# A COMPLETE ANALYSIS OF A MODEL NONLINEAR SINGULAR PERTURBATION PROBLEM HAVING A CONTINUOUS LOCUS OF SINGULAR POINTS

by

Nancy Kopell
and
Seymour V. Parter

Computer Sciences Technical Report #404

January 1981

A COMPLETE ANALYSIS OF A MODEL NONLINEAR SINGULAR PERTURBATION PROBLEM HAVING A CONTINUOUS LOCUS OF SINGULAR POINTS

Nancy Kopell and Seymour V. Parter

January 1981

(Received August 25, 1980)

Sponsored by

U.S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

Office of Naval Research
Arlington, VA 22217

National Science Foundation
Washington, D.C. 20550

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

# A COMPLETE ANALYSIS OF A MODEL NONLINEAR SINGULAR PERTURBATION PROBLEM HAVING A CONTINUOUS LOCUS OF SINGULAR POINTS*

Nancy Kopell[1] and Seymour V. Parter[2]

ABSTRACT

Consider the boundary value problem $\varepsilon y'' = (y^2 - t^2)y'$, $-1 < t < 0$, $y(-1) = A$, $y(0) = B$. Depending on the choice of $A$ and $B$, one can insure the existence of "turning points", $\hat{t}$; $y(\hat{t},\varepsilon)^2 - \hat{t}^2 = 0$. However, due to the nonlinear nature of the problem, one does not know the position or number of such turning points. In the case when $A > 0 = B$ Kedem, Parter and Steuerwalt gave a development of this problem based on an abstract bifurcation analysis which in turn was based on "degree theory". In this paper we give a complete analysis of the problem based entirely on a-priori estimates and the "shooting" method.

AMS (MOS) Subject Classifications: 34B15, 34E15

Key Words: Ordinary differential equations, Singular perturbations, Turning points, Boundary layers, Multiplicity of solutions, Asymptotic behavior

Work Unit Number 1 (Applied Analysis)

---

*Will also appear as University of Wisconsin-Madison, Department of Mathematics, Technical Summary Report #2165.

[1] Department of Mathematics, Northeastern University, Boston, MA 02115.

[2] Department of Mathematics and Computer Sciences Department, University of Wisconsin-Madison, Wisconsin, WI 53706.

---

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041, by the Office of Naval Research under Contract No. N00014-76-C-0341, and by the National Science Foundation under Grant NSF-MCS77-03715-01.

A COMPLETE ANALYSIS OF A MODEL NONLINEAR SINGULAR PERTURBATION PROBLEM HAVING A CONTINUOUS LOCUS OF SINGULAR POINTS*

Nancy Kopell(1) and Seymour V. Parter(2)

## 1. Introduction

The nonlinear boundary value problem

$$\varepsilon y''(t,\varepsilon) = [y^2(t,\varepsilon) - t^2]y'(t,\varepsilon) \ , \tag{1.1}$$

$$y(-1,\varepsilon) = A, \qquad y(0,\varepsilon) = B \ , \tag{1.2}$$

$\varepsilon > 0$, was introduced by Howes and Parter [2] as a model problem having a continuous locus of potential "turning points", i.e. points $\hat{t}$ at which $y^2(\hat{t},\varepsilon) - \hat{t}^2 = 0$. The main questions raised by (1.1), (1.2) concern the multiplicity of solutions for a given $A$ and $B$, and the asymptotic behavior of these solutions as $\varepsilon \to 0$.

In [2], Howes and Parter showed that, if $0 < B < A < 1$, the only possible constant limiting solutions are $y \equiv A$, $y \equiv B$ and $y \equiv 1/\sqrt{3}$. A further analysis was carried out by Kedem, Steuerwalt and Parter [3], who studied the case $A > B = 0$. This work was motivated by a conjecture of Sutton [5], who has made computational experiments based on the methods of [2]. Her conjecture was: Suppose $0 < B < 1/\sqrt{3} < A < 1$. Then for $\varepsilon$ sufficiently small, there exist at least three solutions, having $A$, $B$ and $1/\sqrt{3}$ respectively as limiting values for $t \in (-1,0)$. (Howes [1] has since given a proof of the conjecture.)

In [3] it was shown that there is always at least one solution to (1.1), (1.2). If $A = 1$, $B = 0$, the number of solutions tends to $\infty$ as $\varepsilon \to 0$: in addition to the solution $y = -t$, for each positive integer $j$ and $\varepsilon = \varepsilon(j)$ sufficiently small, there are at least two distinct solutions which cross $y = -t$ exactly $j$ times in $(-1,0)$. The

---

*Will also appear as University of Wisconsin-Madison, Department of Mathematics, Technical Summary Report #2165.

(1) Department of Mathematics, Northeastern University, Boston, MA 02115.

(2) Department of Mathematics and Computer Sciences Department, University of Wisconsin-Madison, Wisconsin, WI 53706.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041, by the Office of Naval Research under Contract No. N00014-76-C-0341, and by the National Science Foundation under Grant NSF-MCS77-03715-01.

limiting behaviour of these solutions is described by step functions. For $A \neq 1$, $B = 0$, necessary and sufficient conditions for existence of solutions were derived in terms of the solution to an associated set of algebraic equations. The main techniques involved a theorem of Rabinowitz [4] and some asymptotic estimates on the solutions of (1.1). The restriction $B = 0$ was necessary mainly in order to use Rabinowitz's result.

In this paper, we extend the results of [3] by removing the restriction $B = 0$. In order to do this, we use completely different methods. [3] was based on an abstract bifurcation analysis which, in turn, was based on degree theory; this paper gives a complete analysis based entirely on a-priori estimates and the "shooting" method. In Section 2, the asymptotic behaviour of solutions is analyzed for the regions $|y| < |t|$ and $|y| > |t|$. It is shown that the asymptotic limits of the crossing points of $y(t,\varepsilon)$ with $y = \pm t$ can be explicitly computed. Section 3 is used to derive an associated algebraic system, satisfied (in the limit) by the intersection of solutions with $y = \pm t$; it also proves the existence of a unique solution to this algebraic system. In Section 4, we return to the question: for given $A$ and $B$, how many solutions are there to (1.1), (1.2)? The answer is given implicitly in terms of the solutions of the algebraic system. For most $A$, $B$ and $\varepsilon$ sufficiently small the number of solutions to (1.1), (1.2) is finite and can (with some work) be explicitly calculated. This number tends to $\infty$ as $A \to 1$, $B \to 0$, and $\varepsilon \to 0$. If $A$, $B$, the number of "crossings" $j$, and the behaviour near $t = -1$ and $t = 0$ is specified, the analysis of Section 4 yields existence and "uniqueness" theorems - provided $\varepsilon$ is sufficiently small. The methods of analysis enables one to get a complete picture of the variation in the solutions as $A$ and $B$ are changed.

Since $-y(t,\varepsilon)$ is a solution of (1.1) whenever $y(t,\varepsilon)$ is, we may restrict ourselves to the case $B < A$. (For $A = B$, $y \equiv A$ is the only solution.) The case $B < A \leq 0$ is relatively easy to handle, and will be discussed in an appendix. Thus, our main effort is concerned with the case $B < A$, $A > 0$. Our major tools are those developed in Section 2, using extensions of the asymptotic estimates of [3] and backwards shooting methods starting at $t = 0$. (The backward integration is used for a technical reason: in this direction, it can be proved that solutions continue to exist for all $t$.)

## 2. Local Results

The first proposition is an algebraic result which will be useful in establishing facts about (1.1). The constant $\bar{t}$ satisfies $-1 < \bar{t} < 0$.

Proposition 2.1: Let $z(w) = \frac{1}{3}w^3 - \bar{t}^2 w$. Then

i) $z(w)$ is an increasing function of $w$ for $w > -\bar{t}$.

ii) If $\bar{y} > -\bar{t}$ and $c > 0$, there is a unique solution $w > -\bar{t}$ to $z(w) = z(\bar{y}) + c$.

Proof: i) $z(w)$ is a cubic with critical points at $w = \pm\bar{t}$. The point $w = -\bar{t} > 0$ is a local minimum, so $z(w)$ is increasing for $w > -\bar{t}$.

ii) By i), $\bar{y} > -\bar{t} \Longrightarrow z(\bar{y}) > z(-\bar{t})$. Hence for $c > 0$, the horizontal line $z = z(\bar{y}) + c$ is above the local minimum, and so it intersects the graph of $z(w)$ at a unique value $w > -\bar{t}$. (See Figure 2.1.) |

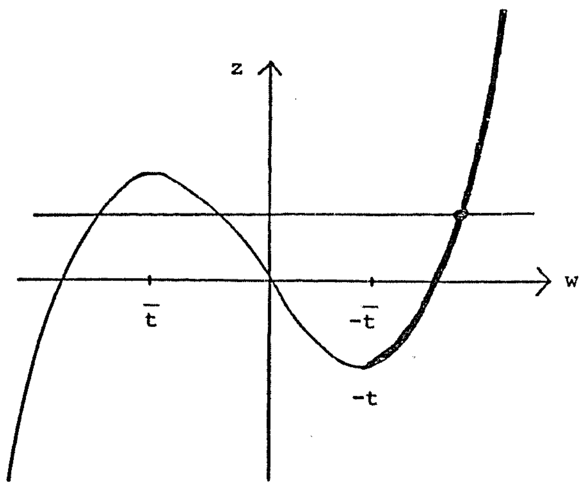



Figure 2.1

The cubic $z = \frac{1}{3} w - \bar{t}^2 w$, $\bar{t} < 0$. The horizontal line is $z = z(\bar{y}) - K$, $\bar{y}$ and $K$ given. The desired solution to $z(w) = z(\bar{y}) - K$ is the intersection of the horizontal line with the darker portion of the cubic, $w > -\bar{t}$.

Let $y(t,\varepsilon)$ be a solution to (1.1).

The next two lemmas describe the limiting forms of the trajectories when the initial conditions are in regions I and II respectively. (See Figure 2.2.)

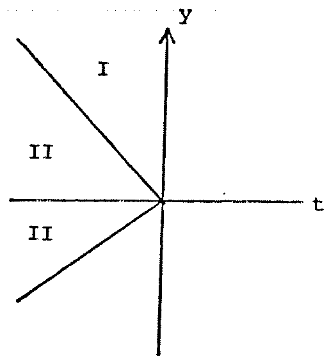



Figure 2.2

The regions (I): $y > |t|$, $t < 0$ and (II): $|y| < |t|$, $t < 0$. In region I, solutions (integrated backward) have derivatives whose absolute value decreases exponentially; in region II, the derivatives increase at an exponential rate.

