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A FAST ALGORITHM FOR THE EUCLIDEAN TRAVELING SALESMAN PROBLEM, OPTIMAL WITH PROBABILITY ONE

J. H. Halton and R. Terada

Computer Sciences Department

University of Wisconsin — Madison

ABSTRACT

This paper presents an algorithm for the *Traveling Salesman Problem* in k-dimensional Euclidean space. For n points independently uniformly distributed in a set $\underline{\underline{E}}$, we show that, for any choice of a function σ of n increasing to infinity with n more slowly than n, we can adjust the algorithm so that, *in probability*, the time taken by the algorithm will be of order less than $n\,\sigma(n)$ as $n \to \infty$. The algorithm puts the n points in a cyclic order, and we also show that, *with probability one*, the length of the corresponding *tour* (that is, the sum of the n distances between adjacent points in the order given) will be asymptotic to the *minimal tour length* as $n \to \infty$. The latter is known (also with probability one) to be asymptotic to $\beta_k\, v(\underline{\underline{E}})^p\, n^q$, where β_k is a constant depending only on the dimension k, $v(\underline{\underline{E}})$ is the volume of the set $\underline{\underline{E}}$, $p = 1/k$, and $q = 1 - p$. Our result is *stronger*, and the algorithm is *faster*, than any other we have been able to find in the literature.

1. INTRODUCTION

Consider a set $\underline{\underline{A}}$ of n points in the k-dimensional Euclidean space $\underline{\underline{R}}^k$ (with the usual topology.) A *tour* of $\underline{\underline{A}}$ is defined to be a *cyclically ordered set* containing $\underline{\underline{A}}$ [that is, a set $\underline{\underline{T}}$ such that $\underline{\underline{A}} \subseteq \underline{\underline{T}} \subseteq \underline{\underline{R}}^k$, with an *ordering relation* τ, such that, for any finite subset of $\underline{\underline{T}}$ — e.g., $\{A, B, C, D, E, F\}$ — a unique, complete cyclic order exists — e.g., $\{A \tau C, C \tau B, B \tau F, F \tau D, D \tau E, E \tau A\}$, which we shall abbreviate to $A \tau C \tau B \tau F \tau D \tau E \tau A$, or just to the string of point-symbols *ACBFDE*.] (Note that a path, which may be intuitively viewed as a tour which crosses itself, can always be described as a cyclically ordered set by removing the single point of intersection from one of the branches. Similarly, a path which is traced more than once may be cyclically ordered by suitably interlacing the points of each passage.) If a *metric* d is defined in $\underline{\underline{R}}^k$ (not necessarily consistent with the topology of $\underline{\underline{R}}^k$), such a tour will have a (possibly infinite) *length* $\ell(\underline{\underline{T}}, \tau)$ [defined as the supremum of the sum of the metric distances between successive points in any finite sub-cycle in the tour --- e. g., $d(A, C) + d(C, B) + d(B, F) + d(F, D) + d(D, E) + d(E, A)$.] Since all tour-lengths are non-negative, they are bounded below by zero; so that there will be an infimum for the lengths of all tours of a given set $\underline{\underline{A}}$: we denote this by $\ell(\underline{\underline{A}})$.

Given a tour $(\underline{\underline{T}}, \tau)$ of $\underline{\underline{A}}$, it will uniquely determine a cyclic ordering of $\underline{\underline{A}}$ (since $\underline{\underline{A}}$ is a finite subset of $\underline{\underline{T}}$), so that $(\underline{\underline{A}}, \tau)$ is itself a tour of $\underline{\underline{A}}$. If we label the points of $\underline{\underline{A}}$ in such a manner that the tour $(\underline{\underline{T}}, \tau)$ imposes the cyclic order $A_0 \tau A_1 \tau A_2 \tau \ldots \tau A_n = A_0$, then the triangle inequality for the metric d ensures that the length $\ell(\underline{\underline{A}}, \tau) = \Sigma_{i=1}^{n} d(A_{i-1}, A_i)$, and it is clear that this cannot exceed the length $\ell(\mathbf{T}, \tau)$. It follows that the infimum of the lengths of all tours of $\mathbf{A}$ is the same as the infimum of the lengths of all tours $(\mathbf{A}, \tau)$: and this is the infimum of $\ell(\mathbf{A}, \tau)$ over all $(n - 1)!$ cyclic orderings of $\mathbf{A}$. Since this last infimum is taken over a *finite* collection of lengths, it is certainly *attained*. We thus see that there will always exist at least one cyclic ordering of $\mathbf{A}$, which we may denote by π, such that $\ell(\mathbf{A}, \pi) = \inf_\tau \ell(\mathbf{A}, \tau) = \ell(\mathbf{A})$. Such a tour will be termed a *minimal tour* of $\mathbf{A}$. The search for minimal tour-lengths and for minimal tours in $\mathbf{R}^k$ is called the Traveling Salesman Problem (k-TSP.)

In this paper, we shall limit ourselves to the problems in which the metric d is the *Euclidean* (or *Pythagorean* or ℓ^2) metric, for which $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_2 = \sqrt{\Sigma_{i=1}^k (x_i - y_i)^2}$. This is called the Euclidean Traveling Salesman Problem (k-ETSP.)

The 2-TSP has been shown to be NP-hard (see Garey, Graham, and Johnson [1976], Papadimitriou [1977], Garey and Johnson [1979]), and this strongly suggests that there is no polynomial-time algorithm for obtaining the exact solution of this problem --- and, by natural extension, we believe that the same is true for the k-TSP with $k \geqslant 3$. Certainly, no such algorithm has been found, so far.

On the other hand, there has been some research on fast heuristic methods for the solution of the 2-TSP: for example, computer programs to find near-optimal solutions for sets of up to 300 points in an acceptable amount of time have been described by Krolak, Felts, and Marble [1970], and by Lin and Kernighan [1973]. Their programs seem to give satisfactory results; but no rigorous analyses of the algorithms are available.

Bellman [1962], and Held and Karp [1962] describe a dynamic programming algorithm for the k-TSP, which determines an exactly minimal tour of a set of s points in a time

$$t_s = 2\,A\,(s-1)\,[2^{s-3}\,(s-2)+1] \quad \text{for} \quad s \geqslant 1, \tag{1.1}$$

where A is a computer-dependent constant (roughly, half the time needed for an addition.) We subsume the use of this algorithm, which we shall refer-to as Algorithm C, in constructing our own, and the estimate (1.1) yields our timing estimate in Theorem 2. (Should a faster algorithm than the above become available, it will lead to an increase in the speed of ours also.)

Since many important computational problems can only be solved by exponential-time algorithms, interest has recently shifted to *probabilistic algorithms*, which, with a high degree of probability, will yield accurate answers in acceptably short times; but for which (with very low probability) either (i) accurate answers may take very long times to obtain, or (ii) answers obtained may not be accurate.

Beardwood, Halton, and Hammersley [1959] studied the statistical properties of the solutions of k-ETSP: in particular, they showed that, if $\underline{\underline{E}}$ is a bounded, Lebesgue-measurable subset of $\underline{\underline{R}}^k$, with k-dimensional Lebesgue measure (or volume) $v(\underline{\underline{E}}) > 0$, and if $\underset{\sim}{P}$ is an infinite sequence of points independently uniformly distributed in $\underline{\underline{E}}$, with $\underset{\sim}{P}^n$ denoting the set consisting of the first n points of $\underset{\sim}{P}$, then there exists a constant β_k, not dependent on $\underline{\underline{E}}$ or $\underset{\sim}{P}$, such that, *with probability one*,

$$\ell(\underset{\sim}{P}^n) \sim \beta_k\, v(\underline{\underline{E}})^p\, n^q \quad \text{as} \quad n \to \infty, \tag{1.2}$$

where $p = 1/k$ and $q = 1 - p$. They also showed that, if the points of $\underset{\sim}{P}$ are instead independently distributed in $\underline{\underline{E}}$ with any fixed probability-distribution and if the absolutely continuous component of this distribution is represented by a probability-density function ρ (whatever the discrete and singular components of the distribution may be); then, again *with probability one*,

$$\ell(\underset{\sim}{P}^n) \sim \beta_k\, n^q \int_{\underline{\underline{E}}} \rho^q \, dv \quad \text{as} \quad n \to \infty. \tag{1.3}$$

When the density is constant, $\rho = 1/v(\underline{\underline{E}})$, (1.3) reverts to (1.2). We take our point of departure in the above paper, which we shall refer-to as BHH. In the course of reviewing the proofs of various results in BHH, we found that the proof of their Lemma 7 had to be modified somewhat (the statement of the lemma remains correct.) This is discussed in Appendix II of Halton and Terada [1978] ---- hereinafter referred-to as HT. The present paper is a revised version of HT.

Karp [1977] has described a probabilistic algorithm for the 2-TSP: it is a recursive algorithm, for which he claims an *expected* running-time of the order of $n\,(\log n)^2$ and an *expected* resulting tour-length asymptotic to $\ell(\underline{\underline{A}})$ as $n \to \infty$. It will be seen below that the algorithm presented here is proved *in probability* to run in a time which is $o[n\,\sigma(n)]$, for an arbitrarily chosen function σ, satisfying

$$\sigma(n) \to \infty \quad \text{and} \quad \sigma(n)/n \to 0 \quad \text{as} \quad n \to \infty \tag{1.4}$$

(see Theorem 2), and it is also proved that the resulting tour-length is asymptotic to $\ell(\underline{\underline{A}})$, *with probability one* (see Theorem 3.) Some questions and discussion of Karp's paper are given in Appendix III of HT; but anyway, our results are stronger. We are not aware of the existence of any other algorithm comparable to ours.

We are grateful to referees for some helpful suggestions which have been incorporated in the present version of HT. Our main results are the same; but we have rearranged the material, made a few changes in the presentation, and, in reviewing the paper, have taken the opportunity to refine and simplify both the algorithm and the proofs of its speed and accuracy.

2. THE MAIN ALGORITHM

Given a set $\underline{\underline{A}}$ of n points in $\underline{\underline{R}}^k$, our algorithm covers it with a cubic lattice of cells, solves the k-ETSP in each cell by Algorithm C, and prescribes how these partial tours should be connected cell-to-cell to form a tour of $\underline{\underline{A}}$. The all-important lattice is defined in such a way that the tour generated has the desirable properties of speed and accuracy claimed in Theorems 2 and 3 below. These are both statistical and asymptotic properties, derived by embedding the given problem in a large class of similar problems in two ways: first, the set $\underline{\underline{A}}$ is viewed as the first n points of an infinite sequence of points; and secondly, the points of the sequence are assumed to be independently uniformly distributed at random in a set $\underline{\underline{E}}$ having the properties:

(a) $\underline{\underline{E}}$ is a Lebesgue-measurable set in $\underline{\underline{R}}^k$, with positive volume $v(\underline{\underline{E}})$;

(b) $\underline{\underline{E}}$ is bounded in $\underline{\underline{R}}^k$: we can find a semi-open hypercube (more briefly, a *cube*)

$$\underline{\underline{C}} = \{\underset{\sim}{x} = (x_1, x_2, \ldots, x_k) \in \underline{\underline{R}}^k: \ b_i \leqslant x_i < b_i + \lambda, \quad \text{for} \quad i = 1, 2, \ldots, k\}, \tag{2.1}$$

with sides of length λ, such that $\underline{\underline{E}} \subseteq \underline{\underline{C}}$;

(c) if the cube $\underline{\underline{C}}$ defined in (b) is divided into a cubic lattice of $M = m^k$ similarly semi-open hypercubic *cells* $\underline{\underline{C}}_j$ ($j = 1, 2, \ldots, M$), each with sides of length λ/m, and if N_2 of these cells contain points both of $\underline{\underline{E}}$ and of its complement $\underline{\underline{E}}^c$, then the boundary of $\underline{\underline{E}}$ is such that, as $M \to \infty$, $N_2 = O(M^q)$, where $q = 1 - 1/k$; so that, in particular, $N_2/M \to 0$. [We see that this property holds whenever the $(k-1)$-dimensional Lebesgue measure of the boundary of $\underline{\underline{E}}$ is finite.]

