, and G is derivative-regular on c since F is. Hence by the inductive hypothesis, S(G,c) is algebraic. Since F and G are each regular on c with positive c-degree, both are delineable on c.

Let H be the unique reductum of F such that F = H on c and $ldcf(H)$ is nonvanishing on c. Since F is delineable on c, so is H. Let h = $cont(H')$. By Lemma 2.5.1, h is nonvanishing on c. There is a factor g of h such that gG = H' on c. Since h is nonvanishing on c, so is g. Hence since G is delineable on c, so is H'. Since H is delineable on c, by Theorem 2.2.8, the least k such

that $psc_k(H_a,H'_a)\neq 0$ is invariant over all $\alpha\in c$, hence by Theorem 2.2.9, HH' is delineable on c. Hence again by Theorem 2.2.9, any H-section and any H'-section of Z(c) are either disjoint or identical. Hence any F-section and any G-section of Z(c) are either disjoint or identical. Hence for any F-section t of Z(c), either t is a section of S(G,c), or t is contained in some sector of S(G,c). In the latter case, by Rolle's theorem, $t = s' \cap V(F)$ for some sector s' of S(G,c).

If an F-section t of Z(c) is a section of S(G,c), t is semi-algebraic by the inductive hypothesis. If $t = s' \cap V(F)$ for a sector s' of S(G,c), then s' is semi-algebraic by the inductive hypothesis, and where $\psi_{s'}$ is a defining formula for s', $\varphi_c \,\&\, \psi_{s'} \,\&\, (F = 0)$ is a defining formula for t, so t is semi-algebraic.

Let s be the top sector of S(F,c), i.e. s is an (f,∞)-section for some f. If s = Z(c), then φ_c defines s, so s is semi-algebraic. If $s\neq Z(c)$, then s is bounded below by an F-section t. If t is a section of S(F,c), then s is the union of regions of S(G,c), and so by the inductive hypothesis is semi-algebraic. If t is not a section of S(G,c), then $t = s' \cap V(F)$ for a sector s' of S(G,c). Since $t \cap V(G) = \emptyset$, and since by Lemma 2.5.1 $cont(F')$ is non-vanishing on c, $t \cap V(F') = \emptyset$, hence t has multiplicity one. Hence where $s'_1 = s' \cap s$, and $s'_2 = s' - (s \cup t)$, F has opposite signs on s'_1 and s'_2. Hence where σ is the sign of F on s'_1, and $\psi_{s'}$ is a defining formula for s', $\varphi_c \,\&\, \psi_{s'} \,\&\, (\sigma F > 0)$ is a defining formula for s'_1, so s'_1 is semi-algebraic. Since $s - s'_1$ is a union of regions of S(G,c), each semi-algebraic by the inductive hypothesis, s is semi-algebraic. The argument for the other possible kinds of sectors of S(F,c) is similar. ▪

Definition. Let c be a region in E^{r-1}, $r \geq 2$. Let A = $\{A_1, \ldots, A_n\}$ be a subset of I_r. A is ***derivative-regular*** on c if (1) each A_i is derivative-regular on c, and (2) A_c is delineable on c.

The next theorem has the same conclusion as Theorem 2.2.17 but stronger hypotheses. Its proof implicitly exhibits the structure of quantifier-free defining formulas constructed by the cad algorithm for the cells of $S(A_c, c)$.

Theorem 2.5.4. Let A = $\{A_1, \ldots, A_n\}$, $n \geq 1$ be a subset of I_r, $r \geq 2$. If A is derivative-regular on a semi-algebraic region c in E^{r-1}, then $S(A_c, c)$ is A-invariant and algebraic.

Proof. If every element of A is cylindrical on c, then $S(A_c, c)$ = { Z(c) } and the assertion is obvious, so assume at least one element of A is delineable on c. Without loss of generality, let $A_1, \ldots, A_m$, $1 \leq m \leq n$, be the elements of A which are delineable on c. By Theorem 2.5.3 and Lemma 2.2.4, for $1 \leq j \leq m$, each $S(A_j, c)$ is an algebraic, A_j-invariant stack over c. Each section of $S(A_c, c)$ is a section of some $S(A_j, c)$ and so a semi-algebraic set. Each sector of $S(A_c, c)$ is either the intersection of a sector of $S(A_{j_1}, c)$ and a sector of $S(A_{j_2}, c)$ for some j_1 and j_2 with $1 \leq j_1 < j_2 \leq m$, or a sector of $S(A_j, c)$ for some j, $1 \leq j \leq m$. Thus each sector of $S(A_c, c)$ is a semi-algebraic set. For any A_j which is delineable on c, $1 \leq j \leq m$, we have $V(A_j) \subset V(A_c)$, hence since A_c is delineable on c, every A_j-section of Z(c) is an A_c-section of Z(c). Since no two A_c-sections of Z(c) meet, it follows that A_j is invariant on every A_c-section of Z(c). Since every sector of $S(A_c, c)$ is a subset of some sector of $S(A_j, c)$, A_j is invariant on each sector of $S(A_c, c)$. Since the cylindrical elements of A are trivially invariant on each region of $S(A_c, c)$, it follows that

$S(A_c,c)$ is A-invariant. ▪

In order to satisfy the stronger hypotheses of Theorem 2.5.4, a substitute for the projection operator of Theorem 2.2.16 is needed. We call this substitute the "augmented projection" operator.

Definition. Let $A = \{A_1, \ldots, A_n\}$, $n \geq 1$, be a set of polynomials in I_r, $r \geq 2$. The *augmented projection* of A, written APROJ(A), is defined as follows. For each i, $1 \leq i \leq n$, let $R_i = RED(A_i)$,

$$D_i = \bigcup_{G \in R_i} DER(G'),$$

and

$$S_i = \bigcup_{H \in D_i} PSC(H,H').$$

Then $P_3(A) = \bigcup_{i=1}^{n} S_i$, and APROJ(A) = $PROJ(A) \cup P_3(A)$.

Theorem 2.5.5. For $A \subset I_r$, $r \geq 2$, if c is an APROJ(A)-invariant region in E^{r-1}, then A is derivative-regular on c.

Proof. Since $PROJ(A) \subset APROJ(A)$, by Theorem 2.2.18, A is regular on c. Hence each A_i is regular on c, and A_c is delineable on c. Consider any particular $A_i \in A$. If A_i is cylindrical on c, then every element of $DER(A_i)$ is cylindrical on c, hence A_i is derivative-regular on c. Suppose A_i is delineable on c. Then there is a unique reductum G of A_i such that $A_i = G$ on c, and $ldcf(G)$ is nonvanishing on c. G is delineable on c since A_i is. Let $k = deg(G)$. If $k = 0$, then DER(G) = $\{pp(G)\}$ = {1}, hence every element of DER(G) is delineable on c. Suppose $k \geq 1$. Since G is delineable on c, $cont(G)$ is nonvanishing on c, hence $pp(G)$ is delineable on c. Since $ldcf(G)$ is nonvanishing on c, $ldcf(der^j(G))$ is nonvanishing on c for $0 \leq j \leq k$, hence

$ldcf(pp(der^j(G)))$ is nonvanishing on c for $0 \leq j \leq k$, hence $ldcf(H)$ is nonvanishing on c for every $H \in DER(G)$. For all $H \in DER(G')$, $PSC(H,H') \subset APROJ(A)$, hence for all $H \in DER(G')$ of degree ≥ 2, the least j such that $(psc_j(H,H'))_a \neq 0$ is invariant over all $\alpha \in c$. Since $ldcf(H)$ is nonvanishing on c for all $H \in DER(G')$, $(psc_j(H,H'))_a = psc_j(H_a,H'_a)$ for all $H \in DER(G')$ of degree ≥ 2 and all $\alpha \in c$, hence for all $H \in DER(G')$ of degree ≥ 2, the least j such that $psc_j(H_a,H'_a) \neq 0$ is invariant over all $\alpha \in c$, and hence by Theorem 2.2.8, every $H \in DER(G')$ is delineable on c. Hence since $pp(G)$ is delineable on c, every $H \in DER(G)G$ is delineable on c.

Consider now any $D \in DER(A_i)$. Let m be such that $D = pp(der^m(A_i))$. If $m > k$, then D is cylindrical on c. If $m \leq k$, then there is a unique $h \in I_{r-1}$ and a unique $H \in DER(G)$ such that h is nonvanishing on c and $D = hH$ on c. Since H is delineable on c, so is D. Hence every element of $DER(A_i)$ is either delineable or cylindrical on c, i.e. every element of $DER(A_i)$ is regular on c, hence A_i is derivative-regular on c. Hence A is derivative-regular on c. ∎

As per Theorem 2.3.1 of Section 2.3 and its preceding discussion, the augmented projection we actually compute will be $CONT(A) \cup APROJ(B)$, where B is a basis for A. We now precisely specify the structure of the defining formulas constructed by the cad algorithm. By Theorems 2.5.3 and 2.5.4, it will be seen in all cases that a formula which we assert to be a defining formula for a particular cell of a cad indeed is.

Definition. Let D be a (B^1) basis-determined cad of E^1. For any cell d of D, we define a *derivative-based defining formula of d* as follows. If D consists of the single 1-cell E^1, then a derivative-based defining formula for E^1 is (0 = 0). If |D| > 1, then assume given isolating intervals for the real roots of ele-

ments of B^1 that are (1) ***strongly disjoint***, i.e. the closures of any two isolating intervals are disjoint, and (2) have rational number endpoints. Consider a 0-cell $c^0 = \{\alpha\}$ in D. α is a root of some irreducible polynomial M(x) in B^1; let n be the degree of M(x). If n>1, then we have an open rational isolating interval (r,s) for α, and a derivative-based defining formula for c^0, expressed in an obvious shorthand, is

$$(r < x < s) \ \& \ (M(x) = 0).$$

If n = 1, then a derivative-based defining formula for c^0 is

$$M(x) = 0.$$

Consider now a 1-cell $c^1 = (\alpha,\beta)$ in D. Let M(x) and N(x) be the integral minimal polynomials of α and β, and let [r,s] and [t,u] be the closures of the given isolating intervals for α and β. We may assume $r \le s < t \le u$. Let $d = deg(M)$ and $e = deg(N)$. Let σ and τ be the signs of M and N on c^1. Then a derivative-based defining formula for c^1 is defined to be

$$(\sigma M(x) > 0) \ \& \ (\tau N(x) > 0), \text{ if } d = 1 \ \& \ e = 1,$$

$$(x < u) \ \& \ (\sigma M(x) > 0) \ \& \ (\tau N(x) > 0), \text{ if } d = 1 \ \& \ e > 1,$$

$$(r < x) \ \& \ (\sigma M(x) > 0) \ \& \ (\tau N(x) > 0), \text{ if } d > 1 \ \& \ e = 1,$$

$$(r < x < u) \ \& \ (\sigma M(x) > 0) \ \& \ (\tau N(x) > 0), \text{ if } d > 1 \ \& \ e > 1.$$

Consider now a 1-cell $c^1 = (-\infty,\beta)$. Then a derivative-based defining formula for c^1 is

$$x < \beta, \text{ if } e = 1,$$

$$(x < u) \ \& \ (\tau N > 0), \text{ if } e > 1.$$

The case $c^1 = (\alpha,\infty]$ is similar.

Definition. A $(B^1, \ldots, B^r)$ basis-determined cad D of E^r, $r \ge 1$, is ***derivative-***

regular in case either (1) r = 1, or (2) r > 1, D' is derivative-regular, and B^r is derivative-regular on each cell of D'.

Definition Let D' be a derivative-regular $(B^1, \ldots, B^{r-1})$ basis-determined cad of E^{r-1}, $r \geq 2$. Assume that derivative-based defining formulas for D' have been defined, let c be a particular cell of D', and let $\varphi_c(x_1, \ldots, x_{r-1})$ be a derivative-based defining formula for c. Let $F \in I_r$ be derivative-regular on c. For any cell d of S(F,c), we define the ***derivative-based defining formula of d*** as follows. Let $k = deg_c(F)$. If k=0, then a derivative-based defining formula for the unique sector Z(c) of S(F,c) is $\varphi_c \ \& \ (0 = 0)$. If k = 1, then S(F,c) consists of two sectors and one section, and F has opposite signs on the sectors. Let σ be the sign of F on the bottom sector. Then

$$\varphi_c \ \& \ (\sigma F > 0)$$

$$\varphi_c \ \& \ (F = 0)$$

$$\varphi_c \ \& \ (\sigma F < 0)$$

are derivative-based defining formulas for the cells of S(F,c), in ascending order. Suppose k > 1, and let G = $pp(F')$. Then (by Lemma 2.5.2) $deg_c(G) = k-1$. Suppose inductively that the derivative-based defining formulas for S(H,c) have been defined, for any primitive $H \in I_r$ such that H is derivative-regular on c and $deg_c(H) \leq k-1$. Then G is derivative-regular on c, so the derivative-based defining formulas for S(G,c) have been defined.

Let t be any section of S(F,c); let m(t) be the multiplicity of t. If m(t) > 1, then t is a section of S(G,c), and its derivative-based defining formula with respect to S(F,c) is its derivative-based defining formula with respect to S(G,c). If m(t) = 1, then there is a unique sector of S(G,c) which meets t; let $\varphi_c(x_1, \ldots, x_{r-1}) \ \& \ \psi(x_1, \ldots, x_r)$ be its derivative-based defining formula.

Then

$$\varphi_c \ \& \ \psi \ \& \ (F = 0)$$

is the derivative-based defining formula for t.

Let s be a sector of S(F,c). Let $v_1, \ldots, v_n$, $n \geq 1$, be the cells of S(G,c) which meet s, listed in ascending order. Let $\varphi_c(x_1, \ldots, x_{r-1}) \ \& \ \psi_i(x_1, \ldots, x_r)$ be the derivative-based defining formula for v_i, $1 \leq i \leq n$. Suppose there are sections t_1 and t_2 of S(F,c) below and above s. Then v_1 and v_n are sectors, and $s = (v_1 \cap s) \cup v_2 \cup \cdots \cup v_{n-1} \cup (v_n \cap s)$. By Rolle's Theorem, there is a section t' of S(G,c) such that $t_1 < t' < t_2$, hence $n \geq 3$. Let σ be the sign of F on s. If $m(t_1) > 1$, set $\psi_1^* = \psi_1$, else if $m(t_1) = 1$, set $\psi_1^* = \psi_1 \ \& \ (\sigma F > 0)$. Then ψ_1^* defines $v_1 \cap s$. Similarly, if $m(t_2) > 1$, set $\psi_n^* = \psi_n$, else set $\psi_n^* = \psi_n \ \& \ (\sigma F > 0)$. Then ψ_n^* defines $v_n \cap s$. Then the derivative-based defining formula for s is

$$\varphi_c \ \& \ (\ \psi_n^* \ \vee \ \psi_2 \ \vee \ \cdots \ \vee \ \psi_{n-1} \ \vee \ \psi_n^*\ ).$$

Suppose next that s is the top sector of S(F,c). Let t_1 be the section of S(F,c) directly below s. Then the derivative-based defining formula for s is

$$\varphi_c \ \& \ (\ \psi_1^* \ \vee \ \psi_2 \ \vee \ \cdots \ \vee \ \psi_{n-1} \ \vee \ \psi_n\ ).$$

Suppose that s is the bottom sector of S(F,c). Let t_2 be the section of S(F,c) directly above s. Then the derivative-based defining formula for s is

$$\varphi_c \ \& \ (\ \psi_1 \ \vee \ \psi_2 \ \vee \ \cdots \ \vee \ \psi_{n-1} \ \vee \ \psi_n^*\ ).$$

Note that for any $d \in S(F,c)$, the derivative-based defining formula for d has the form $\varphi_c(x_1, \ldots, x_{r-1}) \ \& \ \psi_r(x_1, \ldots, x_r)$, for some quantifier-free formula $\psi_r(x_1, \ldots, x_r)$. Hence the first conjunct of the derivative-based defining formula for d is the derivative-based defining formula for c, the

base of the stack S(F,c) in which d lies.

Definition. Let D be the derivative-regular $(B^1, \ldots, B^r)$ basis-determined cad of $E^r, r \geq 2$. For any cell d of D, a *derivative-based defining formula of d* is defined as follows. Assume that the derivative-based defining formulas of cells of D' have been defined. Let $c \in D'$, $d \in S_D(c)$, and let φ_c be the derivative-based defining formula for c. If $S_D(c) = \{Z(c)\}$, then $\varphi_c \ \& \ (0 = 0)$ is the derivative-based defining formula for the unique sector Z(c) of $S_D(c)$. Suppose $|S_D(c)| > 1$, and let t be a section of $S_D(c)$. Then t is a section of S(B,c) for some $B \in B^r$, and its derivative-based defining formula with respect to $S_D(c)$ is its derivative-based defining formula with respect to S(B,c). Let s be a sector of $S_D(c)$. If s is the top (bottom) sector of $S_D(c)$, then s is the top (bottom) sector of S(B,c) for some $B \in B^r$, and its derivative-based defining formula with respect to $S_D(c)$ is its derivative-based defining formula with respect to S(B,c). Otherwise, there are sections t_1 and t_2 of $S_D(c)$ directly below and directly above s, with $t_1 \in S(B_1,c)$ and $t_2 \in S(B_2,c)$ for some $B_1, B_2 \in B^r$. Let $\varphi_c \ \& \ \psi_1$ be the derivative-based defining formula for the sector of $S(B_1,c)$ directly above t_1, and let $\varphi_c \ \& \ \psi_2$ be the derivative-based defining formula for the sector of $S(B_2,c)$ directly below t_2. Then the derivative-based defining formula for s is

$$\varphi_c \ \& \ (\psi_1 \ \& \ \psi_2),$$

unless $B_1 = B_2$, in which case $\psi_1 = \psi_2$, and the derivative-based defining formula for s is just

$$\varphi_c \ \& \ \psi_1.$$

The original cad algorithm constructs derivative-based defining formulas. For further details on how this is done, see algorithm DEFINE of

[COL75]. When we want to construct defining formulas, we use APROJ instead of PROJ, as one can see in the abstract algorithm of Section 2.6.

2.6 The cad algorithm.

CAD(r,A,k;S,F)

[Cylindrical algebraic decomposition. A is a list of $n \geq 0$ integral polynomials in r variables, $r \geq 1$. k satisfies $0 \leq k \leq r$. S is a list of sample points for an A-invariant cad D of E^r. If $k \geq 1$, F is a list of defining formulas for the induced cad of E^k, and if k = 0, F is the empty list.]

(1) [Initialize.] Set $N \leftarrow CONT(A)$. Set $\bar{A} \leftarrow PP(A)$. Set $B \leftarrow$ the finest square-free basis for $\bar{A}$. Set $S \leftarrow ()$ and $F \leftarrow ()$.

(2) [r = 1.] If r > 1 then go to 3. Isolate the real roots of B. Construct a sample point for each cell of D and add it to S. If k = 1, then construct a defining formula for each cell of D and add it to F. Exit.

(3) [r>1.] if k = r then set $P \leftarrow APROJ(B)$ and $k' \leftarrow k-1$; otherwise set $P \leftarrow PROJ(B)$ and $k' \leftarrow k$. Set $P \leftarrow N \cup P$. Call the algorithm recursively with inputs r-1, P and k' to obtain outputs S' and F', which specify a cad D' of E^{r-1}. For each cell c of D' with sample point α, let $B^* = \{B_i(\alpha,x_r) \mid B_i \in B \ \ \& \ deg(B_i(\alpha,x_r)) > 0\}$. Isolate the real roots of B^*, and use the isolating intervals to construct sample points for the sections and sectors of $S(B_c,c)$, adding them to S. If k = r , then construct defining formulas for the sections and sectors of $S(B_c,c)$ (using F') and add them to F, otherwise set $F \leftarrow F'$. Exit ∎

CHAPTER 3

CELL BOUNDARIES AND ADJACENCY

3.1 Overview.

Let $X \subset T$, T a topological space. The *boundary* of X, written ∂X, is defined to be $\bar{X} - X$ ($\bar{X}$ denotes the closure of X). Where L(X) denotes the set of all limit points of X, one can easily show that $\partial X = L(X) - X$.

We remark that this is the usual definition of boundary in algebraic topology (see e.g. [MAS78], p. 135). In general topology, the "frontier" of X, written $fr(X)$, is defined to be $\bar{X} \cap \overline{-X}$, where -X denotes the complement of X (see e.g. [WID70],p. 28). The frontier is sometimes also called the boundary. It follows easily from the definitions that $\partial X = fr(X) - X$.

In this chapter we establish properties of the boundaries of cells of a cad. Our chief results, in Sections 3.2 - 3.4 and 3.6, are that certain kinds of cad's of E^1,E^2, and E^3 have what we call the "boundary property" (defined below). Many of the related properties of cell boundaries which we establish will be used in later chapters. Section 3.5 introduces the concept of adjacency.

Definition. A cad D of E^r, $r \geq 1$ has the *boundary property* if

1) For any cell c of D, there is a (possibly empty) collection of cells of D whose union is ∂c.

2) If r>1, then D' has the boundary property.

If we replace ∂c by $fr(c)$ in this definition, we obtain what might be called the frontier property. But it then follows easily that a cad has the boundary property if and only if it has the frontier property.

Theorem 3.1.1. Let D be a cad of E^r such that if r > 1, then D' has the boundary property. Then D has the boundary property if and only if for all cells c,d of D, if $c \cap \partial d \neq \emptyset$, then $c \subset \partial d$.

Proof. If D has the boundary property, then ∂d is the union of certain cells of D. Hence if $c \cap \partial d \neq \emptyset$, then there is a cell e contained in ∂d, such that $c \cap e \neq \emptyset$. Hence c = e, by disjointess of cells, and so $c \subset \partial d$. Suppose for all cells c,d of D that if $c \cap \partial d \neq \emptyset$, then $c \subset \partial d$. Then for any cell c of D containing a point of ∂d, $c \subset \partial d$. Therefore the union of all cells which contain a point of ∂d is ∂d. Therefore by the hypothesis that if r > 1 then D' has the boundary property, it follows that D has the boundary property. ∎

3.2 Cell boundaries in 1-space.

Theorem 3.2.1. Every r-cell in E^r, $r \geq 1$, is an open subset of E^r.

Proof. Let c be an r-cell in E^r. By definition of r-cell, c is homeomorphic to E^r. The principle of invariance of domain ([WID70]) states that any subset of E^r homeomorphic to an open subset of E^r is itself open. Thus since c is homeomorphic to E^r, and E^r is open, c is open. ∎

Theorem 3.2.2. Let D be a cad of E^1, and let c,d be cells of D. Then $c \cap \partial d \neq \emptyset$ if and only if c is a 0-cell, d is a 1-cell, and $c \subset \partial d$.

Proof. If $c \cap \partial d \neq \emptyset$ then $c \neq d$, and hence c and d are disjoint. Any 0-cell in E^1 is a one-point cell and hence its boundary is empty. Hence if $c \cap \partial d \neq \emptyset$,

then d is a 1-cell of D, and is therefore an open interval. Therefore, the boundary of d consists of the endpoints, if any, of the interval. If such an endpoint belongs to $c \cap \partial d$, then since c and d are disjoint, c must be the 0-cell consisting of just this one endpoint, and hence $c \subset \partial d$. The converse is obvious. ▪

Corollary 3.2.3. Every cad of E^1 has the boundary property.

Proof. An immediate consequence of Theorems 3.2.2 and 3.1.1. ▪

3.3 Cell boundaries in 2-space.

Definition. Let Q,R be regions in E^{r-1}, $r \geq 2$. Let S(Q), S(R) be stacks over Q,R. S(R) *has the boundary property in S(Q)* if for every element s of S(R), there is a (possibly empty) collection of elements of S(Q) whose union is $\partial s \cap Z(Q)$. ▪

Notation. We denote the usual two-point compactification of E by E^*. Thus $E^* = E \cup \{-\infty, \infty\}$. For $r \geq 2$, and any subset X of E^{r-1}, $Z^*(X)$ denotes $X \times E^*$.

Definition. Let R be a region in E^{r-1}, $r \geq 2$. *The* $-\infty$*-section of* $Z^*(R)$ is the graph of the constant function $g(x) = -\infty$ for all x in R. *The* $+\infty$*-section of* $Z^*(R)$ is the graph of the constant function $h(x) = +\infty$ for all x in R. An ∞*-section of* $Z^*(R)$ is either the $+\infty$-section or the $-\infty$-section of $Z^*(R)$. By a *section of* $Z^*(R)$ we shall mean either a section of Z(R) or an ∞-section of $Z^*(R)$.

Definition. Let R be a region in E^{r-1}, $r \geq 2$, and let S be a stack over R. The *extended stack over R*, written S^*, is the union of S and the ∞-sections of

$Z^*(R)$. If S = $S(f_1, \ldots, f_m, R)$, $m \geq 0$, we let f_0 denote the constant function $-\infty$ on R, and f_{m+1} the constant function $+\infty$ on R.

Definition. Let R be a region in E^{r-1}, $r \geq 2$, and let $S = S(f_1, \ldots, f_m, R)$, $m \geq 0$, be a stack over R. For $0 \leq i \leq m$, the *ith sector of* S^* is the (f_i, f_{i+1})–sector of $Z^*(R)$. For $0 \leq j \leq m+1$, the *jth section of* S^* is the f_j - section of $Z^*(R)$. For $0 \leq i \leq m$, the *ith sector of S* is the *ith* sector of S^*, and for $1 \leq j \leq m$, the *jth section of S* is the *jth* section of S^*.

Lemma 3.3.1. Let R be a region in E^{r-1}, $r \geq 2$. Then $\overline{Z^*(R)} = Z^*(\bar{R})$.

Proof. Elementary. ▪

Corollary 3.3.2. Let R be a region in E^{r-1}, $r \geq 2$, and let S(R) be a stack over R. For any region s of $S^*(R)$, $\partial s \subset Z^*(\bar{R})$.

Proof. $\partial s \subset \bar{s} \subset \overline{Z^*(R)} \subset Z^*(\bar{R})$ by Lemma 3.3.1. ▪

Lemma 3.3.3. Let R be a region in E^{r-1}, $r \geq 2$, and let t be an ∞-section of $Z^*(R)$. If t is a $+\infty$-section, then $\partial t \cap Z^*(\bar{R})$ is $\partial R \times \{+\infty\}$. If t is a $-\infty$-section, then $\partial t \cap Z^*(\bar{R})$ is $\partial R \times \{-\infty\}$.

Proof. Obvious. ▪

Lemma 3.3.4. Let Q,R be regions in E^{r-1}, $r \geq 2$, such that $Q \subset \partial R$. Let t_0 be the $-\infty$-section, and t_1 the $+\infty$-section, of $Z^*(R)$. Then $\partial t_0 \cap Z^*(Q)$ is the $-\infty$-section of $Z^*(Q)$, and $\partial t_1 \cap Z^*(Q)$ is the $+\infty$-section of $Z^*(Q)$.

Proof. Obvious. ▪

Lemma 3.3.5. Let R be a region in E^{r-1}, $r \geq 2$, and let s be an f-section of Z(c). Let $U_1 = \{<a,b> \mid a \in R \ \& \ b < f(a)\}$, and

$U_2 = \{<a,b> | a \in R \ \& \ b > f(a)\}$. Then U_1 and U_2 are open in $Z^*(R)$.

Proof. Consider any $<a,b> \in U_1$. Choose $c \in E$ such that $b < c < f(a)$. Let $V_1 = \{x | x \in R \ \& \ f(x) > c\}$. Let $W_1 = \{y | y \in E \ \& \ y > c\}$. Then $V_1 = f^{-1}(W_1)$, hence since W_1 is open in E^* and f is continuous, V_1 is open in R. Furthermore, V_1 contains a. Then $Y_1 = V_1 \times \{y | y \in E \ \& \ y < c\}$ is an open set in $Z^*(R)$ such that $<a,b> \in Y_1 \subset U_1$. Hence U_1 is open in $Z^*(R)$. The proof for U_2 is similar. ▪

Corollary 3.3.6. Assume the hypotheses of Lemma 3.3.5. Then $s = -U_1 \cap -U_2$ is closed in $Z^*(R)$.

Proof. Since U_1 and U_2 are open in $Z^*(R)$, $-U_1$ and $-U_2$ are closed in $Z^*(R)$. Since the intersection of two closed sets is closed, $s = -U_1 \cap -U_2$ is closed in $Z^*(R)$. ▪

Theorem 3.3.7. Let R be a region in E^{r-1}, $r \geq 2$, and let S be a stack over R. Let s be any section of S^*. Then $\partial s \cap Z^*(R) = \emptyset$.

Proof. If s is a section of S, then by Corollary 3.3.6, s is closed in $Z^*(R)$, hence $\partial s \cap Z^*(R) = \emptyset$. If s is an ∞-section of S^*, then $\partial s \cap Z^*(R) = \emptyset$ by Lemma 3.3.3. ▪

Theorem 3.3.8. Let R be a region in E^{r-1}, $r \geq 2$, and let $S = S(f_1, \ldots, f_m, R)$, $m \geq 0$, be a stack over R. Where s_i is the ith sector of S^*, $0 \leq i \leq m$, and t_j is the jth section of S^*, $0 \leq j \leq m+1$,

$$\partial s_i \cap Z^*(R) = t_i \cup t_{i+1}.$$

Proof. For any i, $0 \leq i \leq m$, and any $\gamma \in R$, any open neighborhood of $<\gamma, f_i(\gamma)>$ contains a point $<\gamma, b>$ with $f_i(\gamma) < b < f_{i+1}(\gamma)$, i.e. $<\gamma, b> \in s_i$. Hence $<\gamma, f_i(\gamma)>$ is a limit point of s_i for any $\gamma \in R$, hence $t_i \subset \partial s_i$. By a

similar argument, $t_{i+1} \subset \partial s_i$.

Let $U_1 = \{ <\gamma,b> | \gamma \in R, b < f_i(\gamma)\}$, and $U_2 = \{ <\gamma,b> | \gamma \in R, b > f_{i+1}(\gamma)\}$. By Lemma 3.3.5, U_1 and U_2 are open in $Z^*(R)$. Hence $-U_1 \cap -U_2 = s_i \cup t_i \cup t_{i+1}$ is closed in $Z^*(R)$. Hence $\partial s_i \cap Z^*(R) = t_i \cup t_{i+1}$. ▪

Corollary 3.3.9. Let R be a region in E^{r-1}, $r \geq 2$. Any stack over R has the boundary property in itself.

Proof. Follows immediately from Theorem 3.3.7 and Theorem 3.3.8. ▪

Theorem 3.3.10. Let D be a cad of E^r, $r \geq 2$, such that D' has the boundary property. D has the boundary property iff for every pair of cells c,d in D' with $c \subset \partial d$, $S_D(d)$ has the boundary property in $S_D(c)$.

Proof. Suppose D has the boundary property. Let c,d be cells of D' with $c \subset \partial d$. Let s be any cell of $S_D(d)$. Then ∂s is the union of cells $e_1, \ldots, e_k$ of D. But then $\partial s \cap Z(c)$ is the union of those e_i which are contained in Z(c), since any other e_i is disjoint from Z(c). So $S_D(d)$ has the boundary property in $S_D(c)$. Assume the hypotheses for the converse. By Corollary 3.3.2,

$$\partial s = [\partial s \cap Z(d)] \cup \bigcup_{c \in \partial d} [\partial s \cap Z(c)].$$

By Corollary 3.3.9, there is a collection of cells of $S_D(d)$ whose union is $\partial s \cap Z(d)$, and by hypothesis, for each $c \in \partial d$, there is a collection of cells of $S_D(c)$ whose union is $\partial s \cap Z(c)$. Hence there is a collection of cells of D whose union is ∂s. Hence D has the boundary property. ▪

Definition. Let Q,R be regions in E^{r-1}, $r \geq 2$, such that $Q \subset \partial R$. Let S(Q), S(R) be stacks over Q,R. *$S^*(R)$ has the section (sector) boundary property in $S^*(Q)$* if for every section (sector) s of $S^*(R)$, $\partial s \cap Z^*(Q)$ is the union of a finite number of regions of $S^*(Q)$. *$S^*(R)$ has the unique section boundary*

property in $S^*(Q)$ if for every section s of $S^*(R)$, $\partial s \cap Z^*(Q)$ is a section of $S^*(Q)$. In this case, $\partial s \cap Z^*(Q)$ is called the *boundary section of s in* $S^*(Q)$.

Notation. Let R be a region in E^{r-1}, $r \geq 2$, and let S(R) = $S(f_1, \ldots, f_m, R)$, $m \geq 0$, be a stack over R. Let t_i denote the f_i-section of S(R), $0 \leq i \leq m+1$. Then for any $i, j, 0 \leq i, j \leq m$,

$$[t_i, t_j] = \{ < \alpha, \beta > \in Z^*(R) \mid \alpha \in R,\ f_i(\alpha) \leq \beta \leq f_j(\alpha) \}.$$

(t_i, t_j) is defined similarly.

Notation. Let R be a region in E^{r-1}, $r \geq 2$, and let $S = S(f_1, \ldots, f_m, R)$, $m \geq 0$, be a stack over R. For $0 \leq k \leq 2m+2$, let S_k^* denote the $(k/2)^{th}$ section of S^* if k is even, and the $((k-1)/2)^{th}$ sector of S^* if k is odd. For $1 \leq k \leq 2m+1$, let S_k denote the same region of S as denoted by S_k^*. k is the *index* of S_k or S_k^*.

Lemma 3.3.11. Let R_1, R_2, and R_3 be mutually disjoint regions in E^r, $r \geq 1$, such that $R_1 \subset \partial R_2$ and $R_2 \subset \partial R_3$. Then $R_1 \subset \partial R_3$.

Proof. By hypothesis, $R_1 \subset \overline{R_2}$, and $R_2 \subset \overline{R_3}$. Hence since $X \subset Y$ implies $\overline{X} \subset \overline{Y}$, and $\overline{\overline{X}} = \overline{X}$, for any subsets X and Y of E^r, we have $R_1 \subset \overline{R_3}$. Hence since R_1 and R_3 are disjoint, $R_1 \subset \partial R_3$. ∎

Definition. Let X be a subset of E^{r-1}, $r \geq 2$, and let α be a point of ∂X. Let $f : X \to E^*$ be continuous. Then we say

$$\lim_{x \to \alpha \text{ in } X} f(x) = b$$

if for every open interval I in E^* such that $b \in I$, there exists an open ball B in E^{r-1} centered at α such that $f(B \cap X) \subset I$.