Lemma 2.1. Suppose that $\lim_{\varepsilon\to 0} y(\bar{t},\varepsilon) = \bar{y}$, with $\bar{y} > -\bar{t}$. Suppose also that $\lim_{\varepsilon\to 0} -\varepsilon y'(\bar{t},\varepsilon) = K > 0$. Then

i) $\lim_{\varepsilon\to 0} y(t,\varepsilon) = a$ for $-a < t < \bar{t}$, where $a$ is the solution $> -\bar{t}$ to

$$\frac{1}{3}(\bar{y}^3 - a^3) - \bar{t}^2(\bar{y} - a) = -K \, , \tag{2.1}$$

and

ii)
$$\lim_{\varepsilon\to 0} \varepsilon \,\ell n|y'(-a,\varepsilon)| = -a^2\bar{t} + \frac{\bar{t}^3}{3} - \frac{2a^3}{3} \, . \tag{2.2}$$

Proof: We first show that $y(t,\varepsilon)$ tends to a limit for $\gamma < t < \bar{t}$, some $\gamma < \bar{t}$. (See Figure 2.3.) This follows from the direct integration of (1.1): We get:

$$y'(t,\varepsilon) = \left[\exp\left(-\frac{1}{\varepsilon}\int_t^{\bar{t}} (y^2 - s^2)\,ds\right)\right] y'(\bar{t},\varepsilon) \, . \tag{2.3}$$

We claim that, for any $\delta$ sufficiently small, $\lim_{\varepsilon\to 0} y'(\bar{t} - \delta,\varepsilon) = 0$. For $\bar{y} > -\bar{t}$, let

$0 < \delta < \bar{y} + \bar{t}$. Then

$$\int_{\bar{t}-\delta}^{\bar{t}} (y^2 - s^2)ds > \int_{\bar{t}-\delta}^{\bar{t}} [\bar{y}^2 - (\bar{t} - \delta)^2]ds = \delta[\bar{y}^2 - (\bar{t} - \delta)^2] \equiv \bar{\delta} .$$

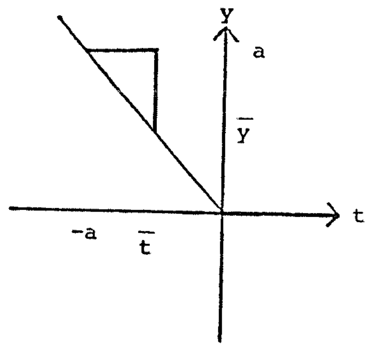



Figure 2.3

The limiting form of a trajectory with initial $(\bar{t},\bar{y})$ in region I, and $\lim_{\varepsilon\to 0} - \varepsilon y'(\bar{t},\varepsilon) = K > 0$.

Thus $|y'(\bar{t} - \delta,\varepsilon)| < e^{-\bar{\delta}/\varepsilon}|y'(\bar{t},\varepsilon)|$, so $y'(\bar{t} - \delta,\varepsilon)$ is exponentially small. Now suppose $\bar{y} = \bar{t}$. We shall show that we can replace the initial conditions $\bar{t}$, $\bar{y}$ by $t^\varepsilon$, $y(t^\varepsilon)$ (i.e. a point on the same solution) with some $t^\varepsilon \to \bar{t}$ as $\varepsilon \to 0$ and $\lim_{\varepsilon\to 0} y(t^\varepsilon) > \bar{y}$; the previous argument will then hold. First, we may assume that $t^\varepsilon < \bar{t}$, $t^\varepsilon \to \bar{t}$ as $\varepsilon \to 0$ such that $y'(t^\varepsilon,\varepsilon) = -K/2\varepsilon$ and $|y'| > K/2\varepsilon$ for $t \quad (t^\varepsilon,\bar{t}]$. (If not, there is an interval $[\bar{t} - \bar{\delta},\bar{t}]$ for which $|y'(t,\varepsilon)| > K/2\varepsilon$; it is then clear that $\tilde{t},y(\tilde{t})$ may be chosen along the trajectory such that $\lim_{\varepsilon\to 0} \tilde{t} = \bar{t}$, $\lim_{\varepsilon\to 0} \tilde{y} > \bar{y}$.) Next, we see from (2.3) that

$$\frac{1}{2} \approx y'(t^\varepsilon,\varepsilon)/y'(\bar{t},\varepsilon) = \exp\{-\frac{1}{\varepsilon}\int_{t^\varepsilon}^{\bar{t}} (y^2 - s^2)ds\} .$$

Either $y^2 - s^2$ is unbounded for $t \quad (t^\varepsilon,\bar{t})$ as $\varepsilon \to 0$, in which case we are through (as above), or else $y^2 - s^2 < 0(1)$. In the latter case, since

$$\int_{t^\varepsilon}^{\bar{t}} (y^2 - s^2)ds \approx \varepsilon \ \ln 2$$

we must have $|t^{\varepsilon} - \bar{t}| > O(\varepsilon)$. Finally

$$y(t^{\varepsilon},\varepsilon) = y(\bar{t},\varepsilon) + \int_{\bar{t}}^{t^{\varepsilon}} y'(t,\varepsilon)dt > y(\bar{t},\varepsilon) + \frac{K}{2\varepsilon}|t^{\varepsilon} - \bar{t}| .$$

Hence, $\lim_{\varepsilon\to 0} y(t^{\varepsilon},\varepsilon) > \bar{y}$.

As long as $y^2 - t^2 > 0$, $|y'(t)|$ decreases (integrating backwards), so $y'(t,\varepsilon)$ remains small and $y$ approaches a constant. Since this is true for any small $\delta$, $y$ must approach a constant for $\gamma < t < \bar{t}$, some $\gamma$. In order to see what that constant is, we integrate (1.1) from $t = \bar{t}$ to $t = \bar{t} - \delta$. Then

$$\varepsilon[y'(\bar{t} - \delta,\varepsilon) - y'(\bar{t},\varepsilon)] = -\int_{\bar{t}-\delta}^{\bar{t}} y'(s)(y^2 - s^2)ds . \tag{2.4}$$

The R.H.S. of (2.4) may be rewritten:

$$\frac{-y^3(s,\varepsilon)}{3}\Bigg]_{\bar{t}-\delta}^{\bar{t}} + s^2 y(s,\varepsilon)\Bigg]_{\bar{t}-\delta}^{\bar{t}} - 2\int_{\bar{t}-\delta}^{\bar{t}} sy(s,\varepsilon)ds . \tag{2.5}$$

By hypothesis, and by the previous calculation, the limit of the L.H.S. is $K$. The R.H.S. approaches

$$\frac{a^3 - \bar{y}^3}{3} + \bar{t}^2[\bar{y} - a] + O(\delta) . \tag{2.6}$$

Since (2.5) is valid for all $\delta$ sufficiently small, we conclude that a must satisfy (2.1). (Note that $a$ is the solution to $z(w) = z(\bar{y}) + K$, which exists by Proposition 2.1.) Furthermore, it is now clear that $\gamma$ may be taken to be $-a$.

ii) To establish (2.2), we again integrate (1.1) from $t = \bar{t}$ to $t = -a$; this time, we first divide by $y'$. We get

$$\varepsilon \ln|y'(-a,\varepsilon)| - \varepsilon \ln|y'(\bar{t},\varepsilon)| = -\int_{-a}^{\bar{t}} (y^2 - s^2)ds . \tag{2.7}$$

Since $y(t,\varepsilon) \to a$ and $\varepsilon \ln|y'(\bar{t},\varepsilon)| \to 0$, the limiting form of (2.7) is (2.2). |

Lemma 2.2. Suppose that $\lim_{\varepsilon\to 0} y(\bar{t},\varepsilon) = \bar{y}$, with $|\bar{y}| < |\bar{t}|$. Suppose also that $\lim_{\varepsilon\to 0} -\varepsilon \ln|y'(\bar{t},\varepsilon)| = L > 0$. Then

i) $\lim_{\varepsilon\to 0} y(t,\varepsilon) = \bar{y}$ for $-b < t < \bar{t}$, where $b$ is the unique solution $> |\bar{y}|$ of

$$\bar{y}^2\bar{t} - \frac{\bar{t}^3}{3} + \bar{y}^2 b - \frac{b^3}{3} = -L \; . \tag{2.8}$$

ii) $\exists\, t_\varepsilon \to -b$ as $\varepsilon \to 0$ such that $y(t_\varepsilon,\varepsilon) = -t_\varepsilon$.

iii)
$$\lim_{\varepsilon\to 0} \varepsilon y'(t_\varepsilon,\varepsilon) = \frac{b^3 - \bar{y}^3}{3} - b^2(b - \bar{y}) \; . \tag{2.9}$$

Proof: i) Once again, we integrate (1.1) to get (2.3). Since, by hypothesis, $y'(\bar{t},\varepsilon)$ is exponentially small, $y'(t,\varepsilon)$ is exponentially small for all $t > -b$, where $b$ satisfies

$$\int_{-b}^{\bar{t}} (\bar{y}^2 - s^2)\,ds = -L \; . \tag{2.10}$$

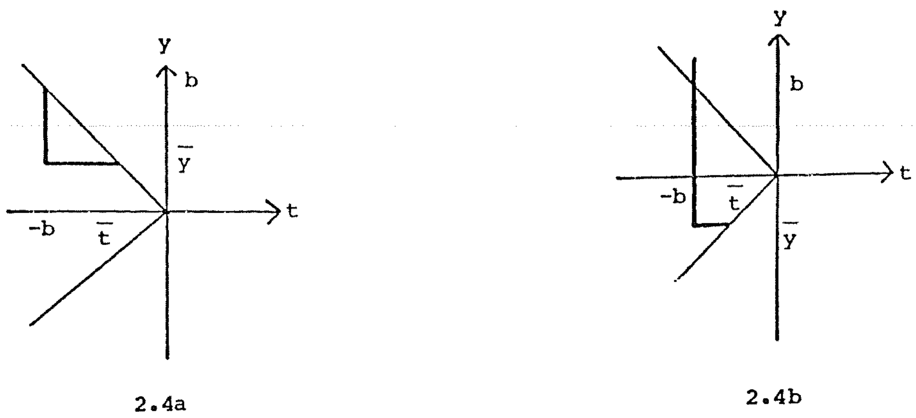
(See Figure 2.4.) We can use (2.10) to solve for $b$: Equation (2.10), when integrated, yields (2.8). Note that (2.8) may be written as

$$\frac{b^3}{3} - \bar{y}^2 b = \frac{(-\bar{t})^3}{3} - \bar{y}^2(-\bar{t}) + L \; . \tag{2.11}$$

By the same arguments as in Proposition 2.1, with $\bar{y}$ and $-\bar{t}$ interchanged, there is a unique solution $b$ to (2.11) for $b > \bar{y}$.

ii) For any $\delta > 0$ sufficiently small, (2.3) shows that $y'(-b-\delta,\varepsilon)$ either grows exponentially large in $\varepsilon$, or else $\exists\, t_\varepsilon$, with $-b - \delta < t_\varepsilon < -b$, such that $y(t_\varepsilon,\varepsilon) = -t_\varepsilon$. (See Figure 2.3.) Since the exponential growth also implies the existence of such a $t_\varepsilon$, and since this is true for all $\delta > 0$, assertion ii) is proved.

iii) The growth of $y$ implies that $y'(t,\varepsilon)$ must stop being exponentially small in $\varepsilon$, i.e. $\exists\, \bar{t}_\varepsilon > t_\varepsilon$, with $\bar{t}_\varepsilon \to b$ as $\varepsilon \to 0$, such that $y'(\bar{t}_\varepsilon,\varepsilon) = -\varepsilon$. Note that, since $y'(t,\varepsilon) \to 0$ for $\bar{t}_\varepsilon < t < \bar{t}$, $\lim_{\varepsilon\to 0} y(t,\varepsilon) = \bar{y}$ for $\bar{t}_\varepsilon < t < \bar{t}$. To establish (2.9), we now use equations (2.4) and (2.5), with $\bar{t} - \delta$ and $\bar{t}$ replaced by $t_\varepsilon$ and $\bar{t}_\varepsilon$ respectively. $\blacksquare$




Figures 2.4a, 2.4b

The limiting form of trajectories with initial $\bar{t}, \bar{y}$ in region II, and $\lim_{\varepsilon\to 0} - \varepsilon \ln|y'(\bar{t},\varepsilon)| = L > 0$. Figure 2.4a is for $\bar{y} > 0$; Figure 2.4b is for $\bar{y} < 0$.

The last lemma of this section describes the initial segment of the trajectory of (1.1) if $y(0,\varepsilon) = B < 0$.

