ON THE SWIRLING FLOW BETWEEN ROTATING COAXIAL DISKS, ASYMPTOTIC BEHAVIOR II.

by

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ABSTRACT

Consider solutions $\langle H(x,\varepsilon),G(x,\varepsilon)\rangle$ of the von Kármán equations for the swirling flow between two rotating coaxial disks

1.1) $$\varepsilon H^{iv} + HH''' + GG' = 0 \quad ,$$

and

1.2) $$\varepsilon G'' + HG' - H'G = 0 \quad .$$

We assume that $|H(x,\varepsilon)| + |H'(x,\varepsilon)| + |G(x,\varepsilon)| \leq B$. This work considers shapes and asymptotic behavior as $\varepsilon \to 0+$. We consider the type of limit functions $\langle \bar{H}(x),\bar{G}(x)\rangle$ that are permissible. In particular, if $\langle H(x,\varepsilon),G(x,\varepsilon)\rangle$ also satisfy the boundary conditions $H(0,\varepsilon) = H(1,\varepsilon) = 0$, $H'(0,\varepsilon) = H'(1,\varepsilon) = 0$ then $\bar{H}(x)$ has no simple zeros. That is, there does not exist a point $z \in [0,1]$ such that $\bar{H}(z) = 0$, $\bar{H}'(z) \neq 0$. Moreover, the case of "cells" which oscillate is studied in detail.

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SIGNIFICANCE AND EXPLANATION

Under appropriate conditions the steady-state flow of fluid between two planes rotating about a common axis perpendicular to them may be described by two functions $H(x,\varepsilon)$, $G(x,\varepsilon)$ which satisfy the coupled system of ordinary differential equations

$$\varepsilon H^{iv} + HH''' + GG' = 0$$
$$\varepsilon G'' + HG' - H'G = 0 \quad .$$

The quantity $\varepsilon > 0$ is related to the kinematic viscosity and $\frac{1}{\varepsilon} = R$ is usually called the Reynolds number.

These equations have received quite a bit of attention. First of all, people who are truly interested in the phenomena modeled by these equations, e.g. fluid dynamicists, are interested in this problem. However, as these equations have been studied by a variety of mathematical methods, they have taken on an independent interest. The major methods employed have been (i) Matched Asymptotic Expansions and (ii) Numerical Computations. In both approaches technical problems have appeared. There may be "turning points," i.e. points at which $H(x,\varepsilon) = 0$. Such points require special and delicate analysis within the theory of (i). As numerical problems, these equations are "stiff" - precisely because ε is small. The occurrence of "turning points" only makes computation more difficult.

For these reasons, these equations have become "test" problems for methods of "matching in the presence of turning points" and "stiff O.D.E. solvers." However, when one has "test problems," one needs to know the answers. Unfortunately here the answers are largely unknown.

In this report we study the asymptotic behavior as ε becomes small. We concentrate on two main cases. First, the case where $|H(x,\varepsilon)| + |H'(x,\varepsilon)| + |G(x,\varepsilon)| \leq B$. These bounds are reasonable because of the physical interpretation of these values as velocities. Finally we consider the case when the limit function $\bar{H}(x)$ oscillates about zero. Such "cell" structure is both interesting and important.

ON THE SWIRLING FLOW BETWEEN ROTATING COAXIAL DISKS, ASYMPTOTIC BEHAVIOR II.**

Heinz Otto Kreiss[1] and Seymour V. Parter[2]

1. Introduction

Consider the von Kármán similarity equations for incompressible axi-symmetric fluid flow between two rotating planes

1.1) $$\varepsilon H^{iv} + HH''' + GG' = 0, \quad 0 \le x \le 1 ,$$

1.2) $$\varepsilon G'' + HG' - H'G = 0, \quad 0 \le x \le 1 .$$

(A thorough discussion of the derivation of these equations is found in [2], [1]).

In [4] we considered the asymptotic behavior (as $\varepsilon \to 0+$) of solutions $\langle H(x,\varepsilon), G(x,\varepsilon)\rangle$ under the basic hypothesis:

H.1) $$|H(x,\varepsilon)| \le B\sqrt{\varepsilon}, \quad |G(x,\varepsilon)| \le B .$$

In this paper we consider the asymptotic behavior under the assumption that

H.2) $$|H(x,\varepsilon)| + |H'(x,\varepsilon)| + |G(x,\varepsilon)| \le C_0 .$$

We recall that if q_r, q_θ, q_x are the components of velocity in cylindrical coordinates (r,θ,x) then

$$q_r = \frac{r}{2} H'(x), \quad q_\theta = \frac{r}{2} G(x), \quad q_x = -H(x) .$$

Thus, assumption H.2 merely asserts that the velocities are bounded in bounded regions, i.e. $r \le R$.