Lemma 3.3.12. Let R be a region in E^{r-1}, $r \geq 2$, suppose S(R) is a stack over R, and let s be an f-section of S(R). For $\alpha \in \partial R$ and $b \in E^*$, $< \alpha, b >$ is the unique limit point of s in $Z^*(\alpha)$ if and only if $\lim_{x \to \alpha \text{ in } R} f(x) = b$.

Proof. Suppose $< \alpha, b >$ is the unique limit point of s in $Z^*(\alpha)$, but $\lim_{x \to \alpha \text{ in } R} f(x) \neq b$. Then there is an open interval I in E^* containing b, such that for any $\delta > 0$, where $B_\delta(\alpha)$ is the ball of radius δ in E^{r-1} centered at α, we have $f(B_\delta(\alpha) \cap R) \not\subset I$. Setting $\delta_i = 1/2^i$, and choosing x_i in $B_{\delta_i}(\alpha) \cap R$ so that $f(x) \notin I$, we obtain a sequence of points in R which converges to α. Let $V = B_1(\alpha)$. $\bar{V} \times E^*$ is compact, so the sequence $< x_i, f(x_i) >$ contained in $\bar{V} \times E^*$ has a convergent subsequence, and its limit has the form $< \alpha, z >$, $z \in E^*$. Since $f(x_i) \notin I$ for all i, $z \notin I$. But since $< x_i, f(x_i) > \in s$ for each i, $< \alpha, z >$ is a second limit point of s in $Z^*(\alpha)$, contradicting the hypothesis that $< \alpha, b >$ is the unique limit point of s in $Z(\alpha)$. Hence $\lim_{x \to \alpha \text{ in } R} f(x) = b$.

Assume $\lim_{x \to \alpha \text{ in } R} f(x) = b$. Let U be any open neighborhood of $< \alpha, b >$. Then U contains an open set of the form $B \times I$, such that B is an open ball in E^{r-1} centered at α and I is an open interval in E^*. By definition of $\lim_{x \to \alpha \text{ in } R} f(x) = b$, there exists an open ball B' centered at α, with $B' \subset B$, such that $f(B' \cap R) \subset I$. Since $\alpha \in \partial R$, α is a limit point of R, hence there exists $u \in B' \cap R$. Then $< u, f(u) > \in B' \times I \subset U$. Hence since $< u, f(u) > \in s$, $< \alpha, b >$ is a limit point of s. The proof that $< \alpha, b >$ is unique is straightforward. ∎

Lemma 3.3.13. Let Q,R be regions in E^{r-1}, $r \geq 2$, such that $Q \subset \partial R$. Let S(Q) and $S(R) = S(f_1, \ldots, f_m, R)$, $m \geq 0$, be stacks over Q,R such that $S^*(R)$

has the unique section boundary property in $S^*(Q)$. For $0 \le j \le m+1$, let z_j be the boundary section in $S^*(Q)$ of the j^{th} section of $S^*(R)$, and suppose z_j is a g_j-section. Then for $0 \le i < j \le m+1$, either $g_i = g_j$ or $g_i < g_j$.

Proof. Suppose for some i and j, $0 \le i < j \le m+1$, that $g_i > g_j$. Then there exist disjoint open intervals I_i, $I_j \subset E^*$ such that $g_i(\alpha) \in I_i$, $g_j(\alpha) \in I_j$, and $I_i > I_j$. By Lemma 3.3.12, $\lim_{x \to a \text{ in } R} f_i(x) = g_i(\alpha)$, and $\lim_{x \to a \text{ in } R} f_j(x) = g_j(\alpha)$. Hence there exist balls B_i, B_j centered at α such that $f_i(B_i \cap R) \subset I_i$ and $f_j(B_j \cap R) \subset I_j$. Let $B = B_i \cap B_j$. $B \cap R \neq \emptyset$, since $\alpha \in \partial R$ and so is a limit point of R. Thus there exists $\gamma \in B \cap R$ such that $f_i(\gamma) > f_j(\gamma)$, a contradiction. Hence $g_i \le g_j$. ∎

Theorem 3.3.14. Let Q,R be regions in E^{r-1}, $r \ge 2$, such that $Q \subset \partial R$. Let S(Q), S(R) = $S(f_1, \ldots, f_m, R)$, $m \ge 0$ be stacks over Q,R such that $S^*(R)$ has the unique section boundary property in $S^*(Q)$. For $0 \le i \le m$, let s_i be the ith sector of S(R). For $0 \le j \le m+1$, let t_j be the f_j-section of $S^*(R)$, and z_j the boundary section of t_j in $S^*(Q)$. Then for $0 \le i \le m$,

$$\partial s_i \cap Z^*(Q) = [z_i, z_{i+1}].$$

Proof. For $0 \le i \le m+1$, $z_i \subset \partial t_i$ by definition of boundary section. Choose a particular i, $0 \le i \le m$. By Theorem 3.3.8, $t_i \subset \partial s_i$, and so by Lemma 3.3.11, $z_i \subset \partial s_i$. Similarly, $t_{i+1} \subset \partial s_i$, and so $z_{i+1} \subset \partial s_i$.

Suppose z_i is a g_i-section of $S^*(Q)$. By Lemma 3.3.13, $g_i \le g_{i+1}$. If $g_i = g_{i+1}$ we are done, so suppose $g_i < g_{i+1}$. Let $< \alpha, b > \in Z(Q)$ with $g_i(\alpha) < b < g_{i+1}(\alpha)$. By Lemma 3.3.12, $\lim_{x \to a \text{ in } R} f_i(x) = g_i(\alpha)$, and $\lim_{x \to a \text{ in } R} f_{i+1}(x) = g_{i+1}(\alpha)$. Let $x_1, x_2, \ldots,$ be a sequence of points in R converging to α. The sequence $< x_1, b >$, $< x_2, b >, \cdots,$ converges to $< \alpha, b >$,

and since for sufficiently large j, $< x_j, b > \in s_i$, $< \alpha, b >$ is a limit point of s_i. Thus $[z_i, z_{i+1}] \subset \partial s_i$.

Suppose that $< \alpha, b > \in Z^*(Q)$ is a limit point of s_i. Then there exists a sequence of points $< y_1, b_1 >, < y_2, b_2 >, \cdots$ in s_i converging to $< \alpha, b >$. For each b_j we have $f_i(y_j) < b_j < f_{i+1}(y_j)$. The sequences $< y_j, f_i(y_j) >$ and $< y_j, f_{i+1}(y_j) >$ converge to $< \alpha, g_i(\alpha) >$ and $< \alpha, g_{i+1}(\alpha) >$ respectively, and so we must have $g_i(\alpha) \leq b \leq g_{i+1}(\alpha)$, and hence $< \alpha, b > \in [z_i, z_{i+1}]$. Hence $\partial s_i \cap Z^*(Q) = [z_i, z_{i+1}]$. ∎

Theorem 3.3.15. Let Q,R be regions in E^{r-1}, $r \geq 2$, such that $Q \subset \partial R$. Let S(Q), S(R) be stacks over Q,R such that $S^*(R)$ has the unique section boundary property in $S^*(Q)$. Then S(R) has the boundary property in S(Q).

Proof. By the unique section boundary property, for any section t of S(R), $\partial t \cap Z^*(Q)$ is either a section of Z(Q) or an ∞-section of $Z^*(Q)$. In the first case, $\partial t \cap Z(Q)$ is the union of one cell of S(Q), and in the second, $\partial t \cap Z(Q)$ is the union of no cells of S(Q). Let S(R) = $S(f_1, \ldots, f_m, R)$, $m \geq 0$. If m=0, then the unique sector s of S(R) is Z(R), hence by Lemma 3.3.1, $\partial s \cap Z(Q) = Z(Q)$, and since Z(Q) = $\bigcup S(Q)$, $\partial s \cap Z(Q)$ is the union of regions of S(Q). Suppose $m \geq 1$. Let s_i be the i^{th} sector of S(R) for $0 \leq i \leq m$. Let t_i be the i^{th} section of $S(R)$, $0 \leq i \leq m+1$. Let z_i be the boundary section of t_i in $S^*(Q)$. By Theorem 3.3.14, $\partial s_i \cap Z^*(Q) = [z_i, z_{i+1}]$. Clearly $[z_i, z_{i+1}]$ is the union of a finite set of regions of $S^*(Q)$. Deleting any ∞-sections from this set, we obtain a set of regions of S(Q) whose union is $\partial s_i \cap Z(Q)$. ∎

Lemma 3.3.16. Let R be a pathwise connected region in E^{r-1}, $r \geq 2$. Then

any section of $Z^*(R)$ is pathwise connected, and any sector of $Z^*(R)$ is pathwise connected.

Proof. Let s be an f-section of $Z^*(R)$. Let $g:R\to R\times E^*$ be defined by $g(x) = \langle I(x), f(x)\rangle$, where I is the identity map on R. g is continuous (by a standard theorem) since I and f are both continuous. g maps R onto s, so s is the continuous image of a pathwise connected set, and hence (by a standard theorem) is itself pathwise connected. Suppose s is an (f,g)–sector of Z(R), and that both f and g take on only finite values. Let $\langle \alpha_1,\beta_1 \rangle$ and $\langle \alpha_2,\beta_2 \rangle$ be points of s with $\alpha_1,\alpha_2\in R$. Let $P:[0,1]\to R$ be a path in R from α_1 to α_2. Let h be the function $(f + g)/2$. Then the map $Q:[0,1]\to s$ defined by $Q(t) = \langle P(t), h(P(t))\rangle$ is a path from $\langle \alpha_1, h(\alpha_1)\rangle$ to $\langle \alpha_2, h(\alpha_2)\rangle$ in s. Joining $\langle \alpha_1,\beta_1 \rangle$ to $\langle \alpha_1, h(\alpha_1)\rangle$, and $\langle \alpha_2,\beta_2 \rangle$ to $\langle \alpha_2, h(\alpha_2)\rangle$, with vertical line segments contained in s, we obtain a path from $\langle \alpha_1,\beta_1 \rangle$ to $\langle \alpha_2,\beta_2 \rangle$ in s. Hence s is pathwise connected. If $graph(f)$ is a finite section of $Z^*(R)$ and $graph(g)$ is the $+\infty$-section of $Z^*(R)$, then, replacing $h(x)$ by the function $f(x) + 1$, we repeat the argument given above. If $graph(f)$ is the $-\infty$-section of $Z^*(R)$ and $graph(g)$ a finite section, then we may use $h(x) = g(x) - 1$. If $graph(f)$ is the $-\infty$-section and $graph(g)$ the $+\infty$-section of $Z^*(R)$, then we may use $h(x) = 0$. ∎

Theorem 3.3.17. Let $A_1, \ldots, A_k$, $k \geq 1$, be non-zero elements of I_2. There are infinitely many pairs $\langle \alpha_1,\alpha_2 \rangle$ of complex numbers which are zeros of A_i, for $1 \leq i \leq k$, if and only if $gcd(A_1, \ldots, A_k)$ has positive degree in either x_1 or x_2.

Proof. Let $B = gcd(A_1, \ldots, A_k)$. If B has positive degree in either x_1 or x_2 then it is obvious that $A_1, \ldots, A_k$ have infinitely many common complex

zeros. Assume that $A_1, \ldots, A_k$ have infinitely many common complex zeros. We will prove by induction on k that B has positive degree in x_1 or x_2. For k = 1 this is obvious. Let $k \geq 2$ and assume it true for k - 1. Let $B' = gcd(A_1, \ldots, A_{k-1})$. $A_1, \ldots, A_{k-1}$ have infinitely many common zeros, so by the inductive hypothesis, B' has positive degree in x_1 or x_2. If the degree of B' in x_2 is zero, then there is a root α_1 of $B'(x_1)$ such that $A_k(\alpha_1,\alpha_2) = 0$ for infinitely many α_2. Hence $(x_1 - \alpha_1)$ divides $A_k(x_1,x_2)$, and so B has positive degree in x_1. Symmetrically, if the degree of A_k in x_2 is zero, then the degree of B in x_1 is positive. So assume that B' and A_k both have positive degree in x_2. Let $R(x_1)$ be their resultant with respect to x_2. For every common zero (α_1,α_2) of B' and A_k, we have $R(\alpha_1) = 0$. If $R(x_1) = 0$, then B has positive degree in x_2 by Theorem 2 of [COL71]. If $R(x_1) \neq 0$, then there are only finitely many α_1 such that, for some α_2, $< \alpha_1,\alpha_2 >$ is a common zero of B' and A_k. So for some α_1, there are infinitely many α_2 for which $< \alpha_1,\alpha_2 >$ is a common zero. But then $(x_1 - \alpha_1)$ divides $B(x_1,x_2)$, so B has positive degree in x_1. This completes the induction ∎

Definition. Let c be a 1-cell in E^1, and let $f : c \to E^1$ be continuous. f is *piecewise monotone* if there is a finite partition $P = \{P_1, \ldots, P_n\}$, $n \geq 1$, of c such that for each i, $1 \leq i \leq n$, f restricted to P_i is monotone.

Theorem 3.3.18. Let c be a 1-cell in E^1, and let F(x,y) be a primitive element of I_2 of positive degree. Let f be a continuous function $c \to E$, such that $F(x,f(x)) = 0$ for all $x \in c$. Then f is piecewise monotone.

Proof. Let $G(x,y), H(x,y) \in I_2$ be distinct, i.e. relatively prime, irreducible factors of F. By Theorem 3.3.17, there are only finitely many $\alpha \in c$ such that $G(\alpha,f(\alpha)) = H(\alpha,f(\alpha)) = 0$. Let G be any irreducible factor of F. Since F is

primitive, G has positive degree in y, hence $G' \neq 0$. Thus by Theorem 3.3.17, there are only finitely many $\alpha \in c$ such that $G(\alpha, f(\alpha)) = G'(\alpha, f(\alpha)) = 0$. For any $K \in I_2$, let K_x denote $\partial K / \partial x$. Let G be any irreducible factor of F of positive degree in x. Then $G_x \neq 0$, and so by Theorem 3.3.17, there are only finitely many $\alpha \in c$ such that $G(\alpha, f(\alpha)) = G_x(\alpha, f(\alpha)) = 0$. Let $\alpha_1 < \cdots < \alpha_n$, $n \geq 0$, be all the $\alpha \in c$ satisfying one of the three conditions above. Let c be the open interval (α_0, α_{n+1}). Let P be the partition $\{(\alpha_0, \alpha_1), \{\alpha_1\}, (\alpha_1, \alpha_2), \ldots, \{\alpha_n\}, (\alpha_n, \alpha_{n+1})\}$ of c. Consider any open interval $I = (\alpha_i, \alpha_{i+1})$ in P. There is exactly one (distinct) irreducible factor G of F such that $G(x, f(x)) = 0$ for all $x \in I$. Furthermore, $G'(x, f(x)) \neq 0$ for $x \in I$, hence by the Implicit Function Theorem, f is differentiable on I and $f'(x) = -G_x(x, f(x)) / G'(x, f(x))$. Either $G_x(x, f(x)) \neq 0$ for all $x \in I$, or $G_x(x, f(x)) = 0$ for all $x \in I$. Since I is connected, by a standard theorem, the graph of f restricted to I is connected, hence by an argument similar to that used in the proof of Lemma 2.2.4, $G'(x, f(x))$ and $G_x(x, f(x))$ are sign-invariant on I. Hence f' is sign-invariant on I, hence f is monotone on I. Since f is trivially monotone on each $\{\alpha_i\}$, f is monotone on each element of P, hence f is piecewise monotone. ▪

Lemma 3.3.19. Let $A \in I_r$, $r \geq 2$, let R be a region in E^{r-1}, and let S(R) be a stack over R. If A vanishes on a sector of S(R) then A vanishes on Z(R).

Proof. For every α in R, $A(\alpha, x_r)$ has infinitely many roots, hence is the zero polynomial. ▪

Lemma 3.3.20. Let $A \in I_2$ be primitive and of positive degree. There is no $\alpha \in E^1$ such that A is cylindrical at α.

Proof. Let

$$A(x,y) = \sum_{i=0}^{m} A_i(x)y^i \ , \ m \geq 1.$$

Suppose A is cylindrical at $\alpha \in E^1$. Then $A_i(\alpha) = 0$ for $0 \leq i \leq m$. Then α is an algebraic number, and where $M(x)$ is its integral minimal polynomial, M(x) divides $A_i(x)$ for $0 \leq i \leq m$, contradicting the primitivity of A(x,y). ∎

Corollary 3.3.21. Let $A \in I_2$ be primitive and of positive degree, let R be a region in E^1, and let S(R) be a stack over R. There is no sector of S(R) on which A vanishes identically.

Proof. Let s be a sector of S(R). If A vanishes on s, then A vanishes on Z(R) by Lemma 3.3.19, hence A is cylindrical at each $\alpha \in R$, contrary to Lemma 3.3.20. Hence A does not vanish on s. ∎

Lemma 3.3.22. Let R be a semi-algebraic region in E^{r-1}, $r \geq 2$, and let t be a semi-algebraic section of Z(R). There exists $F \in I_r$ such that F is primitive, F has positive degree, and $t \subset V(F)$.

Proof. Let $\varphi(x_1, \ldots, x_r)$ be a standard quantifier-free defining formula for t. Let $G_1, \ldots, G_n \in I_r$, $n \geq 1$, denote the nonzero polynomials occurring on the lefthand sides of the standard atomic formulas of φ. Let $F_i = pp(G_i)$, $1 \leq i \leq n$. Let $F = \prod_{i=1}^{n} F_i$. Note that if $deg(F) = 0$, then $F = 1 \in I_r$ and so $V(F) = \emptyset$. Suppose there exist $\alpha \in c$ and $b \in E$ such that $<\alpha,b> \in t$ but $F(\alpha,b) \neq 0$. Then $F_i(\alpha,b) \neq 0$ for $1 \leq i \leq n$. Since each F_i is a continuous function $E^r \rightarrow E$, there exists $b' \neq b$ such that $sign(F_i(\alpha,b')) = sign(F_i(\alpha,b))$ for $1 \leq i \leq n$. Then $<\alpha,b'>$ satisfies φ, hence $<\alpha,b'> \in t$. But this is a contradiction, since $<\alpha,b'> \notin graph(f)$, where t is an f-section. Hence $t \subset V(F)$. Hence $V(F) \neq \emptyset$, hence F has positive degree. ∎

Lemma 3.3.23. Let c be a semi-algebraic 1-cell in E^1, let $\alpha \in \partial c$, and let t be

a semi-algebraic section of Z(c). Then t has a unique limit point in $Z^*(\alpha)$.

Proof. By Lemma 3.3.22, there exists $F \in I_2$ such that $t \in V(F)$, F is primitive, and F has positive degree. Hence where t is an f-section, by Theorem 3.3.18, f is piecewise monotone. Then where c is an open interval $I \subset E$, there is an open subinterval J of I such that α is an endpoint of J and f is monotone on J. Then by a standard theorem, f has a unique limit point in $Z^*(\alpha)$. ▪

Theorem 3.3.24. Let D be a (B^1,B^2) basis-determined cad of E^2. Let c^0,c^1 be a 0-cell and a 1-cell of D' such that $c^0 \subset \partial c^1$. Then $S_D^*(c^1)$ has the unique section boundary property in $S_D^*(c^0)$.

Proof. Let t be any 1-section of $S_D(c^1)$, and suppose $\partial t \cap Z(c^0) \neq \emptyset$, i.e. t has a limit point $<a,b>$ in $Z(c^0)$, where $c^0 = a$. Since D is (B^1,B^2) basis-determined, some $B \in B^2$ vanishes on t, hence since V(B) is closed, B vanishes at $<a,b>$. Let d be the cell of $S_D(c^0)$ containing $<a,b>$. Since D is (B^1,B^2) basis-determined, it is B-invariant, hence B vanishes on d. By Corollary 3.3.21, d cannot be a 1-sector of $S_D(c^0)$. Hence d must be a 0-section of $S_D(c^0)$, i.e. any limit point of t in $Z(c^0)$ is a section of $S_D(c^0)$. By Lemma 3.3.23, t has a unique limit point in $Z^*(c^0)$; by our argument above, this limit point is a section of $S_D^*(c^0)$. By Lemma 3.3.4, if t is an ∞-section of $S_D^*(c^1)$, then $\partial t \cap Z^*(c^0)$ is an ∞-section of $S_D^*(c^0)$. Hence $S_D^*(c^1)$ has the unique section boundary property in $S_D^*(c^0)$. ▪

Corollary 3.3.25. Let D be a (B^1,B^2) basis-determined cad of E^2. D has the boundary property.

Proof. By Corollary 3.2.3, D' has the boundary property. Hence by Theorem 3.2.2, for any cells c,d of D' with $c \subset \partial d$, c is a 0-cell and d a 1-cell. Then by

Theorem 3.3.24, $S_D^*(d)$ has the unique section boundary property in $S_D^*(c)$, hence by Theorem 3.3.15, $S_D(d)$ has the boundary property in $S_D(c)$, hence by Theorem 3.3.10, D has the boundary property. ▪

The following theorem will be used in Section 3.5.

Theorem 3.3.26. Let D be a cad of E^2, and let $s \in D$ be a 2-sector. Suppose $s \in S_D(c)$, c a 1-cell in D', and let $\alpha \in \partial c$. Where $s = (t_1, t_2)$, for sections t_1, t_2 of $S_D^*(c)$, let z_i be the unique limit point of t_i in $Z^*(\alpha)$. Then

$$\partial s \cap Z^*(\alpha) = [z_1, z_2].$$

Proof. By Lemma 3.3.23, every section of $S_D(c)$ has a unique limit point in $Z^*(\alpha)$. Let the stack S be a refinement of $S_D(\alpha)$ defined by the requirement that each limit point in $Z(\alpha)$ of a section of $S_D(c)$ be a section of S. Then $S_D^*(c)$ has the unique section boundary property in S^*, hence by Theorem 3.3.14, $\partial s \cap Z^*(\alpha) = [z_1, z_2]$. ▪

Lemma 3.3.27. Let c be an open interval in E^1, and let $\alpha \in \partial c \cap E$. Let $< \alpha, \beta > \in Z^*(\alpha)$, and let U be any open neighborhood in $E \times E^*$ of $< \alpha, \beta >$. Then there exist open intervals $I \subset E$ and $J \subset E^*$ such that $< \alpha, \beta > \in I \times J \subset U$, and $I \cap c$ is an open interval.

Proof. The open sets in $E \times E^*$ of the form $M \times N$, M an open interval in E, N an open interval in E^*, are a basis for the topology on $E \times E^*$. Hence there exists some such basis element $I \times J$ with $< \alpha, \beta > \in I \times J \subset U$. Clearly $I \cap c$ is an open interval. ▪

Theorem 3.3.28. Let c be a cell in a cad of E^1, and let $\alpha \in \partial c \cap E$. Let s be a section or sector of Z(c). Let $< \alpha, \beta_1 >$ and $< \alpha, \beta_2 > \in \partial s \cap Z^*(\alpha)$. Let $\beta_1 < \beta < \beta_2$. Then $< \alpha, \beta > \in \partial s$.

Proof. By Theorem 3.2.2, if c is a 0-cell, then $\partial c = \emptyset$, so we may assume that c is a 1-cell. Let U be any open neighborhood in $E \times E^*$ of $< \alpha,\beta >$. By Lemma 3.3.27, there exist open intervals $I \subset E$ and $J \subset E^*$ such that $< \alpha,\beta > \in I \times J \subset U$, and K = $I \cap c$ is an open interval. Where J = (b_1,b_2), without loss of generality we may assume $\beta_1 < b_1 < \beta < b_2 < \beta_2$. Hence there exist open intervals $J_1,J_2 \subset E^*$ such that $\beta_1 \in J_1$, $\beta_2 \in J_2$, $J_1 \cap J = \emptyset$, and $J_2 \cap J = \emptyset$. Then $I \times J_1$ is an open neighborhood of $< \alpha,\beta_1 >$, and $I \times J_2$ is an open neighborhood of $< \alpha,\beta_2 >$. Since $< \alpha,\beta_1 >$ is a limit point of s, there is a point $< \gamma_1,\delta_1 > \in (I \times J_1) \cap s$. Similarly, since $< \alpha,\beta_2 >$ is a limit point of s, there is a point $< \gamma_2,\delta_2 > \in (I \times J_2) \cap s$. Since $K \subset c$, $Z(K) \cap s$ is either a section or sector of Z(K), and since K is pathwise connected, by Lemma 3.3.16, $Z(K) \cap s$ is pathwise connected. Since $s \subset Z(c)$, $\gamma_1 \in c$, hence $\gamma_1 \in L$, hence $< \gamma_1,\delta_1 > \in Z(L) \cap s$. Similarly, $< \gamma_2,\delta_2 > \in Z(L) \cap s$. Let $P:[0,1] \to E^2$ be a path in $Z(K) \cap s$ from $< \gamma_1,\delta_1 >$ to $< \gamma_2,\delta_2 >$. Where $P(t) = < x(t),y(t) >$ for any $t \in [0,1]$, the function $y(t)$ is continuous since P is. Since $y(0) = \delta_1 \in J_1$, and every element of J_1 is less than any element of J, $y(0) < \beta$. Similarly, $y(1) > \beta$. By the Intermediate Value Theorem, there exists $u \in (0,1)$ such that $y(u) = \beta$. Then $P(u) = <x(u),\beta >$ with $<x(u),\beta > \in I \times J \subset U$ Hence $<x(u),\beta > \in U \cap s$. Hence $< \alpha,\beta >$ is a limit point of s, hence $< \alpha,\beta > \in \partial s$. ▪

Theorem 3.3.29. Let c be a 2-cell of a (B^1,B^2) basis-determined cad D of E^2, and let $<u,v> \in \partial c \cap E^2$. Then there exists an open interval $J' \subset E$ containing v, such that for any open subinterval J of J' containing v, there is an open interval $I \subset E$ containing u such that $(I \times J) \cap c$ is pathwise connected.

Proof. Let e^1 be the unique 1-cell of D' such that $c \in S_D(e^1)$. By Theorem

3.3.8, there are two 1-sections d_1 and d_2 of $S_D^*(e^1)$ such that $\partial c \cap Z^*(e^1) = d_1 \cup d_2$. Suppose that neither d_1 nor d_2 is an ∞-section, and that $\alpha \in d_1$. Let d_1 be an f_1-section and d_2 an f_2-section. Where $\alpha = \langle u,v \rangle$, let $\varepsilon = f_2(u) - f_1(u) = f_2(u) - v$. Let $J' = (v - \varepsilon/2, v + \varepsilon/2)$. Let J be any open subinterval of J' containing v. By continuity of f_1 and f_2, there exists an interval I in E containing u such that if $x \in I$, then $f_1(x) \in J$ and $f_2(x) \notin J$. Clearly I is pathwise connected. Since $(I \times J) \cap c = \{\langle x,y \rangle | x \in I \ \& \ f_1(x) < y < v + \varepsilon/2\}$, $(I \times J) \cap c$ is a sector of $Z(I)$, hence pathwise connected by Lemma 3.3.16. The changes needed in this argument if $\alpha \in d_2$, or if either d_1 or d_2 is an ∞-section, are minor and evident.

Suppose there is a 0-cell e^0 of D' such that $e^0 \in \partial e^1$. By Theorem 3.3.24, $S_D^*(e^1)$ has the unique section boundary property in $S_D^*(e^0)$, hence d_1 and d_2 have boundary sections z_1 and z_2 in $S_D^*(e^0)$. By Theorem 3.3.14, $\partial c \cap Z^*(e^0) = [z_1,z_2]$. Suppose that $z_1 < z_2$ and $\alpha \in (z_1,z_2)$. Let $e^0 = u$, $z_1 = \langle u,v_1 \rangle$, $z_2 = \langle u,v_2 \rangle$, and $\alpha = \langle u,v \rangle$. Choose $y_1,y_2 \in E$ such that $v_1 < y_1 < v < y_2 < v_2$, and let $J' = (y_1,y_2)$. Let J be any open subinterval of J' containing v; $J = (w_1,w_2)$ with $v_1 < y_1 \leq w_1 < v < w_2 \leq y_2 < v_2$. Let $U_1 = [-\infty,w_1)$, and $U_2 = (w_2,\infty]$. By Lemma 3.3.12, $\lim_{x \to u \text{ in } e^1} f_1(x) = v_1$ and $\lim_{x \to u \text{ in } e^1} f_2(x) = v_2$. Hence there exist open intervals B_1 and B_2 in E containing u such that $f_1(B_1 \cap e^1) \subset U_1$, and $f_2(B_2 \cap e^1) \subset U_2$. Let $I = B_1 \cap B_2$, an open interval in E containing u. Then $(I \times J) \cap c = (I \cap e^1) \times J$, a sector of $Z(I \cap e^1)$, which by Lemma 3.3.16 is pathwise connected since $I \cap e^1$ is.

Still assuming $z_1 < z_2$, suppose that z_1 is finite and that $\alpha = z_1$. Where $\alpha = z_1 = \langle u,v_1 \rangle$ and $z_2 = \langle u,v_2 \rangle$, we have $\lim_{x \to u \text{ in } e^1} f_1(x) = v_1$ and

$\lim_{x \to u \text{ in } e^1} f_2(x) = v_2$. Let $\varepsilon = v_2 - v_1$. Let $J' = [-\infty, v_1 + \varepsilon/2)$. Let J be any open subinterval of J' containing v. There exist open intervals B_1 and B_2 in E containing α such that $f_1(B_1 \cap e^1) \subset J$, and $f_2(B_2 \cap e^1) \subset (v_2 - \varepsilon/2, \infty]$. Let $I = B_1 \cap B_2$. Then $(I \times J) \cap c = \{\langle x,y \rangle | x \in (I \cap e^1) \ \& \ f_1(x) < y < v_1 + \varepsilon/2\}$, hence $(I \times J) \cap c$ is a sector of $Z(I \cap e^1)$, hence by Lemma 3.3.16, $(I \times J) \cap c$ is pathwise connected since $(I \cap e^1)$ is. Minor changes in this argument are needed for the case z_2 finite and $\alpha = z_2$.

Suppose now that $z_1 = z_2$. If $z_1 = z_2$ is infinite, then $\alpha \notin [z_1, z_2]$, so suppose that $z_1 = z_2$ is finite and $\alpha = z_1 = z_2 = \langle u,v \rangle$. Set $J' = (-\infty, +\infty)$, and let J be any open subinterval of J' containing v. There are open intervals B_1 and B_2 in E containing u such that $f_1(B_1 \cap e^1) \subset J$ and $f_2(B_2 \cap e^1) \subset J$. Set $I = B_1 \cap B_2$. Then $(I \times J) \cap c = \{\langle x,y \rangle | x \in (I \cap e^1) \ \& \ f_1(x,y) < y < f_2(x,y)\}$ is a sector of $Z(i \cap e^1)$, hence pathwise connected since $I \cap e^1$ is.

By Corollary 3.3.2, we have considered all points of $\partial c \cap E^2$. This completes the proof. ▪

Lemma 3.3.30. Let c be a 1-cell or 2-cell in a (B^1, B^2) basis-determined cad D of E^2, and let $\alpha \in \partial c \cap E^2$. Let $\langle \alpha, \beta \rangle \in Z^*(\alpha)$, and let U be any open neighborhood in $E^2 \times E^*$ of $\langle \alpha, \beta \rangle$. Then there exist open intervals $I, J \subset E$ and $K \subset E^*$ such that $\langle \alpha, \beta \rangle \in I \times J \times K \subset U$, and $(I \times J) \cap c$ is pathwise connected.

Proof. The open sets in $E^2 \times E^*$ of the form $L \times M \times N$, L, M open intervals in E, N an open interval in E^*, are a basis for the topology on $E^2 \times E^*$. Hence

there exists some such basis element $I' \times J' \times K$ with $< \alpha, \beta > \in I' \times J' \times K \subset U$. We claim that there are open subintervals I and J of I' and J' such that $\alpha \in I \times J$, and $(I \times J) \cap c$ is pathwise connected.

Suppose c is a 1-sector. Then there is a 0-cell e of D', and sections $d_1 < d_2$ of $S_D^*(e)$, such that $c = (d_1, d_2)$. By Theorem 3.3.8, either $\alpha = d_1$ or $\alpha = d_2$. Hence there exist $u \in E$ and $v_1, v_2 \in E^*$ such that $c = \{u\} \times (v_1, v_2)$, and either $\alpha = < u, v_1 >$ or $\alpha = < u, v_2 >$. Clearly $W = J' \cap (v_1, v_2)$ is an open interval and hence pathwise connected. Then taking $I = I'$ and $J = J'$, we have that $(I \times J) \cap c = \{u\} \times W$ is pathwise connected.

Suppose c is a 1-section. Let $J = J'$. By Theorem 3.3.7, Corollary 3.3.2, and Theorem 3.2.1, there is a 1-cell e^1 of D', and a 0-cell e^0 of D' such that $c \in S_D(e^1)$ and $\alpha \in S_D^*(e^0)$. By Theorem 3.3.24, α is the boundary section of c in $S_D^*(e^0)$. Hence if c is an f-section, if $e^0 = u$, and if $\alpha = < u, v >$, then by Lemma 3.3.12, $\lim_{x \to u \text{ in } e^1} f(x) = v$. Then there is an open interval B in E containing u such that $f(B \cap e^1) \subset J$. $I = B \cap I'$ is an open interval of E containing u. Clearly $I \cap e^1$ is an open interval, and hence pathwise connected. Let g denote the restriction of f to $I \cap e^1$. $graph(g)$ is a section of $Z(I \cap e^1)$, hence by Lemma 3.3.16, it is pathwise connected. Hence since $graph(g) = (I \times J) \cap c$, we have found open subintervals I and J of I' and J' such that $\alpha \in I \times J$ and $(I \times J) \cap c$ is pathwise connected.

Suppose now that c is a 2-sector. Let $\alpha = < u, v >$. By Theorem 3.3.29, there is an open interval $J^* \subset E$ containing v, such that for any open subinterval $\mathcal{J}$ of J^* containing v, there is an open interval $\mathcal{I} \subset E$ containing u such that $(\mathcal{I} \times \mathcal{J}) \cap c$ is pathwise connected. Set $\mathcal{J} = J^* \cap J'$. Then setting $J = \mathcal{J}$ and $I = \mathcal{I} \cap I'$, we obtain open subintervals I and J of I' and J' such that

$\alpha \in I \times J$ and $(I \times J) \cap c$ is pathwise connected. ∎

Theorem 3.3.31. Let c be a cell in a cad of E^2, and let $\alpha \in \partial c \cap E^2$. Let s be a section or sector of $Z(c)$. Let $< \alpha,\beta_1 >$ and $< \alpha,\beta_2 > \in \partial s \cap Z^*(\alpha)$, and let $\beta_1 < \beta < \beta_2$. Then $< \alpha,\beta > \in \partial s$.