Lemma 2.3. Suppose $\lim_{\varepsilon\to 0} y(0,\varepsilon) = B < 0$, and $\lim_{\varepsilon\to 0} - \varepsilon y'(0,\varepsilon) = K > 0$. Then

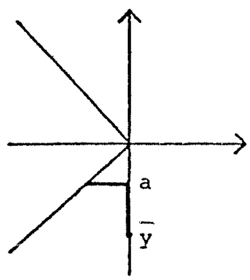
i) $\lim_{\varepsilon\to 0} y(t,\varepsilon) = a$ for $-|a| < t < 0$, where $a$ is the real solution to

$$\frac{1}{3}(a^3 - B^3) = K, \tag{2.12}$$

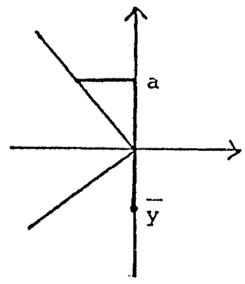
and

ii)
$$\lim_{\varepsilon\to 0} \varepsilon \ln|y'(-|a|,\varepsilon)| = -\frac{2}{3}|a|^3 . \tag{2.13}$$

Proof: This lemma is essentially the same as Lemma 2.1, with $\bar{t} = 0$; there is an added difficulty that the trajectory may cross through the region $|y| < |t|$, so $y'$ may not change in a monotone fashion. (See Figure 2.5.) Hence we cannot immediately use the previous argument that $y(t,\varepsilon)$ approaches a limit for $t$ near $t = 0$.




2.5a




2.5b

<u>Figures 2.5a, 2.5b</u>

The limiting form of trajectories for initial condition $y(0,\varepsilon) = B < 0$, and $\lim\limits_{\varepsilon\to 0} -\varepsilon y'(0,\varepsilon) = K > 0$. In Figure 2.5a, $K$ is small enough that $\lim\limits_{\varepsilon\to 0} y(t,\varepsilon) < 0$ for $t$ near $0$. If the limiting form is as in Figure 2.5b, for $\varepsilon > 0$ the trajectories pass through $|y| < |t|$.

Let $a$ be defined as in (2.12), and let $\delta > 0$ be arbitrarily small. We first claim that $y(t,\varepsilon)$ cannot remain bounded away from $y = a$ for $-\delta < t < 0$ and all $\varepsilon$, $\delta$ sufficiently small. For, integrating (1.1) from $t = 0$ to $t = t_*$ $(-\delta < t_* < 0)$, we get

$$\varepsilon[y'(t_*,\varepsilon) - y'(0,\varepsilon)] = -\int_{t_*}^{0} y'(s)(y^2 - s^2)ds \; . \tag{2.14}$$

By hypothesis, $-\lim\limits_{\varepsilon\to 0} \varepsilon y'(0,\varepsilon) = K > 0$. If $y$ stays bounded away from $a$, the R.H.S. of (2.14) will be strictly less than $K + O(\delta)$. Thus $|y'(t_*,\varepsilon)| > \bar{K}/\varepsilon$ for some $\bar{K} > 0$ and $\delta$ sufficiently small. Since this is true of all $t_*$ such that $-\delta < t_* < 0$, $y$ changes by $O(1/\varepsilon)$, and cannot stay bounded away from $a$ for any finite value of $t$; this contradicts the assumption that $a$ is an upper bound for $y$.

Also, $y$ cannot become strictly larger than $a$, i.e. bounded away from $a$ from above for any $t < 0$ with $t$ sufficiently small. For, if $y$ hits $y = a$, let $t_a(\varepsilon)$ be the value of $t$ when $y = a$. Then from (2.12) and (2.14) we have that

$\lim_{\varepsilon\to 0} \varepsilon y'(t_a,\varepsilon) = 0$. For all $t$ such that $-|a| < t < 0$, $|y'|$ decreases (as (1.1) is integrated backwards), and, as in Lemma 2.1, $y(t,\varepsilon)$ approaches $a$ as $\varepsilon \to 0$.

ii) Same as the proof in Lemma 2.1. |

Corollary 2.1. Suppose $\lim_{\varepsilon\to 0} - \varepsilon y'(0,\varepsilon) = K > 0$, and $\lim_{\delta\to 0} \frac{1}{\delta} \lim_{\varepsilon\to 0} y(\delta,\varepsilon) = 0$ or $\infty$. Then $y(t,\varepsilon)$ exists for all $t < 0$. Furthermore, if the solution hits $y = -t$ at $t < 0$, and

$\lim_{\varepsilon\to 0} \varepsilon \ln y'(t,\varepsilon) \neq 0$ at the crossing (if that crossing is horizontal) or

$\lim_{\varepsilon\to 0} \varepsilon y'(t,\varepsilon) \neq 0$ (if the crossing is vertical), then $y(t,\varepsilon)$ crosses $y = -t$ an infinite number of times.

Proof: If $B > 0$, we apply Lemmas 2.1 and 2.2 alternately, starting with Lemma 2.1 (since $\bar{y} > \bar{t} = 0$). These lemmas explicitly calculate the limiting values of the successive crossing points. (If $K = 0$, there may be only one crossing point, e.g. if $y'(0,\varepsilon) = 0\ \forall\varepsilon$.) If $B = 0$, the hypothesis on $\lim_{\varepsilon\to 0} \frac{1}{\delta} \lim_{\varepsilon\to 0} y(\delta,\varepsilon)$ forces the behaviour near $t = 0$ to be either horizontal or vertical; we can then apply Lemmas 2.1 and 2.2. If $B < 0$, the above hypothesis again rules out solutions $y(t,\varepsilon)$ for which $y = -t$ is the limiting solution for $-1 < t < 0$. We can start with Lemma 2.3 and then apply Lemmas 2.2 and 2.1 alternately. The uniqueness of the point $\beta(\varepsilon)$ is clear from thse lemmas. |

## 3. An Algebraic System

We have seen that, in the limit as $\varepsilon \to 0$, almost all of the solutions to (1.1) are a succession of horizontal and vertical segments; the length of each segment can be explicitly computed given $\lim_{\varepsilon\to 0} y(0,\varepsilon)$ and $\lim_{\varepsilon\to 0} \varepsilon y'(0,\varepsilon)$.

We shall consider a system of algebraic equations which are satisfied in the limit by the successive points $t_i$ at which $y(t,\varepsilon)$ crosses the line $y = -t$. Much of what is in this section was already done in [3] under the condition that $B = 0$. Here we show that those results still hold for $B \neq 0$. The first lemma and theorem apply to a solution $y(t,\varepsilon)$ if $\lim_{\varepsilon\to 0} y(t,\varepsilon) > 0$ for $t < 0$.

Lemma 3.1. Let $t_{i-1}(\varepsilon) > t_i(\varepsilon) > t_{i+1}(\varepsilon)$ be three successive points at which a solution $y(t,\varepsilon)$ crosses $y = -t$. If $t_{i-1}, t_i, t_{i+1}$, denote the limits of these points as $\varepsilon \to 0$, then

$$t_i^2 = \frac{1}{3}\left(t_{i-1}^2 + t_{i-1}t_{i+1} + t_{i+1}^2\right) . \tag{3.1}$$

Proof: There are two possible cases, as shown in Figures 3.1a, 3.1b. Each case can be done by the methods of Section 2. For the case of 3.1a, let $\delta > 0$ be any sufficiently small number, and use (2.4) with $\bar{t} - \delta$ and $\bar{t}$ replaced by $t_i - \delta$ and $t_i + \delta$. Since $y(t_i + \delta,\varepsilon) \to -t_{i-1}$ and $y(t_i - \delta,\varepsilon) \to -t_{i+1}$, by using (2.5) and letting $\delta \to 0$ we get (3.1).

For the case of 3.1b, we use (2.7), with $-a$ and $\bar{t}$ replaced by $t_{i+1}$ and $t_{i-1}$. By previous computations, the L.H.S. tends to zero as $\varepsilon \to 0$. Also, for $t_{i+1} < t < t_{i-1}$, $y(t,\varepsilon) \to t_i$. Evaluating the R.H.S. and setting it equal to zero, we again get (3.1). |

We shall now prove an existence and uniqueness theorem for (3.1), regarded as a system of equations for a set of points $\{t_i\}$. We shall denote by $t_0$ and $t_j$ the right and left endpoints of the set.

Theorem 3.1. There is a unique solution $0 > t_0 > t_1 > \cdots > t_j$ to (3.1) with

$$t_0 < 0, \qquad t_j < t_0 \text{ given} . \tag{3.2}$$

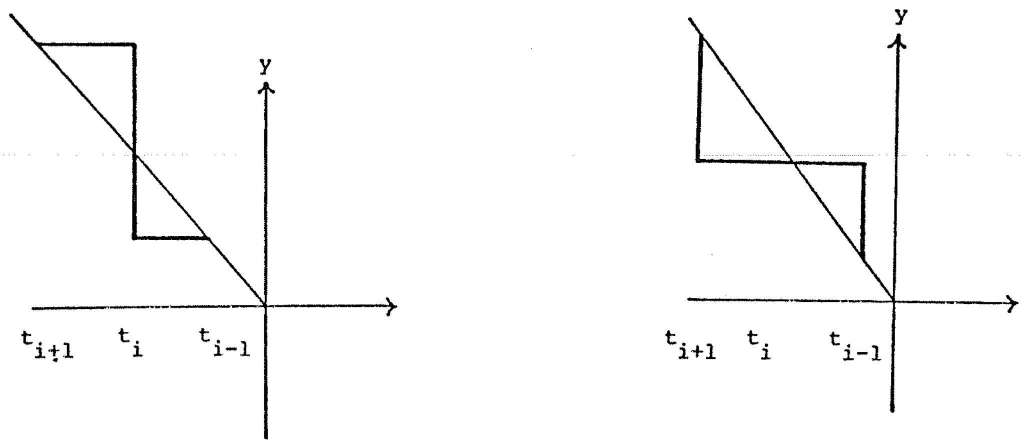



Figure 3.1

Successive crossings of $y = -t$ by limits of solutions to (1.1) as $\varepsilon \to 0$.

Proof: Equation (3.1) gives rise to a mapping of finite sequences of numbers between $t_j$ and 0, namely: For any sequence $\tau_j = t_j < \tau_{j-1} < \cdots < \tau_0 = t_0$, let

$$F_i(\tau) = -\{\frac{1}{3}[\tau_{i+1}^2 + \tau_{i+1}\tau_{i-1} + \tau_{i+1}^2]\}^{1/2} \tag{3.3}$$

for $i = 1,2,\ldots,j$, and $F_0 = t_0$, $F_j = t_j$. It is clear that any fixed point of this map is a solution to (3.1) and (3.2). We shall show that $\{F_i\}$ has a unique fixed point.

Consider two such sequences $\tau = \{\tau_i\}$ and $\sigma = \{\sigma_i\}$. Then

$$F_i^2(\sigma) - F_i^2(\tau) = \frac{1}{3}[(\sigma_{i+1}^2 - \tau_{i+1}^2) + (\sigma_{i+1}\sigma_{i-1} - \tau_{i+1}\tau_{i-1}) + (\sigma_{i-1}^2 - \tau_{i-1}^2)] \tag{3.4}$$

for $i = 1,\ldots,j-1$. Let $\phi_i \equiv \sigma_i - \tau_i$, $W_i \equiv \sigma_i + \tau_i$. Also, write $\hat{W}_i$, $\hat{\phi}_i$ for $F_i(\sigma) + F_i(\tau)$ and $F_i(\sigma) - F_i(\tau)$. Equation (3.4) may then be written as

$$\hat{W}_i\hat{\phi}_i = \frac{1}{3}(W_{i+1} + \frac{1}{2}W_{i-1})\phi_{i+1} + \frac{1}{3}(W_{i-1} + \frac{1}{2}W_{i+1})\phi_{i-1}, \tag{3.5}$$

$i = 1,\ldots,j-1$. Also, $\hat{\phi}_j = 0$, $\hat{\phi}_0 = 0$.