It is clear that the given set $\underline{\underline{A}} \subseteq \underline{\underline{E}} \subseteq \underline{\underline{C}}$; but, beyond this, the choice of $\underline{\underline{E}}$ and $\underline{\underline{C}}$ is free and will depend on our knowledge (or hunch) of the class of problems of which $\underline{\underline{A}}$ is considered to be a sample. In the absence of more precise information, we may take $\underline{\underline{E}} = \underline{\underline{C}}$ and $\underline{\underline{C}}$ to be the smallest cube (2.1) containing $\underline{\underline{A}}$: the determination of $\underline{\underline{C}}$ requires time of the order of kn, which is negligible, in view of Theorem 2.

Underlying the specification of the algorithm is the choice of a function σ of n, satisfying (1.4), but otherwise at our disposal. Because of Theorem 2 and Karp's claim of an expected running time of $O[n\ (\log n)^2]$, we will focus our attention on $\sigma(n)$ increasing with n no faster than $(\log n)^2$. If $p = 1/k$ and $\lceil \ldots \rceil$ denotes the "roof" function [the least upper bound among the integers], we can define the even integer

$$m = 2\left\lceil \frac{\lambda}{2}\left(\frac{2\ n}{v(\underline{\underline{E}})\ \log\ \sigma(n)}\right)^{p}\right\rceil . \qquad (2.2)$$

From this, we can derive that, by (1.4),

$$M = m^k \sim \frac{2\ n\ \lambda^k}{v(\underline{\underline{E}})\ \log\ \sigma(n)} \quad \text{as} \quad n \to \infty. \qquad (2.3)$$

We also observe that

$$m \to \infty, \quad M \to \infty, \quad M/n \to 0, \quad \text{as} \quad n \to \infty. \qquad (2.4)$$

Algorithm A. [A1] Given a set $\underline{\underline{A}}$ of n points in $\underline{\underline{R}}^k$, choose the semi-open hypercube $\underline{\underline{C}}$ defined as in (2.1), the set $\underline{\underline{E}}$ contained in $\underline{\underline{C}}$ and containing $\underline{\underline{A}}$, having properties (a), (b), and (c) above, and a function σ satisfying (1.4). Hence, determine the even integer m (by (2.2)), and $M = m^k$.

[A2] Divide each side of $\underline{\underline{C}}$ into m equal parts, thus creating a cubic lattice of M semi-open hypercubic cells $\underline{\underline{C}}_j$ ($j = 1, 2, \ldots, M$.)

[A3] In each cell $\underline{\underline{C}}_j$, by Algorithm C, find a minimal tour of $\underline{\underline{AC}}_j$ [the intersection of $\underline{\underline{A}}$ and $\underline{\underline{C}}_j$; i. e., the set of points of $\underline{\underline{A}}$ falling in $\underline{\underline{C}}_j$.] The result is a cyclic ordering of the points of $\underline{\underline{AC}}_j$ which may be written as a string of point-symbols

$$\mathcal{S}_j = A_1^{(j)} A_2^{(j)} \dots A_{n_j}^{(j)}, \quad \text{where } \underline{\underline{AC}}_j = \{A_1^{(j)}, A_2^{(j)}, \dots, A_{n_j}^{(j)}\}, \tag{2.5}$$

and we note that

$$\Sigma_{j=1}^{M} n_j = n. \tag{2.6}$$

Of course, if $\underline{\underline{AC}}_j = \emptyset$ [the empty set] for some j, the corresponding string $\mathcal{S}_j$ will be null.

[A4] Using Algorithm B (defined below), determine a cyclic ordering of the M cells, which may, by suitable renumbering, be written as

$$\mathcal{B} = \underline{\underline{C}}_1 \underline{\underline{C}}_2 \dots \underline{\underline{C}}_M, \tag{2.7}$$

[A5] Applying the ordering (2.7) to the strings $\mathcal{S}_j$, form a string

$$\mathcal{S} = \mathcal{S}_1 \mathcal{S}_2 \dots \mathcal{S}_M. \tag{2.8}$$

This represents a cyclic ordering of all the points of $\underline{\underline{A}}$ (see Theorem 1 below), to which corresponds a tour, $(\underline{\underline{A}}, \omega)$, say, of length

$$\ell_0(\underline{\underline{A}}) = \Sigma_{j=1}^{M} \Sigma_{i=1}^{n_j} d(A_{i-1}^{(j)}, A_i^{(j)}), \tag{2.9}$$

where $A_0^{(j)} = A_{n_{j-1}}^{(j-1)}$ and $A_{n_0}^{(0)} = A_{n_M}^{(M)}$.

3. THE CELL-TOUR ALGORITHM

The following algorithm obtains the ordering (2.7) of the cells $\underline{\underline{C}}_j$ in a time of the order of M. Denote the set $\{0, 1, 2, \dots, m-1\}$ by $\underline{\underline{L}}$ and define a lattice of vectors $\underset{\sim}{a} = (a_1, a_2, \dots, a_k)$ with each $a_i \in \underline{\underline{L}}$. Then it is easily seen that there is a one-to-one correspondence between the M vectors $\underset{\sim}{a}$ and the M cells $\underline{\underline{C}}_j$, defined by

$$\underline{\underline{C}}(\underset{\sim}{a}) = \{\underset{\sim}{x} \in \underline{\underline{R}}^k: \; b_i + \tfrac{\lambda}{m} a_i \leqslant x_i < b_i + \tfrac{\lambda}{m}(a_i + 1) \text{ for } i = 1, 2, \dots, k\}. \tag{3.1}$$

Thus, an ordering of the cells will correspond uniquely to an ordering of the lattice vectors $\underset{\sim}{a}$. We write $\underset{\sim}{e}_i$ for the unit vector in the i-th coordinate direction, and we associate with each $\underset{\sim}{a}$ the numbers

$$r_i = r_i(\underset{\sim}{a}) = (-1)^{1+a_1+a_2+\ldots+a_{i-1}} \tag{3.2}$$

for $i = 2, 3, \ldots, k$. We note that the r_i take the values ± 1 only, and that, for any $\underset{\sim}{a}$, $a_i + r_i \in \underline{\underline{L}}$, unless either $a_i = 0$ and $r_i = -1$, or $a_i = m - 1$ and $r_i = +1$. Therefore, for any $\underset{\sim}{a}$, there is at most one value of t such that

$$a_i + r_i \notin \underline{\underline{L}} \quad \text{for} \quad i = k, k-1, \ldots, t+1,$$
$$a_t + r_t \in \underline{\underline{L}}, \quad \text{and} \quad t \geqslant 3. \tag{3.3}$$

Algorithm B. [B1] If there exists an index t satisfying (3.3), then the algorithm identifies the successor of $\underset{\sim}{a}$ as the vector

$$\underset{\sim}{a}' = \underset{\sim}{a} + r_t \underset{\sim}{e}_t; \tag{3.4}$$

that is, the vector with $a'_i = a_i$ for all $i \neq t$ and with $a'_t = a_t + r_t$.

[B2] If (3.3) does not hold for any t, then the successor of $\underset{\sim}{a}$ is determined as follows:

(i) if $a_1 = 1$ and $a_2 = 0$, or if $a_1 > 1$ and a_2 is even, $\underset{\sim}{a}' = \underset{\sim}{a} - \underset{\sim}{e}_1$;

(ii) if $a_1 = 0$ and $a_2 = m - 1$, or if $0 < a_1 < m - 1$ and a_2 is odd, $\underset{\sim}{a}' = \underset{\sim}{a} + \underset{\sim}{e}_1$;

(iii) if $a_1 = 1$, $a_2 \neq 0$, and a_2 is even, or if $a_1 = m - 1$ and a_2 is odd, $\underset{\sim}{a}' = \underset{\sim}{a} - \underset{\sim}{e}_2$;

(iv) if $a_1 = 0$ and $a_2 < m - 1$, $\underset{\sim}{a}' = \underset{\sim}{a} + \underset{\sim}{e}_2$.

In order to apply Algorithms A and B, we need to show that (1) the algorithms do indeed generate a uniquely-defined tour of $\underline{\underline{A}}$, (2) the algorithms are fast, and (3) the tour produced is minimal, or nearly so. These assertions are the burden of Theorems 1, 2, and 3, respectively.

4. THE ALGORITHMS YIELD A TOUR

Theorem 1. Algorithms A and B define a tour of the set $\underline{\underline{A}}$. The length of this tour is less than $\Sigma_{j=1}^{M} \ell(\underline{\underline{AC}}_j) + \lambda M^q \sqrt{(k+3)}$.

Proof. (i) It is clear from (2.1) and (3.1) that

$$\underline{\underline{C}} = \bigcup_{j=1}^{M} \underline{\underline{C}}_j \quad \text{and all } \underline{\underline{C}}_j \text{ are disjoint.} \tag{4.1}$$

Since $\underline{\underline{A}} \subseteq \underline{\underline{C}}$, it follows that each point of $\underline{\underline{A}}$ occurs in one and only one of the $\underline{\underline{C}}_j$, and so is mentioned in exactly one of the strings $\mathcal{S}_j$ generated by Algorithm C, in step [A3]. Therefore, if Algorithm B does indeed yield a cyclic ordering of all M cells $\underline{\underline{C}}_j$, as is asserted in step [A4], and if the corresponding strings $\mathcal{S}_j$ are combined as in step [A5] and (2.8) into a final string $\mathcal{S}$; then this string will mention each point of $\underline{\underline{A}}$ exactly once, and so will define a tour of $\underline{\underline{A}}$.

(ii) In Algorithm B, either step [B1] or step [B2] will be executed, in finding the successor of any vector in the lattice $\underline{\underline{L}}^k$, and the choice is always well-defined. If step [B2] is executed, then it is easily verified that every ~~possible combination of a_1 and a_2 in $\underline{\underline{L}}^2$ occurs in exactly one of the cases (i) - (iv)~~ of [B2]. It is also clear that, in every case,

$$\text{if } \underset{\sim}{a} \in \underline{\underline{L}}, \text{ then } \underset{\sim}{a}' \in \underline{\underline{L}} \text{ and } \underset{\sim}{a}' = \underset{\sim}{a} \pm \underset{\sim}{e}_i \text{ for some } i; \tag{4.2}$$

and the corresponding cells $\underline{\underline{C}}(\underset{\sim}{a})$ and $\underline{\underline{C}}(\underset{\sim}{a}')$ meet in a face [the face defined by $x_t = b_t + \frac{\lambda}{2m}(a_i + a'_i + 1)$]: that is, they are adjacent. Thus, any point of $\underline{\underline{C}}(\underset{\sim}{a})$ may be joined to any point of $\underline{\underline{C}}(\underset{\sim}{a}')$ by a step of length less than $(\lambda/m)\sqrt{(k+3)}$ [since two adjacent cubes form a rectangular *brick* with $(k-1)$ sides of length λ/m and one of length $2\lambda/m$, whose *diameter* is $(\lambda/m)\,[(k-1).1^2 + 1.2^2]^{1/2}$, and by (2.1).]

We have demonstrated that every cell has a well-defined successor cell to which it is adjacent; and it remains to be shown that this relationship defines a single cyclic ordering of the lattice $\underline{\underline{L}}^k$. We proceed inductively.