Proof. Essentially the same as the proof of Theorem 3.3.28. If c is a 0-cell, then $\partial c = \emptyset$, so we may assume that c is a 1-cell or a 2-cell. Let U be any open neighborhood in $E^2 \times E^*$ of $< \alpha,\beta >$. By Lemma 3.3.30, there exist open intervals $I,J \subset E$ and $K \subset E^*$ such that $< \alpha,\beta > \in I \times J \times K \subset U$, and L = $(I \times J) \cap c$ is pathwise connected. Where K = (b_1,b_2), without loss of generality we may assume $\beta_1 < b_1 < \beta < b_2 < \beta_2$. Hence there exist open intervals $K_1,K_2 \subset E^*$ such that $\beta_1 \in K_1$, $\beta_2 \in K_2$, $K_1 \cap K = \emptyset$, and $K_2 \cap K = \emptyset$. Then $I \times J \times K_1$ is an open neighborhood of $< \alpha,\beta_1 >$, and $I \times J \times K_2$ is an open neighborhood of $< \alpha,\beta_2 >$. Since $< \alpha,\beta_1 >$ is a limit point of s, there is a point $< \gamma_1,\delta_1 > \in (I \times J \times K_1) \cap s$, with $\gamma_1 \in I \times J$ and $\delta_1 \in K_1$. Similarly, since $< \alpha,\beta_2 >$ is a limit point of s, there is a point $< \gamma_2,\delta_2 > \in (I \times J \times K_2) \cap s$, with $\gamma_2 \in I \times J$ and $\delta_2 \in K_2$. Since $L \subset c$, $Z(L) \cap s$ is either a section or sector of Z(L), and since L is pathwise connected, by Lemma 3.3.16, $Z(L) \cap s$ is pathwise connected. Since $s \subset Z(c)$, $\gamma_1 \in c$, hence $\gamma_1 \in L$, hence $< \gamma_1,\delta_1 > \in Z(L) \cap s$. Similarly, $< \gamma,\delta_2 > \in Z(L) \cap s$. Let $P:[0,1] \to E^3$ be a path in $Z(L) \cap s$ from $< \gamma_1,\delta_1 >$ to $< \gamma_2,\delta_2 >$. Where $P(t) = < x(t),y(t),z(t) >$ for any $t \in [0,1]$, the function $z(t)$ is continuous since P is. Since $g(0) = \delta_1 \in K_1$, and every point of K_1 is less than any point of K, $z(0) < \beta$. Similarly, $g(1) > \beta$. By the Intermediate Value Theorem, there exists $u \in (0,1)$ such that $z(u) = \beta$. Then $P(u) = <x(u),y(u),\beta >$ with $<x(u),y(u),\beta > \in I \times J \times K \subset U$. Hence $<x(u),y(u),\beta > \in U \cap s$. Hence $< \alpha,\beta >$ is a limit point of s, hence

$< \alpha,\beta > \in \partial s$. ▪

Lemma 3.3.32 Let c,d be cells of a cad of E^{r-1}, r = 2 or r = 3, such that $c \subset \partial d$. Let S(c), S(d) be stacks over c,d. $S^*(d)$ has the unique section boundary property in $S^*(c)$ if and only if for every section s of S(d), if $\partial s \cap Z(c) \neq \emptyset$, then $\partial s \cap Z(c)$ is a section of S(c).

Proof. Suppose for every section s of S(d) that if $\partial s \cap Z(c) \neq \emptyset$, then $\partial s \cap Z(c)$ is a section of S(c). Suppose for some f-section s of S(d) that $\partial s \cap Z(c) = \emptyset$. For any $\alpha \in c$, since $c \subset \partial d$, there is a sequence $x_1, x_2, \ldots$ in d converging to α; let B be a closed ball in E^{r-1} centered at α, which, we may assume without loss of generality, contains $x_1, x_2, \ldots$. $B \times E^*$ is compact, hence the sequence $< x_1, f(x_1) >, < x_2, f(x_2) >, \cdots$ has a convergent subsequence whose limit is $< \alpha, b >$, $b \in E^*$, hence $< \alpha, b >$ is a limit point of s. Hence for all $\alpha \in c$, $\partial s \cap Z^*(\alpha) \neq \emptyset$. Since $\partial s \cap Z(c) = \emptyset$, by Theorem 3.3.28 and Theorem 3.3.31, for each $\alpha \in c$, s has a unique limit point $< \alpha, b >$ in $Z^*(\alpha)$. with either $b = +\infty$ or $b = -\infty$. Hence by Lemma 3.3.12, for every $\alpha \in c$, either $\lim_{x \to \alpha \text{ in } d} f(x) = +\infty$, or $\lim_{x \to \alpha \text{ in } d} f(x) = -\infty$. Define $T(\alpha) = \lim_{x \to \alpha \text{ in } d} f(x)$, for $\alpha \in c$. It is easy to prove that T is continuous on c, and hence by the Intermediate Value Theorem, T is either the $+\infty$-section or the $-\infty$-section of $Z^*(c)$. Then by Lemma 3.3.4, $S^*(d)$ has the unique section boundary property in $S^*(c)$. The converse is clear. ▪

Theorem 3.3.33. Let X be a subset of E^r, $r \geq 2$, and let $\alpha \in E^{r-1}$, with $X \cap Z(\alpha) = \emptyset$. Let $\partial X \cap Z^*(\alpha) = \{\alpha\} \times U$. Then U is closed in E^*.

Proof.

$$\{\alpha\}\times U = \partial X \cap Z^*(\alpha)$$
$$= (\overline{X} - X) \cap Z^*(\alpha)$$
$$= \overline{X} \cap Z^*(\alpha)$$

is closed in $Z^*(\alpha)$. If $\beta \in \overline{U} - U$, then there is a sequence $\{\beta_i\}$ in U converging to β. Then the sequence $\{ < \alpha,\beta_i > \}$ in $\{\alpha\}\times U$ converges to $< \alpha,\beta > \notin \{\alpha\}\times U$, contradicting our conclusion that $\{\alpha\}\times U$ is closed in $Z^*(\alpha)$. Hence $\overline{U} - U = \emptyset$, i.e. $U = \overline{U}$, i.e. U is closed. ▪

Corollary 3.3.34. Let c be a cell of a cad of E^{r-1}, r = 2 or r = 3, and let $\alpha \in \partial c$. Let S(c) be a stack over c, and let s be a region of S(c). Then $\partial s \cap Z^*(\alpha) = \{\alpha\}\times I$, where I is a closed interval in E^*.

Proof. By Theorem 3.3.28 and 3.3.31, I is an interval. By Theorem 3.3.33, I is closed in E^*. ▪

The following lemma will be used in later sections.

Lemma 3.3.35. Let c be a cell of a cad of E^{r-1}, r = 2 or r = 3, and let $\alpha \in \partial c$. Let $F \in I_r$, and let t be an F-section of $Z(c)$. If t has two distinct limit points in $Z^*(\alpha)$, then F is cylindrical at α.

Proof. Let $< \alpha,b_1 >$ and $< \alpha,b_2 >$ be distinct limit points of t in $Z^*(\alpha)$, with $b_1 < b_2$. By Corollary 3.3.34, $\{\alpha\}\times[b_1,b_2] \subset \partial t$. Then since V(F) is closed, $\{\alpha\}\times[b_1,b_2] \subset V(F)$. Hence $F(\alpha,x_r)$ has infinitely many roots, hence $F(\alpha,x_r)$ is the zero polynomial, hence F is cylindrical at α. ▪

Theorem 3.3.36. Let c and d be cells of a cad of E^{r-1}, r = 2 or r = 3, such that $d \subset \partial c$. Let S(c) and S(d) be stacks over c and d, and let s be a region of S(c), such that $\partial s \cap Z^*(d)$ is the union of a finite number of regions in $S^*(d)$. Then there exist h and k such that

$$\partial s \cap Z^*(d) = \bigcup_{i=h}^{k} S_i^*(d).$$

Proof. Let h be the smallest positive integer such that $S_h^*(d) \subset \partial s \cap Z^*(d)$. Let k be the largest positive integer such that $S_k^*(d) \subset \partial s \cap Z^*(d)$. Let $h < i < k$, and assume that $S_i^*(d) \not\subset (\partial s \cap Z^*(d))$, and hence by hypothesis, $S_i^*(d)$ is disjoint from $\partial s \cap Z^*(d)$. Let $\alpha \in d$. Then $\partial s \cap Z^*(\alpha) \cap S_i^*(d) = \emptyset$, but $\partial s \cap Z^*(\alpha) \cap S_h^*(d) \neq \emptyset$, and $\partial s \cap Z^*(\alpha) \cap S_k^*(d) \neq \emptyset$, contradicting Corollary 3.3.34. ▪

Lemma 3.3.37. Let c be a cell of a cad of E^{r-1}, $r \geq 2$, and let $\alpha \in \partial c$. Let S(c) be a stack over c, and let s,s' be regions of S(c) with s below s'. Suppose $\partial s \cap Z^*(\alpha) = \{\alpha\} \times I$, where $\bar{I} = [\beta_1, \beta_2]$ with $\beta_1 \leq \beta_2$, and $\partial s' \cap Z^*(\alpha) = \{\alpha\} \times I'$, where $\bar{I'} = [\beta_1', \beta_2']$ with $\beta_1' \leq \beta_2'$. Then $\beta_1 \leq \beta_1'$, and $\beta_2 \leq \beta_2'$.

Proof. Let $\beta \in I$. Then there is a sequence $\{ < x_i, y_i > \}$ in s which converges to $< \alpha, \beta >$. Then $\{x_i\}$ converges to α, and $\{y_i\}$ converges to β. For each i, there exists $y_i' > y_i$ with $< x_i, y_i' >$ in s'. Since E^* is compact, there is a subsequence of $\{ < x_i, y_i' > \}$ which converges, say to a point $< \alpha', \beta' >$ in $E^{r-1} \times E^*$. Clearly $\alpha' = \alpha$, and since the corresponding subsequence of $\{ < x_i, y_i > \}$ converges to $< \alpha, \beta >$, we have $\beta' \geq \beta$. Therefore $< \alpha, \beta' > \in \partial s' \cap Z^*(\alpha)$, $\beta' \in I'$, and $\beta_2' \geq \beta' \geq \beta$. But β was any element of I, so $\beta_2' \geq \beta_2$. By symmetry, $\beta_1' \geq \beta_1$. ▪

Theorem 3.3.38. Let c and d be cells of a cad of E^{r-1}, $r \geq 2$, such that $d \subset \partial c$. Let S(c) and S(d) be stacks over c and d. Let s and s' be regions of S(c), with s below s', such that $\partial s \cap Z^*(d) = \bigcup_{i=h}^{k} S_i^*(d)$, and $\partial s' \cap Z^*(d) = \bigcup_{i=h'}^{k'} S_i^*(d)$. Then $h \leq h'$, and $k \leq k'$.

Proof. Let $\alpha \in d$. Then

$$S_k^*(d) \cap Z^*(\alpha) \subset (\bigcup_{i=h}^{k} S_i^*(d)) \cap Z^*(\alpha)$$

$$= \partial s \cap Z^*(d) \cap Z^*(\alpha)$$

$$= \partial s \cap Z^*(\alpha).$$

Let $< \alpha,\beta > \in S_k^*(d) \cap Z^*(\alpha)$. Then $< \alpha,\beta > \in \partial s \cap Z^*(\alpha)$. By Lemma 3.3.37, there exists $\beta' \geq \beta$ with

$$< \alpha,\beta' > \in (\partial s' \cap Z^*(\alpha))$$

$$= \bigcup_{i=h'}^{k'} S_i^*(d) \cap Z^*(\alpha),$$

so $< \alpha,\beta' > \in S_i^*(d) \cap Z^*(\alpha)$ with $i \leq k'$. But $\beta \leq \beta'$, so $k \leq i \leq k'$. By symmetry, $h \leq h'$. ∎

Theorem 3.3.39. Let c and d be cells of a cad of E^{r-1}, r = 2 or r = 3, such that $d \subset \partial c$. Let S(c) and S(d) be stacks over c and d such that $S^*(c)$ has the section boundary property in $S^*(d)$. Then $S^*(c)$ has the sector boundary property in $S^*(d)$.

Proof. Let s be a sector in S(c). Let $t_0, \ldots, t_{m+1}$, $m \geq 0$, be the sections of $S^*(c)$, and suppose that $s = (t_i,t_{i+1})$. By the section boundary hypothesis and Theorem 3.3.36, there exist h,k,h',and k' such that

$$\partial t_i \cap Z^*(d) = \bigcup_{j=h}^{k} S_j^*(d),$$

and

$$\partial t_{i+1} \cap Z^*(d) = \bigcup_{j=h'}^{k'} S_j^*(d).$$

By Lemma 3.3.38, $h \leq h'$ and $k \leq k'$. Let $h \leq j \leq k$. Then $S_j^*(d) \subset \partial t_i$, and since $t_i \subset \partial s$ by Theorem 3.3.8, by Lemma 3.3.11, $S_j^*(d) \subset \partial s$. Also,

$S_j^*(d) \subset Z^*(d)$, so $S_j^*(d) \subset \partial s \cap Z^*(d)$. Similarly, for $h' \leq j \leq k'$, $S_j^*(d) \subset \partial s \cap Z^*(d)$. Assume $k < j < h'$, and let $\alpha \in d$. Then $S_k^*(d) \cap Z^*(\alpha) \subset \partial s \cap Z^*(\alpha)$, and $S_{h'}^*(d) \cap Z^*(\alpha) \subset \partial s \cap Z^*(\alpha)$. So by Corollary 3.3.34, $S_j^*(d) \cap Z^*(\alpha) \subset \partial s \cap Z^*(\alpha)$. But α was an arbitrary element of d, so $S_j^*(d) \subset \partial s \cap Z^*(d)$. Therefore,

$$\bigcup_{j=h}^{k'} S_j^*(d) \subset \partial s \cap Z^*(d).$$

Let j < h, and assume $S_j^*(d) \cap \partial s \neq \emptyset$. Let $<\alpha,\beta> \in S_j^*(d) \cap \partial s$. Then $<\alpha,\beta> \in \partial s \cap Z^*(\alpha)$, and t_i is a region of S(c) below s. So by Corollary 3.3.34 and Lemma 3.3.37, there exists $\beta' \leq \beta$ with $<\alpha,\beta'> \in \partial t_i \cap Z^*(\alpha)$. But then $\partial t_i \cap S_{j'}^*(d) \neq \emptyset$ for some $j' \leq j$, contradicting

$$\partial t_i \cap Z^*(d) = \bigcup_{j=h}^{k} S_j^*(d).$$

Hence if $S_j^*(d) \cap \partial s \neq \emptyset$, then $j \geq h$. Similarly, $j \leq k'$, and so

$$\partial s \cap Z^*(d) = \bigcup_{j=h}^{k'} S_j^*(d). \blacksquare$$

Corollary 3.3.40. Let c and d be cells of a cad of E^{r-1}, r = 2 or r = 3, such that $d \subset \partial c$. Let S(c), S(d) be stacks over c,d such that S(c) has the section boundary property in S(d). Then S(c) has the boundary property in S(d).

Proof. By Theorem 3.3.39, $S^*(c)$ has the sector boundary property in $S^*(d)$, and since every region of S(c) is either a section or sector, it follows that S(c) has the boundary property in S(d). ∎

3.4 Cell boundaries in 3-space, part 1.

Theorem 3.4.1. Let $B \in I_3$ be primitive and of positive degree. Let R be a region in E^2 such that B is cylindrical on R. Then R consists of a single point.

Proof. Suppose R is not a single point. Then since E^2 is Hausdorff and R is connected, R contains infinitely many points. Let

$$B(x_1,x_2,x_3) = \sum_{i=1}^{n} B_i(x_1,x_2)x_3^{e_i},$$

with $n \geq 1$, $e_n \geq 1$, $e_n > e_{n-1} > \cdots > e_1 \geq 0$, and $B_i \neq 0$ for $1 \leq i \leq n$. Since B is cylindrical on R, each B_i vanishes at each point of R, hence $B_1, \ldots, B_n$ have infinitely many common zeros. Hence by Theorem 3.3.17, $gcd(B_1, \ldots, B_n)$ has positive degree in x_1 or x_2, contradicting the primitivity of B. Hence R consists of a single point ▪

Definition. Let F be an element of I_r, $r \geq 2$, and let R be a region in E^{r-1}. R is *cylindrical with respect to F* if F is cylindrical on R. R is *noncylindrical with respect to F* if F is noncylindrical at every point of R. Let $A \subset I_r$. R is *cylindrical with respect to A* if R is cylindrical with respect to some $A_i \in A$. R is *noncylindrical with respect to A* if R is noncylindrical with respect to every $A_i \in A$.

Corollary 3.4.2. Let D be a (B^1,B^2,B^3) basis-determined cad of E^3. If c is a cell of D' which is cylindrical with respect to B^3, then c is a 0-cell.

Proof. If c is cylindrical with respect to B^3, then some $B \in B^3$ vanishes on c. Since B is primitive and of positive degree, by Theorem 3.4.1, c consists of a single point, i.e. c is a 0-cell ▪

Theorem 3.4.3. Let D be a (B^1,B^2,B^3) basis-determined cad of E^3. Let c^0 be a 0-cell of D' which is noncylindrical with respect to B^3, and let d be a 1-cell or a 2-cell of D' such that $c^0 \subset \partial d$. Then $S_D^*(d)$ has the unique section boundary property in $S_D^*(c^0)$.

Proof. Essentially the same as the proof of Theorem 3.3.24. Let t be a

section of $S_D(d)$, and suppose $\partial t \cap Z(c^0) \neq \emptyset$, i.e. t has a limit point <a,b> in $Z(c^0)$, where $c^0 = a$. Since D is (B^1,B^2,B^3) basis-determined, some $B \in B^3$ vanishes on t, hence since V(B) is closed, B vanishes on <a,b>. Where d is the cell of $S_D(c^0)$ containing <a,b>, d is B-invariant. Hence if d is a sector and B vanishes on d, then by Lemma 3.3.19, B vanishes on $Z(c^0)$, i.e. B is cylindrical at c^0, contrary to hypothesis. Hence d is a 0-section of $S_D(c^0)$, i.e. any limit point of t in $Z(c^0)$ is a section of $S_D(c^0)$. By Corollary 3.3.34, $\partial t \cap Z^*(c^0) = c^0 \times I$, where I is a closed interval in E^*, and since V(B) is closed, $B(c^0,z) = 0$ for every $z \in I$. Since B is noncylindrical at c^0, I contains only finitely many points, and therefore since I is an interval, I contains exactly one point. Hence if $\partial t \cap Z(c^0) \neq \emptyset$, then $\partial t \cap Z(c^0)$ consists of a section of $S_D(c^0)$, hence by Lemma 3.3.32, $S_D^*(d)$ has the unique section boundary property in $S_D^*(c^0)$. ∎

Theorem 3.4.4. Let Q,R be regions in E^{r-1}, $r \geq 2$, such that $Q \subset \partial R$. Let S(Q), S(R) be stacks over Q,R. Suppose for $\alpha \in Q$, for an f-section of S(R), and for a g-section t of $S^*(Q)$, that $\lim_{x \to \alpha \text{ in } R} f(x) = g(\alpha)$. Then there exists an open interval I in E^* containing $g(\alpha)$, and a bounded open neighborhood B of α in E^{r-1} such that (1) t is the only section of $S^*(Q)$ meeting $\overline{B} \times \overline{I}$, and (2) $f(B \cap R) \subset I$.

Proof. Since the sections of $S^*(Q)$ are disjoint graphs of continuous functions, it is clear that there exists an open interval I in E^* containing $g(\alpha)$, and an open ball B_1 in E^2 centered at α, so that t is the only section of $S^*(Q)$ which meets $\overline{B_1} \times \overline{I}$. By definition of $\lim_{x \to \alpha \text{ in } R} f(x) = g(\alpha)$, there exists an open ball B_2 in E^2 centered at α such that $f(B_2 \cap R) \subset I$. Set B = $B_1 \cap B_2$. Since $B \subset B_1$, t is the only section of $S^*(Q)$ meeting $\overline{B} \times \overline{I}$, and

since $B \subset B_2$, $f(B \cap R) \subset I$. Since B_1 and B_2 are bounded, so is B. ▪

Theorem 3.4.5. Let D be a (B^1,B^2,B^3) basis-determined cad of E^3. Let c^1 and c^2 be a 1-cell and 2-cell of D' such that $c^1 \subset \partial c^2$. Let s be an f-section of $S_D(c^2)$, and let $\alpha \in c^1$. Then s has a unique limit point in $Z^*(\alpha)$, which lies in a section of $S_D^*(c^1)$.

Proof. Since $\alpha \in \partial c^2$, there is a sequence $x_1,x_2, \ldots,$ in c^2 converging to α. Without loss of generality we may assume there is a ball B in E^2 centered at α containing every x_i. The sequence $< x_1,f(x_1)>, < x_2,f(x_2)>, \cdots$ is contained in $s \cap (\overline{B} \times E^*)$; since $\overline{B} \times E^*$ is compact, the sequence has a subsequence which converges to $< \alpha,b >$ for some $b \in E^*$. $< \alpha,b >$ is a limit point of s, and we claim it is the unique limit point of s in $Z^*(\alpha)$. Suppose s has a second limit point $< \alpha,b' >$, $b \neq b'$, in $Z^*(\alpha)$. Assume without loss of generality that $b < b'$. Then by Theorem 3.3.31, every point of $\{\alpha\} \times [b,b']$ is a limit point of s. Since D is (B^1,B^2,B^3) basis-determined, s is a B_i -section, for some $B_i \in B^3$. Since $V(B_i)$ is closed, B_i vanishes on $\{\alpha\} \times [b,b']$. Hence there is a sector u of $S_D(c^1)$ such that B_i vanishes at some point of u. Since D is (B^1,B^2,B^3) basis-determined, B_i vanishes everywhere on u, hence by Lemma 3.3.19, B_i vanishes on $Z(c^1)$. Thus c^1 is cylindrical with respect to B_i, which by Corollary 3.4.2 is impossible since $dim(c^1) > 0$. Hence $< \alpha,b >$ is the unique limit point of s in $Z^*(\alpha)$. If $< \alpha,b >$ is contained in a sector of $S_D^*(c^1)$, and hence in a sector of $S_D(c^1)$, then B_i vanishes on this sector, which by another application of the argument just given leads to a contradiction. Hence $< \alpha,b >$ is contained in a section of $S_D^*(c^1)$. ▪

Theorem 3.4.6. Let D be a (B^1,B^2,B^3) basis-determined cad of E^3. Let c^1 and c^2 be a 1-cell and 2-cell of D' such that $c^1 \subset \partial c^2$. Let s be an f-section of

$S_D(c^2)$, and let $\alpha \in c^1$. Then there exists a section t of $S_D^*(c^1)$, and an open neighborhood M of α in c^1, such that for all $\beta \in M$, the unique limit point of s in $Z^*(\beta)$ is in t.

Proof. Suppose $<\alpha,b>$ is the unique limit point of s in $Z^*(\alpha)$. Thus for some g-section t of $S_D^*(c^1)$, $b = g(\alpha)$. By Lemma 3.3.12, $\lim_{x \to \alpha \text{ in } c^2} f(x) = g(\alpha)$. Then by Theorem 3.4.4, there is a bounded open neighborhood B of α in E^2 such that (1) t is the only section of $S_D^*(c^1)$ meeting $\bar{B} \times \bar{I}$, and (2) $f(B \cap c^2) \subset I$. Let $M = B \cap c^1$. For any $\beta \in M$, let $x_1, x_2, \ldots$ be a sequence of points in c^2 converging to β; without loss of generality we may assume $x_1, x_2, \ldots$ contained in B. Since B is bounded , $\bar{B}$ is closed and bounded, hence compact, and since $\bar{I}$ is a closed subset of the compact space E^*, it is compact. Hence $\bar{B} \times \bar{I}$ is compact. Hence the sequence $<x_1, f(x_1)>$, $<x_2, f(x_2)>, \ldots$ of points in $s \cap (B \times I)$ has a subsequence which converges to a limit $<\beta,z>$ in $\{\beta\} \times \bar{I}$. By Theorem 3.4.5, $<\beta,z>$ is in a section of $S_D^*(c^1)$. Since t is the only section of $S_D^*(c^1)$ meeting $\bar{B} \times \bar{I}$, $<\beta,z> \in t$. So for all $\beta \in M$, the unique limit point of s in $Z^*(\beta)$ is in t. ∎

Theorem 3.4.7. Let D be a (B^1, B^2, B^3) basis-determined cad of E^3. Let c^2 be a 2-cell, c^1 a 1-cell of D' such that $c^1 \subset \partial c^2$. Then $S_D^*(c^2)$ has the unique section boundary property in $S_D^*(c^1)$.

Proof. Let s be any section of $S_D(c^2)$. For any $\alpha \in c^1$, let $T(\alpha)$ be the index of the unique section of $S_D^*(c^1)$ which contains a limit point of s in $Z^*(\alpha)$. By Theorem 3.4.6, T is locally invariant on c^1. Hence by Lemma 2.2.5, T is invariant on c^1. Hence $\partial s \cap Z^*(c^1)$ is a section of $S_D^*(c^1)$, and so by Lemma 3.3.32, $S_D^*(c^2)$ has the unique section boundary property in $S_D^*(c^1)$ ∎

Theorem 3.4.8. Let c,d be cells in E^{r-1}, $r \geq 2$, such that $c \subset \partial d$ and $dim(c) <$

$dim(d)$. Let S(c), S(d) be stacks over c,d such that $S^*(d)$ has the unique section boundary property in $S^*(c)$. Then for any cell s of S(d), and any cell t of $S^*(c)$ or $S^*(d)$ such that $t \cap \partial s \neq \emptyset$, $dim(t) < dim(s)$ and $t \subset \partial s$.

Proof. Let s be a section of S(d). By Theorem 3.3.7, $\partial s \cap Z^*(d) = \emptyset$. By hypothesis, s has a boundary section t in $Z^*(c)$. We have $dim(t) = dim(c) < dim(d) = dim(s)$, and $t \subset \partial s$ by definition of boundary section. Let s be a sector of S(d). By Theorem 3.3.8, $\partial s \cap Z^*(d)$ consists of two sections of $S^*(d)$, hence for any cell t of $S^*(d)$, $\partial t \cap s \neq \emptyset$ implies $dim(t) < dim(s)$ and $t \subset \partial s$. By Theorem 3.3.14, $\partial s \cap Z^*(c)$ is the union of certain cells of $S^*(c)$, and since $dim(t) \leq dim(c)+1 \leq dim(d) < dim(s)$ for all cells t of $S^*(c)$, for any cell t of $S^*(c)$, if $t \cap \partial s \neq \emptyset$, then $t \subset \partial s$ and $dim(t) < dim(s)$ ∎

Corollary 3.4.9. Let D be a (B^1,B^2) basis-determined cad of E^2. For cells s,t of D, if $\partial s \cap t \neq \emptyset$, then $t \subset \partial s$ and $dim(t) < dim(s)$.

Proof. Follows immediately from Theorems 3.2.2, 3.3.24, and 3.4.8 ∎

Definition. A $(B^1, \ldots, B^r)$ *cylindricity-free cad* of E^r, $r \geq 1$, is defined as follows. If r=1, then any (B^1) basis-determined cad of E^1 is (B^1) cylindricity-free. If r>1, then a $(B^1, \ldots, B^r)$ basis-determined cad D of E^r is $(B^1, \ldots, B^r)$ cylindricity-free if every cell of D' is noncylindrical with respect to B^r, and if D' is $(B^1, \ldots, B^{r-1})$ cylindricity-free.

A $(B^1, \ldots, B^r)$ basis-determined cad of $E^r, r \geq 2$, which is not cylindricity-free is said to be a cad *with cylindricity*, or alternatively, a cad that *has cylindricity*.

Theorem 3.4.10. A (B^1,B^2) basis-determined cad D of E^2 is (B^1,B^2) cylindricity-free.

Proof. Since each element of B^2 is primitive and of positive degree, by Lemma 3.3.20, no element of B^2 is cylindrical at any point of E^1, hence no element of B^2 is cylindrical on any cell of D'. Since any (B^1) basis-determined cad of E^1 is (B^1) cylindricity-free, D' is (B^1) cylindricity-free, and hence D is (B^1,B^2) cylindricity-free. ▪

Theorem 3.4.11. Let D be a (B^1,B^2,B^3) cylindricity-free cad of E^3. D has the boundary property.

Proof. By Theorem 3.4.10, D' is (B^1,B^2) cylindricity-free. By Theorems 3.3.10 and 3.3.15, it suffices to show, for all cells c,d of D' such that $c \subset \partial d$, that $S_D^*(d)$ has the unique section boundary property in $S_D^*(c)$. By Corollary 3.4.9, for any $c,d \in D'$ with $c \subset \partial d$, either $dim(c) = 0$ and $dim(d) = 2$, or $dim(c)=0$ and $dim(d)=1$, or $dim(c)=1$ and $dim(d)=2$. In the first two cases, $S_D^*(d)$ has the unique section boundary property in $S_D^*(c)$ by Theorem 3.4.3. In the third case, $S_D^*(d)$ has the unique section boundary property in $S_D^*(c)$ by Theorem 3.4.7. ▪

Corollary 3.4.12. Let D be a (B^1,B^2,B^3) cylindricity-free cad of E^3. For cells s,t of D, if $t \cap \partial s \neq \emptyset$, then $t \subset \partial s$ and $dim(t) < dim(s)$.

Proof. Suppose $t \cap \partial s \neq \emptyset$ for cells $s,t \in D$. By Theorem 3.4.11, D has the boundary property, hence by Theorem 3.1.1, $t \subset \partial s$. By Corollary 3.3.2, there exist cells $c,d \in D'$ such that $t \in S_D(d)$, $s \in S_D(c)$, and $d \subset \bar{c}$. Hence either $d = c$ or $d \subset \partial c$. If $d = c$, then by Theorems 3.3.7 and 3.3.8, s is a sector of $S_D(c)$ and t is a section of $S_D^*(c)$, hence $dim(t) < dim(s)$. If $d \subset \partial c$, then by Corollary 3.4.9, $dim(d) < dim(c)$. Thus by Theorem 3.4.3 and Theorem 3.4.7, $S_D^*(c)$ has the unique section boundary property in $S_D^*(d)$, hence by Theorem 3.4.8, $dim(t) < dim(s)$ ▪

3.5 Adjacency.

Definition. Two disjoint regions in $E^r, r \geq 1$, are ***topologically adjacent***, (we will usually say just ***adjacent***), if their union is a region.

Theorem 3.5.1. Let R_1 and R_2 be disjoint regions in $E^r, r \geq 1$. Then R_1 and R_2 are adjacent if and only if one contains a limit point of the other.

Proof. As shown in [WID70], p. 192, a subset X of E^r is connected if and only if there exists no separation of X, where a ***separation*** of X is a pair (H,K) of nonempty, disjoint subsets of X such that $X = H \cup K$, and $H \cap \partial K = K \cap \partial H = \emptyset$. Thus if neither of R_1 or R_2 contains a limit point of the other, then (R_1, R_2) is a separation of $R_1 \cup R_2$, hence $R_1 \cup R_2$ is not connected, hence R_1 and R_2 are not adjacent. Suppose that either R_1 contains a limit point of R_2 or vice versa. Then (R_1, R_2) is not a separation of $R_1 \cup R_2$. Suppose that nonetheless $R_1 \cup R_2$ has a separation (H,K). Then either $H \cap R_1 \neq \emptyset$ and $K \cap R_1 \neq \emptyset$, or $H \cap R_2 \neq \emptyset$ and $K \cap R_2 \neq \emptyset$. Without loss of generality, suppose the latter. Then $(H \cap R_2, K \cap R_2)$ is a separation of R_2, contradicting the connectivity of R_2. Hence if R_1 contains a limit point of R_2 or vice versa, then $R_1 \cup R_2$ has no separation, i.e. $R_1 \cup R_2$ is connected, i.e. R_1 and R_2 are adjacent ▪

Theorem 3.5.2. Let D be a cad of E^1. Cells c and d of D are adjacent if and only if either $dim(c) < dim(d)$ and $c \subset \partial d$, or $dim(d) < dim(c)$ and $d \subset \partial c$.

Proof. Suppose c and d are adjacent. Then by Theorem 3.5.1, one contains a limit point of the other. Assume without loss of generality that d contains a limit point of c. Then $d \cap \partial c \neq \emptyset$, hence by Theorem 3.2.2, $dim(d) < dim(c)$ and $d \subset \partial c$. The converse is obvious. ▪

Theorem 3.5.3. Let D be a (B^1,B^2) basis-determined cad of E^2. Cells c and d of D are adjacent if and only if either $dim(c) < dim(d)$ and $c \subset \partial d$, or $dim(d) < dim(c)$ and $d \subset \partial c$.

Proof. Suppose c and d are adjacent. Then by Theorem 3.5.1, one contains a limit point of the other. Assume without loss of generality that d contains a limit point of c. Then $d \cap \partial c \neq \emptyset$, hence by Corollary 3.4.9, $dim(d) < dim(c)$ and $d \subset \partial c$. The converse is obvious. ▪

Lemma 3.5.4. Let D be a $(B^1, \ldots, B^r)$ basis-determined cad of E^r, r=2 or r=3. Let c,d be distinct cells of D' such that $dim(d) \leq dim(c)$. Then no cell of $S_D(c)$ contains a limit point of any cell of $S_D(d)$.

Proof. By Theorem 3.2.2 and Corollary 3.4.9, $c \cap \partial d = \emptyset$, hence $c \cap \bar{d} = \emptyset$. Thus $Z(c) \cap Z(\bar{d}) = \emptyset$, and since $Z(\bar{d}) = \overline{Z(d)}$ by Lemma 3.3.1, $Z(c) \cap \overline{Z(d)} = \emptyset$. Hence $Z(c) \cap \partial Z(d) = \emptyset$, and hence no cell of $S_D(c)$ contains a limit point of a cell of $S_D(d)$. ▪

Definition. An *adjacency* of a cad D of $E^r, r \geq 1$, is an ordered pair (c_1,c_2), such that c_1 and c_2 are adjacent cells of D.

Notation. If (c_1,c_2) is an adjacency with $dim(c_1) = i \leq dim(c_2) = j$, or $dim(c_2) = i \leq dim(c_1) = j$, then we say the adjacency is an *(i,j) type adjacency*. We shall sometimes omit the word "type".