The coefficients of $\{\phi_i\}$ in (3.5) are not constant, but we will get bounds on them which insure that $\{F_i\}$ maps any two sequences closer to one another. We divide (3.5) by $\hat{W}_i$, and write the resulting system as

$$\hat{\phi} = L\phi . \tag{3.6}$$

Notice that the matrix $L$ has non-zero entries only just above and just below the diagonal. We first obtain a bound on these coefficients, and then show this implies the desired result. Let $a_i$ denote the entry in the $i^{th}$ row, $(i+1)^{th}$ column; $b_i$ is in the $i^{th}$ row, $(i-1)^{th}$ column.

Lemma 3.2. $|a_i| + |b_i| < 1$.

Proof: $a_i = \frac{1}{3\hat{W}_i}[W_{i-1} + \frac{1}{2} W_{i+1}]$; $b_i = \frac{1}{3\hat{W}_i}[W_{i+1} + \frac{1}{2} W_{i-1}]$. Both $a_i$ and $b_i$ are $> 0$. An elementary calculation shows that

$$a_i + b_i = \frac{1}{2\hat{W}_i}[W_{i+1} + W_{i-1}] \ . \tag{3.7}$$

Now

$$0 < \frac{1}{12}(\tau_{i+1} - \tau_{i-1})^2 = (\frac{1}{3} - \frac{1}{4})\tau_{i+1}^2 + (\frac{1}{3} - \frac{2}{4})\tau_{i+1}\tau_{i-1} + (\frac{1}{3} - \frac{1}{4})\tau_{i+1}^2 \ .$$

Hence

$$\frac{1}{3}\{\tau_{i+1}^2 + \tau_{i+1}\tau_{i-1} + \tau_{i-1}^2\} > \frac{1}{4}\{\tau_{i+1}^2 + 2\tau_{i+1}\tau_{i-1} + \tau_{i-1}^2\} \ .$$

This may be rewritten as

$$|F_i(\tau)| > \frac{1}{2}|\tau_{i+1} + \tau_{i-1}| \ . \tag{3.8}$$

Inequality (3.8) also applies to $\sigma$, and hence

$$|\hat{W}_i| > \frac{1}{2}|W_{i+1} + W_{i-1}| \ .$$

It follows that the R.H.S. of (3.7) is $< 1$. |

We return to Theorem 3.1. $L$ is a tri-diagonal matrix whose entries satisfy $|a_i| + |b_i| < 1$. It follows [6] that if $|\lambda| \geq 1$, then $L - \lambda I$ is nonsingular, so all the eigenvalues of $L$ are $< 1$ in absolute value. From this it can be shown that $\{F_i\}$ has a unique fixed point. In fact, a direct iterative scheme is convergent to this unique fixed point. |

We now deal with solutions which cross $y = +t$ as well, i.e. when $B < 0$. We have seen that there is a unique turning point $\beta(\varepsilon)$, and we shall be concerned now with solutions for which $\beta \equiv \lim_{\varepsilon\to0} \beta(\varepsilon) < 0$. Lemma 2.3 implies that for $t < 0$ sufficiently near $t = 0$, $\lim_{\varepsilon\to0} y(t,\varepsilon) = \beta$.

Lemma 3.3. Suppose $\lim_{\varepsilon\to0} y(t,\varepsilon) = \beta < 0$ for $\beta < t < 0$. Let $t_1(\varepsilon),\dots,t_j(\varepsilon),\dots$ denote the crossings of $y(t,\varepsilon)$ by $y = -t$, and $t_1,t_2,\dots,t_j,\dots$ their limiting values. Then

i) $$t_1^2 = \frac{1}{3}(\beta^2 - \beta t_2 + t_2^2) \ . \qquad (3.9)$$

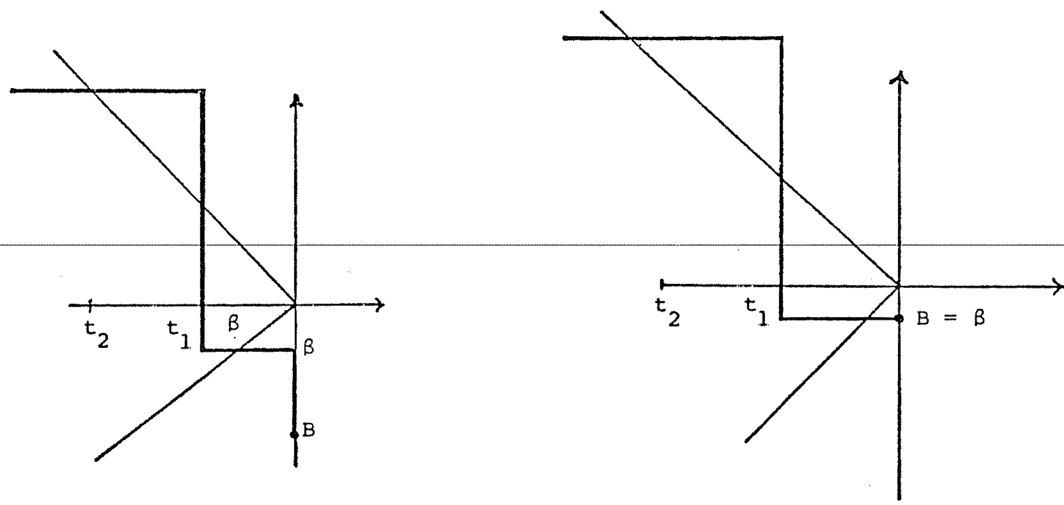
Furthermore, if $B < \beta$, then

ii) $$\beta^2 = \frac{1}{3} t_1^2 \ . \qquad (3.10)$$

Proof: i) Use equation (2.4) with $\bar{t} - \delta$ and $\bar{t}$ replaced by $t_1 - \delta$ and $t_1 + \delta$, with $\delta$ small. The argument of Lemma 2.1 then produces (3.9).

ii) Use equation (2.7), with $-a$ and $\bar{t}$ replaced by $t_1$ and $0$. The hypothesis $B < \beta$ implies that $\lim_{\varepsilon \to 0} \varepsilon \, \ln|y'(0,\varepsilon)| \to 0$, so the L.H.S. $\to 0$. The R.H.S., set equal to zero, yields (3.10). |

The following is the analogue of Theorem 3.1. Its use will be different depending on whether $B < \beta$ or $B = \beta$, with (3.10) holding if $B < \beta$ and not, in general, otherwise. (See Figure 3.2.)




3.2a 3.2b

Figures 3.2a, 3.2b

Crossings of $y = +t$ and $y = -t$ by a limit of solutions if $B < 0$ and $\beta < 0$. For Figure 3.2a, $\beta$ and $t_1$ are related by $3\beta^2 = t_1^2$. For Figure 3.2b, $t_1^2 > 3\beta^2$.

Theorem 3.2. i) Let $t_j$ be given. Then there is a unique solution $0 > \beta > t_1 > t_2 > \cdots > t_j$ to (3.1), $i = 2,\ldots,j = 1$, and (3.9), (3.10).

ii) Let $t_j$ and $\beta$ be given. Then there is a unique solution $0 > \beta > t_1 > \cdots t_j$ to (3.1), $i = 2,\ldots,j - 1$ and (3.9).

Proof: The proof is almost identical to that of Theorem 3.1, so we shall just indicate the differences.

i) Let $\tau$ denote a sequence $\beta > \tau_1 > \tau_2 > \cdots > \tau_j$, $\sigma$ a sequence $\gamma > \sigma_1 > \sigma_2 > \cdots > \sigma_j$. $F_i(\tau)$, $i > 2$ is as before, but

$$F_1(\tau) = -\{\frac{1}{3}\,[\beta^2 - \beta\tau_2 + \tau_2^2]\}^{1/2} \quad ,$$

$$F_0(\tau) = \frac{1}{3}\,\tau_1 \ .$$

$\phi_i$, $W_i$, $\hat{\phi}_i$, $\hat{W}_i$ are as before. Then (3.5) still holds for $i > 2$. For $i = 0,1$, we have

$$\hat{W}_0\hat{\phi}_0 = \frac{1}{3}\,W_1\phi_1 \ , \tag{3.11}$$

$$\hat{W}_1\hat{\phi}_1 = \frac{1}{3}\,(W_2 - \frac{1}{2}\,W_0)\phi_2 + \frac{1}{3}\,(W_0 - \frac{1}{2}\,W_2)\phi_0 \ . \tag{3.12}$$

To show that $|a_i| + |b_i| < 1$ for all $i$, it suffices to show this for $i = 1$ and $i = 0$. For $i = 0$, $b_i = 0$ and $|a_i| = W_1/3\hat{W}_0 = 1/\sqrt{3} < 1$. For $i = 1$,

$$|a_1| + |b_1| = \frac{\frac{1}{2}\,(|\tau_2| + |\sigma_2| - |\beta| - |\gamma|)}{|F_1(\tau)| + |F_1(\sigma)|} \ .$$

As in the previous theorem, it can be shown that $|F_1(\tau)| > \frac{1}{2}\,|\tau_2 - \beta|$ and similarly for $F_1(\sigma)$, which implies that $|a_i| + |b_i| < 1$.

ii) Let

$$F_1(\tau) = -\{\frac{1}{3}\,[\beta^2 - \beta\tau + \tau_2^2]\}^{1/2} \quad ,$$

$$F_i(\tau),\ i > 1 \ \text{as before} \ .$$

The relevant equations are (3.5), $i > 2$, (3.12). The proof that $|a_1| < 1$ is done by the usual methods. $\blacksquare$

## 4. Existence and Uniqueness of Solutions

The solution to the boundary value problem (1.1), (1.2) is not unique, as shown by examples in [1], [3]. However, as we shall see, any two solutions are qualitatively different: they can be distinguished by the number of points at which the solutions cross $y = \pm t$, and the behavior of the solutions at the boundary.

From the estimates of Section 2, it follows that the behavior near each boundary is characterized by $\lim_{\varepsilon\to 0} \varepsilon y'(\tilde{t},\varepsilon)$, where $\tilde{t} = -1$ or $0$; if this limit is non-zero, the initial segment of the solution at that boundary approaches a vertical line whose length goes to zero with this limit. If $\lim_{\varepsilon\to 0} y'(\tilde{t},\varepsilon) = 0$, the initial segment is (in the limit) horizontal.

We shall divide the analysis into four cases, depending on whether the behavior at $t = -1$ and $t = 0$ is vertical or horizontal. In each case we ask the question: for given $A$ and $B$, how many solutions are there to (1.1), (1.2) in the limit as $\varepsilon \to 0$? The answer is given implicitly in terms of the solutions to (3.1), but in a form such that it could be explicitly computed. The analysis provides a complete picture of how the solutions change as $A$ and $B$ are varied, and also how the boundary value $A$ changes as $B$ and $y'(0,\varepsilon)$ are varied.