(iii) First, let $k = 2$. Then (3.3) is impossible, and [B2] is always executed. The rules of succession embodied in cases (i) - (iv) of [B2] generate a tour: this can be described as follows. Begin at (0, 0); by case (iv), move in $+\underset{\sim}{e}_2$ direction until $(0, m - 1)$ is reached; by case (ii), move in $+\underset{\sim}{e}_1$ direction until $(m - 1, m - 1)$ is reached; thereafter, if a_2 is even, we move in the direction of $-\underset{\sim}{e}_1$ from $(m - 1, a_2)$ to $(1, a_2)$ (or to (0, 0), when $a_2 = 0$) (this is case (i)), and if a_2 is odd, we move in the direction of $+\underset{\sim}{e}_1$ from $(1, a_2)$ to $(m - 1, a_2)$ (this is case (ii)); whenever the end of a segment parallel to the first axis is reached, the tour descends to the next one, by moving in the $-\underset{\sim}{e}_2$ direction from $(1, a_2)$ to $(1, a_2 - 1)$ or from $(m - 1, a_2)$ to $(m - 1, a_2 - 1)$. Because m is *even*, what we have described is indeed a *tour* of $\underline{\underline{L}}^2$. [If m were to be *odd*, the point $(1, m - 1)$ would be the successor of both $(0, m - 1)$ and $(2, m - 1)$, while $(m - 1, m - 1)$ would have no predecessor, and the algorithm would not yield a tour.] Figures 1 and 2 illustrate these concepts for the cases of $m = 8$ and 5, respectively.

(Figures are at end of report)

Figure 1. Tour of $\underline{\underline{L}}^2$ by [B2] for the case $m = 8$ (EVEN).

Figure 2. Path generated by [B2] for the (forbidden) case when $m = 5$ (ODD).

Now, consider the application of Algorithm B to $\underline{\underline{L}}^k$, and suppose that the algorithm has already been shown to generate a tour $\bar{\mathcal{J}}$ of $\underline{\underline{L}}^{k-1}$. Denote the vector, whose first $(k - 1)$ coordinates are the same as those of $\underset{\sim}{a}$, by $\bar{\underset{\sim}{a}} = (a_1, a_2, \ldots, a_{k-1})$. Then we see that, if $a_k + r_k \in \underline{\underline{L}}$, by (3.3), the successor of $\underset{\sim}{a}$ is $\underset{\sim}{a} + r_k \underset{\sim}{e}_k$; that is, the path generated by Algorithm B moves parallel to the k-th axis, in the $r_k \underset{\sim}{e}_k$

direction. Indeed, since r_k depends only on the coordinates of $\bar{\mathbf{a}}$ (which do not change when the path moves parallel to $\mathbf{e}_k$), we deduce that, when $r_k = +1$, the path crosses the cube $\mathbb{L}^k$ from $(\bar{\mathbf{a}}, 0)$ to $(\bar{\mathbf{a}}, m - 1)$, and when $r_k = -1$, the path crosses $\mathbb{L}^k$ from $(\bar{\mathbf{a}}, m - 1)$ to $(\bar{\mathbf{a}}, 0)$. On reaching the end of such a segment parallel to the k-th axis, we find that $a_k + r_k \notin \mathbb{L}$, so that (3.3) cannot hold for $t = k$. On perusal of [B1] for $t < k$ and of [B2], we see that the rules of succession in $\mathbb{L}^k$ are identical with those in the tour $\bar{\mathcal{S}}$ of $\mathbb{L}^{k-1}$. Observing further that, if $\mathbf{a}' - \mathbf{a}$ is perpendicular to $\mathbf{e}_k$, then r_k changes sign [since just one of $a_1, a_2, \ldots, a_{k-1}$ changes by ±1], we can infer that the new $a_k + r_k \in \mathbb{L}$ and the path forthwith proceeds to cross $\mathbb{L}^k$ again in the reversed direction $r_k\, \mathbf{e}_k$.

Summing-up, we see that, if a tour congruent to $\bar{\mathcal{S}}$ is drawn on each of the faces $a_k = 0$ and $a_k = m - 1$ of $\mathbb{L}^k$ perpendicular to $\mathbf{e}_k$, then the path generated by Algorithm B in $\mathbb{L}^k$ zig-zags alternately between the two tours, passing from a "zig" whose first $(k - 1)$ coordinates are given by $\bar{\mathbf{a}}$ to a "zag" whose first $(k - 1)$ coordinates are given by the successor of $\bar{\mathbf{a}}$ in the tour $\bar{\mathcal{S}}$. Since $\bar{\mathcal{S}}$ passes through every point of $\mathbb{L}^{k-1}$, the path passes through every point of $\mathbb{L}^k$; and since the number of segments parallel to $\mathbf{e}_k$ equals the number of points in $\mathbb{L}^{k-1}$, namely m^{k-1}, which is even [because m is even], it follows that the number of "zigs" equals the number of "zags", and the path defined by Algorithm B in k dimensions is a *tour* too.

The form of the inductive step is illustrated in Figure 3 for the case of $k = 3$ and $m = 6$. The two extreme tours in two dimensions, congruent to $\bar{\mathcal{S}}$, are seen as alternating double and dotted line-segments. The "zigs" and "zags" parallel to the third axis are single lines (most of the interior points of $\mathbb{L}^3$ are omitted to make the path easier to see.)

(Figure 3 is at end of report)

Figure 3. Tour of $\underline{\underline{L}}^3$ generated by Algorithm B. Follow the arrows on single and double line-segments. Illustrates the inductive process described in part (iii) of the proof of Theorem 1.

(iv) Having shown that Algorithm B does generate a tour of $\underline{\underline{L}}^k$ (in (ii) and (iii) above), and that therefore Algorithm A does generate a tour of $\underline{\underline{A}}$ (in (i)), we are left with the bound on the length $\ell_0(\underline{\underline{A}})$ of this tour. The tour generated is described by the string (2.8). Each "piece" $\mathcal{S}_j$ of $\mathcal{S}$ is shorter by $d(A^{(j)}_{n_j}, A^{(j)}_1)$ than $\ell(\underline{\underline{AC}}_j)$; because, by the definition of the tour-length and (2.5),

$$\ell(\underline{\underline{AC}}_j) = \Sigma^{n_j}_{i=2}\ d(A^{(j)}_{i-1},\ A^{(j)}_i) + d(A^{(j)}_{n_j},\ A^{(j)}_1). \tag{4.3}$$

On the other hand (see (2.9)) the "pieces" of $\mathcal{S}$ are joined by segments $A^{(j)}_0 A^{(j)}_1$, or more properly $A^{(j-1)}_{n_{j-1}} A^{(j)}_1$, joining a point of $\underline{\underline{C}}_{j-1}$ to a point of $\underline{\underline{C}}_j$ (for each of $j = 1, 2, \ldots, M$); and we have shown (in (ii) above) that any such segment cannot be longer than $(\lambda/m)\ \surd(k + 3)$. Thus, by (2.3), if $q = 1 - 1/k$,

$$\begin{aligned} \ell_0(\underline{\underline{A}}) &< \Sigma^M_{j=1}\ \ell(\underline{\underline{AC}}_j) + M\ (\lambda/m)\ \surd(k + 3) \\ &= \Sigma^M_{j=1}\ \ell(\underline{\underline{AC}}_j) + \lambda\ M^q\ \surd(k + 3). \end{aligned} \tag{4.4}$$

Note that the inequality in (4.4) is ***strict***, both because $A^{(j)}_1 \neq A^{(j)}_{n_j}$ [and d is a metric] and because the cubes $\underline{\underline{C}}_j$ are semi-open. *Q. E. D.*

(Figure 4 is at end of report)

Figure 4. Example of a tour of 53 points in $\underline{\underline{R}}^2$ generated by Algorithm A with $\underline{\underline{E}}$ as shown and $m = 6$, $M = 36$.

5. THE ALGORITHMS ARE FAST

Theorem 2. *In probability*, the time taken to execute Algorithms A and B will be asymptotic to $A\, n \sqrt{\sigma(n)} \log \sqrt{\sigma(n)}$ as $n \to \infty$; that is, the time will be $o[n\, \sigma(n)]$.

Proof. The execution time of our algorithm may be divided into several parts: T_1 is the time required to determine $\underline{\underline{E}}$, $\underline{\underline{C}}$, λ, m, and M; T_2 is the time required to determine which points of $\underline{\underline{A}}$ are in each of the cells $\underline{\underline{C}}_j$ $(j = 1, 2, \ldots, M)$; T_3 is the time required to determine the succession of cells (by Algorithm B); T_4 is the time required to obtain the cyclic order of the points in each individual cell (by Algorithm C); and T_5 is the time required to compute the tour-length $\ell_0(\underline{\underline{A}})$. We must prove that each of these five times is of the order of $n \sqrt{\sigma(n)} \log \sqrt{\sigma(n)}$ or less.

(i) We have already mentioned that $\underline{\underline{E}}$, $\underline{\underline{C}}$, and λ will either be known *a priori*, or will be determined in time of the order of n. Now, λ and $v(\underline{\underline{E}})$ will be obtained in time independent of n, and generally, we would say that $\sigma(n)$, and hence m and M, will also be computed (by (2.2) and (2.3)) in constant time. However, if n is really large, it will run to multiple precision, and $\sigma(n)$ may take a time $O(\log n)$ to compute. Nevertheless, we see that, at worst,

$$T_1 = O(n) = o[n \sqrt{\sigma(n)} \log \sqrt{\sigma(n)}]. \tag{5.1}$$

It is clear, also, that, given the tour $(\underline{\underline{A}}, \omega)$ generated by our algorithm, its length $\ell_0(\underline{\underline{A}})$ can be computed in time of the order of n (see (2.9), with (2.6).) Thus,

$$T_5 = O(n) = o[n \sqrt{\sigma(n)} \log \sqrt{\sigma(n)}]. \tag{5.2}$$

(ii) Let us suppose that the coordinates of the n points of $\underline{\underline{A}}$ are each directly-addressable in an array $\mathcal{K}$, occupying some kn locations. Define lists $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_M$, corresponding to the M cells: for instance, the M vectors $\underset{\sim}{a} \in \underline{\underline{L}}^k$ may be lexically ordered to identify the corresponding cells $\underline{\underline{C}}(\underset{\sim}{a})$ and lists $\mathcal{L}(\underset{\sim}{a})$.

In a time of order n, one may make a single pass through $\mathcal{K}$, determining for each point the cell $\underline{\underline{C}}(\underset{\sim}{a})$ in which it lies and entering its address in the corresponding list $\mathcal{L}(\underset{\sim}{a})$. For each $\underset{\sim}{a}$, the list $\mathcal{L}(\underset{\sim}{a})$ of points in $\underline{\underline{AC}}(\underset{\sim}{a})$ will have a length $2\ n(\underset{\sim}{a})$ [where $n(\underset{\sim}{a})$ is the number of points in $\underline{\underline{AC}}(\underset{\sim}{a})$]: each entry in the list will consist of an address in $\mathcal{K}$ and a pointer to the next entry in the list. By (2.6), this will add-up to some $2n$ locations in all. Thus, with moderate storage capacity, we get

$$T_2 = O(n) = o[n\ \sqrt{\sigma(n)}\ \log\ \sqrt{\sigma(n)}]. \qquad (5.3)$$

The procedure is thus to begin with one cell, say $\underline{\underline{C}}(\underset{\sim}{0})$, compute a minimal tour of the points of $\underline{\underline{AC}}(\underset{\sim}{0})$, using the list $\mathcal{L}(\underset{\sim}{0})$ and Algorithm C, and begin a new list $\mathcal{M}$, giving the ordering of the tour $(\underline{\underline{A}}, \omega)$ as a string of addresses in $\mathcal{K}$, by entering the string $\mathcal{M}(\underset{\sim}{0})$ of addresses generated by Algorithm C. We now use Algorithm B to determine the successor cell $\underline{\underline{C}}(\underset{\sim}{0}')$ to $\underline{\underline{C}}(\underset{\sim}{0})$, and use $\mathcal{L}(\underset{\sim}{0}')$ and Algorithm C to generate the next piece $\mathcal{M}(\underset{\sim}{0}')$ of $\mathcal{M}$. We repeat, from cell to cell, until all pieces $\mathcal{M}(\underset{\sim}{0}^{(j)})$ have been constructed and entered in $\mathcal{M}$. The total time needed to compute the cell-succession is then T_3, while the time needed to determine all the individual cell-tours is T_4.

