COMPOUNDING DENSER (d,k) GRAPH ARCHITECTURES FOR COMPUTER NETWORKS

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Compounding Denser (d,k) Graph Architectures for Computer Networks*

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Abstract

An algorithm is described for compounding n+1 copies of a regular (d,k) [or (n,d,k)] graph, G, of degree d and diameter k, with n nodes, to give $(n!)^n$ potentially different new compound graphs, $C_G(n(n+1),d+1,<=2k+1)$. In many cases this gives "denser" graphs (i.e., with more nodes for a given degree and diameter) than the best previously achieved.

Index Terms: (d,k) graphs, Moore graphs, computer networks, packing density, compounding, graphs of computers, (n,d,k) graphs, size of graphs, network architectures.

An important goal for computer networks is to maximize the number of processors within a given diameter, to keep message-passing paths short.

Hoffman and Singleton [1] and their students [14], after hearing Moore [2] give an upper bound, n_b, on n (the number of nodes) in a regular graph of degree d and diameter k:

$$n_b(d,k) = \frac{(d-1)^k - 2}{d - 2}$$

and show that the n=10,d=3,k=2, (10,3,2) Petersen graph [3] achieves this bound, proved that at most 3 such optimal Moore graphs exist [(10,3,2), (50,7,2) and one, (3250,57,2), that has not been proved impossible], except for the (n,d,2) complete graphs and (n,2,k) polygon graphs.

Elspas [4] described several construction methods that gave non-optimal but densest-so-far graphs in a number of cases.

* I am grateful to E. F. Moore, for helping me appreciate the Petersen and other highly symmetric graphs, and to John Halton, Pat Hanrahan, Will Leland and Li Qiao, as well as E.F. Moore for many stimulating discussions. The results of Leland and Li mentioned in the Addenda will be reported by them elsewhere.

Akers [5] gave an elegant construction that generates graphs such that "No proof has been found that these graphs are maximal, but in each case the number of nodes is as great or greater than any reported by Elspas." Friedman [6] described a construction for joining trees at their buds (the nodes with 1 link); Korn [7] improved it slightly; and Storwick [8] developed a still-better variant that joins different trees, sometimes at inner nodes.

Recently, Arden and Lee [9], also combining trees, achieved better results for several d=3 graphs. And Toueg and Steiglitz [10], using a computer program to make a directed search, achieved several still better graphs for d=3 and d=4.

The Graph Compounding Algorithm

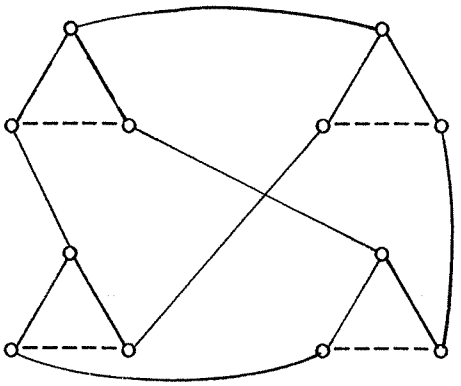
Take any regular (n,d,k) graph, G. Form n+1 copies, $G_{\emptyset}, G_1, \ldots G_n$. For each node, $G_{i,j}$ (i = copy, j = node), i = $\emptyset,1,\ldots n-1$, j = $i, i+1, \ldots n$, k = $i+1, i+2, \ldots n$, l = i; link (in ascending order of i,j,k) the node $G_{i,j}$ to $G_{k,l}$.

The following gives a step-by-step description, and Figure 1 shows a compound on (3,2,1).

Link the 1st node of the 1st copy to the 1st node of the 2d copy;
Link the 2d node of the 1st copy to the 1st node of the 3d copy;
Link the 3d node of the 1st copy to the 1st node of the 4th copy;
.
.
Link the nth node of the 1st copy to the 1st node of the nth copy.

Link the 2d node of the 2d copy to the 2d node of the 3d copy;
Link the 3d node of the 2d copy to the 2d node of the 4th copy;
.
.
Link the nth node of the 2d copy to the 2d node of the nth copy.
.
.
Link the n-1th node of the n-1th copy to the nth node of the nth copy.

Figure 1. The Compound (12,3,3) of (3,2,1)



Theorem: Given any n node graph, G, of degree d and diameter k, a new compound graph, C, can be constructed with parameters:

$n_C = n(n+1)$; $d_C = d+1$; $k_C <= 2k+1$.

The proof is essentially the above construction, plus the following observations.

The total number of nodes in the compounded graph is clearly n(n+1), since the compound graph contains n+1 copies and each copy has n nodes.

The degree of the compound graph is d+1 of the original, since 1 and only 1 link has been added to each node.

The diameter of the compound is <=2k+1, since:

1) the longest distance needed to reach within a cluster from one node to another that then links to the cluster containing any possible destination node is k,

2) the distance needed to move from this origin cluster to the destination cluster is 1,

3) the longest distance needed to reach that destination node once that destination cluster is reached is k.

In almost all cases this compounding algorithm gives appreciably denser graphs than the previously found best [8].

For example:

(10,3,2) compounds to (110,4,5) [improving on (62,4,5)].

(15,4,2) compounds to (240,5,5) [improving on (120,5,5)].

(20,3,3) compounds to (400,4,7) [improving on (188,4,7)].

(50,7,2) compounds to (2550,8,5) [improving on (504,8,5)].

(74,9,2) compounds to (5550,10,5) [improving on (910,10,5)].

But, e.g., (57840,10,9) is poorer than Akers' (92378,10,9) graph.

Addenda: Additional Observations and Possibilities

These results, plus several others that have since come to my attention, suggest that substantial additional movement toward the Moore bounds may be near. It seems important to mention the other results briefly here (they will be reported on elsewhere) in the hope that as many people as possible will become interested in searching for these elusive graphs, since they are of great potential interest for computer network architectures.

Several new compounding operations, by W. Leland, Li Quiao, and Uhr [11], give a variety of interesting new types of graphs, including a number that are denser than those found previously. In addition, Leland [11] has found many denser graphs, up to about n = 500, using a heuristic search program that links the most distant nodes in a tree. And Imase and Itoh [12] have proved that de Bruijn networks [13] (shift registers) are asymptotically better than any other found so far.

At the present time, densest graphs are got by:

heuristically augmented trees with up to (roughly) 500 nodes;

the new compounding operations between 500 and 20,000 nodes;

de Bruijn networks with more than (roughly) 20,000 nodes.

Conclusions

This paper presents a new compounding technique that achieves a number of improvements over previously discovered "best" graphs in terms of density (number of nodes within a given degree and diameter).

A compound will be good to the extent the clusters it uses as its basic building blocks are good. Clusters can be chosen using whatever criteria are deemed desirable, including good local properties like small diameter or average distance of local clusters, local symmetries, or array, lattice or other structures appropriate for particular classes of algorithms. In terms of global diameter, only 2 non-trivial (of the at most 3 possible) optimal (Moore) graphs are known. And very few proved maximal graphs are known - only (15,4,2), (24,5,2) and (20,3,3). The only others that even approach the Moore bound are all (n,d,2). There are good prospects of finding better small clusters, if only by computer searches. A compounding technique can then be used to build relatively dense new graphs with good local properties.

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