We first discuss solutions which, in the limit as $\varepsilon \to 0$, have a vertical segment (shock layer) at both $t = -1$ and $t = 0$. (See Figure 4.1.) We also assume that, if $B < 0$, then $\beta = 0$. For each odd $j$ we consider the solution to (3.1) with $t_0 = 0$ and $t_{j+1} = -1$. (For this bundary behaviour, the number of strictly interior crossing points must be odd.) Let $A_j$, $j \geq 1$ be the open interval $(-t_j^j, -t_{j+2}^j)$, where $t_i^j$ denotes the $i^{th}$ point of the solution to (3.1) having $j + 2$ points $t_0^j < \cdots < t_{j+1}^j = 1$. (If the $j$ is understood, we may sometimes suppress the superscript.) Similarly, let $B_j$ be the open interval $(-\infty,-t_1^j)$. By "strictly interior" points of intersection with $y = -t$, we mean those $t_i^j(\varepsilon)$ for which $-1 < \lim_{\varepsilon\to 0} t_i^j(\varepsilon) < 0$.

Remark: For fixed $\varepsilon$ small, the actual number of interior points of intersection may be greater than the limiting number of "strictly interior" such points. That is, we might have $t_0(\varepsilon) < 0$, $t_0(\varepsilon) \to 0$ as $\varepsilon \to 0$ or $t_{j+1}(\varepsilon) > -1$, $\lim_{\varepsilon\to 0} t_{j+1}(\varepsilon) = -1$. The above method of counting was adopted because it eliminates the division into further special

cases that would be needed if those other points of crossing were counted as well, and simplifies the description of the collection of all solutions.

Theorem 4.1. i) There is a solution to (1.1), (1.2) having a boundary layer at each end, $j$ strictly interior points of intersection with $y = -t$ and no strictly interior points of intersection with $y = t$ if $A \in A_j$, $B \in B_j$ and $\varepsilon$ is sufficiently small. Such a solution is unique. Its limiting behavior as $\varepsilon \to 0$ is a step function: a series of horizontal and vertical lines crossing $y = -t$ at the points $t_i^j$. A necessary condition for such a solution is $A \in c\ell\, A_j$, $B \in c\ell\, B_j$, where "$c\ell$" denotes "closure of".

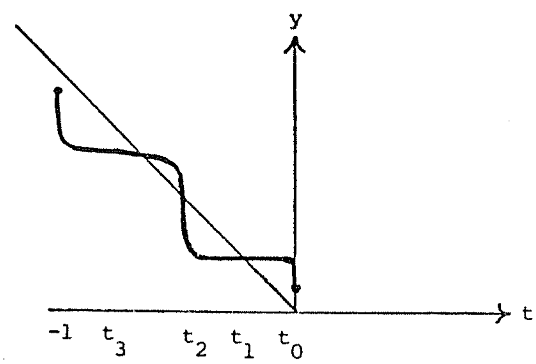



Figure 4.1

A solution to (1.1) with boundary layers at both ends and three interior turning points.

ii) The $\{A_j\}$ and $\{B_j\}$ are nested intervals, i.e. $A_1 \supset A_3 \supset A_5 \supset \cdots$; $B_1 \supset B_3 \supset B_5 \supset \cdots$.

iii) For each $A \neq 1$, (resp. $B > 0$), there are finitely many $j$ for which $A \in A_j$ (resp. $B \in B_j$).

Proof: i) The necessity of the condition $A \in c\ell\, A_j$, $B \in c\ell\, B_j$ follows immediately from the results of Sections 2 and 3. For, by Section 2, the limiting form of a solution is a sequence of horizontal and vertical lines, and, by Section 3, the intersections of these lines with $y = -t$ are the points $t_i^j$. The hypothesis $A \in c\ell\, A_j$, $B \in c\ell\, B_j$ requires that $A$ and $B$ lie on the closure of the vertical segments at $t = -1$ and $t = 0$ respectively, for the limiting solution to (1.1) corresponding to the solution to (3.1) with $t_{j+1} = -1$, $t_0 = 0$; if this hypothesis is not satisfied, the limiting solution cannot be the correct one.

To establish the existence and uniqueness of solutions, consider the solutions $y(t,\varepsilon)$ to (1.1) satisfying

$$y(0,\varepsilon) = B, \qquad y'(0,\varepsilon) = -K/\varepsilon\ . \tag{4.1}$$

By the estimates of Section 2, there is a unique $K_j \neq 0$ such that the solution with $K = K_j$ has the desired limiting value as $\varepsilon \to 0$. ($K_j$ is chosen so that $\lim_{\varepsilon\to 0} y(t,\varepsilon) = -t_1^j$ for $t$ near $0$; this uses $B \in \mathcal{B}_j$. The rest of the limiting behavior follows automatically from the results of Section 3.) We shall show that if $A \in \mathcal{A}_j$ there is a unique $K_j(\varepsilon)$ such that $K_j(\varepsilon) \to K_j$ as $\varepsilon \to 0$, and the solution through $y(0,\varepsilon) = B$ with slope $y'(0,\varepsilon) = -K_j(\varepsilon)/\varepsilon$ satisfies $y(-1,\varepsilon) = A$. (This "local" uniqueness suffices to insure uniqueness of solutions with boundary layers at both ends and $j - 1$ intersections with $y = -t$, since, by Section 3, any such solution must have the limiting behavior described in the theorem.)

Consider the solutions to (1.1), (4.1), with $|K - K_j| < \delta$, $\delta$ arbitrarily small. The next two lemmas are monotonicity results. The first one concerns the limiting values of the solution, and could be phrased as a statement about equations (3.1). (See Lemma 4.2.)

<u>Lemma 4.1</u>. Let $t_j(\varepsilon,K)$ be the value of $t$ where the solution $y(t,\varepsilon,K)$ to (1.1), (4.1) hits $y = -t$ for the $j^{\text{th}}$ time. (By definition, $\lim_{\varepsilon\to 0} t_{j+1}(\varepsilon,K) = -1$.) Then for $\delta$ sufficiently small,

$$\lim_{\varepsilon\to 0} \frac{dt_j}{dK}(\varepsilon,K) < 0\ , \tag{4.2}$$

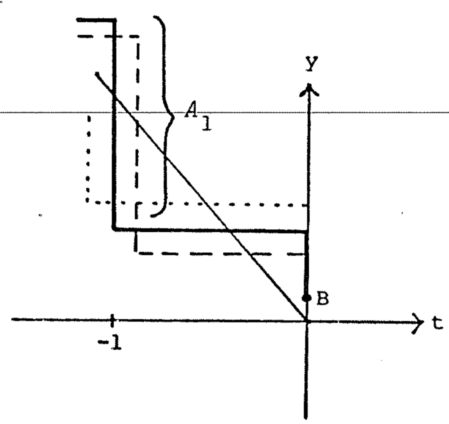
so $\lim_{\varepsilon\to 0} t_j(\varepsilon,K)$ is a monotone decreasing function of $K$.

<u>Proof:</u> The monotonicity follows from the estimates in Section 2. For, using the cubic of Proposition 2.1, it can be seen from Lemma 2.1 that increasing $K$ or increasing $B$ has the effect of increasing $-\lim_{\varepsilon\to 0} t_1(\varepsilon,K)$ and $-\lim_{\varepsilon\to 0} \varepsilon\ \ell n|y'(t_1,\varepsilon,K)|$, in a transversal way:

$$\frac{d}{dK} \lim_{\varepsilon\to 0} t_1(\varepsilon,K) < 0; \quad \frac{d}{dK} \lim_{\varepsilon\to 0} \varepsilon\ \ell n|y'(t_1,\varepsilon,K)| < 0\ .$$

Similarly, from Lemma 2.2, increasing $\lim_{\varepsilon\to 0} |t_1(\varepsilon,K)|$, $\lim_{\varepsilon\to 0} |\varepsilon \, \ell n|y'(t_1,\varepsilon,K)||$ increases $\lim_{\varepsilon\to 0} |t_2(\varepsilon,K)|$ and $\lim_{\varepsilon\to 0} \varepsilon|y'(t_2,\varepsilon,K)|$. Continuing, alternating Lemmas 2.1 and 2.2, we find that (4.2) holds. |

We now return to the proof of Theorem 4.1. Lemma 4.1 implies that for $K < K_j$ (resp. $K > K_j$), $t_{j+1}(\varepsilon,K) > t_{j+1}(\varepsilon,K_j)$ (resp. $t_{j+1}(\varepsilon,K) < t_{j+1}(\varepsilon,K_j)$) for all $\varepsilon$ sufficiently small. Thus, for $\varepsilon$, $\delta$ small, $y(-1,\varepsilon,K)$ is arbitrarily close to the upper limit of $A_j$ for $K < K_j$ and the lower limit of $A_j$ for $K > K_j$ (see Figure 4.2). As we will see, $y(-1,\varepsilon,K)$ is not monotone for $|K - K_j| < \delta$, any <u>fixed</u> $\delta$, for all $\varepsilon$ sufficiently small. However, we shall show that there is a smaller, $\varepsilon$-dependent interval in $K$ such that $y(-1,\varepsilon,K)$ is a monotone function of $K$ on this interval, and the image of this interval tends to $A_j$ as $\varepsilon \to 0$. Furthermore, as $K$ passes the boundaries of this interval, the size of the "vertical" segment at $t = -1$ tends to zero, so the limiting behavior changes from vertical to horizontal. This suffices to prove the uniqueness of the solution with boundary layers at both ends, and $j$ (strictly) interior turning points.




<u>Figure 4.2</u>

The limiting form of nearby solutions to (1.1) with the same value of $B$ and different values of $y'(0,\varepsilon) = -K/\varepsilon$, $(j = 1)$. For $K = K_j$, the limiting solution has a vertical segment at $t = -1$. For $K < K_j$ (resp. $K > K_j$) the limiting solution has a horizontal segment near the upper (resp. lower) endpoint of $A_j$.

By Lemma 4.1, for each $\varepsilon$ sufficiently small, $t_{j+1}(\varepsilon,K)$ is a monotone function of $K$ near $K_j$, so $\exists !\, K_{j1}(\varepsilon)$ such that $t_{j+1}(\varepsilon,K_{j1}(\varepsilon)) = -1$. By definition of $t_{j+1}(\varepsilon,K)$,

$$y(t_{j+1}(\varepsilon,K),\varepsilon,K) = -t_{j+1}(\varepsilon,K) \ . \tag{4.3}$$

Differentiating (4.3) with respect to $K_j$ and evaluating at $K = K_{j1}(\varepsilon)$, we get

$$\frac{\partial y}{\partial K}(-1,\varepsilon,K_{j1}(\varepsilon)) = -[1 + y'(-1,\varepsilon,K_{j1}(\varepsilon))]\,\frac{dt_{j+1}}{dK}(\varepsilon,K_{j1}) \ . \tag{4.4}$$

Now $y'(t_{j+1}(\varepsilon,K),\varepsilon,K)$ is $< 0$ and $O(1/\varepsilon)$. Furthermore, by Lemma 4.1, $dt_{j+1}(\varepsilon,K)/dK < 0$. Hence $(\partial/\partial K)y(-1,\varepsilon,K_{j1}(\varepsilon)) < 0$ and $O(1/\varepsilon)$, and there is an interval $K_{j-}(\varepsilon) < K < K_{j+}(\varepsilon)$ such that $(\partial/\partial K)y(-1,\varepsilon,K) < 0$.

It remains to be shown that the above interval may be chosen so that the image under the map $K \to y(-1,\varepsilon,K)$ tends to $A_j$ as $\varepsilon \to 0$, and that outside this interval, the limit behavior near $t = -1$ is horizontal.