It is clear that Algorithm B is independent of n (except through (2.2) and (2.3)), and that its execution for each cell does not depend on the number of cells. Thus,

$$T_3 = O(M) = O[n/\log\ \sigma(n)] = o[n\ \sqrt{\sigma(n)}\ \log\ \sqrt{\sigma(n)}]. \qquad (5.4)$$

(iii) All that now remains to be estimated is the time T_4, and this will be shown to constitute the major part of the total time, in probability. We know that, if n_j points of $\underline{\underline{A}}$ lie in $\underline{\underline{C}}_j$, then, by (1.1), the time needed by Algorithm C to construct a minimal tour of $\underline{\underline{AC}}_j$ will be

$$\begin{aligned} t(\underline{\underline{AC}}_j) = t_{n_j} &= 2\ A\ (n_j - 1)\ [2^{n_j - 3}\ (n_j - 2) + 1] \quad \text{if} \quad n_j > 0\ , \\ &= 0 \quad \text{if} \quad n_j = 0; \end{aligned} \qquad (5.5)$$

and

$$T_4 = \Sigma_{j=1}^{M}\ t(\underline{\underline{AC}}_j). \qquad (5.6)$$

At this stage, we introduce the *probabilistic structure* of our problem. Since the points of $\mathbf{A}$ are supposed to be independently uniformly distributed at random in the set $\mathbf{E}$, it follows that the probability that exactly s points of $\mathbf{A}$ fall into the cell $\mathbf{C}_j$ will be

$$\binom{n}{s} \alpha_j^{\,s} (1 - \alpha_j)^{n-s}, \tag{5.7}$$

where

$$\alpha_j = v(\mathbf{E}\mathbf{C}_j)/v(\mathbf{E}) \leqslant v(\mathbf{C}_j)/v(\mathbf{E}) = \alpha_0 = \lambda^k/M\, v(\mathbf{E}), \tag{5.8}$$

with equality if and only if $v(\mathbf{E}^c\mathbf{C}_j) = 0$. Similarly, the probability that exactly r points of $\mathbf{A}$ will fall into $\mathbf{C}_i$ and exactly s points into $\mathbf{C}_j$, with $i \neq j$, will be

$$\binom{n}{r+s} \binom{r+s}{s} \alpha_i^{\,r} \alpha_j^{\,s} (1 - \alpha_i - \alpha_j)^{n-r-s}. \tag{5.9}$$

Now partition the index set $\{1, 2, \ldots, M\}$ of the cells into

$$\left.\begin{aligned} \mathbf{H}_0 &= \{j\colon \mathbf{C}_j \subseteq \mathbf{E}^c\}, \\ \mathbf{H}_1 &= \{j\colon \mathbf{C}_j \subseteq \mathbf{E}\}, \\ \mathbf{H}_2 &= \{j\colon \mathbf{C}_j\mathbf{E} \neq \emptyset \;\&\; \mathbf{C}_j\mathbf{E}^c \neq \emptyset\}. \end{aligned}\right\} \tag{5.10}$$

Denote the *cardinality* of any set $\mathbf{F}$ by $N(\mathbf{F})$; and let

$$N(\mathbf{H}_0) = N_0, \quad N(\mathbf{H}_1) = N_1, \quad N(\mathbf{H}_2) = N_2. \tag{5.11}$$

Then property (c) postulated for the set $\mathbf{E}$ tells us that $N_2 = O(M^q)$ as $n \to \infty$; and since, by (2.3), $M = O[n/\delta(n)]$ as $n \to \infty$, where we write

$$\left.\begin{aligned} &\delta(n) = \log \sqrt{\sigma(n)} \quad \text{or} \quad \sigma(n) = e^{2\delta(n)}, \\ &\text{so that, by (1.4),} \quad e^{2\delta(n)}/n \to 0 \quad \text{and} \quad \delta(n)/n \to 0 \quad \text{as} \quad n \to \infty; \end{aligned}\right\} \tag{5.12}$$

then

$$N_2 = O\{[n/\delta(n)]^q\} \quad \text{and} \quad N_2/M \to 0 \quad \text{as} \quad n \to \infty. \tag{5.13}$$

We also observe that, by (2.2) and (2.3),

$$\left.\begin{aligned} &m = \lambda\Big[\frac{n}{v(\mathbf{E})\,\delta(n)}\Big]^p + O(1) = \lambda\Big[\frac{n}{v(\mathbf{E})\,\delta(n)}\Big]^p \Big\{1 + O\Big[\frac{\delta(n)}{n}\Big]^p\Big\} \\ \text{and} \quad &M = m^k = \frac{\lambda^k}{v(\mathbf{E})}\Big[\frac{n}{\delta(n)}\Big] \Big\{1 + O\Big[\frac{\delta(n)}{n}\Big]^p\Big\} \quad \text{as} \quad n \to \infty. \end{aligned}\right\} \tag{5.14}$$

Further, it is clear that

$$\cup_{j\in \underline{\underline{H}}_1} \underline{\underline{C}}_j \subseteq \underline{\underline{E}} \subseteq \cup_{j\in \underline{\underline{H}}_1 \cup \underline{\underline{H}}_2} \underline{\underline{C}}_j, \tag{5.15}$$

whence

$$N_1 \lambda^k/M \leqslant v(\underline{\underline{E}}) \leqslant (N_1 + N_2) \lambda^k/M. \tag{5.16}$$

It now follows from (5.13) that

$$N_1 = \frac{n}{\delta(n)} \{1 + O[\frac{\delta(n)}{n}]^p\} \quad \text{as} \quad n \to \infty. \tag{5.17}$$

(iv) We now seek to obtain asymptotic forms for the *expected value* $\underset{\sim}{E}[T_4]$ and *variance* $\text{var}[T_4]$ of the time T_4. By (5.6),

$$\underset{\sim}{E}[T_4] = \Sigma_{j=1}^M \underset{\sim}{E}[t(\underline{\underline{AC}}_j)] \tag{5.18}$$

and

$$\begin{aligned}\text{var}[T_4] &= \underset{\sim}{E}[(\Sigma_{j=1}^M \{t(\underline{\underline{AC}}_j) - \underset{\sim}{E}[t(\underline{\underline{AC}}_j)]\})^2] \\ &= \Sigma_{i=1}^M \Sigma_{j=1}^M \underset{\sim}{E}[\{t(\underline{\underline{AC}}_i) - \underset{\sim}{E}[t(\underline{\underline{AC}}_i)]\} \{t(\underline{\underline{AC}}_j) - \underset{\sim}{E}[t(\underline{\underline{AC}}_j)]\}].\end{aligned} \tag{5.19}$$

Thus $\underset{\sim}{E}[T_4]$ consists of terms $\underset{\sim}{E}[t(\underline{\underline{AC}}_j)]$, and $\text{var}[T_4]$ consists of products of such terms, together with $\underset{\sim}{E}[\{t(\underline{\underline{AC}}_j)\}^2]$ and $\underset{\sim}{E}[t(\underline{\underline{AC}}_i)\ t(\underline{\underline{AC}}_j)]$ with $i \neq j$. If we adopt the usual notation, for integers n and positive integers ϕ, that

$$(n)_0 = 1, \quad (n)_\phi = n\ (n-1)\ (n-2)\ \dots\ (n-\phi+1) \quad [= 0 \text{ for } \phi > n \geqslant 0], \tag{5.20}$$

we see that, by (1.1),

$$\left.\begin{aligned} t_s &= A\ [2^{s-2}\ (s)_2 - 2^{s-1}\ (s)_1 + 2\ (s)_1 + \tfrac{1}{2} 2^s\ (s)_0 - 2\ (s)_0] \\ \text{and}\quad t_s{}^2 &= A^2\ [16\times 4^{s-4}\ (s)_4 + 8\times 2^{s-3}\ (s)_3 + 2\times 4^{s-2}\ (s)_2 - 4\times 2^{s-2}\ (s)_2 + 4\ (s)_2 \\ &\quad - 4^{s-1}\ (s)_1 + 4\times 2^{s-1}\ (s)_1 - 4\ (s)_1 + \tfrac{1}{4} 4^s\ (s)_0 - 2\times 2^s\ (s)_0 + 4\ (s)_0]; \end{aligned}\right\} \tag{5.21}$$

so that we may write

$$t_s = A \sum_{\psi=0}^2 \sum_{\theta=1}^2 P_{\psi\theta}\ \theta^{s-\psi}\ (s)_\psi \quad \text{and} \quad t_s{}^2 = A^2 \sum_{\psi=0}^4 \sum_{\theta=1}^4 Q_{\psi\theta}\ \theta^{s-\psi}\ (s)_\psi, \tag{5.22}$$

where $P_{22} = 1$, $P_{12} = -1$, $P_{11} = 2$, $P_{02} = \frac{1}{2}$, $P_{01} = -2$, $Q_{44} = 16$, $Q_{32} = 8$, $Q_{24} = 2$, $Q_{22} = -4$, $Q_{21} = 4$, $Q_{14} = -1$, $Q_{12} = 4$, $Q_{11} = -4$, $Q_{04} = \frac{1}{4}$, $Q_{02} = -2$, $Q_{01} = 4$, and all other coefficients vanish. It follows from (5.7), (5.8), and (5.9) that

$$\left.\begin{aligned} \underset{\sim}{E}[t(\underline{\underline{AC}}_j)] &= A \sum_{\psi=0}^2 \sum_{\theta=1}^2 P_{\psi\theta}\ J(n,\ \theta,\ \psi,\ \alpha_j), \\ \underset{\sim}{E}[\{t(\underline{\underline{AC}}_j)\}^2] &= A^2 \sum_{\psi=0}^4 \sum_{\theta=1}^4 Q_{\psi\theta}\ J(n,\ \theta,\ \psi,\ \alpha_j), \\ \underset{\sim}{E}[t(\underline{\underline{AC}}_i)\ t(\underline{\underline{AC}}_j)] &= A^2 \sum_{\phi=0}^2 \sum_{\psi=0}^2 \sum_{\theta_1=1}^2 \sum_{\theta_2=1}^2 P_{\phi\theta_1} P_{\psi\theta_2} K(n;\ \theta_1,\ \theta_2;\ \phi,\ \psi;\ \alpha_i,\ \alpha_j); \end{aligned}\right\} \tag{5.23}$$

where we write

$$J(n, \theta, \psi, x) = \Sigma_{s=1}^{n} \binom{n}{s} x^s (1-x)^{n-s} \theta^{s-\psi} (s)_\psi \tag{5.24}$$

and $$K(n; \theta_1, \theta_2; \phi, \psi; x, y) = \Sigma_{r=1}^{n} \Sigma_{s=1}^{n-r} \binom{n}{r+s} \binom{r+s}{s} x^r y^s (1-x-y)^{n-r-s} \times \theta_1^{r-\phi} (r)_\phi \theta_2^{s-\psi} (s)_\psi. \tag{5.25}$$