Theorem 3.5.5. Let D be a (B^1,B^2,B^3) basis-determined cad of E^3. Suppose that (d,c) is an adjacency of D' with $dim(d) < dim(c)$, $d \subset \partial c$, and d non-cylindrical with respect to B^3. Then $t \in S_D(d)$ and $s \in S_D(c)$ are adjacent if and only if $dim(t) < dim(s)$ and $t \subset \partial s$.

Proof. Suppose s and t are adjacent. Then by Theorem 3.5.1, one contains a

limit point of the other. By Lemma 3.5.4, s cannot contain a limit point of t, and so t contains a limit point of s, hence $t \cap \partial s \neq \emptyset$. The pair $(dim(d), dim(c))$ is either (0,1), (0,2), or (1,2), hence by Theorem 3.4.3 and Theorem 3.4.7, $S_D^*(c)$ has the unique section boundary property in $S_D^*(d)$. Then by Theorem 3.4.8, $t \subset \partial s$ and $dim(t) < dim(s)$. The converse is obvious. ▪

Theorem 3.5.6. Let D be a (B^1, B^2, B^3) cylindricity-free cad of E^3. Cells s and t of D are adjacent if and only if either $dim(t) < dim(s)$ and $t \subset \partial s$, or $dim(s) < dim(t)$ and $s \subset \partial t$.

Proof. By Theorem 3.5.1, if s and t are adjacent, then one contains a limit point of the other. Assume without loss of generality that t contains a limit point of s. Then $t \cap \partial s \neq \emptyset$, hence by Corollary 3.4.12, $t \subset \partial s$ and $dim(t) < dim(s)$. The converse is obvious. ▪

The following result was mentioned in Chapter 1.

Theorem 3.5.7 Let D be a $(B^1, \ldots, B^r)$ cylindricity-free cad of E^r, $1 \leq r \leq 3$. Let c_1 and c_2 be an i-cell and (i+1)-cell of D, for some $i \geq 0$. c_1 and c_2 are adjacent if and only if c_1 is incident on c_2.

Proof. An immediate consequence of Theorems 3.5.1, 3.5.2, 3.4.10, 3.5.3, and 3.5.6. ▪

Theorem 3.5.8. Let S be a stack in E^r, $r \geq 2$. d_1 and d_2 are adjacent regions of S^* if and only if d_1 and d_2 are a successive section and sector of S^* respectively, or a successive sector and section of S^* respectively.

Proof. Follows directly from Theorem 3.5.1, Theorem 3.3.7 and Theorem 3.3.8. ▪

Theorem 3.5.9. Let D be a cad of E^2. Let d^0 be a 0-cell of D, d^2 a 2-cell of D adjacent to d^0. Then there are exactly two 1-cells d_1^1 and d_2^1 of D such that d^0 is adjacent to d_1^1, d^0 is adjacent to d_2^1, d_1^1 is adjacent to d^2, and d_2^1 is adjacent to d^2.

Proof. Where $d^2 \in S_D(c^1)$, $c^1 \in D'$, suppose $d^2 = (e_1^1, e_2^1)$, where e_1^1 and e_2^1 are sections of $S_D(c^1)$. Where $d^0 \in S_D(c^0)$, $c^0 \in D'$, let s_1^1 and s_2^1 be the sectors of $S_D(c^0)$ directly below and above d^0 respectively. By Corollary 3.3.5, Theorem 3.3.2, and Theorem 3.5.8, e_1^1, e_2^1, s_1^1, and s_2^1 are the only possible 1-cells of D adjacent to both d^0 and d^2. Let z_1 and z_2 be the limit points of e_1^1 and e_2^1 in $Z^*(c^0)$. By Theorem 3.3.26, $\partial d^2 \cap Z^*(c^0) = [z_1, z_2]$. Suppose $z_1 = z_2 = d^0$. Then e_1^1 and e_2^1 are adjacent to both d^0 and d^2, and neither s_1^1 nor s_2^1 is adjacent to d^2. Suppose $z_1 < z_2$ and $z_1 = d^0$. Then e_1^1 and s_2^1 are adjacent to both d^0 and d^2, e_2^1 is not adjacent to d^0, and s_1^1 is not adjacent to d^2. The case $z_1 < z_2$ and $z_2 = d^0$ is similar. Suppose $z_1 < d^0 < z_2$. Then s_1^1 and s_2^1 are adjacent to both d^0 and d^2, and neither e_1^1 nor e_2^1 is adjacent to d^0. ∎

Theorem 3.5.10. Let D be a cad of E^r, $r \geq 2$. Let c_1, c_2 be cells of D', with $c_2 \subset \partial c_1$ and $dim(c_2) < dim(c_1)$. Suppose $S_D^*(c_1)$ has the unique section boundary property in $S_D^*(c_2)$. Then for any section d_1 of $S_D(c_1)$, and any cell d_2 of $S_D(c_2)$, d_2 is adjacent to d_1 if and only if d_2 is the boundary section of d_1 in $S_D(c_2)$.

Proof. If d_2 is the boundary section of d_1 in $S_D(c_2)$, then $d_2 \subset \partial d_1$, hence by Theorem 3.5.1, d_2 is adjacent to d_1. If d_2 is a cell of $S_D(c_2)$ which is adjacent to d_1, then by Theorem 3.5.1, one must contain a limit point of the other. By Lemma 3.5.4, d_1 does not contain a limit point of d_2. Hence d_2 contains a

limit point of d_1, hence $d_2 \cap \partial d_1 \neq \emptyset$, hence by Theorem 3.4.8, $d_2 \subset \partial d_1$, hence by the definition of boundary section, d_2 is the boundary section of d_1 in $S_D(c_2)$ ▪

3.6 Cell boundaries in 3-space, part 2.

Theorem 3.6.1. Let I be an open interval of E with endpoint α, α an algebraic number. Let F(x,y) be an element of I_2 of positive degree. Let f be a continuous function $I \to E$, such that $F(x, f(x)) = 0$ for all $x \in I$. Then $graph(f)$ has a unique limit point in $Z^*(\alpha)$.

Proof. Where $A = \{F(x,y), M(x)\}$, M(x) the integral minimal polynomial of α, let B^2 be a basis for A. Let D be an A-invariant (B^1, B^2) basis-determined cad of E^2. D' has a unique 1-cell c such that $c \cap I \neq \emptyset$ and $\alpha \in \partial c$. There is a unique section s of $S_D(c)$ such that s contains $graph(f) \cap Z(c)$. By Theorem 3.3.24, ∂s meets $Z^*(\alpha)$ in a unique point $< \alpha, \beta >$ Clearly s and $graph(f)$ have the same limit points in $Z^*(\alpha)$. ▪

Theorem 3.6.2. Let c^0 be a semi-algebraic 0-cell in E^2 which is adjacent to a semi-algebraic 1-cell c^1 in E^2. Suppose also that either c^1 is a section of $Z(e)$ for some 1-cell $e \subset E^1$, or c^1 is a sector of $Z(\alpha)$ for some 0-cell $\{\alpha\}$ in E^1, α a real algebraic number. Let s be a section of $Z(c^1)$. Then s has a unique limit point in $Z^*(c^0)$.

Proof. Suppose first that c^1 is a section of Z(e). By Lemma 3.3.22, there exist a primitive $H(x,y) \in I_2$ of positive degree such that $c^1 \subset V(H)$, and a primitive $F(x,y,z) \in I_3$ of positive degree such that $s \subset V(F)$. If F has degree zero in y, then let $R(x,z) = F(x,0,z)$, otherwise let $R(x,z)$ be the resultant with respect to y of H and F. Since F is primitive, H and F are relatively

prime, hence by Theorem 2 of [COL71], $R(x,z)\neq 0$. Let h be a function $e\rightarrow E$ such that $c^1 = graph(h)$, and let f be a function $c^1\rightarrow E$ such that $s = graph(f)$. Define $g(x) = f(x,h(x))$ for $x\in e$. Then g is a continuous function $e\rightarrow E$, and $R(x,g(x)) = R(x,f(x,h(x))) = 0$, since $H(x,h(x)) = 0$ and $F(x,h(x),g(x)) = 0$ for all $x\in e$. Let $c^0 = <\alpha,\beta>$. Let $\{<a_i,b_i,c_i>\}$ be a sequence of points of s converging to $<\alpha,\beta,\gamma>$, with $\gamma\in E^*$. Then $\{<a_i,c_i>\}$ is a sequence of points of $graph(g)$ converging to $<\alpha,\gamma>$. Conversely, let $\{<a_i,c_i>\}$ be a sequence of points of $graph(g)$ which converges to some $<\alpha,\gamma>$, with $\gamma\in E^*$. Set $b_i = h(a_i)$. Then $\{<a_i,b_i,c_i>\}$ is a sequence of points of s which has a subsequence converging to $<\alpha,\beta,\gamma>$, since, first, $\{<a_i,b_i>\}$ is a sequence of points of c^1, second, by hypothesis $<\alpha,\beta>\in\partial c^1$, and third, by Lemma 3.3.23, $<\alpha,\beta>$ is the unique limit point of c^1 in $Z^*(\alpha)$. So $<\alpha,\beta,\gamma>$ is a limit point of s if and only if $<\alpha,\gamma>$ is a limit point of $graph(g)$. By Theorem 3.6.1, $graph(g)$ has a unique limit point $<\alpha,\gamma>$ in $Z^*(\alpha)$. Hence $<\alpha,\beta,\gamma>$ is the unique limit point of s in $Z^*(c^0)$.

Suppose now that c^1 is a sector of $Z(\alpha)$. Let $M(x)$ be the integral minimal polynomial of α. If F has degree zero in x, then let $R(y,z) = F(0,y,z)$; otherwise let $R(y,z)$ be the resultant with respect to x of F and M. Let $I\subset E$ be an open interval such that $c^1 = \{\alpha\}\times I$, and let f be a function $c^1\rightarrow E$ such that $s = graph(f)$. Define $g(y) = f(\alpha,y)$, for $y\in I$. Then g is a continuous function $I\rightarrow E$, and $R(y,g(y)) = R(y,f(\alpha,y)) = 0$, since $M(\alpha) = 0$ and $F(\alpha,y,g(y)) = 0$ for all $y\in I$. Let $c^0 = <\alpha,\beta>$. Now an argument similar to that of the preceding paragraph but somewhat simpler shows that for some γ, $<\alpha,\beta,\gamma>$ is the unique limit point of s in $Z^*(c^0)$. ∎

Theorem 3.6.3. Let D be a (B^1,B^2,B^3) basis-determined cad of E^3. Let

(c^0,c^1) be a (0,1) adjacency of D'. Let s be a section of $S_D(c^1)$ such that for some $B \in B^3$ with $s \subset V(B)$, B is not cylindrical on c^0. Then there is a section t of $S_D^*(c^0)$ such that t is the unique limit point of s in $Z^*(c^0)$.

Proof. By Theorem 3.6.2, s has a unique limit point p in $Z^*(c^0)$. Let B be an element of B^3 which vanishes on s. Since V(B) is closed, $p \in V(B)$. If p is contained in a sector of $S_D(c^0)$, then since D is (B^1,B^2,B^3) basis-determined, B vanishes on this sector, hence, by Lemma 3.3.19, B vanishes on $Z(c^0)$, i.e. B is cylindrical on c^0, a contradiction. Hence p is a section t of $S_D^*(c^0)$. ▪

Theorem 3.6.4. For $r \geq 1$, no nonzero element of I_r vanishes at every point of an r-cell in E^r.

Proof. By induction on r. For r=1, any nonzero element F of I_1 has only finitely many roots, hence since any 1-cell c in E^1 is infinite, F cannot vanish at every point of c. Suppose that the assertion is true for r=k-1, $k \geq 2$, and consider now the case r=k. Let c be a k-cell in E^k, and suppose that $F \in I_k, F \neq 0$, vanishes at every point of c. By Theorem 3.2.1, c is open in E^k, so for any $\alpha \in c$, there exists an open ball B in E^{k-1} and an interval (u,v) in E such that $\alpha \in B \times (u,v) \subset c$. B is a (k-1)-cell in E^{k-1}. For every $\beta \in B$, $F(\beta,x_k)$ vanishes at every point of (u,v), hence $F(\beta,x_k)$ is the zero polynomial for each $\beta \in B$. Thus where $F_n = ldcf(F)$, we have $F_n \neq 0$ and F_n vanishes at every point of B, contrary to the inductive hypothesis. Hence no nonzero element of I_r vanishes at every point of c. ▪

Theorem 3.6.5. Suppose $G,H \in I_r, r \geq 1$, have positive degree, and suppose that $V(G) \cap V(H)$ contains an (r-1)-section s of some cad D of E^r. Then G and H have a common factor of positive degree.

Proof. If r=1 the assertion is immediate. Suppose r>1, and suppose

$s \in S_D(c)$ for $c \in D'$. Let $R = R(x_1, \ldots, x_{r-1})$ be the resultant with respect to x_r of G and H. By Theorem 5 of [COL71], R vanishes everywhere on c, and hence by Theorem 3.6.4, R = 0. Hence, since G and H are both of positive degree, by Theorem 2 of [COL71], they have a common factor of positive degree.

Corollary 3.6.6. Let D be a (B^1,B^2,B^3) basis-determined cad of E^3. For any 2-section d of D, there is a unique $B \in B^3$ such that $d \subset V(B)$.

Theorem 3.6.7. Let D be a (B^1,B^2,B^3) basis-determined cad of E^3. Let (c^0,c^2) be a (0,2) adjacency of D'. Let d be a section of $S_D(c^2)$ such that where $B \in B^3$ is the unique element of B^3 with $d \subset V(B)$, B is not cylindrical on c^0. Then there is a section z of $S_D^*(c^0)$ such that $\partial d \cap Z^*(c^0) = z$.

Proof. Let $x_1, x_2, \cdots$ be a sequence of points in c^2 converging to $c^0 = <\alpha,\beta>$, such that $x_1, x_2, \ldots$ is contained in some ball B in E^2 centered at c^0. Then where d is an f-section, the sequence $<x_1, f(x_1)>, <x_2, f(x_2)>, \cdots$ in d has a subsequence which converges to a limit $<\alpha,\beta,\gamma>$ in $Z^*(c^0)$, since $\overline{B} \times E^*$ is compact. Hence d has a limit point $z = <\alpha,\beta,\gamma>$ in $Z^*(c^0)$. As argued in the proof of Theorem 3.6.3, z is a section of $S_D^*(c^0)$. By Lemma 3.3.35, z is the unique limit point of d in $Z^*(c^0)$. ∎

By Theorem 3.5.9, we may make the following definition:

Definition. Let D be a cad of E^2, let c^0 be a 0-cell of D, and let c^2 be a 2-cell of D adjacent to c^0. The *c^0-bounding 1-cells of c^2* are the unique two 1-cells c_1^1 and c_2^1 of D such that c_1^1 is adjacent to c^0 and c^2, and c_2^1 is adjacent to c^0 and c^2.

Definition. Let D be a cad of E^2, let c^0 and c^2 be adjacent 0- and 2-cells of D, and let c_1^1 and c_2^1 be the c^0-bounding 1-cells of c^2. c^2 is *c^0-horizontally stratified* if there is a neighborhood B in E^2 of c^0 such that for every $<a,b> \in c^2 \cap B$, there are $a_1, a_2 \in E$, $a_1 < a < a_2$, with $<a_1,b> \in c_1^1$, $<a_2,b> \in c_2^1$, and $(a_1,a_2)\times\{b\} \subset c^2$ (or, for every $<a,b>$, $a_2 < a < a_1$ and $(a_2,a_1)\times\{b\} \subset c^2$).

Definition. Let D be a cad of E^2, let c^0 and c^2 be adjacent 0- and 2-cells of D, and let c_1^1 and c_2^1 be the c^0-bounding 1-cells of c^2. c^2 is *c^0-vertically stratified* if there is a neighborhood B in E^2 of c^0 such that for every $<a,b> \in c^2 \cap B$, there are $b_1, b_2 \in E$, $b_1 < b < b_2$, with $<a,b_1> \in c_1^1$, $<a,b_2> \in c_2^1$, and $\{a\}\times(b_1,b_2) \subset c^2$ (or, for every $<a,b>$, $b_2 < b < b_1$ and $\{a\}\times(b_2,b_1) \subset c^2$).

Theorem 3.6.8. Let D be a cad of E^2, let c^0 and c^2 be adjacent 0- and 2-cells of D, and let c_1^1 and c_2^1 be the c^0-bounding 1-cells of c^2. If c^2 is horizontally stratified, then at least one of c_1^1 and c_2^1 is a section.

Proof. If neither c_1^1 nor c_2^1 is a section, then c_1^1 is the 1-sector directly below c^0, and c_2^1 is the 1-sector directly above c^0 (or vice versa). But then for any ball B centered at c^0, and for any $<\alpha,\beta> \in B \cap c^2$, the line $y = \beta$ can meet at most one of c_1^1 and c_2^1, contradicting the assumption that c^2 is c^0-horizontally stratified. Hence at least one of c_1^1 and c_2^1 is a section. ▪

Theorem 3.6.9. Let D be a cad of E^2, let c^0 and c^2 be adjacent 0- and 2-cells of D, and let c_1^1 and c_2^1 be the c^0-bounding 1-cells of c^2. c^2 is c^0-vertically stratified if and only if both c_1^1 and c_2^1 are sections.

Proof. Suppose c^2 is c^0-vertically stratified. Then there exists a ball B cen-

tered at c^0, and $<\alpha,\beta> \in B \cap c^2$, such that there is a point $<\alpha,\beta_1> \in c_1^1$, and a point $<\alpha,\beta_2> \in c_2^1$. But neither the 1-sector below c^0 nor the 1-sector above c^0 contains a point whose x-coordinate is α, hence both c_1^1 and c_2^1 are sections.

Suppose both c_1^1 and c_2^1 are sections. Then for any $<\alpha,\beta> \in c^2$, there exist $<\alpha,\beta_1> \in c_1^1$ and $<\alpha,\beta_2> \in c_2^1$ with $\beta_1 < \beta < \beta_2$ and $\{\alpha\}\times(\beta_1,\beta_2) \subset c^2$ (or $\beta_2 < \beta < \beta_1$). Hence c^2 is c^0-vertically stratified. ▪

Notation. Let $c^0 = <\alpha,\beta>$ be a 0-cell in E^2. Let M(x) be the integral minimal polynomial of $\alpha - \beta$, and let N(x) be the integral minimal polynomial of $\alpha + \beta$. We write $STRAT(c^0)$ to denote the subset $\{M(y-x),N(y+x)\}$ of I_2.

Lemma 3.6.10. Let c^0 be a 0-cell of a cad D of E^2. Suppose $STRAT(c^0) \subset H \subset I_2$. Let $\overline{D}$ be an H-invariant cad of E^2 which refines D. Then every 2-cell of $\overline{D}$ adjacent to c^0 is either c^0-horizontally stratified or c^0-vertically stratified.

Proof. Let c^2 be a 2-cell of $\overline{D}$ adjacent to c^0, and let c_1^1 and c_2^1 be the c^0-bounding 1-cells of c^2. If both c_1^1 and c_2^1 are sections, then by Theorem 3.6.9, c^2 is c^0-vertically stratified. Suppose c_1^1 is a sector. Suppose further that c_1^1 is the 1-sector above c^0, and that c^2 is to the right of c^0. Let e^0, e^1 be the cells of the $\overline{D}$-induced cad of E^1 such that $c^0 \in S_{\overline{D}}(e^0)$ and $c^2 \in S_{\overline{D}}(e^1)$. We claim that c_2^1 is a section; clearly it must be the topmost section of $S_{\overline{D}}(e^1)$. The real variety of M(y-x) consists of the collection of lines of the form $y - x = \gamma_i$, where γ_i is a real root of M(x), i.e. a collection of parallel lines of slope one, exactly one of which passes through $<\alpha,\beta>$. Similarly, the real variety of N(y+x) consists of a collection of parallel lines of slope -1,

exactly one of which passes through $<\alpha,\beta>$. Hence M(y-x) has a 1-section d^1 which is a line segment of slope one, in $S_{\mathcal{D}}(e^1)$. Thus either $c_2^1 = d^1$, or c_2^1 is above d^1 in $S_{\mathcal{D}}(e^1)$. Clearly there exists a ball B centered at c^0 such that for any $<u,v>\in B \cap c^2$, there exists $<u_1,v> \in c_1^1$ and $<\bar{u}_2,v> \in d^1$ with $u_1 < u < \bar{u}_2$. Suppose d^1 is an f-section and c_2^1 a g-section. Since $g(\bar{u}_2) \geq v = f(\bar{u}_2)$, and since $g(u) < v$, by the Intermediate Value Theorem, there exists $u_2\in(u,\bar{u}_2]$ such that $g(u_2) = v$, i.e. $<u_2,v>\in c_2^1$. Clearly $(u_1,u_2)\times\{v\} \subset c^2$. Hence c^2 is c^0-horizontally stratified. The three other possible arrangements of c_1^1 and c_2^1 with respect to c^0 are treated similarly. ▪

Notation. For $F\in I_3$, let F_x, F_y, and F_z denote the partial derivatives of F with respect to x,y, and z respectively. For a differentiable function $f: X \to E$, $X \subset E^2$, let f_x and f_y denote the partial derivatives of f with respect to x and y.

Theorem 3.6.11. Let c be a 2-cell in E^2, let d be an f-section of Z(c), and let $F\in I_3$ be such that $F(x,y,f(x,y)) = 0$ for all $<x,y>\in c$. If none of F_x, F_y, or F_z vanish at any point of d, then f is differentiable, and f_x and f_y are each either positive or negative on c.

Proof. Since F_z does not vanish at any point of d, by the Implicit Function Theorem f is differentiable on c, and

$$f_x = \frac{-F_x}{F_z}$$

$$f_y = \frac{-F_y}{F_z}$$

on c. Since none of F_x , F_y, F_z vanish at any point of d, d is F_x-invariant, F_y-invariant, and F_z-invariant (by arguments similar to that given in Lemma 2.2.4). Hence f_x and f_y are sign-invariant and nonzero on c. ▪

Corollary 3.6.12. Let D be a cad of E^2, let c^0 be a 0-cell of D, and let c^2 be a 2-cell of D adjacent to c^0 which is c^0-horizontally stratified. Let d^2 be an f-section of Z(c), and let $F \in I_3$ be such that $F(x,y,f(x,y)) = 0$ for all $\langle x,y \rangle \in c^2$. If none of F_x, F_y, or F_z vanish at any point of d^2, then for any interval $I \subset E$ and any $b \in E$ such that $I \times \{b\} \subset c^2$, $f(x,b)$ is monotone on I.

Proof. Consider any interval $I \subset E$ and any $b \in E$ such that $I \times \{b\} \subset c^2$. Suppose without loss of generality that f_x is positive on c^2, hence $f_x(x,b)$ is positive on I. Let $g(x)$ be the function $I \to E$ defined by $g(x) = f(x,b)$. Then g is differentiable on I, and $g'(x) = f_x(x,b)$. Hence g' is positive on I, hence g is monotone on I, i.e. $f(x,b)$ is monotone on I. ▪

A corresponding Corollary can obviously be proved in case c^2 is c^0-vertically stratified.

Notation. Let F,G be elements of I_3. We write $F(x,y,z)$ and $G(x,y,z)$ for $F(x_1,x_2,x_3)$ and $G(x_1,x_2,x_3)$. If both F and G have positive degree in x, then $Res_x(F,G)$ denotes R(y,z), the resultant of F and G with respect to x. If one of F and G has positive degree in x, say F, then $Res_x(F,G)$ denotes G(0,y,z). If neither F nor G has positive degree in x, then $Res_x(F,G)$ is undefined. $Res_y(F,G)$ and $Res_z(F,G)$ are defined according to similar conventions.

Notation. For $B(x,y,z) \in I_3$ of positive degree, we write RES(B) to denote the subset $\{Res_x(B,B_x), Res_x(B,B_y), Res_x(B,B_z)\}$ of I_2.

Theorem 3.6.13. Let c^0 and c^2 be adjacent 0- and 2-cells in E^2. Let d^2 be an f-section of $Z(c^2)$, and let $B \in I_3$ be such that $B(x,y,f(x,y)) = 0$ for all $\langle x,y \rangle \in c^2$. Suppose also that B has positive degree, B is irreducible, and B is cylindrical at c^0. If c^2 is RES(B)-invariant, then none of B_x, B_y, or B_z van-

ishes at any point of d^2.

Proof. Suppose B_x vanishes at $<a,b,c>\in d^2$. If $deg(B_x) = 0$, then where T(x,y) = $Res_z(B,B_x)$, T(x,y) = $B_x(x,y,0)$, and T(a,b) = 0. If $deg(B_x) > 0$, then T(x,y) is the resultant with respect to z of B and B_x, hence by Theorem 5 of [COL71], T(a,b) = 0. Then since c^2 is RES(B)-invariant and T vanishes at $<a,b>\in c^2$, T vanishes at every point of c^2. Hence by Theorem 3.6.4, T(x,y) = 0. Suppose $deg(B_x) = 0$. Then since T(x,y) = $B_x(x,y,0)$ = 0, if follows that $B_x(x,y,z) = 0$, hence B has degree 0 in x. But then where $c^0 = <\alpha,\beta>$, since B is cylindrical at c^0, we have $B(x,\beta,z) = 0$, contradicting the primitivity of B. Hence $deg(B_x) > 0$. Then by Theorem 2 of [COL71], B and B_x have a common factor of positive degree in z. But this cannot occur, since the degree of B_x in x is less than the degree of B in x, and B is irreducible. Hence B_x does not vanish at any point of d^2. A similar argument shows that B_y does not vanish at any point of d^2. Suppose B_z vanishes at $<a,b,c>\in d^2$. Then where U(x,y) = $Res_z(B,B_z)$, by the same argument used above, U(x,y) = 0. If $deg(B_z) = 0$, then $B_z(x,y,z) = 0$, which is impossible since B has positive degree in z. Hence $deg(B_z) > 0$, hence as argued above, B and B_z have a common factor of positive degree in z, which is impossible. Hence none of B_x, B_y, or B_z vanish at any point of d^2. •

Notation. For a stack S, let |S| denote the number of elements of S.

Theorem 3.6.14. Let D be a cad of E^2, and let $A \subset I_2$. There exists a coarsest A-invariant cad D^* of E^2 such that D^* is a refinement of D.

Proof. Clearly there exists some A-invariant cad $\overline{D}$ of E^2 which is a refinement of D. Initialize D^* to $\overline{D}$. We now coarsen D^* in two stages as follows. Let D^{**} denote the cad of E^1 induced by D^*. In the first stage, for

each cell c of D^{**}, we "eliminate sections" in $S_{D^*}(c)$ as much as possible. By this we mean: let t be any section of $S_{D^*}(c)$, let s_0 be the sector below t, and let s_1 be the sector above t. Let $s = s_0 \cup t \cup s_1$. If t is not a section of D, and if s is A-invariant, then we delete s_0, t, and s_1 from $S_{D^*}(c)$ and add s. In the second stage, we "eliminate cylinders" of D^* as much as possible. By this we mean: let c, c', and c'' be a successive 1-cell, 0-cell, and 1-cell of D^{**}, i.e. c is adjacent to c' and c' is adjacent to c''. Let $S = S_{D^*}(c)$, let $S' = S_{D^*}(c')$, and let $S'' = S_{D^*}(c'')$. If $c' \notin D'$, if $|S| = |S'| = |S''| = 2m+1$, $m \geq 0$, and if $S_i \cup S'_i \cup S''_i$ is an A-invariant region for $1 \leq i \leq 2m+1$, then we "paste together" S, S', and S'' into a new stack $\bar{S}$ over $c \cup c' \cup c''$, with $|\bar{S}| = 2m+1$. If at the conclusion of the two stages above D^* is not the coarsest A-invariant cad of E^2 which refines D, then there must be some section t of D^*, between sectors s_0 and s_1, such that t is not a section of D and $s = s_0 \cup t \cup s_1$ is A-invariant. But then there is a subset t' of t which is a section of $\bar{D}$, and subsets s'_0 and s'_1 of s_0 and s_1 which are the sectors of $\bar{D}$ below and above t', such that t' is not a section of D and $s' = s'_0 \cup t' \cup s'_1$ is A-invariant. But then t' would have been eliminated in our first stage above. Hence there can be no such section t. Hence D^* is the coarsest A-invariant cad of E^2 which refines D. ∎

Definition. Let D be a cad of E^r, $r \geq 2$. Let $\bar{D}$ be a refinement of D'. The *refinement of D determined by* $\bar{D}$ is the following cad of E^r: The union of all cells in E^r of the form $(S_j(d) \cap Z(c))$, where c is a cell of $\bar{D}$, d is a cell of D' containing c, and $S_j(d)$ is the jth element of $S_D(d)$.

Definition. Let D be a (B^1, B^2, B^3) basis-determined cad of E^3. Let $H \subset I_2$.

Let c^0 and c^2 be an adjacent 0-cell and 2-cell of D', and let d be a B-section of $S_D(c^2)$, $B \in B^3$. Let $\bar{D}$ be the coarsest H-invariant refinement of D', and let D^* be the refinement of D determined by $\bar{D}$. The *H-set of z-values of d over* c^0 is a subset K of E^* defined as follows:

(1) Where c_1^1 and c_2^1 are the c^0-bounding 1-cells of c^2, let d_i be the boundary section of d in $S_D^*(c_i^1)$, for i=1,2, and let $< \alpha,\beta,z_i >$ be the unique limit point of d_i in $Z^*(c^0)$. Then z_1 and z_2 are in K.

(2) For each 1-section s^1 of $\bar{D}$ such that s^1 is adjacent to c^0 and $s^1 \cap c^2 \neq \emptyset$, where t^1 is the unique section of $S_{D^*}(s^1)$ contained in d, let $< \alpha,\beta,z >$ be the limit point of t^1 in $Z^*(c^0)$. Then z is in K.

Theorem 3.6.15. Let D be a (B^1,B^2,B^3) basis-determined cad of E^3. Let $H \subset I_2$. Let c^0 and c^2 be an adjacent 0-cell and 2-cell of D', and let d be a B-section of $S_D(c^2)$, $B \in B^3$. For any z in the H-set of z-values of d over c^0, $< \alpha,\beta,z >$ is a limit point of d.

Proof. Let $\bar{D}$ be the coarsest H-invariant refinement of D', and D^* the refinement of D determined by $\bar{D}$. Let K be the H-set of d-values over c^0. For any $z \in K$, there is a 1-section e of D^* such that $< \alpha,\beta,z > \in \partial e$, and either $e \subset \partial d$ or $e \subset d$. Hence by Lemma 3.3.11, $< \alpha,\beta,z > \in \partial d$, hence $< \alpha,\beta,z >$ is a limit point of d. ▪

Theorem 3.6.16. Let Q and R be regions in E^{r-1}, $r \geq 2$, such that $Q \subset \partial R$. Let s be a section of $Z^*(R)$ such that $\partial s \cap Z^*(Q)$ is a section of $Z^*(Q)$. Let s be the f-section of $Z^*(R)$. Then f has a unique extension to a continuous function $h:(Q \cup R) \rightarrow E^*$, and for every $\alpha \in Q$, $h(\alpha) = \lim_{x \rightarrow \alpha \text{ in } R} f(x)$.

Proof. Let $\partial s \cap Z^*(Q) = t$, where t is the g-section of $Z^*(Q)$. Then by Lemma 3.3.12, for every $\alpha \in Q$, $\lim_{x \rightarrow \alpha \text{ in } R} f(x) = g(\alpha)$, so f is continuous at $\alpha \in \bar{R}$

(see e.g. [KEL55], p.100). By [KEL55], p.100, since E^{r-1} is a regular topological space, f has a unique extension to a continuous function $h:(Q \cup R) \to E^*$. By definition of h, for every $\alpha \in Q$, $h(\alpha)$ is a limit point of s, and hence $h(\alpha) = g(\alpha)$. So $\lim_{x \to \alpha \text{ in } R} f(x) = h(\alpha)$ for every $\alpha \in Q$. ∎

Theorem 3.6.17. Let D be a (B^1,B^2,B^3) basis-determined cad of E^3, with B^3 an irreducible basis. Let c^0 be a 0-cell of D' such that $B \in B^3$ is cylindrical on c^0, let c^2 be a 2-cell of D' adjacent to c^0, and let d be a B-section and f-section of $S_D(c^2)$. Let $H = STRAT(c^0) \cup RES(B)$. Let K be the H-set of z-values of d over c^0. Let $z_1 = \min(K)$ and $z_2 = \max(K)$. Then $\partial d \cap Z^*(c^0) \subset c^0 \times [z_1,z_2]$.

Proof. Let $\bar{D}$ be the coarsest H-invariant refinement of D'. A point p of $Z^*(c^0)$ is a limit point of d if and only if there is a sequence in d converging to it. Any sequence of points in d has an infinite subsequence in $Z(e) \cap d$, for some cell e in $\bar{D}$ adjacent to c^0 and contained in c^2, with either dim(e)=1 or dim(e)=2. Consider any sequence $P = p_1,p_2,\ldots$ of points in d converging to $p \in Z^*(c^0)$. Let e be a cell of $\bar{D}$ adjacent to c^0 and contained in c^2, such that P has an infinite subsequence in $Z(e) \cap d$. Suppose dim(e) = 1. Then the limit p of P in $Z^*(c^0)$ must be the limit point in $Z^*(c^0)$ of the unique 1-section of $S_{D^*}(e)$ contained in d, where D^* is the refinement of D determined by $\bar{D}$. Hence the z-coordinate γ of p is in K, by definition of K, and so $\gamma \in [z_1,z_2]$.