To see this, we note that the monotonicity persists as long as $(\partial/\partial K)y(-1,\varepsilon,K) < 0$. Also, the conditions $y = y(t,\varepsilon,K)$, $(\partial/\partial K)y(t,\varepsilon,K) = 0$ define the envelope of the trajectories $y(t,\varepsilon,K)$ parameterized by $K$. Thus, monotonicity persists until a curve $y(t,\varepsilon,K)$ touches the envelope at $t = -1$. From the variational equation of (1.1) around $y(t,\varepsilon,K_{j1})$, it can be seen that $(\partial/\partial K)y(t,\varepsilon,K)|_{K=K_{j1}} < 0$ for $|t - 1| < O(\sqrt{\varepsilon})$. Since the width of the boundary layer is $O(\varepsilon)$, it follows that the curves $y(t,\varepsilon,K)$ for nearby $K$ intersect near the top and bottom of $A_j$ for $\varepsilon$ small. (Geometrically, the envelope of the limiting functions (as $\varepsilon \to 0$) is a pair of curves through the "corners" separating the horizontal and vertical segments. For $\varepsilon > 0$, the envelope is a pair of curves converging to the limiting pair.) Outside the interval $K_{j-}(\varepsilon) < K < K_{j+}(\varepsilon)$, and for $|K - K_j|$ sufficiently small, the size of the "boundary layer jump" $\lim_{t\to -1}\lim_{\varepsilon\to 0} |y(-1,\varepsilon,K(\varepsilon)) - y(t,\varepsilon,K(\varepsilon))|$ is zero.

Having proven i), we now turn to the proof of ii). We first show that $B_j \supset B_{j+2} \supset \cdots$. It suffices to prove that $t_1^{j+2} > t_1^j$.

<u>Lemma 4.2</u>. Suppose $\bar{t}_1 < t_1$. Let $\{\bar{t}_i\}$, $\{t_i\}$ be the solution to (3.1) with $t_0 = -1$, $t_1 = \bar{t}_1$ or $t_1$, respectively. Then for each $i$, $\bar{t}_i < t_i$.

Proof: $\exists!\ \bar{K}$ (resp. $K$) such that the solution to (1.1) satisfying $y(0,\varepsilon) = 0$, $y'(0,\varepsilon) = -\varepsilon/\bar{K}$ (resp. $-\varepsilon/K$) crosses $y = -t$, in the limit as $\varepsilon \to 0$, at $\bar{t}_1$ (resp. $t_1$); if $\bar{t}_1 < t_1$, then $\bar{K} > K$. We know from Lemma 4.1 that the successive crossing points are, in the limit, monotone functions of $K$, and from Section 3 that these crossing points approach solutions of (3.1). |

We now return to the proof of ii). It follows immediately that $t_1^{j+2} > t_1^j\ \forall_j$. For if not, we would have $t_{j+1}^{j+1} > t_{j+1}^{j+3}$, for some $j$; but, by hypothesis, $t_{j+1}^{j+1} = -1$ and $t_{j+1}^{j+3} > -1$.

A similar argument shows that $A_j \supset A_{j+2}$. That is, the point $-1$ and the next nearest points $t_j^{j+1}$ or $t_{j+2}^{j+1}$ determine a solution to (3.1). As in Lemma 4.2, $t_j^{j+1} > t_{j+2}^{j+3}$ and $t_{j+2}^{j+1} < t_{j+4}^{j+3}$, so $A_j \supset A_{j+2}$.

In order to prove iii), first consider $B > 0$. Since $t_1^j$ is monotone in $j$, it suffices to know that $t_1^j \to 0$ as $j \to \infty$, since $B$ will then eventually be outside $B_j$ for large enough $j$. This, in turn follows by considering solutions to (1.1) which, in the limit as $\varepsilon \to 0$, crosses $y = -t$ for the first time at $t_1^j$. It is then clear that if $t_1^j$ is bounded away from $0$, we could not have the required $t_{j+1}^{j+1} = -1$. (Indeed, we would have $t_{j+1}^{j+1} \to -\infty$.)

A similar argument holds for $A \neq 1$: i.e. $t_j^{j+1} \to -1$, $t_{j+2}^{j+1} \to -1$, so any $A \neq 1$ must eventually fall outside $A_j$ for sufficiently large $j$. |

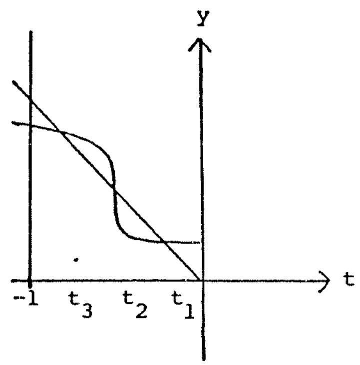
Note that this argument does not rule out a countable number of "limiting solutions" for $A = 1$, $B < 0$. However, the bound on $\varepsilon$ is necessary to insure the existence of a real solution with $j$ strictly interior crossings with $y = -t$ may go to zero as $j \to \infty$.

Next, we deal with solutions (other than the constant solutions) which, in the limit as $\varepsilon \to 0$, have horizontal segments near both $t = -1$ and $t = 0$.

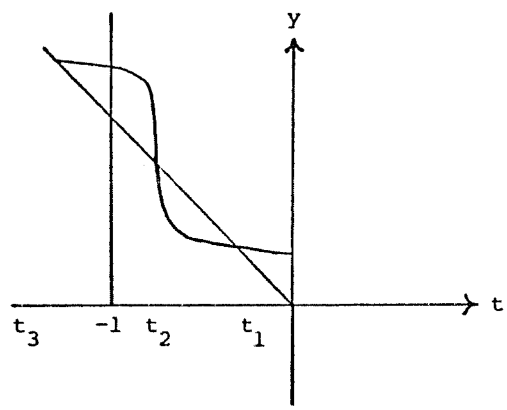
Theorem 4.2. Suppose $B > 0$. For each odd $j > 3$, consider the solution to (3.1) with $t_j = -A$ (if $A < 1$) or $t_{j+1} = -A$ (if $A > 1$), and $t_1 = -B$ (if $B > 0$) or $t_0 = -B$ (if $B = 0$). (See Figures 4.3a and 4.3b.)

i) There is a unique solution to (1.1), (1.2) which approaches horizontal segments at $t = 0$ and $t = -1$, and which has $j$ strictly interior turning points if the following is satisfied; if $A < 1$, then $t_{j+1} < -1$; if $A > 1$, then $-1 < t_j < 0$. Also

$(t_2)^2 > 3B^2$. A necessary condition for such a solution is $t_{j+1} < -1$ if $A < 1$ and $(t_2)^2 > 3B^2$.




4.3a




4.3b

Figures 4.3a, 4.3b

Solutions to (1.1) with $j = 3$ and "horizontal" segments near $t = -1$ and $t = 0$. Figure 4.3a shows $A < 1$; Figure 4.3b has $A > 1$.

ii) If for some $j$, $\left(t_2^j\right)^2 < 3B^2$, then $\left(t_2^{j+2}\right)^2 < 3B^2$. If $A < 1$ and $t_{j+1}^j > -1$, then $t_{j+1}^{j+2} > -1$. If $A > 1$ and $t_j^j \notin (-1,0)$, then $t_{j+2}^{j+2} \notin (-1,0)$.

iii) If $A \neq 1$ or $B > 0$, there are only finitely many solutions to (1.1), (1.2).

Proof: i) We first establish the necessity of the conditions in i). As in Theorem 4.1, the conditions come from the requirement that the limiting solutions intersect $y = -t$ exactly at the $\{t_i^j\}$, where $\{t_i^j\}$ is the solution to (3.1) with the boundary conditions given in i).

The condition $(t_2)^2 > 3B^2$ requires that there be no negative solution to the $i = 1$ equation of (3.1) (with $t_1 = -B$, $t_2$ given by the appropriate solution to (3.1)). Equivalently, there is no "next point" to the right of $t_1$ which is still $< 0$. (For the limiting solution to (1.1), (1.2) must have $-B$ as the first point of intersection with $y = -t$ if $B > 0$; for $B = 0$ the condition is vacuous.) Similarly, the condition on $t_{j+1}$ (if $A < 1$) demands that $t = -A$ be the last (limiting) point of intersection before $t = -1$; if $A > 1$, the condition on $t_j$ requires that $t = -A$ be the first point of intersection for $t < -1$.

We now go to the existence and uniqueness of the solutions. Using Lemmas (2.1) and (2.2), we see that $(t_2)^2 > 3B^2$ implies $\exists !\ L_j \neq 0$ such that the solution to (1.1) with

$$y(0,\varepsilon) = B, \quad y'(0,\varepsilon) = -e^{-L/\varepsilon} \tag{4.4}$$

and $L = L_j$ has a limiting behavior whose first vertical strip is through $t_2^j$. (It follows that the limiting behavior of this $y(t,\varepsilon,L_j)$ is a series of horizontal and vertical strips cutting $y = -t$ through all the $\{t_i^j\}$.) We shall show that for each $\varepsilon$ sufficiently small, $\exists !\ L_j(\varepsilon)$ such that $L_j(\varepsilon) \to L_j$ as $\varepsilon \to 0$, and the solution through $y(0,\varepsilon) = B$, $y'(0,\varepsilon) = -e^{-L_j(\varepsilon)/\varepsilon}$ satisfies $y(-1,\varepsilon) = A$. As in Theorem 4.1, we need a monotonicity result.

<u>Lemma 4.3</u>. Let $t_i(\varepsilon,L)$ be the value of $t$ where the solution $y(t,\varepsilon,L)$ satisfying (4.4) hits $y = -t$ for the $i^{th}$ time. (By definition, $\lim\limits_{\varepsilon\to 0} t_j(\varepsilon,L_j) = -A$ if $A < 1$, and $t_{j+1}(\varepsilon,L_j) = -A$ if $A > 1$.) Then, for $L$ near $L_j$ and $i > 1$, $\lim\limits_{\varepsilon\to 0} \frac{dt_i}{dL}(\varepsilon,L) < 0$.

<u>Proof</u>: As in Lemma 4.2, we use Lemmas 2.1 and 2.2 alternately to see that increasing $L$ increases $-t_i$ for each $i > 1$. |

We return to the proof of Theorem 4.2. Lemma 4.3 implies that for $L < L_j$ (resp. $L > L_j$), $t_i(\varepsilon,L) > t_i(\varepsilon,L_j)$ (resp. $t_i(\varepsilon,L) < t_i(\varepsilon,L_j)$) for all $\varepsilon$ sufficiently small. Also, by hypothesis, $\lim\limits_{\varepsilon\to 0} y(-1,\varepsilon,L) = -\lim\limits_{\varepsilon\to 0} t_j(\varepsilon,L)$ (resp. $-\lim\limits_{\varepsilon\to 0} t_{j+1}(\varepsilon,L)$) for $A < 1$ (resp. $A > 1$). Hence, for fixed $\delta$ small, and all $\varepsilon$ sufficiently small,

$$y(-1,\varepsilon,L_j - \delta) < A < y(-1,\varepsilon,L_j + \delta) \ .$$

To finish i) it suffices to show that for $|L - L_j| < \delta$, $\partial y/\partial L > 0$. This is somewhat easier than the analogous proof for Theorem 4.1, since the monotonicity holds for $|L - L_j| < \delta$, where $\delta$ may be chosen independently of $\varepsilon$. (See Figure 4.4.) By definition of $t_i(\varepsilon,L)$, we have

$$y(t_i(\varepsilon,L),\varepsilon,L) = -t_i(\varepsilon,L) \ .$$

Differentiating this with respect to $L$ yields

$$\frac{\partial y}{\partial L}(t_i(\varepsilon,L),\varepsilon,L) = -[1 + y'(t_i(\varepsilon,L),\varepsilon,L]\frac{dt_i}{dL} \ .$$

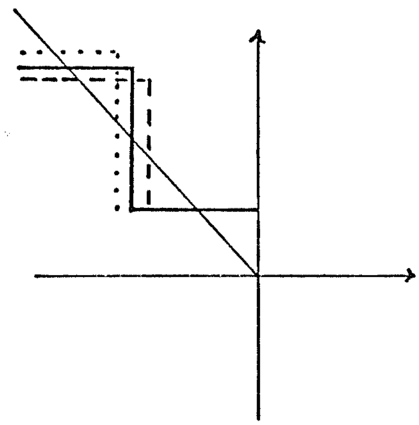


Figure 4.4

The limiting forms of solutions to (1.1) with the same B but different $y'(0,\varepsilon,L) = -e^{-L/\varepsilon}$ (nearby L).