[The sums are over indices from 1 to n because the time for index 0 is zero, not t_0 (compare (5.5).] We may evaluate these sums as follows:

$$J(n, \theta, \psi, x) = (n)_\psi x^\psi \Sigma_{s=1}^{n} \binom{n-\psi}{s-\psi} (x\,\theta)^{s-\psi} (1-x)^{n-s}$$
$$= (n)_\psi x^\psi \{1 + x(\theta - 1)\}^{n-\psi} - \delta_{\psi 0} (1-x)^n, \tag{5.26}$$

where δ_{ij} is the *Kronecker delta* (= 1 if $i = j$; = 0 if $i \neq j$); and similarly,

$$K(n; \theta_1, \theta_2; \phi, \psi; x, y) = (n)_{\phi+\psi} x^\phi y^\psi \Sigma_{u=2}^{n} \binom{n-\phi-\psi}{u-\phi-\psi} (1-x-y)^{n-u} \times \Sigma_{s=1}^{u-1} \binom{u-\phi-\psi}{s-\psi} (x\,\theta_1)^{u-s-\phi} (y\,\theta_2)^{s-\psi}$$
$$= (n)_{\phi+\psi} x^\phi y^\psi \Sigma_{u=2}^{n} \binom{n-\phi-\psi}{u-\phi-\psi} (1-x-y)^{n-u} \{(x\,\theta_1 + y\,\theta_2)^{u-\phi-\psi} - \delta_{\psi 0} (x\,\theta_1)^{u-\phi} - \delta_{\phi 0} (y\,\theta_2)^{u-\psi}\}$$
$$= (n)_{\phi+\psi} x^\phi y^\psi \{[1 + x(\theta_1 - 1) + y(\theta_2 - 1)]^{n-\phi-\psi} - \delta_{\psi 0} [1 + x(\theta_1 - 1) - y]^{n-\phi} - \delta_{\phi 0} [1 - x + y(\theta_2 - 1)]^{n-\psi} + \delta_{\phi 0} \delta_{\psi 0} (1-x-y)^n\}. \tag{5.27}$$

[We note that $\binom{a}{b} = 0$ whenever $b < 0$ or $b > a$.]

Now, by a simple inductive argument on m, we observe that, for all non-negative integers m, n, and ζ with $n \geqslant \zeta$,

$$0 \leqslant n^m - (n-\zeta)_m \leqslant m\, n^{m-1} [\zeta + \tfrac{1}{2}(m-1)]. \tag{5.28}$$

Thus, since $$e^{nz} - (1+z)^{n-\zeta} = \Sigma_{m=1}^{\infty} \frac{1}{m!} [n^m - (n-\zeta)_m]\, z^m, \tag{5.29}$$

we have, for all $z \geqslant 0$, that

$$0 \leqslant e^{nz} - (1+z)^{n-\zeta} \leqslant \Sigma_{m=1}^{\infty} \{\frac{\zeta}{(m-1)!} + \frac{1/2}{(m-2)!}\} n^{m-1} z^m = (\zeta\, z + \tfrac{1}{2} n\, z^2)\, e^{nz};$$

whence

$$e^{nz} (1 - \zeta\, z - \tfrac{1}{2} n\, z^2) \leqslant (1+z)^{n-\zeta} \leqslant e^{nz}. \tag{5.30}$$

Thus, for all those J-terms and K-terms in the sums (5.23) for which $\phi \geq 1$ and $\psi \geq 1$,

$$(n)_\psi \, \alpha_j^{\psi} \, e^{n\alpha_j(\theta-1)} \{1 - \psi\alpha_j(\theta - 1) - \tfrac{1}{2} n \alpha_j^2 (\theta - 1)^2\}$$
$$\leq J(n, \theta, \psi, \alpha_j) \leq (n)_\psi \, \alpha_j^{\psi} \, e^{n\alpha_j(\theta-1)} \quad (5.31)$$

and
$$(n)_{\phi+\psi} \, \alpha_i^{\phi} \, \alpha_j^{\psi} \, e^{n\alpha_i(\theta_1-1)+n\alpha_j(\theta_2-1)} \{1 - (\phi + \psi) [\alpha_i (\theta_1 - 1) + \alpha_j (\theta_2 - 1)]$$
$$- \tfrac{1}{2} n [\alpha_i (\theta_1 - 1) + \alpha_j (\theta_2 - 1)]^2\}$$
$$\leq K(n; \theta_1, \theta_2; \phi, \psi; \alpha_i, \alpha_j) \leq (n)_{\phi+\psi} \, \alpha_i^{\phi} \, \alpha_j^{\psi} \, e^{n\alpha_i(\theta_1-1)+n\alpha_j(\theta_2-1)}; \quad (5.32)$$

and since
$$\alpha_0 = \frac{\delta(n)}{n} \{1 + O[\frac{\delta(n)}{n}]^p\} \quad \text{as} \quad n \to \infty, \quad \text{and} \quad \alpha_j \leq \alpha_0; \quad (5.33)$$

$(n)_\psi = n^\psi \, [1 + O(1/n)]$, $\alpha_j = O[\delta(n)/n]$, and $n \, \alpha_j^2 = O\{[\delta(n)]^2/n\}$; whence

$$J(n, \theta, \psi, \alpha_j) = (n\alpha_j)^\psi \, e^{n\alpha_j(\theta-1)} \, (1 + O\{[\delta(n)]^2/n\}) \quad (5.34)$$

and similarly,

$$K(n; \theta_1, \theta_2; \phi, \psi; \alpha_i, \alpha_j) = (n\alpha_i)^\phi \, (n\alpha_j)^\psi \, e^{n\alpha_i(\theta_1-1)+n\alpha_j(\theta_2-1)} \, (1 + O\{[\delta(n)]^2/n\}) \quad (5.35)$$

We note, further, that the correction terms for $\phi = 0$ and $\psi = 0$ in (5.26) and (5.27) are always of lower asymptotic order than the main terms, found in (5.34) and (5.35).

In calculating $\underset{\approx}{E}[T_4]$, we may distribute the sum over the cells $\underset{=}{C}_j$ among the several J-terms of the corresponding expression of (5.23). For $j \in \underset{=}{H}_0$, there is no contribution; for $j \in \underset{=}{H}_1$, each term equals $J(n, \theta, \psi, \alpha_0)$ and there are N_1 such terms; and for $j \in \underset{=}{H}_2$, when n is sufficiently large, we see by (5.34) that the contributions are somewhat smaller, since the J-terms are monotonically increasing with α_j, and $\alpha_j \leq \alpha_0$, by (5.33). Thus, by (5.13) and (5.17),

$$\sum_{j=1}^{M} J(n, \theta, \psi, \alpha_j) = N_1 \, J(n, \theta, \psi, \alpha_0) + \sum_{j \in \underset{=}{H}_2} J(n, \theta, \psi, \alpha_j)$$
$$= n \, [\delta(n)]^{\psi-1} \, e^{(\theta-1)\delta(n)} \, (1 + O\{[\delta(n)]^{p+1}/n^p\}), \quad (5.36)$$

since $e^{(\theta-1)n\alpha_0} = e^{(\theta-1)\delta(n)} \, e^{O\{[\delta(n)]^{p+1}/n^p\}} = e^{(\theta-1)\delta(n)} \, (1 + O\{[\delta(n)]^{p+1}/n^p\})$ and $[\delta(n)]^2/n = o\{[\delta(n)]^{p+1}/n^p\}$, and since $q = 1 - p$ and $[\delta(n)/n]^p = o\{[\delta(n)]^{p+1}/n^p\}$.

It follows at once from (5.18), (5.23), and (5.36) that

$$\begin{aligned}\mathop{\mathbb{E}}[T_4] &= A\ P_{22} \sum_{j=1}^{M} J(n,\ 2,\ 2,\ \alpha_j)\ \{1 + O[1/\delta(n)]\} \\ &= A\ n\ \delta(n)\ e^{\delta(n)}\ \{1 + O[1/\delta(n)]\},\end{aligned} \tag{5.37}$$

since $[\delta(n)]^{p+1}/n^p = o[1/\delta(n)]$ [because $[\delta(n)]^{p+2}/n^p \to 0$ as $n \to \infty$, by (1.4) and (5.12)] and similarly, by (5.19) and (5.23),

$$\begin{aligned}\mathrm{var}[T_4] &= \sum_{j=1}^{M} \left(\mathop{\mathbb{E}}[\{t(\underline{\underline{AC}}_j)\}^2] - \{\mathop{\mathbb{E}}[t(\underline{\underline{AC}}_j)]\}^2\right) + 2 \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} \left(\mathop{\mathbb{E}}[t(\underline{\underline{AC}}_i)\ t(\underline{\underline{AC}}_j)]\right. \\ &\quad \left. - \mathop{\mathbb{E}}[t(\underline{\underline{AC}}_i)]\ \mathop{\mathbb{E}}[t(\underline{\underline{AC}}_j)]\right) = \sum_{j=1}^{M} A^2\ \left(\sum_{\psi=0}^{4} \sum_{\theta=1}^{4} Q_{\psi\theta}\ J(n,\ \theta,\ \psi,\ \alpha_j)\right. \\ &\quad \left. - \{\sum_{\psi=0}^{2} \sum_{\theta=1}^{2} P_{\psi\theta}\ J(n,\ \theta,\ \psi,\ \alpha_j)\}^2\right) + 2 \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} A^2 \\ &\quad \times \sum_{\phi=0}^{2} \sum_{\psi=0}^{2} \sum_{\theta_1=1}^{2} \sum_{\theta_2=1}^{2} P_{\phi\theta_1}\ P_{\psi\theta_2}\ \left(K(n;\ \theta_1,\ \theta_2;\ \phi,\ \psi;\ \alpha_i,\ \alpha_j)\right. \\ &\quad \left. - J(n,\ \theta_1,\ \phi,\ \alpha_i)\ J(n,\ \theta_2,\ \psi,\ \alpha_j)\right).\end{aligned} \tag{5.38}$$

By breaking up this expression into a single sum Σ_j and a double sum $\Sigma_i\ \Sigma_{j>i}$, and calculating the asymptotic form of each term, we can obtain the form of $\mathrm{var}[T_4]$. By (5.34), we see that the sum representing $\mathop{\mathbb{E}}[\{t(\underline{\underline{AC}}_j)\}^2]$ is dominated by the term with coefficient Q_{44}: $16\ (n\alpha_j)^4\ e^{3n\alpha_j}\ (1 + O\{[\delta(n)]^2/n\})$, the term of next-highest order being in $(n\alpha_j)^2\ e^{3n\alpha_j}$ (from Q_{24}.) The sum representing $\mathop{\mathbb{E}}[t(\underline{\underline{AC}}_j)]\}^2$ is dominated by the term with coefficient $P_{22}{}^2$: $(n\alpha_j)^4\ e^{2n\alpha_j}\ (1 + O\{[\delta(n)]^2/n\})$, which is therefore asymptotically negligible. In the double sum, we observe that, by (5.26) and (5.27),

$$\begin{aligned}&K(n;\ \theta_1,\ \theta_2;\ \phi,\ \psi;\ \alpha_i,\ \alpha_j) - J(n,\ \theta_1,\ \phi,\ \alpha_i)\ J(n,\ \theta_2,\ \psi,\ \alpha_j) \\ &\quad = (n)_{\phi+\psi}\ \alpha_i{}^{\phi}\ \alpha_j{}^{\psi}\ \left([1 + \alpha_i(\theta_1-1) + \alpha_j(\theta_2-1)]^{n-\phi-\psi} - \delta_{\psi 0}\ [1 + \alpha_i(\theta_1-1) - \alpha_j]^{n-\phi}\right. \\ &\qquad \left. - \delta_{\phi 0}\ [1 - \alpha_i + \alpha_j(\theta_2-1)]^{n-\psi} + \delta_{\phi 0}\ \delta_{\psi 0}\ [1 - \alpha_i - \alpha_j]^{n}\right) \\ &\quad - (n)_{\phi}\ (n)_{\psi}\ \alpha_i{}^{\phi}\ \alpha_j{}^{\psi}\ \left([1 + \alpha_i(\theta_1-1)]^{n-\phi} - \delta_{\phi 0}\ [1 - \alpha_i]^{n}\right) \\ &\qquad\qquad \times \left([1 + \alpha_j(\theta_2-1)]^{n-\psi} - \delta_{\psi 0}\ [1 - \alpha_j]^{n}\right).\end{aligned} \tag{5.39}$$