Suppose dim(e) = 2. Let $P' = <a_1,b_1,f(a_1,b_1)>,\ldots$ be the subsequence of P in $Z(e) \cap d$ which converges to p. Let c_1^1 and c_2^1 be the c^0-bounding 1-cells of e. By Lemma 3.6.10, e is either c^0-horizontally stratified or c^0-vertically stratified; assume without loss of generality c^0-horizontally

stratified. Hence since $\pi(P') = <a_1,b_1>, \cdots$ converges to c^0, there exists N such that for all $i \geq N$, there exist u_i, v_i such that $u_i < a_i < v_i$, $<u_i,b_i> \in c_1^1$, $<v_i,b_i> \in c_2^1$, and $(u_i,v_i)\times\{b_i\} \subset e$. Since e is RES(B)-invariant, by Theorem 3.6.13 and Corollary 3.6.12, $f(x,b_i)$ is monotone on (u_i,v_i) for each i. By Theorems 3.4.7 and 3.6.16, f can be extended to a continuous map $\bar{f}$ on $(e \cup c_1^1 \cup c_2^1)$. Clearly $\bar{f}(x,b_i)$ is monotone on $[u_i,v_i]$ for each $i \geq N$. Hence for each $i \geq N$, either $\bar{f}(u_i,b_i) \leq \bar{f}(a_i,b_i) \leq \bar{f}(v_i,b_i)$, or $\bar{f}(u_i,b_i) \geq \bar{f}(a_i,b_i) \geq \bar{f}(v_i,b_i)$; assume without loss of generality the former. Let P_1 and P_2 denote the sequences $<u_N,b_N,\bar{f}(u_N,b_N)>, \ldots$ and $<v_N,b_N,\bar{f}(v_N,b_N)>, \ldots$ respectively. Where $c^0 = <\alpha,\beta>$, P_1 has a subsequence which converges to some $<\alpha,\beta,\gamma_1>$, and P_2 has a subsequence which converges to some $<\alpha,\beta,\gamma_2>$. γ_1 and γ_2 are in K by definition of K. Where P converges to $p = <\alpha,\beta,\gamma>$, since $\bar{f}(u_{N+j},b_{N+j}) \leq \bar{f}(a_{N+j},b_{N+j}) \leq \bar{f}(v_{N+j},b_{N+j})$, for any $j \geq 0$, we must have $\gamma_1 \leq \gamma \leq \gamma_2$. Hence $z_1 \leq \gamma \leq z_2$, hence $\gamma \in [z_1,z_2]$. ∎

Theorem 3.6.18. Let D be (B^1,B^2,B^3) basis-determined cad of E^3, with B^3 an irreducible basis. Let c^0 be a 0-cell of D' such that $B \in B^3$ is cylindrical on c^0, let c^2 be a 2-cell of D' adjacent to c^0, and let d be a B-section and f-section of $S_D(c^2)$. Let $H = STRAT(c^0) \cup RES(B)$. Let K be the H-set of z-values of d over c^0. Let $z_1 = \min(K)$ and $z_2 = \max(K)$. then $\partial d \cap Z^*(c^0) = c^0 \times [z_1,z_2]$.

Proof. By Theorem 3.6.17, $\partial d \cap Z^*(c^0) \subset c^0 \times [z_1,z_2]$. By Theorems 3.6.15 and 3.3.31, every point of $c^0 \times [z_1,z_2]$ is a limit point of S, hence $\partial d \cap Z^*(c^0) = c^0 \times [z_1,z_2]$. ∎

Definition. A (B^1,B^2,B^3) basis-determined cad D of E^3, B^3 an irreducible basis, is (B^1,B^2,B^3) *cylindricity-refined* if for each 0-cell $c^0 = <\alpha,\beta>$ of D',

each $d \in D'$ which is adjacent to c^0, and each section t of $S_D(d)$, where $\partial t \cap Z^*(c^0) = c^0 \times [z_1, z_2]$, $z_1 \leq z_2$, $< \alpha, \beta, z_1 >$ and $< \alpha, \beta, z_2 >$ are sections of $S_D^*(c^0)$.

Clearly any (B^1, B^2, B^3) cylindricity-free cad of E^3 is trivially (B^1, B^2, B^3) cylindricity-refined. The clustering cad algorithm produces a cylindricity-refined cad of E^3, whereas the original cad algorithm may not.

Theorem 3.6.19. Let D be a (B^1, B^2, B^3) cylindricity-refined cad of E^3. Suppose (c^0, c^2) is a (0,2) adjacency of D' such that c^0 is cylindrical with respect to B^3. Let d be a B-section of $S_D(c^2)$. Then there is a collection T of elements of $S_D(c^0)$ such that $\partial d \cap Z(c^0) =$ union T.

Proof. An immediate consequence of the definition of cylindricity-refined cad.

Theorem 3.6.20. Let D be a (B^1, B^2, B^3) cylindricity-refined cad of E^3. Suppose (c^0, c^2) is a (0,2) adjacency of D'. Then $S_D^*(c^2)$ has the section boundary property in $S_D^*(c^0)$.

Proof. If c^0 is not cylindrical with respect to B^3, then the assertion follows from Theorem 3.4.3. If c^0 is cylindrical with respect to B^3, then by Theorems 3.6.7 and 3.6.19, for every section d of $S_D(c^2)$, there is a collection T of elements of $S_D(c^0)$ such that $\partial d \cap Z(c^0) =$ union T. Then clearly for every section d of $S_D(c^2)$, there is a collection T^* of elements of $S_D^*(c^0)$ such that $\partial d \cap Z^*(c^0) =$ union T^*. Hence by Lemma 3.3.4, $S_D^*(c^2)$ has the section boundary property in $S_D^*(c^0)$ ∎

Theorem 3.6.21. Let D be a (B^1, B^2, B^3) cylindricity-refined cad of E^3. Let (c^0, c^1) be a (0,1) adjacency of D'. Then $S_D^*(c^1)$ has the unique section boun-

dary property in $S_D^*(c^0)$.

Proof. If c^0 is not cylindrical with respect to B^3, then the assertion follows from Theorem 3.4.3. If c^0 is cylindrical with respect to B^3, then the assertion is an immediate consequence of Theorem 3.6.3, the definition of cylindricity-refined cad, and Lemma 3.3.4. ▪

Theorem 3.6.22. Let D be a (B^1,B^2,B^3) cylindricity-refined cad of E^3. D has the boundary property.

Proof. By Corollary 3.3.25, D' has the boundary property. Let (c,d) be an adjacency of D', with $dim(c) < dim(d)$. By Theorem 3.5.3, $c \subset \partial d$. By Theorems 3.4.7, 3.6.20, and 3.6.21, either $S_D^*(d)$ has the unique section boundary property in $S_D^*(c)$, or $S_D^*(d)$ has the section boundary property in $S_D^*(c)$. Hence by Theorem 3.3.15 and Corollary 3.3.40, $S_D(d)$ has the boundary property in $S_D(c)$. By Theorem 3.5.3, for any cells c,d of D', $c \subset \partial d$ if and only if c and d are adjacent. Hence by Theorem 3.3.10, D has the boundary property ▪

CHAPTER 4

THE CLUSTERING CYLINDRICAL ALGEBRAIC DECOMPOSITION ALGORITHM

4.1 Overview

Sections 4.2 - 4.5 present the theoretical results underlying the clustering cad algorithm. The material of Section 4.2 is applicable to r-space for any r; Sections 4.3 - 4.5 deal with those points that apply individually to 1-space, 2-space, and 3-space. In Section 4.6 and 4.7 we discuss sample point and defining formula construction in the clustering cad algorithm. Section 4.8 gives abstract algorithms for 2-space and 3-space, and discusses various aspects of them.

To contrast the clustering cad algorithm with the original, one may begin with the following two points. On the one hand, the clustering algorithm produces more information than the original. In addition to an A-invariant cad of E^r, it produces a decomposition of E^r into maximal A-invariant regions. (A is the set of r-variate input polynomials to the cad algorithm). Each such region R is the union of certain cells of the cad. A collection of cells whose union is a particular R is a "cluster"; R is the "underlying region" of that cluster. These maximal A-invariant regions are of geometric significance. For example, if A consists of a single bivariate polynomial F(x,y), then the number of clusters on which F = 0 is the number of connected components of the curve.

On the other hand, the clustering cad algorithm produces less information than the original. One no longer obtains a sample point for every cell of a cad. It does, however, produce a collection of sample points sufficient to enable one to do RCF quantifier elimination, as we now briefly describe. (See [COL75] for further information on quantifier elimination.) Suppose we are given a standard prenex formula

$$\Psi(x_1, \ldots, x_k) = (Q_{k+1}x_{k+1})\ldots(Q_r x_r)\varphi(x_1, \ldots, x_r),\ k \geq 0.$$

In order to eliminate quantifiers, we first determine an A-invariant cad D of E^r, where A is the set of polynomials occurring on the left-hand sides of the standard atomic formulas of φ. We then need to

1. determine the truth value of φ on each cell of D, and
2. construct a defining formula for each cell of the D-induced cad of E^k.

With the original cad algorithm, we determine the truth value of φ on a given cell of the D by evaluating φ at its sample point. With the clustering cad algorithm, although we may not have a sample point for a particular cell c_1, we always will have a sample point for a cell c_2 in the same cluster as c_1. The truth value of φ is the same on both c_1 and c_2, since each element of A has the same sign on both c_1 and c_2, and hence each standard atomic formula of φ has the same truth value on c_1 and c_2. Thus we can determine the truth value of φ at any cell c of D without necessarily having a sample point for c.

Chapter 6 will compare the performance of the two algorithms on several examples.

4.2 Clusters, maximal clusters, and algebraic components.

Notation. Suppose $A \subset I_r$, $r \geq 1$. We call an A-invariant region in E^r an *A-region* in E^r, an A-invariant decomposition an *A-decomposition*, an A-invariant stack an *A-stack*, and an A-invariant cad an *A-cad*.

Definition. Suppose $A \subset I_r$, $r \geq 2$. Two disjoint A-regions are *A-adjacent* if their union is an A-region.

Theorem 4.2.1. For $A \subset I_r$, $r \geq 1$, let R_1, R_2, and R_3 be regions such that R_1 is A-adjacent to R_2, R_2 is A-adjacent to R_3, and R_1 is adjacent to R_3. Then R_1 is A-adjacent to R_3.

Proof. By the first two hypotheses, each $A_i \in A$ is invariant on $R_1 \cup R_3$, and by the third hypothesis, $R_1 \cup R_3$ is a region ▪

Definition. Let $r \geq 1$. A nonempty collection C of disjoint cells in E^r is a *cluster* in case the union of C is a region. The union of a cluster C is its *underlying region*, written R(C). For $A \subset I_r$, a cluster C is an *A-cluster* if R(C) is an A-region. For any cad D of E^r, a *cluster of D* is a cluster each of whose cells is a cell of D. For an A-cad D of E^r, a cluster of D which is also an A-cluster is an *A-cluster of D.*

Definition. For any cluster C, a *subcluster* of C is a subset of C which is itself a cluster.

Definition. For $A \subset I_r$, $r \geq 2$, a region c in E^{r-1} is *A-regularizing* if A is regular on c. A cluster C in E^{r-1} is *A-regularizing* if R(C) is A-regularizing. A decomposition of E^{r-1} is *A-regularizing* in case each of its regions is A-regularizing.

Definition. A $(B^1, \ldots, B^r)$ basis-determined cad of E^r, $r \geq 1$, is *regular* in

case either $r = 1$, or $r \geq 2$, B^r is regular on each cell of D', and D' is a regular $(B^1, \ldots, B^{r-1})$ basis-determined cad of E^{r-1}.

Definition. A *clustering* of a cad D of E^r, $r \geq 1$, is a partition of D into clusters. For $A \subset I_r$, an *A-clustering* of an A-cad D of E^r is a clustering of D into A-clusters. If $r \geq 2$ and D' is A-regularizing, then an *A-regularizing clustering* of D' is a clustering of D' such that each cluster is A-regularizing. A *clustered cad* is a cad D together with a clustering of D.

Definition. Let D be a cad of E^r, $r \geq 1$. Let L be a clustering of D. An adjacency (c_1, c_2) of D is an *L-inner adjacency* of D in case c_1 and c_2 are both in the same cluster of L; otherwise (c_1, c_2) is an *L-outer adjacency* of D.

Theorem 4.2.2. Let A be a subset of I_r, $r \geq 2$, and let B be a basis for PP(A). Let c be a region in E^{r-1} which is both B-regularizing and CONT(A)-invariant. Then A is regular on c, and $S(B_c, c)$ is A-invariant.

Proof. It is easy to see that if B is regular on c, then any subset of B is regular on c. The proof is completed using the same argument as used in the proof of Theorem 2.3.1. ▪

Theorem 4.2.3. Suppose $F \in I_r$ is regular on a region R in E^{r-1}, $r \geq 2$. Let R' be a subregion of R. If F is delineable on R, then F is delineable on R', and $|S(F,R)| = |S(F,R')|$. If F is cylindrical on R, then F is cylindrical on R'.

Proof. Obvious.

Corollary 4.2.4. Let D be a regular $(B^1, \ldots, B^r)$ basis-determined cad of E^r, $r \geq 2$. Let L' be a B^r-regularizing clustering of D'. Let c and d be cells of D' in the same L'-cluster C'. Let R = R(C'). Then $B_c^r = B_R^r = B_d^r$.

Proof. Suppose $B \in B^r$ is delineable on c but cylindrical on R. Then since c is a subregion of R, by Theorem 4.2.3, B is cylindrical on c, a contradiction. If B is delineable on R, then since c is a subregion of R, by Theorem 4.2.3, B is delineable on c. Hence B is delineable on c if and only if it is delineable on R. Similarly one shows B delineable on d if and only if B delineable on R. Hence $B_c^r = B_R^r = B_d^r$. ▪

Corollary 4.2.5. Let D be a regular $(B^1, \ldots, B^r)$ basis-determined cad of $E^r, r \geq 2$. Let L' be a B^r-regularizing clustering of D'. Let c and d be cells of D' in the same L'-cluster C'. Let R = union C'. Then $|S_D(c)| = |S(B_R^r, R)| = |S_D(d)|$.

Proof. Where C' is the L'-cluster containing c and d, let R = R(C'). We have $S_D(c) = S_D(B_c^r, c)$, and $S_D(d) = S(B_d^r, d)$. By Corollary 4.2.4, $S_D(c) = S(B_R^r, c)$, and $S_D(d) = S(B_R^r, d)$. Hence by Theorem 4.2.3,

$$\begin{aligned} |S_D(c)| &= |S(B_R^r, c)| \\ &= |S(B_R^r, R)| \\ &= |S(B_R^r, d)| \\ &= |S_D(d)| \text{ ▪} \end{aligned}$$

Theorem 4.2.6. Let D be a regular $(B^1, \ldots, B^r)$ basis-determined cad of E^r, $r \geq 2$. Let L' be a B^r-regularizing clustering of D', and let C' = $\{c_1, \ldots, c_s\}$ be an L'-cluster, let R = R(C'). Let $t = |S(B_R^r, R)| = |S_D(c_i)|$, $1 \leq i \leq s$. Then for $1 \leq k \leq t$,

$$S_k(B_R^r, R) = \bigcup_{j=1}^{s} S_k(B_{c_j}^r, c_j).$$

Proof. By Corollary 4.2.4, if suffices to establish that

$$S_k(B_R^r,R) = \bigcup_{j=1}^{s} S_k(B_R^r,c_j)$$

for each k. But since $R = \bigcup_{j=1}^{s} c_j$, this is immediate. ∎

Theorem 4.2.7. For $A \subset I_r$, let D be a regular $(B^1, \ldots, B^r)$ basis-determined A-cad of E^r, such that D' is CONT(A)-invariant. Let L' be a B^r-regularizing and CONT(A)-invariant clustering of D'. Let (c_1,c_2) be an L'-inner adjacency. Then for any k, $1 \le k \le |S_D(c^1)|$, $S_k(B_{c_1}^r,c_1)$ is A-adjacent to $S_k(B_{c_2}^r,c_2)$.

Proof. Let $R = c_1 \cup c_2$. By Corollary 4.2.4, $B_{c_1}^r = B_R^r = B_{c_2}^r$. For any k, $1 \le k \le |S_D(c^1)|$, since L' is B^r-regularizing, $S_k(B_R^r,R)$ is well-defined, and by Theorem 4.2.6,

$$S_k(B_R^r,R) = S_k(B_{c_1}^r,c_1) \cup S_k(B_{c_2}^r,c_2).$$

Since $S_k(B_R^r,R)$ is a region, $S_k(B_{c_1}^r,c_1)$ and $S_k(B_{c_2}^r,c_2)$ are adjacent. By Theorem 4.2.2, $S_k(B_R^r,R)$ is A-invariant, hence $S_k(B_{c_1}^r,c_1)$ and $S_k(B_{c_2}^r,c_2)$ are A-adjacent. ∎

Theorem 4.2.8. For $A \subset I_r$, let D be a regular $(B^1, \ldots, B^r)$ basis-determined A-cad of E^r, such that B^r is a basis for PP(A), and D' is CONT(A)-invariant. Let L' be a B^r-regularizing and CONT(A)-invariant clustering of D'. Let (c_1,c_2) be an L'-inner adjacency. Let $s_1 \in S_D(c_1)$ and $s_2 \in S_D(c_2)$. Suppose either s_1 and s_2 are both sections, or s_1 and s_2 are both sectors. Then s_1 and s_2 are adjacent if and only if $s_1 = S_k(B_{c_1}^r,c_1)$ and $s_2 = S_k(B_{c_2}^r,c_2)$, for some k, $1 \le k \le |S_D(c_1)|$.

Proof. Suppose s_1 and s_2 are adjacent. Let $R = c_1 \cup c_2$. Suppose $s_1 \subset S_i(B_R^r,R) = t_i$, and $s_2 \subset S_j(B_R^r,R) = t_j$. By Theorem 4.2.6, either both of t_i and t_j are sections of $S(B_R^r,R)$, or both are sectors of $S(B_R^r,R)$. If $i \neq j$,

then clearly t_i and t_j must be adjacent. Hence by Theorem 3.5.8, one of t_i and t_j is a section and the other a sector, a contradiction. Hence i = j, and by Theorem 4.2.6, $s_1 = S_i(B^r_{c_1},c_1)$, and $s_2 = S_i(B^r_{c_2},c_2)$. The converse follows immediately from Theorem 4.2.7. ∎

Corollary 4.2.9. Assume the hypotheses of Theorem 4.2.8. Then s_1 and s_2 are A-adjacent if and only if $s_1 = S_k(B^r_{c_1},c_1)$ and $s_2 = S_k(B^r_{c_2},c_2)$, for some k, $1 \leq k \leq |S_D(c_1)|$.

Proof. Follows immediately from Theorems 4.2.8 and 4.2.7.

Definition. For $A \subset I_r, r \geq 2$, let D be an A-cad of E^r such that D' is A-regularizing. Let L' be an A-regularizing clustering of D'. Let C be an L'-cluster. A *maximal A-cluster of D in Z(R(C))* is an A-cluster of D whose underlying region is contained in Z(R(C)), and which is not a proper subset of any other A-cluster of D whose underlying region is contained in Z(R(C)).

Definition. For $A \subset I_r, r \geq 2$, let D be an A-cad of E^r and L' a clustering of D'. An *L'-initial A-cluster of D* is a maximal A-cluster of D in Z(R(C)) for some L'-cluster C. The *L'-initial A-clustering of D* is the A-clustering of D consisting of all L'-initial A-clusters of D.

Theorem 4.2.10. For $A \subset I_r$, let D be a regular $(B^1, \ldots, B^r)$ A-cad of E^r, such that B^r is a basis for PP(A) and D' is CONT(A)-invariant. Let L' be a B^r-regularizing and CONT(A)-invariant clustering of D'. For any cluster C'of L', for any c_1, c_2 of C', for any $k \geq 1$, $S_k(B^r_{c_1},c_1)$ and $S_k(B^r_{c_2},c_2)$ are in the same L'-initial A-cluster.

Proof. Let R = union C'. By Theorem 4.2.2, for any $k \geq 1$, $S_k(B^r_R,R)$ is A-invariant. Let C' = $\{c_1, \ldots, c_s\}$. by Theorem 4.2.6, for any $k \geq 1$,

$$S_k(B^r_R,R) = \bigcup_{j=1}^{s} S_k(B^r_{c_j},c_j),$$

hence

$$C_k = \{S_k(B^r_{c_1},c_1), \ldots, S_k(B^r_{c_s},c_s)\}$$

is an A-cluster of D. Since $S_k(B^r_R,R)$ is contained in Z(R), C_k is a subcluster of an L'-initial A-cluster of D. ▪

Definition. For $A \subset I_r, r \geq 1$, and for an A-cad D of E^r, an A-cluster of D is *maximal* if it is not a proper subset of any other A-cluster of D.

Definition. Suppose $A \subset I_r$, $r \geq 1$. An *A-algebraic component* of E^r, also called an *A-component* of E^r, is a maximal A-region in E^r, i.e. an A-region in E^r such that no proper superset of it in E^r is an A-region. A subset of E^r is an *algebraic component* of E^r with respect to $A \subset I_r$, $r \geq 1$, if it is an A-component.

Theorem 4.2.11. For $A \subset I_r$, $r \geq 1$, and for an A-cad D of E^r, an A-cluster of D is maximal if and only if its underlying region is an A-component.

Proof. Suppose C is an A-cluster of D such that R(C) is not an A-component. Let R = R(C). Then there exists an A-region R' properly containing R. Let $R^* = R' - R$. Let $c_1, \ldots, c_k$, $k \geq 1$, be the cells of D which meet R^*, and let $Q = \bigcup_{i=1}^{k} c_i$. Then $R \cup Q$ is an A-region which is the underlying region of an A-cluster of D which properly contains C. Hence C is not a maximal A-cluster of D. Suppose C is an A-cluster of D which is not maximal. Then there exists an A-cluster C' of D which properly contains C, hence R(C) is properly contained in the A-region R(C'), hence R(C) is not an A-component. ▪

Definition. Suppose $A \subset I_r$, $r \geq 1$, and let D be an A-cad of E^r. The *coarsest*

A-clustering of D is the unique A-clustering of D each of whose clusters is a maximal A-cluster.

Our objective in the clustering cad algorithm is, given $A \subset I_r$, to construct both an A-invariant cad D of E^r, and the coarsest A-clustering of D.

Definition. For $A \subset I_r$, $r \geq 2$, let D be an A-cad of E^r and L' a clustering of D'. An *L'-sufficient set of adjacencies of D'* is a set S' of adjacencies of D', with the following property: if L is an A-clustering of D such that

1. Each L'-initial A-cluster of D is a subcluster of some L-cluster, and
2. For each adjacency (c_1,c_2) in S', if a cell d_1 of $S_D(c_1)$ is A-adjacent to a cell d_2 of $S_D(c_2)$, then d_1 and d_2 are in the same L-cluster,

then L is the coarsest A-clustering of D.

Theorem 4.2.12. For $A \subset I_r$, $r \geq 2$, let D be an A-cad of E^r and L' be a clustering of D'. The set of all L'-outer adjacencies of D' is L'-sufficient.

Proof. Let S' be any set of L'-outer adjacencies of D' which is not L'-sufficient. Let L be an A-clustering of D such that (1) every L'-initial A-cluster of D is a subcluster of some L-cluster, and (2) for each adjacency (c_1,c_2) in S', if $d_1 \in S_D(c_1)$ is A-adjacent to $d_2 \in S_D(c_2)$, then d_1 and d_2 are in the same L-cluster. Suppose L is not the coarsest A-clustering of D. Then there exist cells e_1 and e_2 of D such that e_1 and e_2 are A-adjacent but not in the same L-cluster. Let c_1 and c_2 be the unique cells of D' such that $e_1 \in S_D(c_1)$ and $e_2 \in S_D(c_2)$. c_1 and c_2 do not belong to the same L'-cluster, because if they did, then e_1 and e_2 would belong to the same L'-initial cluster of D. Hence (c_1,c_2) is an L'-outer adjacency which does not belong to S'. Therefore S' is not the set of all L'-outer adjacencies of D'. Hence the set of

all L'-outer adjacencies of D' is L'-sufficient. ▪

4.3 Clustering in 1-space.

Given $A \subset I_1$, and a (B^1) basis-determined A-cad D of E^1, it is straightforward to construct the coarsest A-clustering of D. For any adjacent 0-cell c^0 and 1-cell c^1, some element of B^1, and hence some nonzero $A_i \in A$ vanish at c^0, but no nonzero $A_i \in A$ vanishes on c^1, hence c^0 and c^1 are not A-adjacent. Thus the clustering of D in which each cell of D is the unique element of the cluster to which it belongs is the coarsest A-clustering L of D. It follows that every adjacency of cells of D is an L-outer adjacency.

4.4 Clustering in 2-space.

The following theorem will be useful in restricting the amount of adjacency testing that must be done in the clustering cad algorithm.

Theorem 4.4.1. For $A \subset I_2$, let D be a (B^1,B^2) basis-determined A-cad of E^2, such that B^2 is a basis for PP(A). Let (c_1,c_2) be an adjacency of D' with c_1 a 0-cell and c_2 a 1-cell. Let $s_1 \in S_D(c_1)$ and $s_2 \in S_D(c_2)$ be A-adjacent. Then s_1 and s_2 are either both sections or both sectors.

Proof. Suppose that s_1 is a section and s_2 a sector. Since D is (B^1,B^2) basis-determined, some $B \in B^2$ vanishes on s_1, hence some nonzero $A_i \in A$ vanishes on s_1. Since $dim(c_2) = 1$, we have $dim(s_2) = 2$. By Theorem 3.6.4, no nonzero element of A vanishes on s_2, hence s_1 and s_2 are not A-adjacent, a contradiction. Suppose s_1 is a sector and s_2 is a section. Then (s_1,s_2) is a (1,1) adjacency of D, which contradicts Theorem 3.5.3. Hence s_1 and s_2 are either both sections, or both sectors. ▪

4.5 Clustering in 3-space

Definition. The *dimension* of a cluster in E^r, $r \geq 1$ is the dimension of the largest cell it contains. A cluster of dimension i is an *i-cluster*.

Notation. Given an (i,j) adjacency between an i-cell of an m-cluster ($i \leq m$) and a j-cell of an n-cluster ($j \leq n$), we write this as a type

$$(i,j)[m,n]$$

adjacency.

Lemma 4.5.1. Let $F \in I_r$, and let R_1 and R_2 be adjacent F-invariant regions in E^r. Suppose also that F is nonzero on R_1 and F is nonzero on R_2. Then $R_1 \cup R_2$ is F-invariant.

Proof. Let $R = R_1 \cup R_2$. Since R_1 and R_2 are adjacent, R is a region such that $F(x) \neq 0$ for all $x \in R$. Hence by Lemma 2.2.3, R is F-invariant. ▪

Theorem 4.5.2. For $A \subset I_2$, let D be a (B^1,B^2) basis-determined A-cad of E^2, such that B^2 is a basis for PP(A). Let L be the coarsest A-clustering of D. Then every L-outer adjacency of D is of one of the following types:

(0,1)[0,1]

(0,1)[0,2]

(0,1)[1,1]

(0,1)[1,2]

(0,2)[0,2]

(0,2)[1,2]

(1,2)[1,2]

Proof. By Theorem 3.5.3, every adjacency of D is of type (i,j)[m,n] with i < j. Since furthermore we must have $0 \leq i \leq m$ and $0 \leq j \leq n$, the only types of adjacencies not mentioned in the theorem are are (0,1)[2,2], (0,2)[2,2],

(0,1)[2,1], and (1,2)[2,2]. By Theorem 3.6.4, no nonzero element of A vanishes on a 2-cell in E^2, hence no nonzero element of A vanishes on any cell of a 2-cluster. However some element of B^2, and hence some nonzero element of A, must vanish on a 0-cell of a (B^1,B^2) basis-determined cad, hence a 0-cell cannot be in a 2-cluster. Thus (0,1)[2,2], (0,2)[2,2], and (0,1)[2,2] L-outer adjacencies cannot occur. Suppose $c_1 \in D$ belongs to a 2-dimensional L-cluster C_1, $c_2 \in D$ belongs to a 2-dimensional L-cluster C_2, and c_1 is adjacent to c_2. For each nonzero $A_i \in A$, by Lemma 4.5.1, $c_1 \cup c_2$ is A_i-invariant. Hence c_1 is A-adjacent to c_2, hence by maximality of L-clusters, (c_1,c_2) is an L-inner adjacency. Hence L-outer adjacencies of type (i,j)[2,2] cannot occur. ∎

Definition. Let D be a cad of E^2 and L a clustering of D. A (0,2)[1,2] L-outer adjacency (c^0,c^2) between a 0-cell c^0 of a 1-cluster C_1 of L, and a 2-cell c^2 of a 2-cluster C_2 of L, is *nonisolated* if there is a 1-cell c^1 of C_1 which is adjacent both to c^0 and c^2; otherwise (c^0,c^2) is *isolated*. A (0,1)[1,2] L-outer adjacency (c^0,c^1) between a 0-cell c^0 of a 1-cluster C_1 of L, and a 1-cell c^1 of a 2-cluster C_2 of L, is *nonisolated* if there is a 1-cell d^1 of C_1 and a 2-cell d^2 of C_2, such that c^0 is adjacent to d^1, c^1 is adjacent to d^2, and d^1 is adjacent to d^2; otherwise (c^0,c^1) is *isolated*.

Theorem 4.5.3. For $A \subset I_3$, let D be a regular, (B^1,B^2,B^3) basis-determined A-cad of E^3, such that B^3 is a basis for PP(A) and D' is CONT(A)-invariant. Let L' be a B^3-regularizing and CONT(A)-invariant clustering of D'. Suppose (c^0,c^2) is a nonisolated (0,2)[1,2] L'-outer adjacency of D', with $c^0 \in C_1 \in L'$ and $c^2 \in C_2 \in L'$. Let the 1-cell $c^1 \in C_1$ be adjacent to both c^0 and c^2. Then for every A-adjacency (s_0,s_2), with $s_0 \in S_D(c^0)$ and $s_2 \in S_D(c^2)$, there is an A-

adjacency (s_1,s_2), with $s_1 \in S_D(c^1)$, such that s_0 and s_1 are in the same L'-initial A-cluster.

Proof. We first prove the following proposition: for any $k_1 \geq 1$, $k_2 \geq 1$,

$$(S_{k_1}(B^3_{c^0},c^0),S_{k_2}(B^3_{c^2},c^2))$$

is an A-adjacency if and only if

$$(S_{k_1}(B^3_{c^1},c^1),S_{k_2}(B^3_{c^2},c^2))$$

is an A-adjacency. Since $dim(C_1) > 0$, $dim(C_2) > 0$, and L' is B^3-regularizing, by Theorem 3.4.1, no element of B^3 is cylindrical on any of c^0, c^1, or c^2. Suppose $s_1 = S_{k_1}(B^3_{c^1},c^1)$ is A-adjacent to $s_2 = S_{k_2}(B^3_{c^2},c^2)$. By Theorem 4.2.7, s_1 is A-adjacent to $s_0 = S_{k_1}(B^3_{c^0},c^0)$. Then by Theorem 3.5.5, $s_0 \subset \partial s_1$ and $s_1 \subset \partial s_2$, hence by Lemma 3.3.11, $s_0 \subset \partial s_2$, hence by Theorem 3.5.1, s_0 is adjacent to s_2, hence s_0 is A-adjacent to s_2 by Theorem 4.2.1.

Suppose $s_0 = S_{k_1}(B^3_{c^0},c^1)$ is A-adjacent to $s_2 = S_{k_2}(B^3_{c^2},c^2)$. By Theorems 3.4.3 and 3.4.7, $S^*_D(c^2)$ has the unique section boundary property in both $S^*_D(c^0)$ and $S^*_D(c^1)$, and $S^*_D(c^1)$ has the unique section boundary property in $S^*_D(c^0)$. We claim that for any section t_2 of $S^*_D(c^2)$, the index of the boundary section t_0 of t_2 in $S^*_D(c^0)$ is the same as the index of the boundary section t_1 of t_2 in $S^*_D(c^1)$. Let z_0 be the boundary section of t_1 in $S^*_D(c^0)$. Since $z_0 \subset \partial t_1$, and $t_1 \subset \partial t_2$, by Lemma 3.3.11, $z_0 \subset \partial t_2$, hence since $\partial t_2 \cap Z^*(c^0) = t_0$, we have $t_0 = z_0$, hence by Theorem 4.2.8, t_0 and t_1 have the same index.

Suppose s_2 is a section. Then by Theorem 3.5.10, s_0 is its boundary section in $S^*_D(c^0)$, hence by our claim above, $s_1 = S_{k_1}(B^3_{c^1},c^1)$ is its boundary section in $S^*_D(c^1)$, hence by Theorem 3.5.10, s_1 is adjacent to s_2. By Theorem 4.2.7, s_1 is A-adjacent to s_0, hence by Theorem 4.2.1, s_1 is A-adjacent to s_2.

Suppose s_2 is a sector (u,v) of $S_D(c^2)$, where u and v are sections of $S_D^*(c^2)$. Let u^0 and u^1 be the boundary sections of u in $S_D^*(c^0)$ and $S_D^*(c^1)$, and let v^0 and v^1 be the boundary sections of v in $S_D^*(c^0)$ and $S_D^*(c^1)$. By Theorem 3.3.14, $\partial s_2 \cap Z^*(c^0) = [u^0,v^0]$, and $\partial s_2 \cap Z^*(c^1) = [u^1,v^1]$. By our claim above, u^0 and u^1 have the same index i, and v^0 and v^1 have the same index $j \geq i$. Since s_0 is adjacent to s_2, by Theorem 3.5.5 it follows that $i \leq k_1 \leq j$. Hence again by Theorem 3.5.5, $s_1 = S_{k_1}(B^3_{c^1},c^1)$ is adjacent to s_2. As above, it follows that s_1 is A-adjacent to s_2. This completes the proof of our proposition.

Now let (s_0,s_2) be an A-adjacency with $s_0 \in S_D(c^0)$ and $s_2 \in S_D(c^2)$. There exist $k_1 \geq 1$, $k_2 \geq 1$, such that

$$s_0 = S_{k_1}(B^3_{c^0},c^0) \text{ ,and}$$

$$s_2 = S_{k_2}(B^3_{c^2},c^2).$$

By our proposition above, $s_1 = S_{k_1}(B^3_{c^1},c^1)$ is A-adjacent to s_2. By Theorem 4.2.10, s_0 and s_1 are in the same L'-initial A-cluster. ▪

Theorem 4.5.4. For $A \subset I_3$, let D be a regular, (B^1,B^2,B^3) basis-determined A-cad of E^3, such that B^3 is a basis for PP(A), and D' is CONT(A)-invariant. Let L' be a B^3-regularizing and CONT(A)-invariant clustering of D'. Suppose (c^0,c^1) is a nonisolated (0,1)[1,2] L'-outer adjacency of D', with $c^0 \in C_1 \in L'$ and $c^1 \in C_2 \in L'$. Let the 1-cell $d^1 \in C_1$ and the 2-cell $d^2 \in C_2$ be such that c^0 is adjacent to d^1, c^1 is adjacent to d^2, and d^1 is adjacent to d^2. Then for every A-adjacency (s_0,s_1), with $s_0 \in S_D(c^0)$ and $s_1 \in S_D(c^1)$, there is an A-adjacency (t_1,t_2), with $t_1 \in S_D(d^1)$ and $t_2 \in S_D(d^2)$, such that s_0 and t_1 are in the same L'-initial A-cluster, and s_1 and t_2 are in the same L'-initial A-cluster.