By Lemma 4.2, $dt_i/dL < 0$. Also, for $\varepsilon$, $\delta$ sufficiently small, $y'(t_i(\varepsilon,L),\varepsilon,L)$ is exponentially small, so $(\partial/\partial L)y(t_i(\varepsilon,L),\varepsilon,L) \approx -\frac{dt_i}{dL} < 0$. Furthermore, as $\varepsilon \to 0$, $y(t_i(\varepsilon,L),\varepsilon,L) \to \lim_{\varepsilon\to 0} y(-1,\varepsilon,L)$. Hence we may also conclude that $(\partial/\partial L)y(-1,\varepsilon,L) < 0$.

ii) To show that $\left(t_2^j\right)^2 < 3B^2 \Longrightarrow \left(t_2^{j+2}\right)^2 < 3B^2$, it suffices to show that $|t_2^{j+2}| < |t_2^j|$. This follows from the arguments in the proof of Theorem 4.1, ii). The assertion for $A < 1$ follows from the fact that the first point to the left of $t_j$ in the solution to (3.1) with $t_j = -A$, $t_1 = -B$ becomes closer to $t_j$ as $j$ increases. Similarly, if $A > 1$, the assertion claimed in ii) follows from the fact that the first point to the right of $t_{j+1}$ in the solution to (3.1) with $t_{j+1} = -A$, $t_1 = -B$ is closer to $t_{j+1}$ as $j$ increases.

iii) This follows immediately from ii). |

The two combinations left to be done (for $B > 0$ or, if $B < 0$, $\beta = 0$) of horizontal and vertical limiting behavior near $t = 0$, $t = -1$ are proved by the methods of the two previous theorems. Hence, we shall merely state the results. Theorem 4.3 concerns solutions which, in the limit, are horizontal near $t = 0$ and vertical at $t = -1$. Theorem 4.4 concerns the reverse combination.

Theorem 4.3. Suppose $B > 0$. For each odd $j > 3$, consider the solution $\{t_i^{j+1}\}$ to (3.1) with $t_{j+1} = -1$, $t_1 = -B$. Let $C_j(B)$ be the interval $(-t_j, -t_{j+2})$.

i) There is a unique solution to (1.1), (1.2) having a boundary layer at $t = -1$, no boundary layer at $t = 0$, and $j$ strictly interior turning points, if $A \in C_j(B)$ and $(t_2)^2 > 3B^2$. A necessary condition for such a solution is $A \in c\ell\, C_j(B)$, $(t_2)^2 > 3B^2$.

ii) If $A \neq 1$ or $B > 0$, there are at most finitely many $j$ for which there is such a solution. |

Theorem 4.4. Suppose $B > 0$ or, if $B < 0$, $\beta = 0$. For each odd $j > 3$, consider the solution $\{t_i^j\}$ to (3.1) with $t_j = -A$, $t_0 = 0$. Let $D_j(A)$ be the interval $(-\infty, -t_1)$. Then

i) There is a solution to (1.1), (1.2) with a boundary layer at $t = 0$, no boundary layer at $t = -1$ and $j$ strictly interior crossings with $y = -t$ if $B \in D_j(A)$ and $t_{j+1} < -1$ (for $A < 1$) or $-1 < t_j < 0$ (for $A > 1$). A necessary condition for such a solution is $B \in c\ell\, D_j(A)$, $t_{j+i} < -1$ (for $A < 1$) or $-1 < t_j < 0$ for $A > 1$.

ii) If $A \neq 1$ or $B > 0$, there are at most finitely many $j$ for which there is such a solution. |

The techniques of Section 3 used in Theorems 4.3 and 4.4 work only for $j > 3$. There are solutions for $j = 1$, and these cases may be done by going back to the results of Section 2 from which those of Section 3 were derived:

Proposition 4.1. Suppose $0 < B < \frac{1}{\sqrt{3}}$. Let $\bar{L}_1 = \frac{1}{3} - B^2$, and $y_1(t,\varepsilon)$ the solution to (1.1) with $y_1(0,\varepsilon) = B$, $y_1'(0,\varepsilon) = -1$. Let $C_1(B)$ be the open, first vertical segment of the limiting solutions of $y(t,\varepsilon)$ as $\varepsilon \to 0$. (See Figure 4.5.) Then if $A \in C_1(B)$, there is a unique solution to (1.1), (1.2) with a boundary layer at $t = -1$, no boundary layer at $t = 0$, and intersecting $y = -t$ exactly once. Furthermore, if $y(t,\varepsilon)$ is this solution, then $y'(0,\varepsilon) = -e^{-\bar{L}_1(\varepsilon)/\varepsilon}$, where $\bar{L}_1(\varepsilon) \to \bar{L}_1$ as $\varepsilon \to 0$. If $B > \frac{1}{\sqrt{3}}$ there is no such solution. |

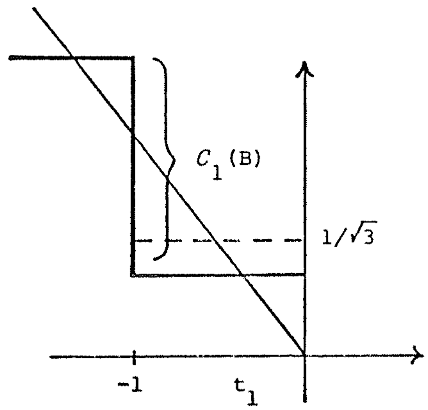



Figure 4.5

A limiting solution for the case $j = 1$, Theorem 4.3.

Proposition 4.2. Suppose $A > \frac{1}{\sqrt{3}}$, $B < A$. Then there is a unique solution to (1.1), (1.2) with a boundary layer at $t = 0$, no boundary layer at $t = -1$, having exactly one (resp. no) strictly interior point of intersection with $y = -t$ if $A < 1$ (resp. $A > 1$), and no interior points of intersection with $y = t$. (See Figure 4.6.) Furthermore, if $y(t,\varepsilon)$ is this solution, then $y'(0) = -K_1(\varepsilon)/\varepsilon$, where $K_1(\varepsilon) \to K_1 \equiv (A^3 - B^3)/3$. If $A < \frac{1}{\sqrt{3}}$, there is no such solution. |

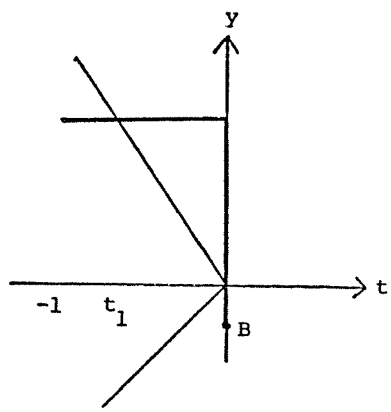



Figure 4.6

A limiting solution for the case $j = 1$, Theorem 4.4.

The four previous theorems deal with solutions to (1.1) for which there are no strictly interior crossing points with $y = t$, i.e. $B > 0$, or $B < 0$ and $\beta = 0$. We now indicate how those theorems must be modified if $B < 0$ and $\beta < 0$. The next theorem refers to solutions which, in the limit as $\varepsilon \to 0$, have a vertical segment at $t = 0$; the final one deals with solutions that approach a horizontal line near $t = 0$.

Theorem 4.5. Suppose $B < 0$. Consider the solution $\{\beta, t_i^j\}$ to (3.1), (3.9), (3.10) with $t_{j+1} = -1$ if the limiting behavior near $t = -1$ is vertical, or $t_j = -A$ (resp. $t_{j+1} = -A$) if the limiting behavior near $t = -1$ is horizontal and $A < 1$ (resp. $A > 1$). Let $\bar{A}_j$ be the interval $(-t_j, -t_{j+2})$ if the limiting solution is vertical near $t = -1$. Let $\bar{B}_j = (-\infty, \beta)$. If the limiting solution is horizontal near $t = 0$, let $\bar{D}_j(A) = (-\infty, \beta)$. Then

i) Theorems 4.1 i) and 4.4 i) continue to hold if $A_j$, $B_j$ and $D_j(A)$ are replaced by $\bar{A}_j$, $\bar{B}_j$ and $\bar{D}_j(A)$, and the word "odd" changed to "even".

ii) $\bar{A}_2 \supset \bar{A}_4 \supset \bar{A}_6 \cdots$, $\bar{B}_2 \subset \bar{B}_4 \subset \bar{B}_6 \cdots$ .

iii) For each $A \neq 1$, there are finitely many $j$ for which $A \in \bar{A}_j$. |

Theorem 4.6. Suppose $B < 0$. Consider the solution $\{\beta, t_i^j\}$ to (3.1), (3.9) with $\beta = B$ and $t_{j+1}$ or $t_j$ given as in Theorem 4.5. Let $\bar{A}_j$ be as in Theorem 4.5. Then

i) Theorems 4.2, 4.3 i) continue to hold if $A_j$ is replaced by $\bar{A}_j$, $(t_2)^2 > 3B^2$ replaced by $(t_1)^2 > 3B^2$, and "odd" is replaced by "even".

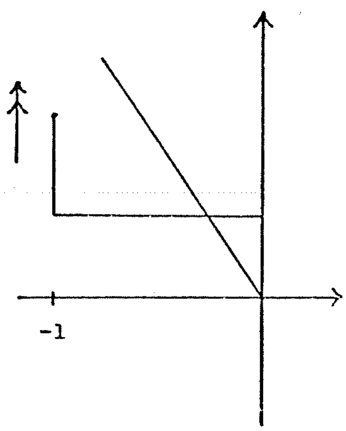
ii) As in Theorem 4.5.

iii) As in Theorem 4.5. |

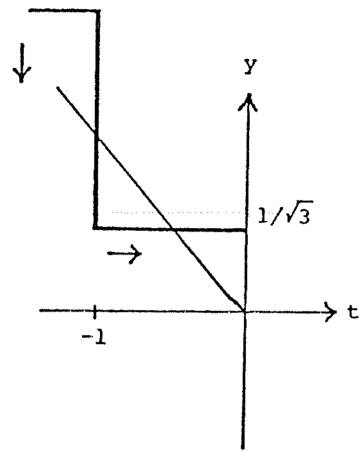
Remarks: 1. For each of the four combinations of horizontal and vertical behavior near the endpoints (6 cases for $B < 0$) there is a slight difference between necessary and sufficient conditions; the necessary conditions involve certain closed intervals, while the sufficient conditions hold on the interior of those intervals. The difference is due to the fact that for each fixed $\varepsilon > 0$, there is a region around certain values of $A$ and $B$ for which the number of solutions is not necessarily the same as the number of limiting solutions as $\varepsilon \to 0$; the size of each such region goes to $0$ with $\varepsilon$.