We now note again that $(n)_{\phi+\psi} = n^{\phi+\psi}\ [1 + O(1/n)]$, $(n)_{\phi} = n^{\phi}\ [1 + O(1/n)]$, and $(n)_{\psi} = n^{\psi}\ [1 + O(1/n)]$; and $\{1 + O[\delta(n)/n]^p\}^{\zeta} = 1 + O[\delta(n)/n]^p$, for any ζ; while

$$[(1 + x + y)/(1 + x)(1 + y)]^n = [1 - xy + xy(x + y) - \ldots]^n$$
$$= 1 - nxy + O\{[\delta(n)]^3/n^2\}, \quad \text{if } x = O[\delta(n)/n] \text{ and } y = O[\delta(n)/n]. \tag{5.40}$$

Thus $K(n;\ \theta_1,\ \theta_2;\ \phi,\ \psi;\ \alpha_i,\ \alpha_j) - J(n,\ \theta_1,\ \phi,\ \alpha_i)\ J(n,\ \theta_2,\ \psi,\ \alpha_j)$
$$= n^{-1}\ (n\alpha_i)^{\phi+1}\ (n\alpha_j)^{\psi+1}\ \left((\theta_1-1)(\theta_2-1)e^{n\alpha_i(\theta_1-1)+n\alpha_j(\theta_2-1)}\ (1 + O\{[\delta(n)]^2/n\})\right), \tag{5.41}$$

so long as $\phi \geqslant 1$ and $\psi \geqslant 1$; and the dominant term in the sum representing $\underset{\sim}{E}[t(\underline{\underline{AC}}_i)\ t(\underline{\underline{AC}}_j)] - \underset{\sim}{E}[t(\underline{\underline{AC}}_i)]\ \underset{\sim}{E}[t(\underline{\underline{AC}}_j)]$ is again that with coefficient $P_{22}{}^2$: $-n^{-1}(n\alpha_i)^3(n\alpha_j)^3\ e^{n\alpha_i+n\alpha_j} \times (1 + O\{[\delta(n)]^2/n\})$. It follows that the total contribution of the double sum $\Sigma_i\ \Sigma_{j>i}$ in any case cannot exceed M^2 times this, or, by (5.14) and (5.33), $O\{n\ [\delta(n)]^4\ e^{2\delta(n)}\}$. Now, the single sum Σ_j is dominated by $\sum_{j=1}^{M} 16\ A^2\ (n\alpha_j)^4\ e^{3n\alpha_j}$, and, arguing as in obtaining (5.36) and (5.37), we see that this is $16\ A^2\ n\ [\delta(n)]^3\ e^{3\delta(n)}\ \{1 + O[1/\delta(n)]\}$. Since this overshadows the contribution of $\Sigma_i\ \Sigma_{j>i}$, we obtain, finally, that
$$\text{var}[T_4] = 16\ A^2\ n\ [\delta(n)]^3\ e^{3\delta(n)}\ \{1 + O[1/\delta(n)]\}. \tag{5.42}$$

(v) We may now complete the proof of Theorem 2. First, we note that, for any $\varepsilon > 0$ and all sufficiently large n, by (5.37), say $n \geqslant n_0(\varepsilon)$,
$$|\underset{\sim}{E}[T_4] - A\ n\ \delta(n)\ e^{\delta(n)}| \leqslant \frac{\varepsilon}{2}\ A\ n\ \delta(n)\ e^{\delta(n)}. \tag{5.43}$$

Next, we use *Chebyshev's inequality* with (5.42) and (5.43) to obtain that
$$\text{Prob}\left(\left|\frac{T_4}{A\ n\ \delta(n)\ e^{\delta(n)}} - 1\right| \leqslant \varepsilon\right) \geqslant \text{Prob}\left(\left|\frac{T_4 - \underset{\sim}{E}[T_4]}{A\ n\ \delta(n)\ e^{\delta(n)}}\right| \leqslant \frac{\varepsilon}{2} \text{ and } \left|\frac{\underset{\sim}{E}[T_4]}{A\ n\ \delta(n)\ e^{\delta(n)}} - 1\right| \leqslant \frac{\varepsilon}{2}\right)$$
$$= \text{Prob}\left(\left|\frac{T_4 - \underset{\sim}{E}[T_4]}{A\ n\ \delta(n)\ e^{\delta(n)}}\right| \leqslant \frac{\varepsilon}{2}\right) \quad \text{for all } n \geqslant n_0(\varepsilon)$$
$$\geqslant 1 - \frac{\text{var}\ [T_4]}{[(A\varepsilon/2)\ n\ \delta(n)\ e^{\delta(n)}]^2} \quad \text{[Chebyshev]}$$
$$\sim 1 - \left(\frac{64}{\varepsilon^2}\right)\frac{\delta(n)\ e^{\delta(n)}}{n} \to 1 \quad \text{as} \quad n \to \infty. \tag{5.44}$$

Thus, $T_4/An\delta(n)e^{\delta(n)} \to 1$ *in probability* as $n \to \infty$; or

$$T_4 \sim A\ n\ \delta(n)\ e^{\delta(n)} \textit{ in probability as } n \to \infty. \tag{5.45}$$

Now, we have already shown that T_1, T_2, T_3, and T_5 are all $o[n \sqrt{\sigma(n)} \log \sqrt{\sigma(n)}]$ with *certainty* as $n \to \infty$ (see (5.1) - (5.4).) Therefore, since $\delta(n) = \log \sqrt{\sigma(n)}$ and $e^{\delta(n)} = \sqrt{\sigma(n)}$, by (5.12), it follows that the total time taken by the algorithms to compute a tour of $\underline{\underline{A}}$ will be

$$\textstyle\sum_{r=1}^{5} T_r \sim A\ n\ \sqrt{\sigma(n)} \log \sqrt{\sigma(n)}, \tag{5.46}$$

in probability, as $n \to \infty$. *Q. E. D.*

6. THE ALGORITHMS ARE ACCURATE

Theorem 3. *With probability one* [that is, *almost surely*: *a. s.*], the length $\ell_0(\underline{\underline{A}})$ of the tour of the n points of $\underline{\underline{A}}$ generated by Algorithms A and $\underline{B}$ is asymptotic to the minimal tour length $\ell(\underline{\underline{A}}) \sim \beta_k\ v(\underline{\underline{E}})^p\ n^q$.

Proof. (i) Since $\ell(\underline{\underline{A}})$ is defined to be *minimal*, and since (by Theorem 1) the algorithms define a *tour* of the set A, we have that its length,

$$\ell_0(\underline{\underline{A}}) \geqslant \ell(\underline{\underline{A}}). \tag{6.1}$$

(ii) By (5.14),

$$\lambda\ M^q\ \sqrt{(k + 3)} \sim \frac{\lambda^k}{v(\underline{\underline{E}})^q} \left(\frac{n}{\delta(n)}\right)^q \sqrt{(k + 3)} = o(n^q) \quad \text{as} \quad n \to \infty; \tag{6.2}$$

so that, by (4.4),

$$\ell_0(\underline{\underline{A}}) < \textstyle\sum_{j=1}^{M} \ell(\underline{\underline{AC}}_j) + o(n^q) \quad \text{as} \quad n \to \infty. \tag{6.3}$$

Now consider a minimal tour $(\underline{\underline{A}}, \pi)$ of $\underline{\underline{A}}$ and let $\underline{\underline{P}}$ denote the polygonal path $A_1 A_2 \ldots A_j$ [that is, let $X \in \underline{\underline{P}}$ iff $(\exists j \in \{1, 2, \ldots, n\})\ (\exists x)\ 0 \leqslant x \leqslant 1$ and $X = x\ A_{j-1} + (1 - x)\ A_j$], where the points of $\underline{\underline{A}}$ are so numbered that $A_0\, \pi\, A_1\, \pi\, A_2\, \pi \ldots \pi\, A_n = A_0$. Let $\underline{\underline{P}}_j$ be the union of the closures of all connected pieces of $\underline{\underline{PC}}_j$ containing at least one point of $\underline{A}$. Then the number of such pieces will be

$$h_j \leqslant n_j = N(\underline{\underline{AC}}_j), \tag{6.4}$$

and the sum of the lengths of these pieces, ℓ_j, will satisfy

$$\textstyle\sum_{j=1}^{M} \ell_j \leqslant \ell(\underline{A}). \tag{6.5}$$

The end-points of the pieces of $\underline{\underline{P}}_j$ all lie in the boundary of $\underline{\underline{C}}_j$, which consists of $2k$ *faces* $\underline{\underline{F}}_{jf}$ (f = 0, 1, ..., $2k$ - 1; with $\underline{\underline{F}}_{j,2k} = \underline{\underline{F}}_{j0}$): let $\underline{\underline{E}}_{jf}$ be the set of end-points in $\underline{\underline{F}}_{jf}$. We shall form a tour $(\underline{\underline{U}}_j, \upsilon_j)$ of all the end-points, consisting of a tour (V_{jf}, ν_{jf}) of $\underline{\underline{E}}_{jf}$, for each f, each connected to the next tour $\underline{\underline{V}}_{j(f+1)}$ by a chord, whose length cannot exceed the diameter of $\underline{\underline{C}}_j$,

$$\Delta(\underline{\underline{C}}_j) = \lambda\, M^{-p} \sqrt{k}. \tag{6.6}$$

It is proved in BHH, by a non-trivial combinatorial argument, that a tour $(\underline{\underline{T}}_j, \tau_j)$ of $\underline{\underline{AC}}_j$ may be constructed by alternately traversing parts of $\underline{\underline{U}}_j$ and pieces of $\underline{\underline{P}}_j$, in such a way that $\underline{\underline{P}}_j$ is traversed only once and $\underline{\underline{U}}_j$ not more than twice. This means that

$$\ell(\underline{\underline{AC}}_j) \leqslant \ell(\underline{\underline{T}}_j, \tau_j) \leqslant \ell_j + 2\, \ell(\underline{\underline{U}}_j, \upsilon_j). \tag{6.7}$$

[The interested reader may find the abovementioned proof under Lemma 2 in the Appendix of BHH.] We note that

$$\ell(\underline{\underline{U}}_j, \upsilon_j) \leqslant \textstyle\sum_{f=1}^{2k} \ell(\underline{\underline{V}}_{jf}, \nu_{jf}) + 2\, k\, \Delta(\underline{\underline{C}}_j). \tag{6.8}$$