Proof. We first prove the following proposition: for any $k_1 \geq 1$, $k_2 \geq 1$,

$$(S_{k_1}(B^3_{c^0},c^0),S_{k_2}(B^3_{c^1},c^1))$$

is an A-adjacency if and only if

$$(S_{k_1}(B^3_{d^1},d^1),S_{k_2}(B^3_{d^2},d^2))$$

is an A-adjacency. Choose k_1 and k_2, and let

$$s_0 = S_{k_1}(B^3_{c^0},c^0)$$

$$s_1 = S_{k_2}(B^3_{c^1},c^1)$$

$$t_1 = S_{k_1}(B^3_{d^1},d^1)$$

$$t_2 = S_{k_2}(B^3_{d^2},d^2).$$

Note that since c^0 is adjacent to d^1 we have $c^0 \subset \partial d^1$, and since d^1 is adjacent to d^2, we have $d^1 \subset \partial d^2$, hence by Lemma 3.3.11, $c^0 \subset \partial d^2$, hence c^0 is adjacent to d^2. Suppose s_0 is A-adjacent to s_1. Then $s_0 \subset \partial s_1$, and since by Theorem 4.2.7 s_1 is A-adjacent to t_2, $s_1 \subset \partial t_2$, hence by Lemma 3.3.11, $s_0 \subset \partial t_2$, hence s_0 is adjacent to t_2, hence by Theorem 4.2.1, s_0 is A-adjacent to t_2. Then by Theorem 4.5.3, t_1 is A-adjacent to t_2.

Suppose t_1 is A-adjacent to t_2. By Theorem 3.4.3, $S^*_D(c^1)$ has the unique section boundary property in $S^*_D(c^0)$ and $S^*_D(d^2)$ has the unique section boundary property in $S^*_D(c^0)$. We claim that for any $i \geq 0$, where u_i is the i^{th} section of $S^*_D(c^1)$, where v_i is the i^{th} section of $S^*_D(d^2)$, where y is the boundary section of u_i in $S^*_D(c^0)$, and where z is the boundary section of v_i in $S^*_D(c^0)$, y = z. By Theorem 3.5.10, $y \subset \partial u_i$. By Theorem 4.2.7, u_i and v_i are adjacent, hence by Theorem 3.5.5, $u_i \subset \partial v_i$, hence by Lemma 3.3.11, $y \subset \partial v_i$, hence $\partial v_i \subset Z^*(c^0) = z$, we have y = z.

Since t_1 is A-adjacent to t_2, by Theorem 4.5.3, s_0 is A-adjacent to t_2. Suppose t_2 is a section. Then by Theorem 3.5.10, s_0 is the boundary section of t_2 in $S^*_D(c^0)$. If t_2 is a section, then s_1 is a section with the same index,

hence by our claim above, s_0 is the boundary section of s_1 in $S_D^*(c^0)$. Hence by Theorem 3.5.10, s_0 is adjacent to s_1. Since s_0 is A-adjacent to t_2, and by Theorem 4.2.7, t_2 is A-adjacent to s_1, s_0 is A-adjacent to s_1 by Theorem 4.2.1.

Suppose t_2 is a sector (v_j,v_{j+1}) of $S_D(d^2)$, where v_j and v_{j+1} are the j^{th} and $(j+1)^{st}$ sections of $S_D^*(d^2)$. Let w_j and w_{j+1} be the boundary sections of v_j and v_{j+1} in $S_D^*(c^0)$. By Theorem 3.3.14, $\partial t_2 \cap Z^*(c^0) = [w_j,w_{j+1}]$. Since s_0 is adjacent to t_2, by Theorem 3.5.5, $s_0 \subset [w_j,w_{j+1}]$. s_1 is the sector (u_j,u_{j+1}) of $S_D(c^1)$, where u_j and u_{j+1} are the j^{th} and $(j+1)^{st}$ sections of $S_D^*(c^1)$. By our claim above and Theorem 3.3.14, $\partial s_1 \cap Z^*(c^0) = [w_j,w_{j+1}]$, hence $s_0 \subset \partial s_1$, hence by Theorem 3.5.5, s_0 is adjacent to s_1. Then as argued above, s_0 is A-adjacent to s_1. This completes the proof of our proposition.

Now let (s_0,s_1) be an A-adjacency with $s_0 \in S_D(c^0)$ and $s_1 \in S_D(c^1)$. There exist $k_1 \geq 1$, $k_2 \geq 1$, such that

$$s_0 = S_{k_1}(B^3_{c^0},c^0) \text{ ,and}$$

$$s_1 = S_{k_2}(B^3_{c^1},c^1).$$

By Theorem 4.5.5, $t_1 = S_{k_1}(B^3_{d^1},d^1)$ is A-adjacent to $t_2 = S_{k_2}(B^3_{d^2},d^2)$. By Theorem 4.2.10, s_0 and t_1 are in the same L'-initial A-cluster, and s_1 and t_2 are in the same L'-initial A-cluster. ▪

Theorem 4.5.5. For $A \subset I_3$, let D be a regular (B^1,B^2,B^3) basis-determined A-cad of E^3, such that B^3 is a basis for PP(A), and D' is $CONT(A)$-invariant. Let L' be a B^3-regularizing and $CONT(A)$-invariant clustering of D'. Let S' be the set of all L'-outer adjacencies except for nonisolated (0,1)[1,2] and nonisolated (0,2)[1,2]. Then S' is L'-sufficient.

Proof. Suppose L is an A-clustering of D such that (1) each L'-initial cluster is a subcluster of some L-cluster, and (2) for each adjacency (e_1,e_2) of S',

any cell of $S_D(e_1)$ that is A-adjacent to a cell of $S_D(e_2)$ is in the same L-cluster as that cell. Let (c^0,c^1) be a nonisolated (0,1)[1,2] L'-outer adjacency. Then there is a (1,2) L'-outer adjacency (d^1,d^2) with (c^0,d^1) and (c^1,d^2) L'-inner adjacencies. Hence by Theorem 4.5.4, for any A-adjacent cells $s_1 \in S_D(c^0)$ and $s_2 \in S_D(c^1)$, there exist A-adjacent cells $t_1 \in S_D(d^1)$ and $t_2 \in S_D(d^2)$, such that s_1 and t_1 are in the same L'-initial cluster, and s_2 and t_2 are in the same L'-initial cluster. Since $(d^1,d^2) \in S'$, by (2), t_1 and t_2 are in the same L-cluster C. By (1), the L'-initial cluster containing t_1 is a subcluster of C, and the L'-initial cluster containing t_2 is a subcluster of C. Hence s_1 and s_2 are in C. Let (c^0,c^2) be a nonisolated (0,2)[1,2] L'-outer adjacency. Then there is a (1,2) L'-outer adjacency (d^1,c^2) with (c^0,d^1) an L'-inner adjacency. We have $(d^1,c^2) \in S'$, and by an application of Theorem 4.5.4 similar to the application of Theorem 4.5.4 above, we obtain that any cell of $S_D(c^0)$ that is A-adjacent to a cell of $S_D(c^2)$ is in the same L-cluster as that cell. Thus L has the property, for any L'-outer adjacency (e_1,e_2), that any cell of $S_D(e_1)$ which is A-adjacent to a cell of $S_D(e_2)$ is in the same L-cluster as that cell. Since L' is a B^3-regularizing clustering of D', by Theorem 4.2.12, the set of all L'-outer adjacencies is L'-sufficient. Hence L is the coarsest A-clustering of D, and hence S" is L'-sufficient. ▪

Theorem 4.5.6. For $A \subset I_3$, suppose D is a (B^1,B^2,B^3) cylindricity-refined A-cad of E^3, such that B^3 is a basis for PP(A). Let (c_1,c_2) be an adjacency of D' such that either c_1 is noncylindrical with respect to B^3, or (c_1,c_2) is type (0,1). Suppose $s_1 \in S_D(c_1)$ and $s_2 \in S_D(c_2)$ are A-adjacent. Let L' be any clustering of D'. Then there is some $t_1 \in S_D(c_1)$, in the L'-initial A-cluster containing s_1, and some $t_2 \in S_D(c_2)$, in the L'-initial A-cluster containing s_2, such

that t_1 and t_2 are A-adjacent, and either both t_1 and t_2 are sections, or both t_1 and t_2 are sectors.

Proof. By Theorems 3.4.3, 3.4.7, and 3.6.21, $S_D^*(c^2)$ has the unique section boundary property in $S_D^*(c^1)$. Suppose s_2 is a section. Then by Theorem 3.5.10, s_1 must be the boundary section of s_2 in $S_D^*(c^1)$, so we are done. Suppose s_2 is a sector. If s_1 is a sector then we are done, so suppose s_1 is a section. Since D is (B^1,B^2,B^3) basis-determined, there is some $B \in B^3$ vanishing on s_1, hence there is some nonzero $A_i \in A$ vanishing on s_1, hence since s_1 and s_2 are A-adjacent, A_i vanishes on s_2. By Theorem 3.6.4, no nonzero element of A vanishes on a 3-sector, hence $dim(s_2) \leq 2$. Since $dim(c_1) < dim(c_2)$, and $dim(s_2) = dim(c_2) + 1$, we must have $dim(s_2) = 2$ and $dim(s_1) = 0$.

By Lemma 3.3.19, any A_i which vanishes on s_2 vanishes everywhere on $Z(c_2)$. By Theorem 3.5.3, $c_1 \subset \partial c_2$, hence by Lemma 3.3.1, $Z(c_1) \subset \partial Z(c_2)$, hence any A_i which vanishes everywhere on $Z(c_2)$ also vanishes everywhere on $Z(c_1)$, since $V(A_i)$ is closed. Hence if A_i vanishes on s_1 and s_2, then it vanishes everywhere on $Z(c_1)$ and $Z(c_2)$.

By Theorems 3.5.8 and 3.5.9, either there is a 1-section t_1 of $S_D(c_2)$ adjacent to both s_1 and s_2, or there is a 1-sector u_1 of $S_D(c_1)$ adjacent to both s_1 and s_2. Suppose the first case. By the preceding paragraph, any A_i which vanishes on s_1 and s_2 also vanishes on t_1. Consider any A_i which vanishes on t_1. By Theorem 3.5.10, s_1 is the boundary section of t_1 in $S_D^*(c_1)$, hence $s_1 \subset V(A_i)$ since $V(A_i)$ is closed. Hence A_i vanishes on s_1 and s_2. Thus t_1 and s_1 are A-adjacent. Also, t_1 and s_2 are A-adjacent, hence they are in the same L'-initial A-cluster, so we are done with this case. In the second case, by the preceding paragraph, each A_i which vanishes on s_1 and s_2 also vanishes on u_1. Consider any A_i which vanishes on u_1. By Lemma 3.3.19, A_i

vanishes on $Z(c_1)$, hence A_i vanishes on s_1, hence A_i vanishes on s_1 and s_2. Thus u_1 and s_2 are A-adjacent. Also, u_1 and s_1 are A-adjacent, hence in the same L'-initial A-cluster, so we are done. ▪

Theorem 4.5.7. For $A \subset I_3$, let D be a (B^1,B^2,B^3) basis-determined A-cad of E^3 with cylindricity, such that B^3 is a basis for PP(A). Let (c^0,c^2) be a (0,2) adjacency of D', such that c^0 is cylindrical with respect to B^3. If $s_1 \in S_D(c^0)$ and $s_2 \in S_D(c^2)$ are A-adjacent, then s_2 is a section.

Proof. Suppose $B_i \in B$ is cylindrical on c^0. Since B_i vanishes on every cell of $S_D(c^0)$, there is some nonzero $A_j \in A$ which vanishes on every cell of $S_D(c^0)$. Suppose s_2 is a sector. Then since A_j vanishes on s_1, and s_1 is A-adjacent to s_2, A_j vanishes on s_2. but this contradicts Theorem 3.6.4, since any sector of $S_D(c^2)$ is a 3-cell in E^3. Hence s_2 must be a section. ▪

Definition. For $A \subset I_r$, $r \geq 2$, let D be an A-cad of E^r and L' a clustering of D'. An *L'-adequate set of A-adjacencies of D* is a set S of A-adjacencies of D, with the following property: if L is an A-clustering of D such that

(1) Each L'-initial A-cluster of D is a subcluster of some L-cluster, and

(2) For each A-adjacency $(s_1,s_2) \in S$, s_1 and s_2 are in the same L-cluster,

then L is the coarsest L-clustering of D.

Corollary 4.5.8. For $A \subset I_3$, let D be a (B^1,B^2,B^3) cylindricity-refined A-cad of E^3, such that B^3 is a basis for PP(A). Let L' be a B^3-regularizing clustering of D', and let S' be an L'-sufficient set of adjacencies of D'. Let S be a set of A-adjacencies of D defined as follows:

(1) For each adjacency $(c_1,c_2) \in S'$ such that either c_1 is noncylindrical with respect to B^3, or (c_1,c_2) is type (0,1), if a section d_1 of $S_D(c_1)$ is A-

adjacent to a section d_2 of $S_D(c_2)$, then $(d_1,d_2)\in S$, and if a sector e_1 of $S_D(c_1)$ is A-adjacent to a sector e_2 of $S_D(c_2)$, then $(e_1,e_2)\in S$, and

(2) if (c_1,c_2) is a (0,2) adjacency of S' such that c_1 is cylindrical with respect to B^3, then if a cell d_1 of $S_D(c_1)$ is A-adjacent to a section d_2 of $S_D(c_2)$, then $(d_1,d_2)\in S$.

Then S is L'-adequate.

Proof. Suppose L is an A-clustering of D such that

(1) Each L'-initial A-cluster of D is a subcluster of some L-cluster, and

(2) For each A-adjacency $(s_1,s_2)\in S$, s_1 and s_2 are in the same L-cluster.

We claim that L is the coarsest A-clustering of D. Since S' is L'-sufficient, it suffices to show, for every (c_1,c_2) of S', that if $s_1\in S_D(c_1)$ is A-adjacent to $s_2\in S_D(c_2)$, then s_1 and s_2 are in the same L-cluster. Suppose that either c_1 is noncylindrical with respect to B^3, or (c_1,c_2) is type (0,1). Then by Theorem 4.5.6, there is a t_1 in the same L'-initial A-cluster as s_1, and a t_2 in the same L'-initial A-cluster as s_2, such that either both t_1 and t_2 are sections, or both t_1 and t_2 are sectors. By hypothesis, $(t_1,t2)\in S$, hence t_1 and t_2 are in the same L-cluster C. Then the L'-initial A-cluster containing t_1 must be a subcluster of C, and the L'-initial A-cluster containing t_2 must be a subcluster of C. Hence s_1 and s_2 are both in C. Suppose (c_1,c_2) is a (0,2) adjacency, such that c_1 is cylindrical with respect to B^3. By Theorem 4.5.7, s_2 must be a section, hence by hypothesis, $(s_1,s_2)\in S$, hence s_1 and s_2 are in the same L-cluster. Hence L is the coarsest A-clustering of D, hence S is L'-adequate. ▪

4.6 Sample point construction.

There are at least two conditions which determine what cell sample points the clustering cad algorithm constructs for a cad of E^r. First, there will always be at least one cell in each cluster for which a sample point has been constructed. Second, if r < 3 and the cad D of E^r is to be extended to a cad of E^{r+1}, then where D is an A-cad for $A \subset I_r$, where L is the coarsest A-clustering of D, and where S is an L-sufficient set of outer adjacencies in E^r, for each $(c_1,c_2) \in S$ with $dim(c_1) < dim(c_2)$, a sample point for c_1 will be constructed.

There are two reasons for the first condition. First, if the cad D of E^r is to be used for quantifier elimination for a formula whose unquantified matrix is $\varphi(x_1, \ldots, x_r)$, then it is necessary to have a sample point for each cluster of D in order to carry out the quantifier elimination. Second, if the cad of E^r is to be extended to a cad K of E^{r+1}, then for each cluster C of D, where α is the sample point for C and R = R(C), we will determine how many sections and sectors there are in $S_K(R)$ by evaluating certain polynomials in I_{r+1} at α, then isolating the real roots of the resulting polynomials in $Q(\alpha)[x_{r+1}]$. The reason for the second condition is that the sample points whose construction is specified there are required by the adjacency algorithms.

Although sample points for more than one cell of a cluster may be constructed, each cluster has one of its cells with a constructed sample point designated as the *representative cell* of the cluster. The representative cell of an i-cluster will be some i-cell of the cluster, which we can expect to have a more easily constructed sample point than any lower dimensional cell of the cluster.

Suppose r = 1, and that we are constructing an A-cad D of E^1 for some $A \subset I_1$ with the clustering cad algorithm. The coarsest A-clustering L of D is obtained by putting each cell of D in a cluster by itself, hence by the first condition above, we must construct a sample point for every cell of D, as in the original cad algorithm. For $r \geq 2$, however, we will in general construct sample points for only a proper subset of the cells of D.

4.7 Defining formula construction.

Let D be a derivative-regular $(B^1, \ldots, B^r)$ basis-determined cad of E^r, $r \geq 1$, for which we wish to construct defining formulas. Let L' be a B^r-regularizing clustering of D'. Let c_1 and c_2 be cells of a cluster C of L'. Let $\varphi_1(x_1, \ldots, x_{r-1})$ be a defining formula for c_1, and let $\varphi_2(x_1, \ldots, x_{r-1})$ be a defining formula for c_2. Let S = $S_D(c_1)$, and T = $S_D(c_2)$. Then for any k, $1 \leq k \leq |S|$, there exists a quantifier-free formula $\psi_k(x_1, \ldots, x_r)$ such that the derivative-based defining formula for S_k is $\varphi_1 \ \& \ \psi_k$, and the derivative-based defining formula for T_k is $\varphi_2 \ \& \ \psi_k$. It is clear that where c_i is the representative cell of C with derivative-based defining formula φ_i, if we construct defining formulas for $S_D(c_i)$, then for any $c_j \neq c_i$ in C with defining formula φ_j, we do not need to explicitly construct defining formulas for $S_D(c_j)$. For each k, $1 \leq k \leq S_D(c_i)$, we replace φ_i by φ_j in $\varphi_i \ \& \ \psi_k$ to obtain a defining formula $\varphi_j \ \& \ \psi_k$ for the kth element of $S_D(c_j)$.

4.8 The 2-space and 3-space clustering algorithms

For ease of exposition, the versions of CLCAD2 and CLCAD3 we present in this section do not correspond in some respects with the versions implemented in the SAC-2 computer algebra system. For precise information on

the implemented versions, SAC-2 listings must be consulted (see [COL80] for information on SAC-2).

CLCAD2(A,k;C,I,S,F)

[Clustered cylindrical algebraic decomposition of 2-space. A is a list of $n \geq 0$ bivariate integral polynomials. k satisfies $0 \leq k \leq 2$. C is a list of the clusters of the coarsest A-clustering L of a regular (B^1, B^2) basis-determined A-cad D of E^2, where $B^1 \subset I_1$ is a basis and $B^2 \subset I_2$ is a finest squarefree basis for PP(A). I is a list of the (0,1)[0,1], (0,1)[0,2], (0,1)[1,1], (0,2)[0,2], isolated (0,1)[1,2], isolated (0,2)[1,2], and (1,2)[1,2] L-outer adjacencies of D. S is a list of sample points for certain cells of D, such that S contains a sample point for the representative cell of every cluster in C, and S contains a sample point for the lower-dimensional cell of each L-outer adjacency in I. If $k \geq 1$, then F is a list of defining formulas for the cad of k-dimensional space induced by D. If k = 0, then F is the null list.]

(1) [Initialize.] Set $N \leftarrow CONT(A)$. Set $\bar{A} \leftarrow PP(A)$. Set $B \leftarrow$ the finest squarefree basis for $\bar{A}$.

(2) [Determine D'.] if k = 2 then set $P \leftarrow APROJ(B)$; otherwise set $P \leftarrow PROJ(B)$. Set $P \leftarrow N \cup P$. Set $\bar{A}' \leftarrow PP(P)$ Set $B' \leftarrow$ the finest squarefree basis for $\bar{A}'$. Isolate the real roots of B', and use the isolating intervals to construct sample points for each cell of D', recording these in S'. If k = 1, then construct a defining formula for each cell of D' and record it in F'; otherwise set F' <- ().

(3) [Determine D and the sign patterns of its cells.] For each cell c of D' with sample point α, let $B^* = \{B_i(\alpha, x_2) \mid B_i \in B\}$. Isolate the real roots

of B^*, thereby determining $|S_D(c)|$. Determine the sign of of each element of A on each section and sector of $S_D(c)$ and save for use in Step 6 below.

(4) [Construct defining formulas for D, if desired.] If k = 2 then for each cell c of D', construct defining formulas for the sections and sectors of $S_D(c)$ (using F") and record them in F. Otherwise set $F \leftarrow F'$.

(5) [Determine topological adjacencies of D.] Let $c_1, c_2, \ldots, c_m$ be the cells of D' in increasing (left-to-right) order. For each pair c_i, c_{i+1} of adjacent cells in D' , $1 \leq i \leq m-1$, use the section boundary adjacency algorithm to determine all topological adjacencies between a section of $S_D(c_i)$ and a section of $S_D(c_{i+1})$, and between a sector of $S_D(c_i)$ and a sector of $S_D(c_{i+1})$. Infer all other topological adjacencies of D.

(6) [Determine maximal A-clusters and L-outer adjacencies.] Initialize C to the L'-initial A-clusters of D in $Z(c_1)$, where L' denotes the clustering of D' which puts each cell in a cluster by itself. For i = 1 ,..., m-1 do the remaining actions of this step: Add the L'-initial A-clusters of D in $Z(c_{i+1})$ to C. For each previously determined topological adjacency (s_i, s_{i+1}) between an element s_i of $S_D(c_i)$ and an element s_{i+1} of $S_D(c_{i+1})$, use the sign information saved in Step 3 to determine whether the adjacency is A-algebraic. If so, then combine the cluster containing s_1 and the cluster containing s_2. If not, then if appropriate, add the adjacency to I.

(7) [Final sample point construction.] If the representative cell sample point of any cluster is currently unconstructed, then construct it

and add it to S. If the sample point for the lower-dimensional cell of any adjacency recorded in I is currently unconstructed, then construct it and add it to S. Exit ▪

As was mentioned in Section 4.6, every cluster manipulated by the clustering cad algorithm has a representative cell whose dimension is the same as the dimension of the cluster. We now describe how these representative cells are chosen. In E^1, a cluster contains exactly one cell, which will be its representative cell. For an A-cad of E^r, $r = 2$ or $r = 3$, let L' denote the inductively determined clustering of D'. When we construct each L'-initial A-cluster of D, we arbitrarily designate one of its cells of highest dimension as its representative cell. Suppose now that two clusters of D are to be combined ("pasted together"). If their representative cells have different dimensions, then we retain the one of higher dimension as the representative cell of the new cluster. If their representative cells have the same dimension, we arbitrarily choose one as the representative cell of the new cluster.

We now indicate briefly how we construct a (B^1,B^2,B^3) cylindricity-refined cad D of E^3. Suppose $B \in B^3$ is cylindrical on a 0-cell $c^0 = <a,b>$ of D'. Let $B^* = \{B_i(a,b,z) \mid B_i \in B^3 \ \& \ B_i(a,b,z) \neq 0\}$. We now augment B^*. Suppose, for example, that c^1 is a 1-section of D' adjacent to c^0, and that d^1 is a B-section of $Z(c^0)$. Suppose $G \in B^2$ vanishes on c^1. We compute R(x,z) = $pp(Res_y(G,B))$. As in the proof of Theorem 3.6.2, if the limit point of d^1 in $Z^*(c^0)$ is finite, there will be a root γ of $R(a,z)$ such that $<a,b,\gamma>$ is this limit point, so we add R(a,z) to B^*. We proceed similarly to add other

polynomials in $Q(a)[z]$ to B^*, so that among their real roots are all the z-coordinates of points of $Z(c^0)$ which are required to be sections of $S_D(c^0)$ by the definition of cylindricity-refined cad. Finally we isolate the real roots of all elements of B^*, and for each such real root z_i, we make $<a,b,z_i>$ a section of $S_D(c^0)$.

It may happen that by the above process we obtain more sections of $Z(c^0)$ than are required by the definition of cylindricity-refined cad. But let us note that if D is a (B^1,B^2,B^3) cylindricity-refined cad of E^3, and if $B \in B^3$ is cylindrical on a 0-cell c^0 of D', then we may add any (finite) number of additional 0-sections to $S_D(c^0)$ and obtain a cad D^* which is still (B^1,B^2,B^3) cylindricity-free. This is because each new section we add is contained in V(B), so D^* is B^3-invariant, and hence D^* is a (B^1,B^2,B^3) cylindricity-refined cad of E^3.

CLCAD3(A,k;C,I,S,F)

[Clustered cylindrical algebraic decomposition of 3-space. A is a list of $n \geq 0$ trivariate integral polynomials. k satisfies $0 \leq k \leq 3$. C is a list of the clusters of the coarsest A-clustering L of a regular (B^1,B^2,B^3) cylindricity-refined A-cad D of E^3, where $B^1 \subset I_1$ and $B^2 \subset I_2$ are bases, and $B^3 \subset I_3$ is a finest squarefree basis for PP(A). I is a list of certain of the L-outer adjacencies of D. S is a list of sample points for certain cells of D, such that S contains a sample point for the representative cell of every cluster in C. If $k \geq 1$, then F is a list of defining formulas for the cad of k-dimensional space induced by D. If k = 0, then F is the null list.]

(1) [Initialize.] Set $N \leftarrow CONT(A)$. Set $\bar{A} \leftarrow PP(A)$. Set $B \leftarrow$ the finest square-

free basis for $\bar{A}$.

(2) [Determine D' and a B-regularizing clustering of D'.] if k = 3 then set $P \leftarrow APROJ(B)$ and $k' \leftarrow k-1$; otherwise set $P \leftarrow PROJ(B)$ and $k' \leftarrow k$. Set $P \leftarrow N \cup P$. Call CLCAD2 with inputs P and k' to obtain outputs C', I', S', and F'.

(3) [Determine D and the sign patterns of its cells.] For each representative cell c of a cluster K of C', do the remaining actions of this step. Where α is the sample point for c, set $B^* \leftarrow \{B_i(\alpha,z) \mid B_i \in B \ \& \ B_i(\alpha,z) \neq 0\}$. If c is a 0-cell which is cylindrical with respect to B, add additional elements of $Q(\alpha)[z]$ to B^* as described earlier in this section, so that D will be (B^1,B^2,B^3) cylindricity-refined. Isolate the real roots of B^*, thereby determining $|S_D(c)|$. Determine the sign of each element of A on each section and sector of $S_D(c)$, and hence on each section and sector of $S_D(d)$ for any d in K, and save for use in Step 6 below.

(4) [Construct defining formulas for D, if desired.] if k < 3 then set $F \leftarrow F'$ and go to 5. For each representative cell c of a cluster K in C', construct defining formulas for the elements of $S_D(c)$ (using F'). From these infer the defining formulas for $S_D(d)$, for every d in K, as described in Section 4.7. Record these defining formulas in F.

(5) [Determine a set T of topological adjacencies of D, such that the subset of all A-adjacencies of T is L'-adequate, where L' denotes the coarsest P-clustering of D'.] Use the section boundary adjacency algorithm to determine all section-section and sector-sector adjacencies between $S_D(c_i)$ and $S_D(c_j)$, for each $(c_i,c_j) \in I'$ such that either c_i is noncylindrical with respect to B or (c_i,c_j) is type (0,1), and to

determine all adjacencies between an element of $S_D(c_i)$ and a section of $S_D(c_j)$ for each $(c_i,c_j)\in I'$ such that $dim(c_i) = 0$, $dim(c_j) = 2$, and c_i is cylindrical with respect to B.

(6) [Determine maximal A-clusters and L-outer adjacencies.] Initialize $I\leftarrow(),C\leftarrow()$. For each adjacency $(c_i,c_j)\in I'$, do the remaining actions of this step. If not previously done, add the L'-initial A-clusters of D in $Z(R(K_i))$ to C, where K_i denotes the cluster of C' containing c_i. Similarly, if not previously done, add the L'-initial A-clusters of D in $Z(R(K_j))$ to C, where K_j is the cluster of C' containing c_j. For each previously determined topological adjacency (s_i,s_j) between an element s_i of $S_D(c_i)$ and an element s_j of $S_D(c_j)$, use the sign information saved in Step 3 to determine whether the adjacency is A-algebraic. If so, then combine the cluster containing s_i and the cluster containing s_j. If not, then add (s_i,s_j) to I.

(7) [Final sample point construction.] If the representative cell sample point of any cluster in C is currently unconstructed, then construct it and add it to S. Exit ▪

CHAPTER 5

THE SECTION BOUNDARY ADJACENCY ALGORITHM

5.1 Overview

Let D be a (B^1,B^2) basis-determined cad of E^2, and let (c_1,c_2) be a (0,1) adjacency of D'. The 2-space section boundary adjacency algorithm finds all adjacencies between sections of $S_D(c_1)$ and sections of $S_D(c_2)$, and between sectors of $S_D(c_1)$ and sectors of $S_D(c_2)$. By Theorem 3.3.24, $S_D^*(c_2)$ has the unique section boundary property in $S_D^*(c_1)$, hence finding the section-section adjacencies between the two stacks amounts to finding the boundary section in $S_D(c_1)$ (if any) of each section of $S_D(c_2)$. By Theorem 3.3.14, it is clear that from knowledge of boundary sections, we can infer the adjacencies between sectors of $S_D(c_1)$ and sectors of $S_D(c_2)$. Section 5.2 gives the theorem on which section-section adjacency determination relies, and an abstract algorithm for the 2-space section boundary adjacency algorithm.

The general strategy of the 3-space section boundary adjacency algorithm is to reduce determination of the adjacencies between sections of two adjacent stacks in E^3 to determination of the adjacencies between sections of a certain pair of adjacent stacks in E^2. This is accomplished by "projecting" the adjacent stacks in E^3 into E^2 in an appropriate manner.

For some $A \subset I_3$, let D be a (B^1,B^2,B^3) cylindricity-refined A-cad of E^3, such that B^3 is the irreducible basis for PP(A). We may distinguish four kinds of adjacencies (c_1,c_2) between cells of D': a (1,2) adjacency, a (0,1) adjacency, a (0,2) adjacency where the 0-cell is noncylindrical with respect

to B^3, and a (0,2) adjacency where the 0-cell is cylindrical with respect to B^3.

For the first three of these kinds, Theorems 3.4.3, 3.4.7, and 3.6.21 insure that $S_D^*(c^2)$ has the unique section boundary property in $S_D^*(c^1)$. Thus, as in 2-space, finding the section-section adjacencies between the two stacks amounts to finding the boundary section in $S_D(c_1)$ (if any) of each section of $S_D(c_2)$, and we then apply Theorem 3.3.14 to infer the adjacencies between sectors of $S_D(c_1)$ and sectors of $S_D(c_2)$.

When (c_1,c_2) is a (0,2) adjacency of D' with c_1 a cylindrical 0-cell (for the remainder of this chapter, we will say "cylindrical" instead of "cylindrical with respect to B^3"), the situation is more complicated. Let s be a 2-section of $S_D(c_2)$. Since D is (B^1,B^2,B^3) basis-determined, $s \subset V(B)$ for some unique $B \in B^3$ (by Corollary 3.6.6). If B is not cylindrical on c_1, then by Theorem 3.6.7, there is a unique section t of $S_D^*(c_1)$ such that $t = \partial s \cap Z^*(c_1)$, and the algorithm determines t just as though c_1 were non-cylindrical. If B is cylindrical on c_1, then the sections and sectors of $S_D^*(c^1)$ which are adjacent to s are determined by application of Theorem 3.6.18. Since by Theorem 4.5.7, there can be no A-adjacency between a sector of $S_D(c_2)$ and a cell of $S_D(c_1)$, the 3-space section boundary adjacency algorithm does not attempt to determine any topological adjacencies between a sector of $S_D(c_2)$ and a cell of $S_D(c_1)$.

Sections 5.3-5.5 discuss individually these various kinds of adjacencies between cells of D'.

5.2 The 2-space algorithm.

Theorem 5.2.1 Let $A \subset I_2$ and $\alpha,\alpha',\beta,b_1,b_2 \in E$ be such that $\alpha < \alpha'$, $b_1 < \beta < b_2$, $d^0 = <\alpha,\beta>$ is the unique section of $S(A_Q,Q)$ in $Q\times[b_1,b_2]$, where $Q = \{\alpha\}$. A is regular on $R = (\alpha,\alpha']$, and no nonzero $A_i \in A$ vanishes at any point of $[\alpha,\alpha']\times\{b_1\}$ or $[\alpha,\alpha']\times\{b_2\}$. Then for any section d^1 of $S(A_R,R)$, d^0 is the unique limit point of d^1 in $Z(\alpha)$ if and only if

$$d^1 \cap \{\alpha'\}\times(b_1,b_2) \neq \emptyset.$$

Proof. Let d^1 be an f-section. By Theorem 3.6.16, f can be extended to a continuous map $h: Q \cup R \to E^*$. Suppose that $d^1 \cap \{\alpha'\}\times(b_1,b_2) \neq \emptyset$. Then we have $f(\alpha') \in (b_1,b_2)$. Let $z^0 = <\alpha,\beta_1>$ be the unique limit point of d^1 in $Z^*(Q)$, and suppose $\beta_1 \neq \beta$. There exists a nonzero $A_i \in A$ which vanishes on d^1, and since $V(A_i)$ is closed, A_i vanishes at z_0. A_i is not cylindrical at α since $A_i(\alpha,b_1) \neq 0$, hence z^0 is a section of $S(A_Q,Q)$, and so either $\beta_1 < b_1$ or $\beta_1 > b_2$; assume without loss of generality the former. Then $h(\alpha) < b_1$ and $h(\alpha') > b_1$, hence by the Intermediate Value Theorem, there exists $\alpha^* \in (\alpha,\alpha')$ such that $h(\alpha^*) = f(\alpha^*) = b_1$. Hence where $d^1 \subset V(A_i)$ for some nonzero $A_i \in A$, $A_i(\alpha^*,b_1) = 0$, contrary to hypothesis. Hence $\beta_1 = \beta$.