To see why this is so, it is useful to consider (1.1) as an initial value problem, with $B = y(0,\varepsilon)$ and $y'(0,\varepsilon)$ specified. As an example, we shall consider $B$ fixed, $B \in \mathcal{B}_1$, $B \notin \mathcal{B}_3$ (i.e. $B < \frac{1}{\sqrt{3}}$ and close to $\frac{1}{\sqrt{3}}$), and increase $|y'(0,\varepsilon)|$ from 0. For $y' = 0$, $y(t,\varepsilon) \equiv B$. If $y'(0,\varepsilon) = -e^{-L/\varepsilon}$, and $L$ is now decreased from $\infty$, there is a last value of $L$ (namely $\bar{L}_1 = 1/3 - B^2$) for which $\lim_{\varepsilon \to 0} y \equiv B$. For fixed $\varepsilon$ small, $y(-1,\varepsilon,L)$ stays $o(1)$ from $y = B$ for $L > \bar{L}_1$. As $L$ passes $\bar{L}_1$, $y(-1,\varepsilon,L)$ moves rapidly up. (By "rapid", we mean that a change of order $O(1)$ in $y(-1,\varepsilon,L)$ is produced by a $o(1)$ change in $L$.) This increase continues until $y(-1,\varepsilon,L)$ hits some point close to the upper endpoint of $\mathcal{C}_1(B)$, and $y(-1,\varepsilon,L)$ starts to decrease. Since the upper limit is not exactly the boundary of $\mathcal{C}_1(B)$ for fixed $\varepsilon$, the number of solutions to (1.1), (1.2) for $A$ very close to this boundary may be two more or less than the number of limiting solutions.

After $L$ passes $\bar{L}_1$, a decrease in $L$ leads to a slow decrease in $y(-1,\varepsilon,L)$. (By "slow", we mean that a change of $O(1)$ in $y(-1,\varepsilon,L)$ is produced by an $O(1)$ change in $L$.) (See Figure 4.7.) The decrease continues until $L$ reaches a value $\approx \bar{L}_3$ where the solution to (1.1) with $y(0,\varepsilon,L) = B$, $y'(0,\varepsilon,L) = -e^{-\bar{L}_3/\varepsilon}$ has a limiting behavior with a vertical segment at $t = -1$ and three strictly interior turning points. It then undergoes a rapid increase in $y(-1,\varepsilon,L)$ for $L \approx \bar{L}_3$ and, as before, stops at a value of $y$ near the upper endpoint of $\mathcal{C}_3(B)$. This alternation of rapid increases and slow declines in $y(-1,\varepsilon,L)$ continues a finite number of times for $L \in [0,\infty]$. (Indeed, for $B$ close enough to $\frac{1}{\sqrt{3}}$, the first rise near $L = \bar{L}_1$ is the only such rise; the number of rises can be computed by solving (3.1) with $t_0 = 0$, $t_1 = B$, and finding the last $j$ such that $t_{2j} > -1$.) The rapid rises correspond to solutions which are approximately vertical near $t = -1$; the slow declines correspond to solutions which are close to horizontal near $t = -1$.




4.7a




4.7b

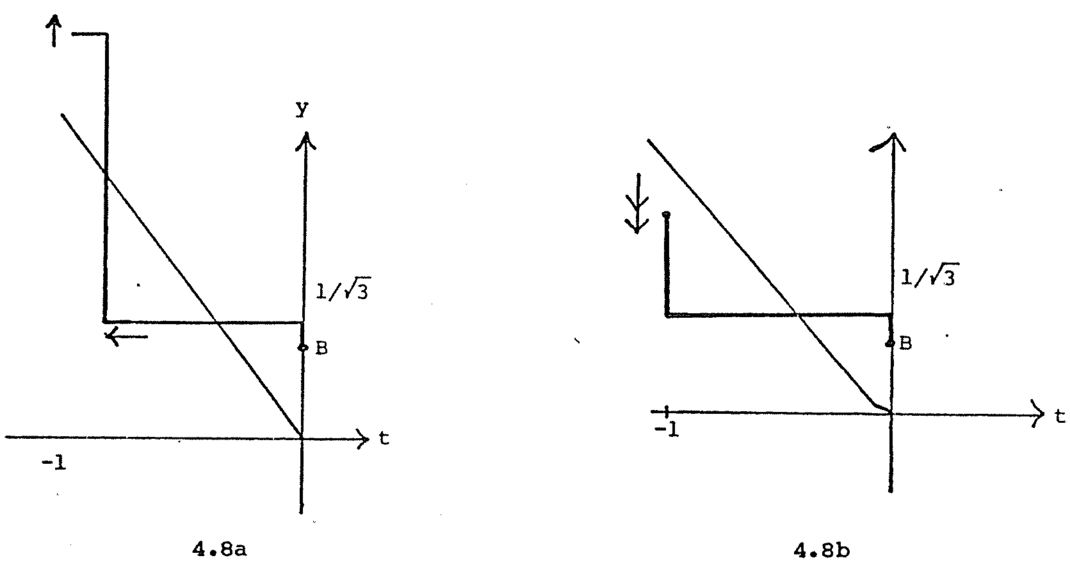
Figures 4.7a, 4.7b

The limiting forms of solutions with $y(0,\varepsilon) = B$, $y'(0,\varepsilon) = -e^{-L/\varepsilon}$, with varying L. The arrow indicates motion as L is decreased; the double arrow indicates rapid. Figure 4.7a: $L \approx \bar{L}_1$; Figure 4.7b: $\bar{L}_3 < L < \bar{L}_1$.

Thereafter, as $|y'|$ increases, we consider $y'(0,\varepsilon) = -K/\varepsilon$. Increasing K (see Figure 4.8a) at first increases $y(-1,\varepsilon,K)$ slowly (slow and rapid have the same meaning as for $y(-1,\varepsilon,L)$). The slow increase ends when $y(-1,\varepsilon,K)$ reaches a point near the upper value of $A_1$, at $K \approx K_1$; for K near $K_1$, there is then a rapid decrease in $y(-1,\varepsilon,K)$, (see Figure 4.8b) which ends near the bottom of $A_1$. For larger K, the value of $y(-1,\varepsilon,K)$ thereafter increases with K, and is unbounded. If $B \in A_j$, $B \notin A_{j+2}$ for some j, there would be a finite sequence of slow increases and rapid declines in $y(-1,\varepsilon,K)$ before the final monotone increase. Once again, near every value of A for which $(\partial/\partial K)y(-1,\varepsilon,K) = 0$, the asymptotic count of solutions to (1.1), (1.2) may be off by two.

2. Consider (1.1), (1.2) now as a boundary value problem, with A and B as variables. For example, suppose we fix B and $\varepsilon > 0$ with $\varepsilon << 1$. As we vary A, the number of solutions changes by two (either up or down) as certain critical values of A are passed; these points correspond to changes in the sign of $(\partial/\partial K)y(-1,\varepsilon,K)$ or

$(\partial/\partial L)y(-1,\varepsilon,L)$ and have been computed in the limit as $\varepsilon \to 0$. The disappearance (or appearance) of two solutions is accomplished by a bifurcation: two solutions coalesce and then disappear. The prototype example is $B < \frac{1}{\sqrt{3}}$, $A$ variable. As $A$ decreases past $\frac{1}{\sqrt{3}}$, the left boundary layer of the solution which is asymptotically $\equiv A$ (for $-1 < t < 0$) tends to zero in length, and the solution coalesces with another solution which is asymptotically $\equiv \frac{1}{\sqrt{3}}$ (for $-1 < t < 0$). (See Figure 4.9.) For $A < \frac{1}{\sqrt{3}}$ and $\varepsilon$ small enough, neither of these solutions exists.




Figures 4.8a, 4.8b

The limiting forms of solutions with $y(0,\varepsilon) = B$, $y'(0,\varepsilon) = -K/\varepsilon$, with varying $K$. The arrows indicate motion of $y(-1,\varepsilon,K)$ as $K$ increases. Figure 4.8a: $K < K_1$; Figure 4.8b: $K \approx K_1$.

3. The methods of this paper can also be used to get a complete analysis of solutions for $t > 0$.

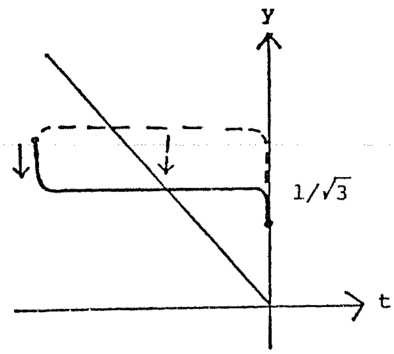



Figure 4.9

A pair of solutions to (1.1), (1.2) with $B < 1/\sqrt{3}$. As $A$ crosses $1/\sqrt{3}$ (in the limit as $\varepsilon \to 0$). the solutions coalesce and disappear.

## Appendix: The case $B < A < 0$.

Proposition A. Suppose $B < A < 0$. Then there is a unique solution to (1.1), (1.2). The limiting behavior of the solution for $-1 < t < 0$ is as follows:

$$\text{if } B < -\frac{1}{\sqrt{3}} < A, \quad \lim_{\varepsilon \to 0} y(t,\varepsilon) = -\frac{1}{\sqrt{3}},$$

$$\text{if } B < A < -\frac{1}{\sqrt{3}}, \quad \lim_{\varepsilon \to 0} y(t,\varepsilon) = A,$$

$$\text{if } -\frac{1}{\sqrt{3}} < B < A, \quad \lim_{\varepsilon \to 0} y(t,\varepsilon) = B.$$

(See Figure A.1.)

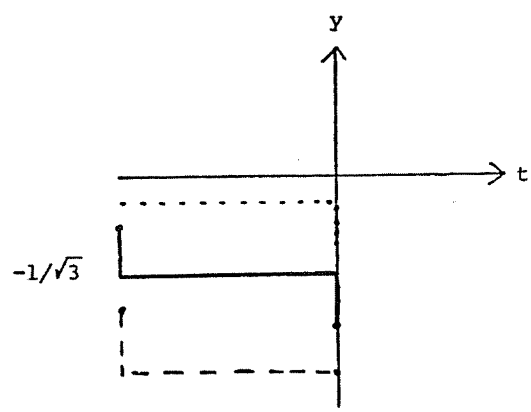



Figure A.1

The limiting forms of solutions for the three different cases of Proposition A.

Proof: It follows from the estimates of Section 2 that, in each case, the above constant is the only possible limiting behavior. To show that there is a unique solution corresponding to these boundary conditions, we may use the arguments of Theorem 4.1 and Theorem 4.2. Alternatively, we may apply the maximum principle to the difference of two solutions. If $y_1, y_2$ are solutions, then $\overline{w} = y_1 - y_2$ satisfies

$$\varepsilon w'' - (y_1^2 - t^2)w' - [y_2'(y_2 + y_1)]w = 0 \ ,$$

$$\overline{w}(-1) = \overline{w}(0) = 0 \ .$$

Since $[y_2'(y_2 + y_2)] > 0$ we see that

$$\overline{w} \equiv 0. \qquad |$$

## REFERENCES


1. Howes, F., "Singularly perturbed superquadratic boundary value problems whose solutions exhibit boundary and shock layer behavior", to appear: J. Nonlinear Anal.

2. ________________ and S. V. Parter, "A model nonlinear problem having a continuous locus of singular points", Studies in Appl. Math. 58 (1978), 249-262.

3. Kedem, G., S. V. Parter and M. Steuerwalt, "The solutions of a model nonlinear singular perturbation problem having a continuous locus of singular points", Studies in Appl. Math. 63 (1980), 119-146.

4. Rabinowitz, P., "A global theorem for nonlinear eigenvalue problems and applications", in Contributions to Nonlinear Functional Analysis, E. H. Zarantonello, ed., Academic Press, NY (1971), 11-36.

5. Sutton, F., Personal communication.

6. Varga, R. S., Matrix Iterative Analysis, Prentice Hall, Englewood Cliffs, NJ (1962).


NK/SVP/scr