(iii) To construct $(\underline{\underline{V}}_{jf}, \nu_{jf})$, we proceed as follows. First, we dissect the face $\underline{\underline{F}}_{jf}$ of $\underline{\underline{C}}_j$, which is a $(k-1)$-dimensional hypercube of side $\lambda\, M^{-p}$, into L equal cells of side $\lambda\, M^{-p}\, L^{-p'}$, where $p' = 1/(k-1)$, just as in applying our algorithms, taking $L^{p'}$ to be an even integer; so that we may construct a tour of the cell-centers of length $L\, \lambda\, M^{-p}\, L^{-p'}$, using <u>Algorithm B</u>. Then, in each cell, we insert any point of $\underline{\underline{E}}_{jf}$ therein into the tour, by connecting it to-and-fro to the nearest point of the tour, thereby increasing the length of the path by no more than $\lambda\, M^{-p}\, L^{-p'} \sqrt{(k-1)}$ for each point of $\underline{\underline{E}}_{jf}$. Then

$$\ell(\underline{\underline{V}}_{jf}, \nu_{jf}) \leqslant \lambda\, M^{-p}\, [L^{1-p'} + h_{jf}\, L^{-p'} \sqrt{(k-1)}], \tag{6.9}$$

where

$$h_{jf} = N(\underline{\underline{E}}_{jf}), \quad \textstyle\sum_{f=1}^{2k} h_{jf} = 2\, h_j. \tag{6.10}$$

We now observe that

$$\frac{\partial}{\partial L}\,[L^{1-p'} + h_{jf}\, L^{-p'}\sqrt{(k-1)}]\,\begin{Bmatrix} <0 & \text{if} & L < L_0 \\ =0 & \text{if} & L = L_0 \\ >0 & \text{if} & L > L_0 \end{Bmatrix}, \tag{6.11}$$

where

$$L_0 = [\sqrt{(k-1)}/(k-2)]\, h_{jf}; \tag{6.12}$$

so that the minimum for L a multiple of 2^{k-1} occurs when

$$L = 2^{k-1}\left\lceil 2^{1-k}[\sqrt{(k-1)}/(k-2)]\, h_{jf}\right\rceil; \tag{6.13}$$

whence, by (6.9),

$$\ell(\underline{\underline{V}}_{jf},\, \nu_{jf}) \leqslant \lambda\, M^{-p}\left(\frac{\sqrt{(k-1)}}{k-2}\right)^{-p'} h_{jf}^{\;-p'}\,[\frac{(k-1)^{3/2}}{k-2}\, h_{jf} + 2^{k-1}], \tag{6.14}$$

or, more simply, $$\ell(\underline{\underline{V}}_{jf},\, \nu_{jf}) \leqslant \lambda\, M^{-p}\,(R_k\, h_{jf}^{\;q'} + S_k\, h_{jf}^{\;-p'}), \tag{6.15}$$

where $q' = 1 - p'$, and R_k and S_k are constants depending only on k.

(iv) We may now combine the foregoing results to yield the following results [the subscripts attached to $\leqslant$ and $<$ signs refer to the justifying assertion; e.g., the first $\leqslant_1$ refers to (6.1) and the first $<_3$ refers to (6.3).]

$$\begin{aligned} 0 &\leqslant_1 \ell_0(\underline{\underline{A}}) - \ell(\underline{\underline{A}}) <_3 \textstyle\sum_{j=1}^{M} \ell(\underline{\underline{AC}}_j) + o(n^q) - \ell(\underline{\underline{A}}) \leqslant_7 \sum_{j=1}^{M} \ell_j + 2\sum_{j=1}^{M} \ell(\underline{\underline{U}}_j,\, \upsilon_j) + o(n^q) \\ &- \ell(\underline{\underline{A}}) \leqslant_5 2\textstyle\sum_{j=1}^{M} \ell(\underline{\underline{U}}_j,\, \upsilon_j) + o(n^q) \leqslant_8 2\sum_{j=1}^{M}\sum_{f=1}^{2k} \ell(\underline{\underline{V}}_{jf},\, \nu_{jf}) \\ &+ 2\,k\textstyle\sum_{j=1}^{M} \Delta(\underline{\underline{C}}_j) + o(n^q) =_6 2\sum_{j=1}^{M}\sum_{f=1}^{2k} \ell(\underline{\underline{V}}_{jf},\, \nu_{jf}) + 2\,k^{3/2}\,\lambda\, M^q \\ &+ o(n^q) \leqslant_{15} 2\,\lambda\, M^{-p}\left[R_k \textstyle\sum_{j=1}^{M}\sum_{f=1}^{2k} h_{jf}^{\;q'} + S_k \sum_{j=1}^{M}\sum_{f=1}^{2k} h_{jf}^{\;-p'}\right] \\ &+ 2\,k^{3/2}\,\lambda\, M^q + o(n^q). \end{aligned} \tag{6.16}$$

We must remark that the bound (6.15) becomes infinite if $h_{jf} = 0$; but if there are no end-points in $\underline{\underline{F}}_{jf}$, then there is no need to tour that face, and $\ell(\underline{\underline{V}}_{jf},\, \nu_{jf})$ becomes zero. Thus not only $h_{jf}^{\;q'}$ but also $h_{jf}^{\;-p'}$ should be interpreted as 0 in (6.15) and (6.16), when $h_{jf} = 0$. Therefore, since h_{jf} must be a non-negative integer, we may replace $h_{jf}^{\;-p'}$ in (6.16) by 1 without decreasing the bound. Further, when $0 < q' < 1$, we may apply Hölder's inequality to the sum of $h_{jf}^{\;q'}$ to yield that, because $p' + q' = 1$,

$$\begin{aligned} \textstyle\sum_{j=1}^{M}\sum_{f=1}^{2k} h_{jf}^{\;q'} &= \textstyle\sum_{j=1}^{M}\sum_{f=1}^{2k} 1 \times h_{jf}^{\;q'} \leqslant \left[\sum_{j=1}^{M}\sum_{f=1}^{2k} 1^{1/p'}\right]^{p'} \left[\sum_{j=1}^{M}\sum_{f=1}^{2k} (h_{jf}^{\;q'})^{1/q'}\right]^{q'} \\ &= (2\,k\,M)^{p'}\left[\textstyle\sum_{j=1}^{M}\sum_{f=1}^{2k} h_{jf}\right]^{q'} \leqslant_{10,4} (2\,k\,M)^{p'}\,(2\,n)^{q'}. \end{aligned} \tag{6.17}$$

(When $k = 2$ and so $p' = 1$ and $q' = 0$, then the sum on the left of (6.17) becomes $2\ k\ M$, and the bound on the right becomes $2\ k\ M$ also; so that (6.17) still holds.) Applying these results to (6.16), we obtain that

$$0 \leqslant \ell_0(\underline{\underline{A}}) - \ell(\underline{\underline{A}}) < 2\ \lambda\ M^{-p}\ [R_k\ (k\ M)^{p'}\ n^{q'} + S_k\ (2\ k\ M)] + 2\ k^{3/2}\ \lambda\ M^q + o(n^q); \quad (6.18)$$

and since, by (5.14), $M = O[n/\delta(n)] = o(n)$, we have that $M^q = o(n^q)$ and $M^{p'-p}\ n^{q'} = o(n^{p'-p}\ n^{q'}) = o(n^q)$. Thus, finally,

$$0 \leqslant \ell_0(\underline{\underline{A}}) - \ell(\underline{\underline{A}}) < o(n^q) \quad \text{as} \quad n \to \infty. \quad (6.19)$$

(v) To complete the proof of our theorem, we observe that BHH have proved that (1.2) holds *with probability one*, when the set $\underline{\underline{A}}$ is taken to be $\underset{\sim}{P}^n$, the first n points of the infinite sequence $\underset{\sim}{P}$, distributed independently and uniformly in the set $\underline{\underline{E}}$. Since, under these circumstances,

$$\ell(\underline{\underline{A}}) = O(n^q) \quad \text{as} \quad n \to \infty, \quad (6.20)$$

we may conclude that

$$\ell_0(\underline{\underline{A}}) \sim \ell(\underline{\underline{A}}) \quad \text{as} \quad n \to \infty. \quad (6.21)$$

—— *Q. E. D.*

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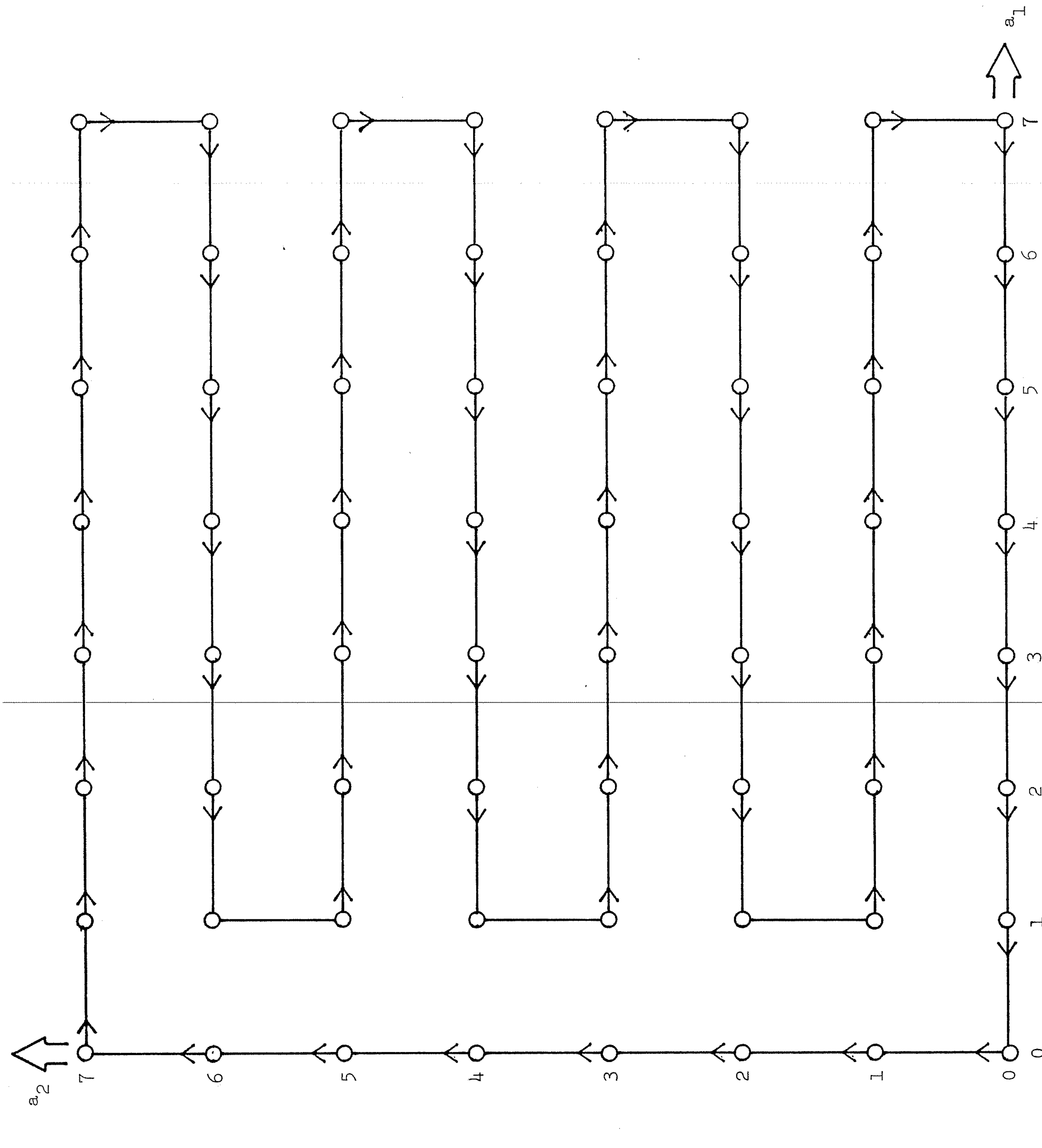