Suppose d^0 is the unique limit point of d^1 in $Z^*(Q)$. If $d^1 \cap \{\alpha'\}\times(b_1,b_2) = \emptyset$, then either $f(\alpha') \leq b_1$ or $f(\alpha') \geq b_2$; assume without loss of generality the latter. Then $h(\alpha) < b_2$ and $h(\alpha') \geq b_2$. Hence there exists $\alpha^* \in (\alpha,\alpha']$ such that $h(\alpha^*) = f(\alpha^*) = b_2$. Hence where $d^1 \subset V(A_i)$, $A_i \neq 0$, we have $A_i(\alpha^*,b_2) = 0$, contrary to hypothesis. Hence $d^1 \cap \{\alpha'\}\times(b_1,b_2) \neq \emptyset$. ∎

Clearly we can prove a corresponding theorem for $\alpha' < \alpha$, i.e. R to the left of Q. Analogously, a second version of the two-space section boundary adjacency algorithm we present below can be given.

The following theorem is used in step (1) of SBAA2.

Theorem 5.2.2. Suppose $A \subset I_2$ and $b \in E$. Then there exist $b_1, b_2 \in E$, $b_1 < b < b_2$, such that A is regular on $[b_1, b)$ and $(b, b_2]$.

Proof. Let P = PROJ(A). Since there are only finitely many $\alpha_i \in E$ at which some nonzero element of P vanishes, we can choose $b_1, b_2 \in E$ with $b_1 < b < b_2$ such that no nonzero element of P vanishes on $[b_1, b)$ and no nonzero element of P vanishes on $(b, b_2]$. Hence $[b_1, b)$ and $(b, b_2]$ are P-invariant regions in E^1, hence by Theorem 2.2.15, A is regular on each. ∎

For any $H \subset I_1$, or for any $H \subset Q(\alpha)[x]$ for some real algebraic number α, by the *real roots of H* we mean the set of all $\gamma \in E$ such that some $H_i \in H$ has γ as a root.

SBAA2$(A, \alpha, b; L)$

[Section boundary adjacency algorithm for E^2. $A \subset I_2$. α is a real algebraic number. Let $\bar{A} = \{A_i(\alpha, y) \mid A_i \in A \ \& \ A_i(\alpha, y) \neq 0\}$, and let $\beta_1, \ldots, \beta_n$, $n \geq 0$, be the real roots of $\bar{A}$. b is a list $(b_0, \ldots, b_n)$ of rational numbers such that $b_{j-1} < \beta_j < b_j$, for $1 \leq j \leq n$. Let $\alpha' > \alpha$ be such that A is regular on $R = (\alpha, \alpha']$. Let $Q = \{\alpha\}$. L is a list of all section-section and sector-sector adjacencies between $S(A_Q, Q)$ and $S(A_R, R)$.]

(1) [Determine rational $\alpha' > \alpha$ such that A is regular on $R = (\alpha, \alpha']$, and for $0 \leq j \leq n$, no nonzero $A_i \in A$ vanishes at any point of $[\alpha, \alpha'] \times \{b_j\}$.] Set $P \leftarrow PROJ(A)$. Add M(x), the integral minimal polynomial of α, to P. Isolate the real roots of P, thereby obtaining an open isolating interval (u,v) for α. Set $\alpha' \leftarrow v$. (A is regular on $(\alpha, \alpha']$.) While there is some j, $0 \leq j \leq n$, and some nonzero $A_i \in A$, such that

$A_i(x,b_j)$ has a real root in $[\alpha,\alpha']$, set α' to a rational approximate midpoint of the interval $(\alpha,\alpha']$. (Since $A_i(\alpha,b_j)\neq 0$ for $0 \leq j \leq n$, this loop will terminate.)

(2) [Initialize loop.] Set $A^* \leftarrow \{A_i(\alpha',y) \mid A_i \in A \ \& \ A_i(\alpha',y)\neq 0\}$. Set $m \leftarrow$ the number of real roots of A^* in $(-\infty,b_0)$. (m is the number of sections of $S(A_R,R)$ whose limit point in $Z^*(\alpha)$ is $< \alpha, -\infty >$). Record (in L) that sector 0 of $S(A_Q,Q)$ is adjacent to sector m of $S(A_R,R)$.

(3) [Process each section of $S(A_Q,Q)$.] For j = 1,...,n do the following four things: First, set $m_j \leftarrow$ the number of real roots of A^* in (b_{j-1},b_j). Second, record that the j^{th} section of $S(A_Q,Q)$ is adjacent to sections $m+1$,..., $m+m_j$ of $S(A_R,R)$. Third, record that sector j of $S(A_Q,Q)$ is adjacent to sector $m+m_j$ of $S(A_R,R)$. Fourth, set $m \leftarrow m+m_j$. Exit ▪

Where $\bar{A}$ is as in the SBAA2 specifications, we note that the rational numbers $b_0, \ldots, b_n$ are easily obtained from the endpoints of isolating intervals for the real roots of $\bar{A}$. (Here, as throughout the thesis, isolating intervals are assumed by definition to have rational endpoints.)

5.3 Adjacencies in 3-space over a (1,2) adjacency.

Notation. Where Q denotes the rational numbers, for any $r \geq 0$, let Q_r denote $Q[x_1, \ldots, x_r]$, the ring of rational polynomials in r variables.

Definition. For any $F \in Q_r$, the *similar integral polynomial for F*, written *sip*(F), is the unique integer primitive, positive element of I_r similar to F. (An element G of I_r is *integer primitive* if there is no integer different from

+1 and -1 which divides each coefficient of G.)

Definition. Let R be a region in E^{r-1}, $r \geq 2$, and let S be a stack over R. Let R' be a subregion of R. The stack $\{s \cap Z(R') \mid s \in S\}$ is the *restriction of S to R'*, written $S|_{R'}$.

Definition. Let Q and R be regions in E^{r-1}, $r \geq 2$, such that $Q \subset \partial R$, and let S(Q) and S(R) be stacks over Q and R. Let U and V be regions in E^{s-1}, $s \geq 2$, such that $U \subset \partial V$, and let S(U) and S(V) be stacks over U and V. The pair (S(Q),S(R)) is said to be *isomorphic* to the pair (S(U),S(V)), written $(S(Q),S(R)) \approx (S(U),S(V))$, if |S(Q)| = |S(U)|, |S(R)| = |S(V)|, and for any $k_1 \geq 1$, $k_2 \geq 1$, $S_{k_1}(Q)$ is adjacent to $S_{k_2}(R)$ if and only if $S_{k_1}(U)$ is adjacent to $S_{k_2}(V)$.

Theorem 5.3.1. Let D be a regular (B^1,B^2,B^3) basis-determined cad of E^3. Let (c^1,c^2) be a (1,2) adjacency of D' with c^1 a 1-sector and c^2 to the right of c^1. Let $<a,b>$ be a point of c^1 with b rational. Let $B^* = \{sip(B(x,b,z)) \mid B \in B^3\}$. Let $a' > a$ be a rational number such that where $I = (a,a']$, B^* is regular on I. Then B^3 is regular on $R = I \times \{b\}$, and where $p = \{a\}$, $Q = \{<a,b>\}$,

$$(S(B_p^*,p),S(B_I^*,I)) \approx (S_D(c^1),S_D(c^2)).$$

Proof. Clearly B^3 is regular on R, and

$$(S(B_p^*,p),S(B_I^*,I)) \approx (S_D(c^1)|_Q , S_D(c^2)|_R).$$

We claim that

$$(S_D(c^1)|_Q , S_D(c^2)|_R) \approx (S_D(c^1),S_D(c^2)).$$

Let $S(c^1) = S_D(c^1)$, $S(c^2) = S_D(c^2)$, $T(Q) = S_D(c^1)|_Q$, and $T(R) = S_D(c^2)|_R$. Obviously $|T(Q)| = |S(c^1)|$ and $|T(R)| = |S(c^2)|$. For any $k_1 \geq 1$, $k_2 \geq 1$, if

$T_{k_1}(Q)$ is adjacent to $T_{k_2}(R)$, clearly $S_{k_1}(c^1)$ is adjacent to $S_{k_2}(c^2)$. Conversely, suppose that s = $S_{k_1}(c^1)$ is adjacent to t = $S_{k_2}(c^2)$, and suppose that both s and t are sections. Let s' = $T_{k_1}(Q)$, and t' = $T_{k_2}(R)$. By Theorem 3.6.2, t' has a unique limit point $<\alpha,b,\gamma> \in Z^*(Q)$. Since t' $\subset$ t, $<\alpha,b,\gamma>$ is a limit point of t. By Theorem 3.4.7, $S(c^2)$ has the unique section boundary property in $S(c^1)$, hence by Theorem 3.5.10, s is the boundary section of t in $S^*(c^1)$. Hence $<\alpha,b,\gamma>$ = s $\cap$ Z(Q), i.e. $<\alpha,b,\gamma>$ = s', hence s' is adjacent to t'. It is clear from this argument that T(R) has the unique section boundary property in T(Q), hence by Theorems 3.3.14 and 3.5.10, we see that for any $k_1 \geq 1$, $k_2 \geq 1$, if $S_{k_1}(c^1)$ is adjacent to $S_{k_2}(c^2)$, then $T_{k_1}(Q)$ is adjacent to $T_{k_2}(R)$. Hence $(S_D(c^1)|_Q\ ,\ S_D(c^2)|_R) \approx (S_D(c^1),S_D(c^2))$. ▪

Clearly with minor changes we obtain a version of Theorem 5.3.1 for the case c^2 to the left of c^1. Also, if c^1 is a 1-section and c^2 is a 2-sector, we have two similar Theorems, one if c^2 is below c^1, and another if it is above. The following is one of these.

Theorem 5.3.2. Let D be a regular (B^1,B^2,B^3) basis-determined cad of E^3. Let (c^1,c^2) be a (1,2) adjacency of D' with c^1 a 1-section and c^2 above c^1. Let $<\alpha,b>$ be a point of c^1 with α rational. Let $B^* = \{sip(B(\alpha,y,z))\,|\,B\in B^3\}$. Let $b' > b$ be a rational number such that where $J = (b,b']$, B^* is regular on J. Then B^3 is regular on $R = \{\alpha\}\times J$, and where $p = \{b\}$, $Q = \{<\alpha,b>\}$,

$$(S(B^*_p,p),S(B^*_J,J)) \approx (S_D(c^1),S_D(c^2)).$$

We now describe the action of the 3-space section boundary adjacency algorithm, SBAA3, for the case (c^1,c^2) a (1,2) adjacency, c^1 a 1-sector, and c^2 to the right of c^1 (the case covered by Theorem 5.3.1). At the time SBAA3

is called, CLCAD2 will have already constructed a sample point $\langle a,b \rangle$ for c^1 with b rational. We form the set B^* of Theorem 5.3.1, set $P = PROJ(B^*)$, and add the integral minimal polynomial for a to P. We then isolate the real roots of P. Among the isolating intervals obtained is an open isolating interval (u,v) for a. Setting $a' = v$, we have that B^* is regular on $I = (a,a']$. Let $\overline{B} = \{B(a,b,z) \mid B \in B^3\}$. We isolate the real roots $\beta_1, \ldots, \beta_n$, $n \geq 0$, of $\overline{B}$, and from the endpoints of the isolating intervals obtain a list $b = (b_0, \ldots, b_n)$ of rational numbers such that $b_{j-1} < \beta_j < b_j$ for $1 \leq j \leq n$. We then call SBAA2 with inputs B^*, a, and b to obtain the section-section and sector-sector adjacencies between $S(B_p,p)$ and $S(B_I,I)$, where $p = \{a\}$. By Theorem 5.3.1, from these adjacencies we immediately obtain the section-section and sector-sector adjacencies between $S_D(c^1)$ and $S_D(c^2)$.

The other three possible dispositions of c^1 and c^2 are handled similarly by SBAA3.

5.4 Adjacencies in 3-space over a (0,1) adjacency.

Notation. We let π_x denote the projection that sends a point <x,y,z> of E^3 to <y,z>. We let π_y denote the projection that sends a point <x,y,z> of E^3 to <x,z>.

Theorem 5.4.1. Let F be a nonzero element of I_2, and let G = pp(F). Then there are real algebraic numbers $a_1, \ldots, a_k$, $k \geq 0$, such that V(F) = $V(G) \cup Z(a_1) \cup Z(a_2) \cdots \cup Z(a_k)$.

Proof. Where H = $cont(F)$, we have F = GH. Thus where $a_1, \ldots, a_k$, $k \geq 0$, are the real roots of H, V(F) = $V(G) \cup Z(a_1) \cup Z(a_2) \cdots \cup Z(a_k)$. ∎

Theorem 5.4.2. Let D be a cad of E^2. Let (c^0,c^1) be a (0,1) adjacency of D such that $c^0 = \langle a,b \rangle$, c^1 is a 1-sector, and c^0 is below c^1. Let $B^3 \subset I_3$ be a basis which is regular on c^1. Let M(x) be the integral minimal polynomial of a. Let $B^* = \{pp(Res_x(M,B)) \mid B \in B^3\}$. Then there is an open interval (b,b') in E, such that $\{a\} \times (b,b') \subset c^1$, B^* is regular on (b,b'), and where $d^0 = \{b\}$, $d^1 = (b,b')$, if a section s of $S(B^3_{c^1},c^1)$ is adjacent to a section t of $S(B^3_{c^0},c^0)$, then $s' = \pi_x(s) \cap Z(d^1)$ is a section of $S(B^*_{d^1},d^1)$, $t' = \pi_x(t)$ is a section of $S(B^*_{d^0},d^0)$, and s' is adjacent to t'.

Proof. By Theorem 5.2.2, there is an open interval (b,b') in E such that B^* is regular on (b,b'). Clearly b' can be chosen so that $a \times (b,b')$ is contained in c^1.

Let $d^0 = \{b\}$ and $d^1 = (b,b')$. Every section s of $S(B^3_{c^1},c^1)$ is contained in $V(B) \cap V(M)$, for some $B \in B^3$, where M(x) is viewed as a polynomial in x,y, and z. We have $deg_x(M) > 0$. If $deg_x(B) = 0$, then by definition, $Res_x(M,B) = B(0,y,z)$, hence $Res_x(M,B)$ vanishes on $\pi_x(s)$. If $deg_x(B) > 0$, then since $s \subset V(B) \cap V(M)$, by Theorem 5 of [COL71], $Res_x(M,B)$ vanishes on $\pi_x(s)$. We have $Res_x(M,B) \neq 0$, for if $Res_x(M,B) = 0$, then by Theorem 2 of [COL71], M and B have a common factor of positive degree in x, contradicting the primitivity of B. Since $\pi_x(x)$ is a section, for any $a_1 \in E$, there exists at most one point at which $\pi_x(s)$ meets $Z(a_1)$. Hence by Theorem 5.4.1, $V(pp(Res_x(M,B)))$ contains all except possibly finitely many points of $\pi_x(s)$. Then since $V(pp(Res_x(M,B)))$ is closed, it contains all points of $\pi_x(s)$. Since no element of B^* is cylindrical on d^1, every element of B^* is delineable on d^1, in particular, $pp(Res_x(M,B))$ is delineable on d^1. Hence $\pi_x(s) \cap Z(d^1)$ is a section of $S(B^*_{d^1},d^1)$.

Now suppose that s is adjacent to a section t of $S(B^3_{c^0},c^0)$. Then t is a limit point of s, hence every open ball in E^3 centered at t contains a point of s. For any open ball U in E^2 centered at t' = $\pi_x(t)$, there is an open ball V in E^3 centered at t such that U = $\pi_x(V)$. Hence where p is the point of s contained in V, $\pi_x(p)$ is contained in U, hence t' is a limit point of $\pi_x(s)$. It is then easy to see that t' is a limit point of s' = ($\pi_x(s) \cap Z(d^1)$), hence t' is adjacent to s'.

As shown earlier, there exists $H \in B^*$ which vanishes on s', hence since V(H) is closed, V(H) vanishes on t'. Since H is primitive, it is not cylindrical on d^0, hence t' is a section of $S(B^*_{d^0},d^0)$. ▪

Clearly only minor changes are needed in the above theorem and proof in case c^0 is above c^1. The next theorem deals with the case of c^1 a section.

Theorem 5.4.3. Let D be a cad of E^2. Let (c^0,c^1) be a (0,1) adjacency of D such that $c^0 = \langle a,b \rangle$ and c^1 is a 1-section. Let e^1 be the unique 1-section of D' such that $c^1 \subset S_D(e^1)$, and assume that e^1 is to the right of a. Let $B^3 \subset I_3$ be a basis which is regular on c^1. Let G(x,y) be an element of I_2 of positive degree such that $c^1 \subset V(G)$. Let $B^* = \{pp(Res_y(G,B) \mid B \in B^3\}$. Then there is an open interval $(a,a') \subset e^1$ such that B^* is regular on (a,a'), and where $d^0 = \{a\}$, $d^1 = (a,a')$, if a section s of $S(B^3_{c^1},c^1)$ is adjacent to a section t of $S(B^3_{c^0},c^0)$, then s' = $\pi_y(s) \cap Z(d^1)$ is a section of $S(B^*_{d^1},d^1)$, t' = $\pi_y(t)$ is a section of $S(B^*_{d^0},d^0)$, and s' is adjacent to t'.

Proof. We alter the proof of Theorem 5.4.2 so that G(x,y) plays the role of M(x), and we eliminate y instead of x using resultants. Apart from these changes, the proof of Theorem 5.4.3 is the same as the proof of Theorem 5.4.2 ▪

Theorems 5.4.2 and 5.4.3 alone do not provide an algorithm for determination of adjacencies between $S(B^3_{c^0},c^0)$ and $S(B^3_{c^1},c^1)$. In general, $(S(B^*_{d^0},d^0),S(B^*_{d^1},d^1))$ will not be isomorphic to $(S(B^3_{c^0},c^0),S(B^3_{c^1},c^1))$, for the stacks of the first pair will typically contain more sections than those of the second. To obtain an adjacency algorithm, we will apply SBAA2 to determine the section-section adjacencies between $S(B^*_{d^0},d^0)$ and $S(B^*_{d^1},d^1)$, then extract from this set, in a manner to be explained, the section-section adjacencies between $S(B^3_{c^0},c^0)$ and $S(B^3_{c^1},c^1)$, then infer from these the sector-sector adjacencies between $S(B^3_{c^0},c^0)$ and $S(B^3_{c^1},c^1)$.

The extraction is specified as follows: For every section s of $S(B^3_{c^0},c^0)$, the projection of s (under whichever of π_x or π_y is appropriate) is a section of $S(B^*_{d^0},d^0)$. For every section t of $S(B^3_{c^1},c^1)$, the projection of t is a section of $S(B^*_{d^1},d^1)$. The adjacencies we wish to extract are those where the section of $S(B^*_{d^0},d^0)$ involved is the projection of a section of $S(B^3_{c^0},c^0)$, and the section of $S(B^*_{d^1},d^1)$ involved is the projection of a section of $S(B^3_{c^1},c^1)$. In other words, the adjacencies we want to extract are those which are the projection of a section-section adjacency between $S(B^3_{c^0},c^0)$ and $S(B^3_{c^1},c^1)$. We now describe how this is done.

We may assume without loss of generality that c^0 and c^1 are as in Theorem 5.4.2. Assume we have chosen $b' > b$ such that the conclusions of Theorem 5.4.2 hold. We choose some rational $b^* \in (b,b')$. Let

$$H^0 = \{B(a,b,z) \mid B \in B^3 \ \& \ B(a,b,z) \neq 0\}$$

$$H^1 = \{B(a,b^*,z) \mid B \in B^3\}$$

$$K^0 = \{B'(b,z) \mid B' \in B^*\}$$

$$K^1 = \{B'(b^*,z) \mid B' \in B^*\}.$$

Every real root of H^0 is a real root of K^0, and every real root of H^1 is a real root of K^1. The real roots of H^0, H^1, K^0, and K^1 are in one-one correspondence with the z-coordinates of sample points for the sections of certain stacks, as follows:

$$H^0 \leftrightarrow S(B^3_{c^0}, c^0)$$

$$H^1 \leftrightarrow S(B^3_{c^1}, c^1)$$

$$K^0 \leftrightarrow S(B^*_{d^0}, d^0)$$

$$K^1 \leftrightarrow S(B^*_{d^1}, d^1).$$

Since our two projection operators π_x and π_y have no effect on z-coordinates, our task reduces to being able to decide whether a particular root of K^0 is a root of H^0, and whether a particular root of K^1 is a root of H^1.

Consider, say, H^0 and K^0. We apply a particular root isolation algorithm (available in the SAC-2 system) to H^0 which provides us with a list of pairs of the form (I,P), where I is an open isolating interval for a real root of H^0, and P(x) is a squarefree algebraic polynomial which has exactly one root in I. For K^0, we obtain a list of pairs of the form (J,Q), with similar specifications.

Let α be a particular root of H^0, represented by the pair (I,P). Suppose we wish to test whether the root of K^0 represented by the pair (J,Q) is α. Let $K = I \cap J$. If $K = \emptyset$, then obviously the answer is no. Suppose $K = (u,v) \neq \emptyset$. If $sign(P(u)) = sign(P(v))$, then P does not have a root in K, hence $\alpha \notin K$, hence $\alpha \notin J$, and hence (J,Q) does not represent α. If $sign(P(u)) \neq sign(P(v))$, then $\alpha \in K$, hence $\alpha \in J$. But every root of H^0 is a root of K^0, hence α is a root of K^0, hence α is the unique root of K^0 in J. Hence (J,Q) represents α.

Having determined the section-section adjacencies between $S(B^3_{c^0},c^0)$ and $S(B^3_{c^1},c^1)$, we infer the sector-sector adjacencies between them by exactly the same method used by SBAA2 to infer sector-sector adjacencies from section-section adjacencies.

5.5 Adjacencies in 3-space over a (0,2) adjacency.

Theorem 5.5.1. Let D be a (B^1,B^2,B^3) cylindricity-refined cad of E^3. Let (c^0,c^2) be a (0,2) adjacency of D'. Let $c^1 \in D'$ be a c^0-bounding 1-cell of c^2. Let s be a section of $S_D(c^2)$, and suppose that where B is the unique element of B^3 for which $s \subset V(B)$, that B is not cylindrical on c^0. Then for any section t of $S_D(c^0)$, s and t are adjacent if and only if there exists a section u of $S_D(c^1)$ such that u is adjacent to both s and t.

Proof. Suppose there exists such a section u. Since t and u are adjacent, one must contain a limit point of the other, and similarly for u and s. Then $t \subset \partial u$, and by Theorem 3.4.8, $u \subset \partial s$, hence by Lemma 3.3.11, $t \subset \partial s$, hence t is adjacent to s. Suppose conversely that s and t are adjacent. By Theorem 3.4.7, $S^*_D(c^2)$ has the unique section boundary property in $S^*_D(c^1)$. Let u be the boundary section of s in $S^*_D(c^1)$. By Theorems 3.4.3 and 3.6.21, $S^*_D(c^1)$ has the unique section boundary property in $S^*_D(c^0)$; let w be the boundary section of u in $S^*_D(c^0)$. By Theorem 3.6.7, w = t, hence u is adjacent to both s and t. ▪

SBAA3 utilizes Theorem 5.5.1 as follows. Let D, c^0, c^2, and c^1 be as specified in the hypotheses of the theorem. Assume that the section-section adjacencies between $S_D(c^0)$ and $S_D(c^1)$, and between $S_D(c^1)$ and $S_D(c^2)$, have previously been determined. Let L_1 and L_2 denote the respective lists of adjacencies. Then for every occurrence of an adjacency (d^0,d^1) in L_1, and

an adjacency (e^1,e^2) in L_2 with $d^1 = e^1$, we record the section-section adjacency (d^0,e^2) between $S_D(c^0)$ and $S_D(c^2)$. By Theorem 5.5.1, we obtain exactly the set of section-section adjacencies between $S_D(c^0)$ and $S_D(c^2)$ in this way. When c^0 is noncylindrical, by Theorems 3.4.3, 3.4.7, and 3.3.14, it is clear that sector-sector adjacencies between $S_D(c^0)$ and $S_D(c^2)$ may be inferred from the section-section adjacencies between the two stacks.

Suppose now that D is a (B^1,B^2,B^3) cylindricity-refined cad of E^3, (c^0,c^2) is a (0,2) adjacency of D', and c^0 is cylindrical. Let s be a section of $S_D(c^2)$, let B be the unique element of B^3 such that $s \subset V(B)$, and suppose B is cylindrical on c^0. We wish to determine all elements of $S_D(c^0)$ which are adjacent to s. Let $H = STRAT(c^0) \cup RES(B)$, and let K be the H-set of z-values of s over c^0. Let $c^0 = <a,b>$. CLCAD3 will construct a cylindricity-refined cad of E^3 such that for every $z \in K$, $<a,b,z>$ is a section of $S_D^*(c^0)$. Hence, by Theorem 3.6.18, where z_1 = min(K), and z_2 = max(K), the elements of $S_D(c^0)$ between $<a,b,z_1>$ and $<a,b,z_2>$ inclusive are precisely the elements of $S_D(c^0)$ adjacent to s.

By definition of K, for each $z \in K$, there exists a 1-cell c^1 in E^2 such that $<a,b,z>$ is the unique limit point in $Z^*(c^0)$ of some 1-section t of $Z^*(c^1)$, and hence $<a,b,z>$ is the unique section of $S_D^*(c^0)$ adjacent to t. SBAA3 essentially uses the algorithm described in Section 5.4 to find the (unique) adjacency between t and a section of $S_D^*(c^0)$, for each such section t. In this way a collection of sections of $S_D^*(c^0)$ is obtained. The one of lowest index will be $<a,b,z_1>$, the one of highest index will be $<a,b,z_2>$, and so we will have determined all elements of $S_D(c^0)$ which are adjacent to s.

We now describe this process in somewhat more detail. From the definition of H-set of z-values, we see that the 1-cell c^1 referred to above is of one of two kinds. First, it may be a c^0-bounding 1-cell of c^2. In this case, $c^1 \in D'$, and t will be the boundary section of s in $S_D^*(c^1)$. We will have previously applied SBAA3 to determine adjacencies between $S_D(c^1)$ and $S_D(c^2)$, and between $S_D(c^0)$ and $S_D(c^1)$. t is either the $+\infty$-section of $Z^*(c^1)$, the $-\infty$-section of $Z^*(c^1)$, or a section of $S_D(c^1)$. In the first case, $\partial t \cap Z^*(c^0) = \langle a, b, +\infty \rangle$, in the second case, $\partial t \cap Z^*(c^0) = \langle a, b, -\infty \rangle$, and in the third case, $\partial t \cap Z^*(c^0)$ is a section u of $S_D^*(c^0)$. In the third case, u is determined from the adjacencies found by SBAA3 between $S_D(c^0)$ and $S_D(c^1)$.

Where $\bar{D}$ is the coarsest H-invariant refinement of D', the second possibility for c^1 is that it is a 1-section of $\bar{D}$ which is adjacent to c^0 and which meets c^2. We determine all the cells c^1 of this second kind as follows. Let $P = PROJ(H \cup B^2) \cup \{M(x)\}$, where $M(x)$ is the integral minimal polynomial of a. We isolate the real roots of P, obtaining an open isolating interval (u,v) for a (with rational number endpoints). Set $a' = v$. Then $H \cup B^2$ is regular on $(a,a']$. Let U = $\{B(a',y) \mid B \in B^2\}$, and let V = $\{Q(a',y) \mid Q \in H \cup B^2\}$. Let d_1^1 and d_2^1 be the c^0-bounding 1-cells of c^2. Assume that both d_1^1 and d_2^1 are sections of D'; the minor changes necessary in our discussion if either is a sector will be evident. Furthermore, assume without loss of generality that for some 1-cell e in the D-induced cad of E^1, d_1^1 is below d_2^1 in $S_D(e)$. There is a unique point $\langle a', \beta_1 \rangle$ of $Z(a') \cap d_1^1$, and a unique point $\langle a', \beta_2 \rangle$ of $Z(a') \cap d_2^1$, and $\beta_2 > \beta_1$. β_1 is a real root of both U and V, and β_2 is a real root of both U and V. The real roots of U are in one-one correspondence with the y-coordinates of sample

points for the sections of $S_D(e)$. Since we know the indices of d_1^1 and d_2^1 in $S_D(e)$, we immediately know which two consecutive real roots of U are β_1 and β_2. By a "root matching" process similar to that described at the end of Section 5.4, we find where β_1 and β_2 occur in the ordered sequence of real roots of V. There will be additional real roots $\gamma_1, \ldots, \gamma_k$, $k \geq 0$, of V such that $\beta_1 < \gamma_1 < \cdots < \gamma_k < \beta_2$. For each γ_i, $<\alpha', \gamma_i>$ is a sample point of a 1-section of $\bar{D}$ which is adjacent to c^0 and which meets c^2, and every such 1-section of $\bar{D}$ gives rise to a γ_i. Thus the γ_i's determine all the 1-cells c^1 of the second kind.

For each γ_i, i.e. for each c^1, we find a $Q \in H$ of positive degree which vanishes at $<\alpha', \gamma_i>$, and hence for which $c^1 \subset V(Q)$. We then apply the algorithm described in Section 5.4 to determine section-section adjacencies between $S_D(c^2)|_{c^1}$ and $S_D(c^0)$. t is the element of $S_D(c^2)|_{c^1}$ whose index is the same as the index of s in $S_D(c^2)$. Thus we can use the list of section-section adjacencies between $S_D(c^2)|c^1$ and $S_D(c^0)$ to determine $\partial t \cap Z^*(c^0)$.

CHAPTER 6

PERFORMANCE OF THE ALGORITHMS

Before turning to the examples, we mention two reasons why the clustering cad algorithm may prove to be faster in a particular case than the original. The first has to do with sample point construction. Define the *degree* of an algebraic point to be the degree of the smallest algebraic extension of the rationals containing all of its coordinates. In the original cad algorithm, much of the time is spent in constructing sample points of high degree. The clustering cad algorithm has two advantages in this regard, first, that fewer sample points are constructed, and second, that the sample points which are constructed tend to be those of lower degree. Second, if r = 3, clusters will be constructed for the induced cad of E^2, and in extending the cad of E^2 to a cad of E^3, the clustering algorithm will perform certain computations once for each cluster in E^2, whereas the original algorithm would perform them once for each cell in E^2.

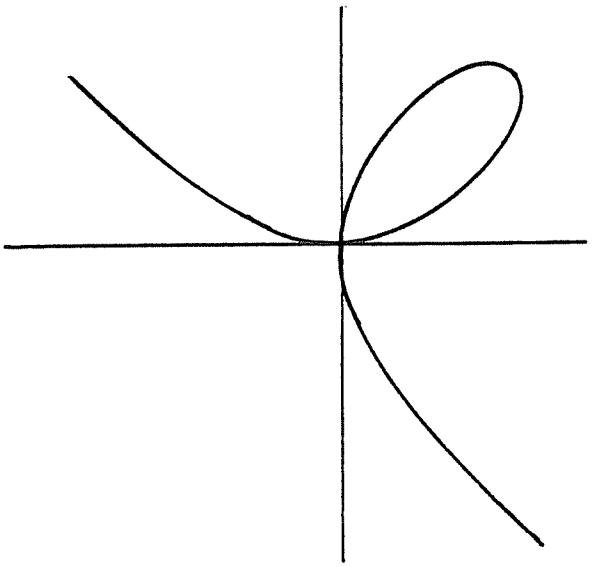
6.1 The folium of Descartes

The purpose of our first example is to illustrate the output produced by the clustering cad algorithm. This output constitutes the last eighteen pages of Section 6.1. We begin with a brief commentary on the output.

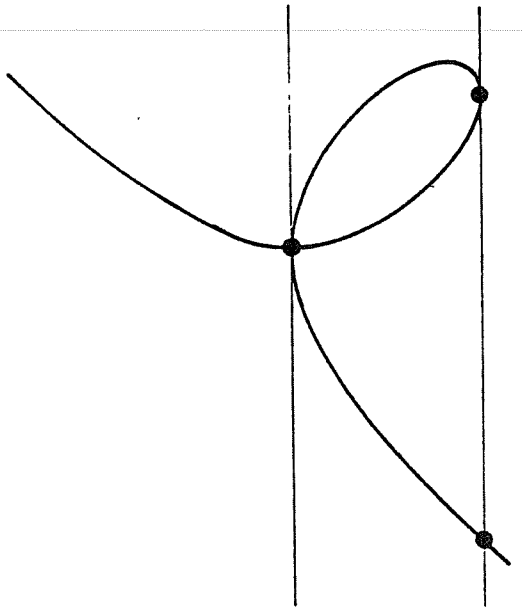
The folium of Descartes ([SEI68], p. 37) is the curve in E^2 which is the locus of

$$F(x,y) = y^3 - 3xy + x^3 = 0.$$

A sketch of the curve is:



Where A = { F }, the A-invariant cad D of E^2 constructed by both cad algorithms may be diagrammed as follows:



In the diagram, each "dot" is a 0-cell, each "arc" is a 1-cell, and each "white region" is a 2-cell. We see that the induced cad of E^1 consists of five cells, which we may label from left to right as c_1, c_2, c_3, c_4, and c_5. c_1, c_3, and c_5 are 1-cells; c_2 and c_4 are 0-cells. $S_D(c_1)$ has three cells, $S_D(c_2)$ has three cells, $S_D(c_3)$ has seven cells, $S_D(c_4)$ has five cells, and $S_D(c_5)$ has three cells, so D has 21 cells. Clearly the coarsest A-clustering of D has four clusters, (the union of) one of which is the folium itself.

The original cad algorithm takes 28 seconds (on a VAX 11/780 computer with the UNIX operating system) to construct D, and the clustering cad algorithm takes 18 seconds to construct D and the four clusters. In both cases, defining formulas for the cells of D were constructed.

In the output, cells are referred to by their *index*. The index of a cell in a cad of E^r, $r \geq 1$, is an r-tuple of positive integers defined inductively as

follows. Any cell of a cad H of E^1 is the i^{th} cell in the left-to-right ordering of cells of H, for some $i \geq 1$; its index is then (i). If r > 1, any cell d of a cad H of E^r is in a stack $S = S_H(c)$, for some $c \in H'$. If $(i_1, \ldots, i_{r-1})$ is the index of c, and if $d = S_j$, $j \geq 1$, then $(i_1, \ldots, i_{r-1}, j)$ is the index of d.

At various places in the output below the sample points of cells are displayed. There are two parts to the display of a cell sample point: first, an exact description of it as specified in Section 2.4, and second, an approximation to five decimal places. Cells for which no sample point representation in the sense of Section 2.4 was constructed by the clustering cad algorithm are identified by the line "*** SAMPLE POINT UNCONSTRUCTED ***".