Figure 1

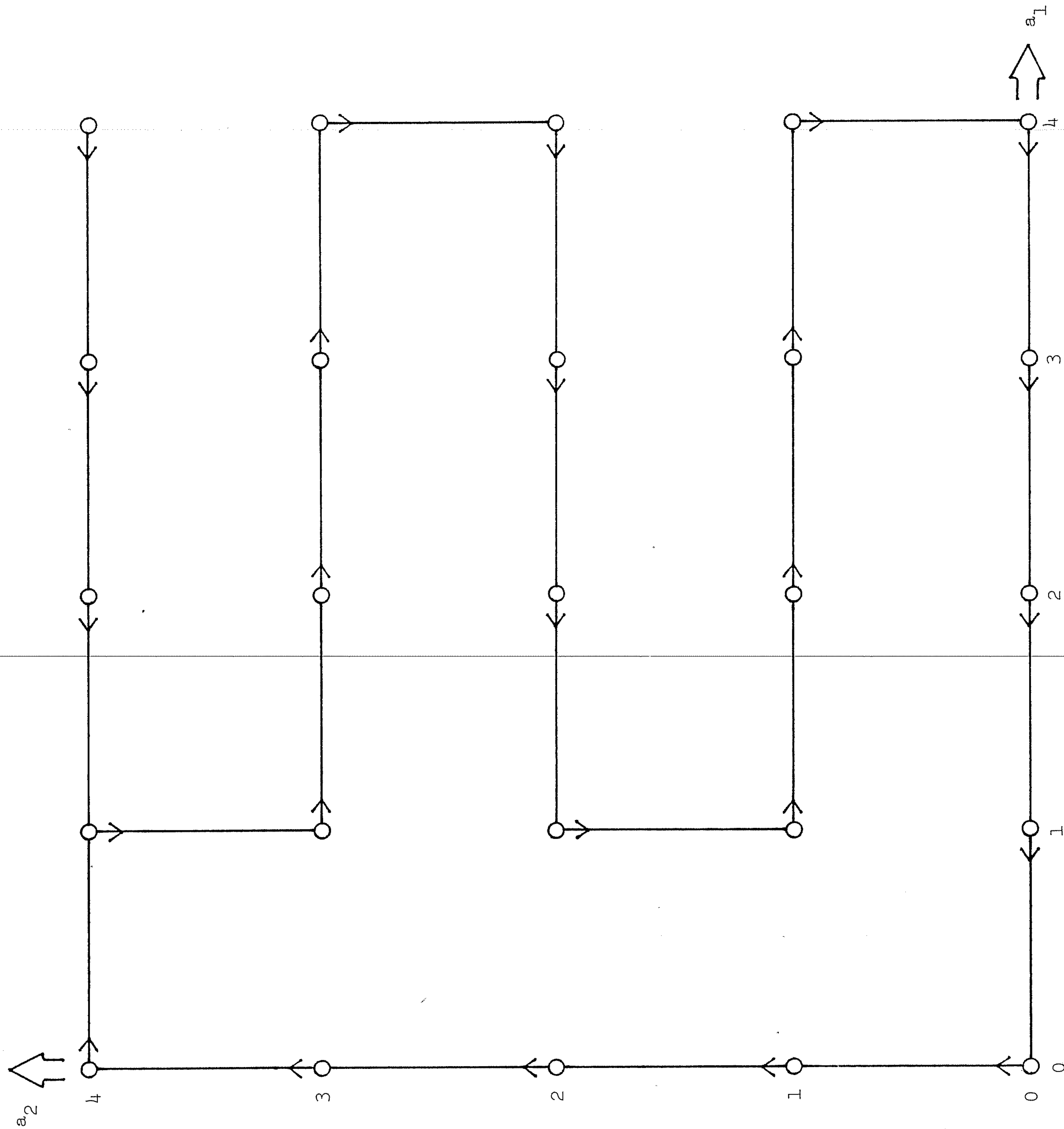

Figure 2

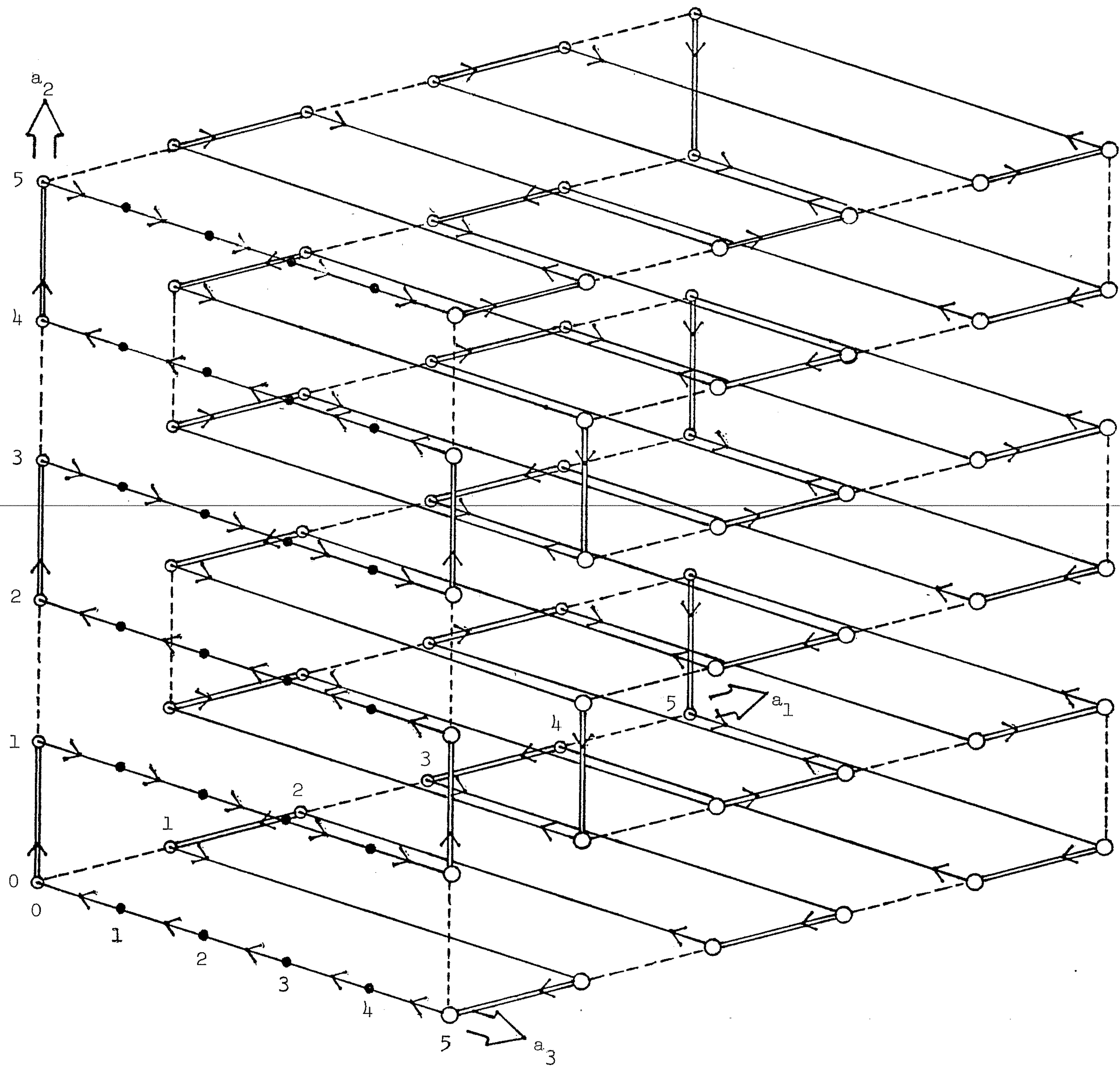

Figure 3

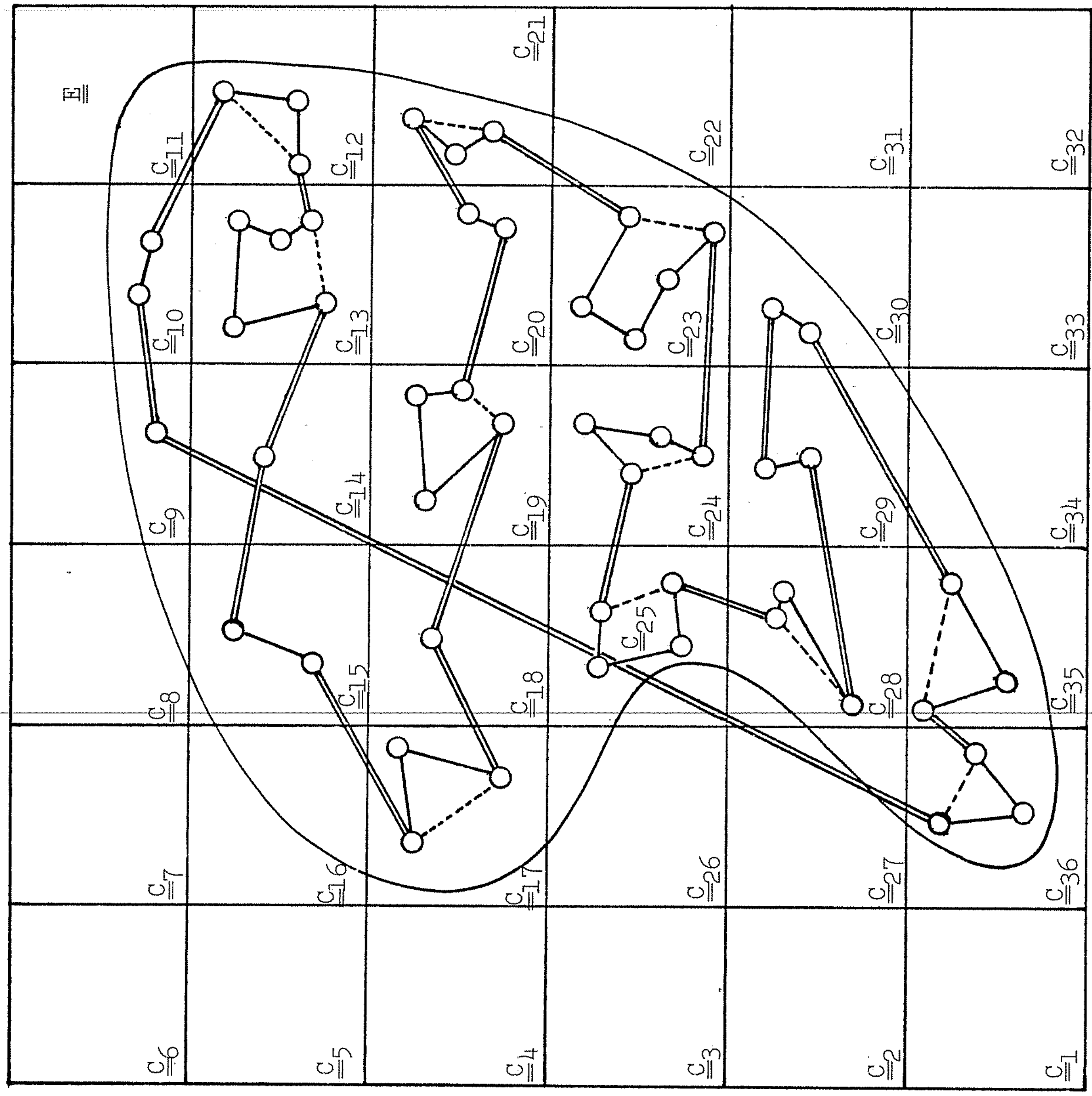
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Figure 4