For each cluster in E^2, the output below first gives the index and dimension of its representative cell (abbreviated as "REP CELL"), then a list of the cells which belong to the cluster, then the polynomials in A (if any) which vanish on each cell of the cluster, and finally the sample point of the representative cell.

In the display of defining formulas for cells, a vertical bar ("|") is used to denote "or".

This concludes our commentary; the output now follows.

BIVARIATE INPUT POLYNOMIALS TO CAD ALGORITHM

$$Y^3 - 3XY + X^3$$

DIMENSION UP TO WHICH DEFINING FORMULAS TO BE CONSTRUCTED

2

FINEST SQUAREFREE BASIS FOR BIVARIATE INPUT POLYNOMIALS

$$Y^3 - 3XY + X^3$$

CONTENTS OF BIVARIATE INPUT POLYNOMIALS

()

AUGMENTED PROJECTION OF BIVARIATE BASIS

$$1$$

$$-27X^6 + 108X^3$$

$$4X$$

UNIVARIATE INPUT POLYNOMIALS TO CAD ALGORITHM

1

$$-27X^6 + 108X^3$$

4 X

FINEST SQUAREFREE BASIS FOR UNIVARIATE INPUT POLYNOMIALS

X

$$X^3 - 4$$

INDUCED CAD OF 1-SPACE

TOTAL OF 5 CELLS IN 1-DIMENSIONAL SPACE

CELL NO. 1 INDEX = (1) DIMENSION = 1

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

- 1

SAMPLE POINT - APPROXIMATE
(-1.00000)

DEFINING FORMULA

(X < 0)

CELL NO. 2 INDEX = (2) DIMENSION = 0

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

0

SAMPLE POINT - APPROXIMATE
(0.00000)

DEFINING FORMULA

(X = 0)

CELL NO. 3 INDEX = (3) DIMENSION = 1

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

1

SAMPLE POINT - APPROXIMATE
(1.00000)

DEFINING FORMULA

$$((X > 0) \& (X^{3} - 4 < 0) \& (X - 2 < 0))$$

CELL NO. 4 INDEX = (4) DIMENSION = 0

SAMPLE POINT - EXACT
LET A BE THE UNIQUE ROOT OF

$$U^{3} - 4$$

BETWEEN 1 AND 2

SAMPLE POINT COORDINATES AS ELEMENTS OF Q(A) ARE

A

SAMPLE POINT - APPROXIMATE
(1.58740)

DEFINING FORMULA

$$((X - 1 > 0) \& (X - 2 < 0) \& (X^{3} - 4 = 0))$$

CELL NO. 5 INDEX = (5) DIMENSION = 1

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

2

SAMPLE POINT - APPROXIMATE
(2.00000)

DEFINING FORMULA

$$((X - 1 > 0) \& (X^{3} - 4 > 0))$$

CELL INDICES FOR CAD OF 2-SPACE

1.	(1,1)	2-CELL
2.	(1,2)	1-CELL
3.	(1,3)	2-CELL
4.	(2,1)	1-CELL
5.	(2,2)	0-CELL
6.	(2,3)	1-CELL
7.	(3,1)	2-CELL
8.	(3,2)	1-CELL
9.	(3,3)	2-CELL
10.	(3,4)	1-CELL
11.	(3,5)	2-CELL

12.	(3,6)	1-CELL
13.	(3,7)	2-CELL
14.	(4,1)	1-CELL
15.	(4,2)	0-CELL
16.	(4,3)	1-CELL
17.	(4,4)	0-CELL
18.	(4,5)	1-CELL
19.	(5,1)	2-CELL
20.	(5,2)	1-CELL
21.	(5,3)	2-CELL

CLUSTERS FOR 2-SPACE

TOTAL OF 4 CLUSTERS IN 2-DIMENSIONAL SPACE

CLUSTER NO. 1 REP CELL INDEX = (5,3) REP CELL DIMENSI
ON = 2

CONSTITUENT CELLS
(5,3)
(3,3)
(4,3)
(3,7)
(1,3)
(2,3)
(4,5)

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

2

- 1

SAMPLE POINT - APPROXIMATE
(2.00000 , -1.00000)

CLUSTER NO. 2 REP CELL INDEX = (5,2) REP CELL DIMENSI
ON = 1

CONSTITUENT CELLS
(5,2)
(3,6)
(3,4)

(3,2)
(1,2)
(2,2)
(4,2)
(4,4)

POLYNOMIALS VANISHING ON THE CLUSTER

$$Y^3 - 3XY + X^3$$

SAMPLE POINT - EXACT
LET A BE THE UNIQUE ROOT OF

$$U^3 - 6U + 8$$

BETWEEN -4 AND -2
SAMPLE POINT COORDINATES AS ELEMENTS OF Q(A) ARE

$$\begin{matrix} 2 \\ A \end{matrix}$$

SAMPLE POINT - APPROXIMATE
(2.00000 , -2.95137)

CLUSTER NO. 3 REP CELL INDEX = (5,1) REP CELL DIMENSI
ON = 2

CONSTITUENT CELLS
(5,1)
(3,1)
(1,1)
(2,1)
(4,1)

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

$$\begin{matrix} 2 \\ - 8 \end{matrix}$$

```
     SAMPLE POINT - APPROXIMATE
     ( 2.00000 , -8.00000 )

CLUSTER NO. 4      REP CELL INDEX = (3,5)     REP CELL DIMENSI
ON = 2

     CONSTITUENT CELLS
          (3,5)

     SAMPLE POINT - EXACT
          SAMPLE POINT COORDINATES ARE


           1



           1

     SAMPLE POINT - APPROXIMATE
     ( 1.00000 , 1.00000 )




CAD OF 2-SPACE

                    TOTAL OF 21 CELLS IN 2-DIMENSIONAL SPACE

CELL NO. 1      INDEX = (1,1)     DIMENSION = 2

     SAMPLE POINT - EXACT
          SAMPLE POINT COORDINATES ARE


          - 1



          - 4

     SAMPLE POINT - APPROXIMATE
     ( -1.00000 , -4.00000 )

     DEFINING FORMULA
```

$$((X < 0) \& (Y^3 - 3 X Y + X^3 < 0))$$

CELL NO. 2 INDEX = (1,2) DIMENSION = 1

SAMPLE POINT - EXACT
LET A BE THE UNIQUE ROOT OF

$$U^3 + 3 U - 1$$

BETWEEN 0 AND 2
SAMPLE POINT COORDINATES AS ELEMENTS OF Q(A) ARE

$$-1$$

$$A$$

SAMPLE POINT - APPROXIMATE
(-1.00000 , 0.32218)

DEFINING FORMULA

$$((X < 0) \& (Y^3 - 3 X Y + X^3 = 0))$$

CELL NO. 3 INDEX = (1,3) DIMENSION = 2

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

$$-1$$

$$5$$

SAMPLE POINT - APPROXIMATE
(-1.00000 , 5.00000)

DEFINING FORMULA

$$((X < 0) \& (Y^3 - 3 X Y + X^3 > 0))$$

CELL NO. 4 INDEX = (2,1) DIMENSION = 1

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

0

- 1

SAMPLE POINT - APPROXIMATE
(0.00000 , -1.00000)

DEFINING FORMULA

$$((X = 0) \& (Y^3 - 3 X Y + X^3 < 0))$$

CELL NO. 5 INDEX = (2,2) DIMENSION = 0

*** SAMPLE POINT UNCONSTRUCTED ***

DEFINING FORMULA

$$((X = 0) \& (Y^3 - 3 X Y + X^3 = 0))$$

CELL NO. 6 INDEX = (2,3) DIMENSION = 1

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

0

2

SAMPLE POINT - APPROXIMATE
(0.00000 , 2.00000)

DEFINING FORMULA

$$((X = 0) \& (Y^3 - 3 X Y + X^3 > 0))$$

CELL NO. 7 INDEX = (3,1) DIMENSION = 2

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

1

- 2

SAMPLE POINT - APPROXIMATE
(1.00000 , -2.00000)

DEFINING FORMULA

$$(((X > 0) \& (X^3 - 4 < 0) \& (X - 2 < 0)) \&$$
$$(((Y < 0) \& (Y^2 - X > 0)) \& (Y^3 - 3 X Y$$
$$+ X^3 < 0)))$$

CELL NO. 8 INDEX = (3,2) DIMENSION = 1

SAMPLE POINT - EXACT
LET A BE THE UNIQUE ROOT OF

$$U^3 - 3\,U + 1$$

BETWEEN -2 AND -7/4
SAMPLE POINT COORDINATES AS ELEMENTS OF Q(A) ARE

$$\begin{matrix} 1 \\ \\ A \end{matrix}$$

SAMPLE POINT - APPROXIMATE
(1.00000 , -1.87939)

DEFINING FORMULA

$$(((X > 0) \,\&\, (X^3 - 4 < 0) \,\&\, (X - 2 < 0)) \,\&$$

$$(((Y < 0) \,\&\, (Y^2 - X > 0)) \,\&\, (Y^3 - 3\,X\,Y$$

$$+ X^3 = 0)))$$

CELL NO. 9 INDEX = (3,3) DIMENSION = 2

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

$$\begin{matrix} 1 \\ \\ 0 \end{matrix}$$

SAMPLE POINT - APPROXIMATE
(1.00000 , 0.00000)

DEFINING FORMULA

$$(((\ X > 0\ ) \ \&\ (\ X\ - 4 < 0\ ) \ \&\ (\ X - 2 < 0\ )) \ \&$$

$$((((\ Y < 0\ ) \ \&\ (\ Y^{2} - X > 0\ )) \ \&\ (\ Y^{3} - 3\ X\ Y$$

$$+ X^{3} > 0\ )) \mid ((\ Y < 0\ ) \ \&\ (\ Y^{2} - X = 0\ )) \mid (($$

$$((\ Y < 0\ ) \ \&\ (\ Y^{2} - X < 0\ )) \mid (\ Y = 0\ ) \mid (($$

$$Y > 0\ ) \ \&\ (\ Y^{2} - X < 0\ ))) \ \&\ (\ Y^{3} - 3\ X\ Y + X^{3}$$

$$> 0\ ))))$$

CELL NO. 10 INDEX = (3,4) DIMENSION = 1

SAMPLE POINT - EXACT
LET A BE THE UNIQUE ROOT OF

$$U^{3} - 3\ U + 1$$

BETWEEN 1/4 AND 1/2
SAMPLE POINT COORDINATES AS ELEMENTS OF Q(A) ARE

$$1$$

$$A$$

SAMPLE POINT - APPROXIMATE
(1.00000 , 0.34730)

DEFINING FORMULA

$$(((\ X > 0\ ) \ \&\ (\ X^{3} - 4 < 0\ ) \ \&\ (\ X - 2 < 0\ )) \ \&$$

$$((((\ Y < 0\ ) \ \&\ (\ Y^{2} - X < 0\ )) \mid (\ Y = 0\ ) \mid ($$

$$2 \qquad\qquad 3$$

(Y > 0) & (Y - X < 0))) & (Y - 3 X Y

$$+ X^3 = 0)))$$

CELL NO. 11 INDEX = (3,5) DIMENSION = 2

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

1

1

SAMPLE POINT - APPROXIMATE
(1.00000 , 1.00000)

DEFINING FORMULA

$$(((X > 0) \& (X^3 - 4 < 0) \& (X - 2 < 0)) \&$$

$$(((((Y < 0) \& (Y^2 - X < 0)) | (Y = 0) |$$

$$((Y > 0) \& (Y^2 - X < 0))) \& (Y^3 - 3 X Y$$

$$+ X^3 < 0)) | ((Y > 0) \& (Y^2 - X = 0)) | (($$

$$(Y > 0) \& (Y^2 - X > 0)) \& (Y^3 - 3 X Y + X^3$$

$$< 0))))$$

CELL NO. 12 INDEX = (3,6) DIMENSION = 1

SAMPLE POINT - EXACT

LET A BE THE UNIQUE ROOT OF

$$U^3 - 3\,U + 1$$

BETWEEN 3/2 AND 2
SAMPLE POINT COORDINATES AS ELEMENTS OF Q(A) ARE

$$1$$

$$A$$

SAMPLE POINT - APPROXIMATE
(1.00000 , 1.53209)

DEFINING FORMULA

$$(((X > 0) \,\&\, (X^3 - 4 < 0) \,\&\, (X - 2 < 0)) \,\&$$

$$(((Y > 0) \,\&\, (Y^2 - X > 0)) \,\&\, (Y^3 - 3\,X\,Y$$

$$+ X^3 = 0)))$$

CELL NO. 13 INDEX = (3,7) DIMENSION = 2

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

$$1$$

$$3$$

SAMPLE POINT - APPROXIMATE
(1.00000 , 3.00000)

DEFINING FORMULA

$$(((\ X > 0\ ) \ \&\ (\ X\ - 4 < 0\ ) \ \&\ (\ X - 2 < 0\ )) \ \&$$

$$(((\ Y > 0\ ) \ \&\ (\ Y^{2} - X > 0\ )) \ \&\ (\ Y^{3} - 3\ X\ Y$$

$$+\ X^{3} > 0\ )))$$

CELL NO. 14 INDEX = (4,1) DIMENSION = 1

SAMPLE POINT - EXACT
LET A BE THE UNIQUE ROOT OF

$$U^{3} - 4$$

BETWEEN 1 AND 2
SAMPLE POINT COORDINATES AS ELEMENTS OF Q(A) ARE

$$A$$

$$-\ 4$$

SAMPLE POINT - APPROXIMATE
(1.58740 , -4.00000)

DEFINING FORMULA

$$(((\ X - 1 > 0\ ) \ \&\ (\ X - 2 < 0\ ) \ \&\ (\ X^{3} - 4 = 0\ )$$

$$)\ \&\ (((\ Y < 0\ ) \ \&\ (\ Y^{2} - X > 0\ )) \ \&\ (\ Y^{3}$$

$$- 3\ X\ Y + X^{3} < 0\ )))$$

CELL NO. 15 INDEX = (4,2) DIMENSION = 0

*** SAMPLE POINT UNCONSTRUCTED ***

DEFINING FORMULA

$$(((\ X - 1 > 0\ ) \ \&\ (\ X - 2 < 0\ ) \ \&\ (\ X^{3} - 4 = 0\ ) \ ) \ \&\ (((\ Y < 0\ ) \ \&\ (\ Y^{2} - X > 0\ )) \ \&\ (\ Y^{3} - 3\ X\ Y + X^{3} = 0\ )))$$

CELL NO. 16 INDEX = (4,3) DIMENSION = 1

SAMPLE POINT - EXACT
LET A BE THE UNIQUE ROOT OF

$$U^{3} - 4$$

BETWEEN 1 AND 2
SAMPLE POINT COORDINATES AS ELEMENTS OF Q(A) ARE

$$A$$

$$0$$

SAMPLE POINT - APPROXIMATE
(1.58740 , 0.00000)

DEFINING FORMULA

$$(((\ X - 1 > 0\ ) \ \&\ (\ X - 2 < 0\ ) \ \&\ (\ X^{3} - 4 = 0\ ) \ ) \ \&\ ((((\ Y < 0\ ) \ \&\ (\ Y^{2} - X > 0\ )) \ \&\ (\ Y^{3} - 3\ X\ Y + X^{3} > 0\ )) \ | \ ((\ Y < 0\ ) \ \&\ (\ Y^{2} - X = 0\ )) \ | \ (((\ Y < 0\ ) \ \&\ (\ Y^{2} - X < 0\ )) \ | \ (\ Y = 0\ ))$$

| ((Y > 0) & (Y - X < 0)))))

CELL NO. 17 INDEX = (4,4) DIMENSION = 0

SAMPLE POINT - EXACT
LET A BE THE UNIQUE ROOT OF

$$U^3 - 4$$

BETWEEN 1 AND 2
SAMPLE POINT COORDINATES AS ELEMENTS OF Q(A) ARE

$$A$$

$$1/2\ A^2$$

SAMPLE POINT - APPROXIMATE
(1.58740 , 1.25992)

DEFINING FORMULA

$$(((X - 1 > 0) \& (X - 2 < 0) \& (X^3 - 4 = 0)) \& ((Y > 0) \& (Y^2 - X = 0)))$$

CELL NO. 18 INDEX = (4,5) DIMENSION = 1

SAMPLE POINT - EXACT
LET A BE THE UNIQUE ROOT OF

$$U^3 - 4$$

BETWEEN 1 AND 2
SAMPLE POINT COORDINATES AS ELEMENTS OF Q(A) ARE

A

5

SAMPLE POINT - APPROXIMATE
(1.58740 , 5.00000)

DEFINING FORMULA

$$(((X - 1 > 0) \& (X - 2 < 0) \& (X^{3} - 4 = 0)) \& ((Y > 0) \& (Y^{2} - X > 0)))$$

CELL NO. 19 INDEX = (5,1) DIMENSION = 2

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

2

- 8

SAMPLE POINT - APPROXIMATE
(2.00000 , -8.00000)

DEFINING FORMULA

$$(((X - 1 > 0) \& (X^{3} - 4 > 0)) \& (Y^{3} - 3 X Y + X^{3} < 0))$$

CELL NO. 20 INDEX = (5,2) DIMENSION = 1

SAMPLE POINT - EXACT

LET A BE THE UNIQUE ROOT OF

$$U^3 - 6\,U + 8$$

BETWEEN -4 AND -2
SAMPLE POINT COORDINATES AS ELEMENTS OF Q(A) ARE

$$\begin{matrix} 2 \\ \\ A \end{matrix}$$

SAMPLE POINT - APPROXIMATE
(2.00000 , -2.95137)

DEFINING FORMULA

$$(((X - 1 > 0) \,\&\, (X^3 - 4 > 0)) \,\&\, (Y^3 - 3\,X\,Y + X^3 = 0))$$

CELL NO. 21 INDEX = (5,3) DIMENSION = 2

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

$$\begin{matrix} 2 \\ \\ -1 \end{matrix}$$

SAMPLE POINT - APPROXIMATE
(2.00000 , -1.00000)

DEFINING FORMULA

$$(((X - 1 > 0) \,\&\, (X^3 - 4 > 0)) \,\&\, (Y^3$$

3

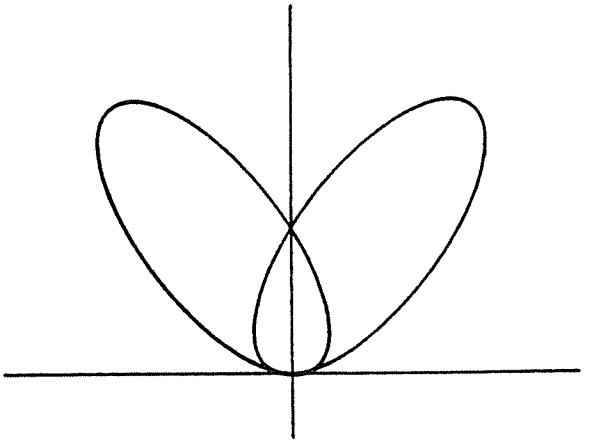
```
- 3 X Y + X  > 0 ) )
```

6.2 A tacnode.

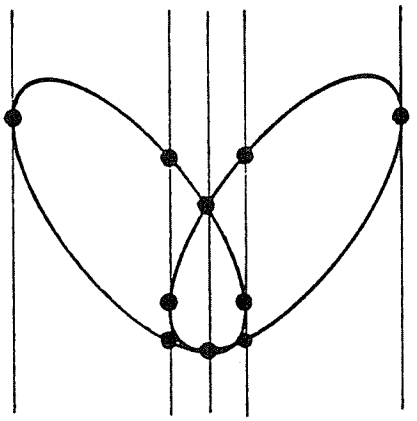
The locus of the following equation is a *tacnode* ([WAL50], p. 58):

$$F(x,y) = y^4 - 2y^3 + y^2 - 3x^2y + 2x^4 = 0.$$

A sketch of the curve is:



Where A = { F }, the A-cad D of E^2 constructed by both cad algorithms (with no defining formula construction) may be diagrammed as follows:



D has 55 cells. Clearly the coarsest A-clustering of D has five clusters.

The original cad algorithm takes 1508 seconds to construct D, and the clustering cad algorithm takes 107 seconds to construct D and the five clusters. (No defining formulas were constructed in the runs for which these times were recorded). The clustering cad algorithm does not construct sample points for the following cells:

(4,2)
(4,6)
(6,2)
(6.4)
(8,2)
(8,6)

When we examine the sample points constructed by the original cad algorithm for these cells, we find that (6,2) and (6,4) have degree one, and (4,2), (4,6), (8,2), and (8,6) all have degree 12. The highest degree of a sample point constructed by the clustering cad algorithm is six; all told it constructs 16 sample points of degree six. Typical of these is the sample point $< \alpha,\beta >$ for the cell (10,2). The primitive element γ is the unique root between 1 and 2 of the irreducible polynomial

$$2048x^6 - 4608x^4 + 37x^2 + 12$$

The representation of α as an element of $Q(\gamma)$ is just

$$x$$

and the representation of β as an element of $Q(\gamma)$ is

$$-(1024/827)x^4 + (2904/827)x^2 + (89/827).$$

By contrast, the sample point $< \alpha,\beta >$ for (4,2) constructed by the original cad algorithm has as its primitive element γ the unique root between -3/16 and -5/32 of the irreducible polynomial

$$4194304x^{12} - 16777216x^{11} + 16252928x^{10} + 10354688x^9$$
$$+ 94023680x^8 - 747077632x^7 + 2135581952x^6 - 3371340736x^5$$
$$+ 3113787697x^4 - 1566148632x^3 + 298107528x^2 + 45313632x$$
$$- 16210224 .$$

The representation for α as an element of $Q(\gamma)$ is

$$\frac{492381547976267162630350034350899 2x^{11}}{569941863719637299060416947113967 9}$$
$$- \frac{212226145481884072877620509785456 64x^{10}}{949903106199395498434028245189946 5}$$

$$+\frac{13863544300957625983486370798878 72x^{9}}{9499031061993954984340282451899465}$$

$$+\frac{22459157500220841804159378123905024x^{8}}{9499031061993954984340282451899465}$$

$$+\frac{43369715451956551314293537826628800x^{7}}{18998062123987909968680564903798 93}$$

$$-\frac{23095837927534598754481717659357017 6x^{6}}{1899806212398790996868056490379893}$$

$$+\frac{25341667823617963486074930343442272 68x^{5}}{9499031061993954984340282451899465}$$

$$-\frac{29787257797888264686019262183901136 91x^{4}}{949903106199395498434028245 1899465}$$

$$+\frac{35381005820845594436303977226654960 5223x^{3}}{18238139639028393569933342307646972 80}$$

$$-\frac{14674169912537006831664266762010585 093x^{2}}{30396899398380655949888903846078 2880}$$

$$-\frac{56983959866975898644326574721546786 9x}{15198449699190327974944451923039144 0}$$

$$+\frac{18095127776574386283323602276950486 9}{75992248495951639874722259615195720}$$

The representation for β as an element of $Q(\gamma)$ is

$$-\frac{49238154797626716263035003435089 92x^{11}}{5699418637196372990604169471139679}$$

$$+\frac{21222614548188407287762050978545664x^{10}}{9499031061993954984340282451899465}$$

$$-\frac{13863544300957625983486370798878 72x^{9}}{949903106199395498434028245 1899465}$$

$$-\frac{22459157500220841804159378123905024x^{8}}{9499031061993954984340282451899465}$$

$$-\frac{43369715451956551314293537826628800x^{7}}{18998062123987909968680564903798 93}$$

$$+\frac{23095837927534598754481717659357017 6x^{6}}{1899806212398790996868056490379893}$$

$$- \frac{25341667823617963486074930343442272 68x^5}{9499031061993954984340282451899465}$$

$$+ \frac{29787257797888264686019262183901136 91x^4}{9499031061993954984340282451899465}$$

$$- \frac{35381005820845594436303977226654960522 3x^3}{18238139639028393569933342307646972 80}$$

$$+ \frac{14674169912537006831664266762010585093x^2}{30396899398380655949888903846078288 0}$$

$$+ \frac{72182409566166226619271026644585930 9x}{15198449699190327974944451923039144 0}$$

$$- \frac{18095127776574386283323602276950486 9}{75992248495951639874722259615195720}$$

6.3 A quartic surface.

Our next example is a cad D of E^3 such that the induced cad of E^2 contains a cylindrical 0-cell. Where

$$F(x,y,z) = y^3z + xy^2 - x^3,$$

we take A = { F }. F is primitive, and one easily sees that F is cylindrical at $<0,0> \in E^2$. (This example was proposed by S. McCallum).

The augmented projection of A is P = $\{y^3, xy^2 - x^3\}$. CONT(P) = { x }, and the finest squarefree basis for PP(P) is $\{y, y + x, y - x\}$, so the polynomials determining the induced cad of E^2 are x, y, $y - x$, and $y + x$.

Thus D' has 17 cells as follows:

1. (1,1) 2-CELL
2. (1,2) 1-CELL
3. (1,3) 2-CELL
4. (1,4) 1-CELL
5. (1,5) 2-CELL
6. (1,6) 1-CELL
7. (1,7) 2-CELL
8. (2,1) 1-CELL

9.	(2,2)	0-CELL
10.	(2,3)	1-CELL
11.	(3,1)	2-CELL
12.	(3,2)	1-CELL
13.	(3,3)	2-CELL
14.	(3,4)	1-CELL
15.	(3,5)	2-CELL
16.	(3,6)	1-CELL
17.	(3,7)	2-CELL

<0,0> is cell (2,2) of D'. D, constructed by the clustering cad algorithm, has 51 cells as follows:

1.	(1,1,1)	3-CELL
2.	(1,1,2)	2-CELL
3.	(1,1,3)	3-CELL
4.	(1,2,1)	2-CELL
5.	(1,2,2)	1-CELL
6.	(1,2,3)	2-CELL
7.	(1,3,1)	3-CELL
8.	(1,3,2)	2-CELL
9.	(1,3,3)	3-CELL
10.	(1,4,1)	2-CELL
11.	(1,5,1)	3-CELL
12.	(1,5,2)	2-CELL
13.	(1,5,3)	3-CELL
14.	(1,6,1)	2-CELL
15.	(1,6,2)	1-CELL
16.	(1,6,3)	2-CELL
17.	(1,7,1)	3-CELL
18.	(1,7,2)	2-CELL
19.	(1,7,3)	3-CELL
20.	(2,1,1)	2-CELL
21.	(2,1,2)	1-CELL
22.	(2,1,3)	2-CELL
23.	(2,2,1)	1-CELL
24.	(2,2,2)	0-CELL
25.	(2,2,3)	1-CELL
26.	(2,2,4)	0-CELL
27.	(2,2,5)	1-CELL
28.	(2,2,6)	0-CELL
29.	(2,2,7)	1-CELL
30.	(2,3,1)	2-CELL
31.	(2,3,2)	1-CELL
32.	(2,3,3)	2-CELL
33.	(3,1,1)	3-CELL
34.	(3,1,2)	2-CELL
35.	(3,1,3)	3-CELL

36.	(3,2,1)	2-CELL
37.	(3,2,2)	1-CELL
38.	(3,2,3)	2-CELL
39.	(3,3,1)	3-CELL
40.	(3,3,2)	2-CELL
41.	(3,3,3)	3-CELL
42.	(3,4,1)	2-CELL
43.	(3,5,1)	3-CELL
44.	(3,5,2)	2-CELL
45.	(3,5,3)	3-CELL
46.	(3,6,1)	2-CELL
47.	(3,6,2)	1-CELL
48.	(3,6,3)	2-CELL
49.	(3,7,1)	3-CELL
50.	(3,7,2)	2-CELL
51.	(3,7,3)	3-CELL

The coarsest A-clustering of D has three clusters, as follows:

CLUSTER NO. 1 REP CELL INDEX = (3,3,1) REP CELL DIMEN
SION = 3

CONSTITUENT CELLS
(3,3,1)
(3,7,3)
(3,1,1)
(1,7,3)
(1,5,3)
(1,3,1)
(1,1,1)
(1,2,1)
(1,4,1)
(1,6,3)
(2,1,1)
(2,3,3)
(3,2,1)
(3,5,3)
(3,6,3)

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

1

- 16

SAMPLE POINT - APPROXIMATE
(1.00000 , -0.50000 , -16.00000)

CLUSTER NO. 2 REP CELL INDEX = (3,7,2) REP CELL DIMEN
SION = 2

CONSTITUENT CELLS
(3,7,2)
(1,7,2)
(1,5,2)
(1,6,2)
(2,3,2)
(3,5,2)
(3,6,2)
(3,3,2)
(3,1,2)
(1,3,2)
(1,1,2)
(1,2,2)
(2,1,2)
(3,2,2)
(2,2,7)
(2,2,5)
(2,2,3)
(2,2,1)
(2,2,2)
(2,2,4)
(2,2,6)

POLYNOMIALS VANISHING ON THE CLUSTER

$Y^3 Z + X Y^2 - X^3$

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

1

2

```
   - 3/8

   SAMPLE POINT - APPROXIMATE
   ( 1.00000 , 2.00000 , -0.37500 )

CLUSTER NO. 3    REP CELL INDEX = (3,7,1)    REP CELL DIMEN
SION = 3

   CONSTITUENT CELLS
      (3,7,1)
      (1,7,1)
      (1,5,1)
      (1,6,1)
      (2,3,1)
      (3,5,1)
      (3,3,3)
      (3,1,3)
      (1,3,3)
      (1,1,3)
      (1,2,3)
      (2,1,3)
      (3,2,3)
      (3,4,1)
      (3,6,1)

   SAMPLE POINT - EXACT
      SAMPLE POINT COORDINATES ARE

       1


       2


      - 1

   SAMPLE POINT - APPROXIMATE
   ( 1.00000 , 2.00000 , -1.00000 )
```

The clustering cad algorithm takes 88 seconds to construct D and the three clusters.

When given A as input, the original cad algorithm will construct an A-cad H of E^3 with 45 cells. D and H differ only in that $S_H(<0,0>)$ consists of

the single 1-cell Z(<0,0>), whereas S_D(<0,0>) consists of seven cells. As described in Chapters 3 and 4, the additional cells are needed to do clustering in E^3. The original cad algorithm takes 18 seconds to construct H.

We now show the cells of S_D(<0,0>) as constructed by the clustering cad algorithm.

CELL NO. 23 INDEX = (2,2,1) DIMENSION = 1

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

0

0

- 1/2

SAMPLE POINT - APPROXIMATE
(0.00000 , 0.00000 , -0.50000)

DEFINING FORMULA

((X = 0) & (Y + X = 0) & ((Z < 0) & (
27 Z^2 - 4 > 0)))

CELL NO. 24 INDEX = (2,2,2) DIMENSION = 0

*** SAMPLE POINT UNCONSTRUCTED ***

DEFINING FORMULA

((X = 0) & (Y + X = 0) & ((Z < 0) & (
27 Z^2 - 4 = 0)))

CELL NO. 25 INDEX = (2,2,3) DIMENSION = 1

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

0

0

- 1/8

SAMPLE POINT - APPROXIMATE
(0.00000 , 0.00000 , -0.12500)

DEFINING FORMULA

$$((X=0)\&(Y+X=0)\&((((Z<0)\&(27Z^2-4<0))|(Z=0)|((Z>0)\&(27Z^2-4<0)))\&(Z<0)))$$

CELL NO. 26 INDEX = (2,2,4) DIMENSION = 0

*** SAMPLE POINT UNCONSTRUCTED ***

DEFINING FORMULA

$$((X=0)\&(Y+X=0)\&(Z=0))$$

CELL NO. 27 INDEX = (2,2,5) DIMENSION = 1

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

0

0

1/4

SAMPLE POINT - APPROXIMATE
(0.00000 , 0.00000 , 0.25000)

DEFINING FORMULA

$$((X = 0) \& (Y + X = 0) \& ((Z > 0) \& (((Z < 0) \& (27 Z^2 - 4 < 0)) | (Z = 0) | ((Z > 0) \& (27 Z^2 - 4 < 0)))))$$

CELL NO. 28 INDEX = (2,2,6) DIMENSION = 0

*** SAMPLE POINT UNCONSTRUCTED ***

DEFINING FORMULA

$$((X = 0) \& (Y + X = 0) \& ((Z > 0) \& (27 Z^2 - 4 = 0)))$$

CELL NO. 29 INDEX = (2,2,7) DIMENSION = 1

SAMPLE POINT - EXACT
SAMPLE POINT COORDINATES ARE

0

0

3/2

SAMPLE POINT - APPROXIMATE
(0.00000 , 0.00000 , 1.50000)

DEFINING FORMULA

((X = 0) & (Y + X = 0) & ((Z > 0) & (

27 Z^2 - 4 > 0)))

6.4 Three simultaneous surfaces

In this example the input to the clustering cad algorithm is the set A of the following three trivariate polynomials:

$$F_1(x,y,z) = 3z - xy - 4$$

$$F_2(x,y,z) = z - y + x - 1$$

$$F_3(x,y,z) = z^2 - 2xz + y - 1.$$

The clustering cad algorithm takes 5573 seconds, or about 93 minutes, to construct an A-invariant cad D of E^3, and the coarsest A-clustering L of D. D has 1541 cells and L has 30 clusters. The D-induced cad of E^2 has 201 cells and 33 clusters, and the D-induced cad of E^1 has 17 cells. No defining formulas were constructed. There are no clusters of L on which all three of the polynomials vanish. Hence we may conclude that the system of polynomial equations:

$$F_1(x,y,z) = 0 \ \& \ F_2(x,y,z) = 0 \ \& \ F_3(x,y,z) = 0$$

has no real solutions.

CHAPTER 7

SUMMARY

Let us summarize what has been done in this thesis. We have given a new and complete presentation of the mathematical results which underlie the original cad algorithm, and which continue to play an essential role in the clustering cad algorithm. We have shown that in certain kinds of cad's, cell boundaries have the property of being the union of other cells of the cad. We have developed the clustering cad algorithm, which is usually more efficient than the original algorithm, and which produces clusters of geometrical interest in their own right. We have developed an algorithm for determining whether certain pairs of cells in a cad are adjacent (this adjacency algorithm is the joint work of G. Collins, D. Arnon, and S. McCallum). Finally, we have written computer programs which provide the first complete implementation of a cad algorithm, and given examples of their application. Since these programs are an extension to the highly portable SAC-2 computer algebra system, they should be usable by any interested persons.

There is much that could be done to continue the work reported in this thesis. Of the many things that could be mentioned, one of the most apparent is that we have hardly begun to investigate the application of our programs to RCF quantifier elimination problems. What we hope we have accomplished with this thesis is to advance the state of algorithms and software for investigations pertaining to the geometric structure of semi-algebraic sets.

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