From the results of [4] we see that those solutions $\langle H(x,\varepsilon), G(x,\varepsilon)\rangle$ which satisfy H.1 also satisfy H.2. However, in this paper we are concerned specifically with the case where

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$$H(x,\varepsilon) \not\to 0, \quad 0 \le x \le 1 .$$

Thus we will consider sequences of solutions $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$ which satisfy

H.3) There is a point $x_0 \in (0,1)$ and a value $\delta > 0$ such that

$$0 < \delta \le |H(x_0,\varepsilon_n)| \le C_0 .$$

When studying a singular perturbation problem such as (1.1), (1.2), it is natural to consider the reduced equations

$$\bar{H}\bar{H}''' + \bar{G}\bar{G}' = 0 , \tag{1.3}$$

$$\bar{H}\bar{G}' - \bar{H}'\bar{G} = 0 , \tag{1.4}$$

and the relationship of $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$ to (an appropriately chosen) pair $\langle \bar{H}(x), \bar{G}(x)\rangle$. The solutions of (1.3), (1.4) are given by

$$\bar{G}(x) = \tau_0 \bar{H}(x) , \tag{1.5a}$$

and, if $\tau_0 \neq 0$ then,

$$\bar{H}(x) = H_0 + (H_1/\tau_0) \sin \tau_0(x-x_0) + \frac{H_2}{\tau_0^2} (1 - \cos \tau_0(x-x_0)) . \tag{1.5b}$$

On the other hand, if $\tau_0 = 0$ then

$$\bar{H}(x) = H_0 + H_1(x-x_0) + \frac{1}{2} H_2(x-x_0)^2 . \tag{1.5c}$$

In fact, H.2 implies that there are sequences $\varepsilon_n \to 0+$ and a continuous function $h(x)$ such that

$$\max\{|H(x,\varepsilon_n) - h(x)|;\ 0 \le x \le 1\} \to 0 \quad \text{as} \quad \varepsilon_n \to 0+ .$$

In section 2 we discuss the convergence of $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$ to a solution $\langle \bar{H}(x), \bar{G}(x)\rangle$ on those intervals on which $\bar{H}(x)$ does not vanish.

In section 3 we consider the local behavior of $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$ near a point β at which $\bar{H}(\beta) = 0$ but $\bar{H}'(\beta) \neq 0$.

In section 4 we show that if $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$ satisfy the boundary conditions

1.6a) $$H(0,\varepsilon_n) = H(1,\varepsilon_n) = 0, \text{ (no penetration)} \quad ,$$

1.6b) $$H'(0,\varepsilon\ ) = H'(1,\varepsilon\ ) = 0, \text{ (no slip)} \quad ,$$

then $\bar{H}(x)$ cannot have a simple zero, β.

The results of section 4 assert that if we insist on the boundary conditions (1.6a), (1.6b) and the bounds H.2, we cannot expect the limit function $\bar{H}(x)$ to have nodal zeros. In the case where one assumes H.1 the results of [4] show that (after selecting a subsequence ε_n,) $G(x,\varepsilon_n) \to G_\infty$, a constant, $0 < \delta' < x < 1-\delta'$. Furthermore, if $G_\infty \neq 0$ one can show that

$$H(x,\varepsilon\ )/\sqrt{\varepsilon_n} \to \text{constant}, \quad 0 < \delta' \leq x \leq 1-\delta' \quad .$$

On the other hand, the computation of Mellor, Chapple and Stokes [9] and the computation of Roberts and Shipman [10] produced solutions in which $H(x,\varepsilon_n)/\sqrt{\varepsilon_n}$ oscillates about zero.

For these reasons the discussion in section 5 is concerned with the following general situation. Let $\langle H(x,\varepsilon_n), G(x,\varepsilon)\rangle$ be solutions of (1.1) and (1.2) which satisfy

1.7a) $$\left|\left(\frac{d}{dx}\right)^\nu H(x,\varepsilon)\right| \leq B\varepsilon^{-\rho_\nu}, \quad \nu = 0,1,2,3,4,5 \quad .$$

1.7b) $$\left|\left(\frac{d}{dx}\right)^\nu G(x,\varepsilon)\right| \leq B\varepsilon^{-\sigma_\nu}, \quad \nu = 0,1,2,3,4 \quad ,$$

for certain fixed constants B, ρ_ν, σ_ν with

1.7c) $$\rho_0 > -1 \quad .$$

Let $0 < \varepsilon < 1/2$. Suppose there are two intervals $[\alpha_0,\beta_0]$, $[\alpha_1,\beta_1]$ with

1.8) $$0 < \alpha_0 < \beta_0 < \alpha_1 < \beta_1 < 1$$

and a constant $R > 0$ such that

1.9a) $$H(x,\varepsilon)\varepsilon^{\rho_0} \geq R, \quad \alpha_0 \leq x \leq \beta_0 \quad ,$$

1.9b) $$H(x,\varepsilon)\varepsilon^{\rho_0} \leq -R, \quad \alpha_1 \leq x \leq \beta_1 \quad ,$$

We then discuss the possible limit behavior of $\langle H(x,\varepsilon), G(x,\varepsilon)\rangle$ as $\varepsilon \to 0+$. In particular, with an appropriate definition of "limit cell", we are able to show that there are at most four cells in the case where the limit function oscillates about zero.

2. Convergence of $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$.

Let $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$ be solutions of (1.1), (1.2) which also satisfy H.2. Let $x_0 \in (0,1)$ be a point at which

$$0 < \delta \leq |H(x_0,\varepsilon_n)| \leq C_0$$

for all $\varepsilon = \varepsilon_n$. The major result of this section is the following:

Theorem 2.1: Let η be a constant with $0 < \eta \leq \delta/2$. Let $[a,b] \subset [0,1]$ be the largest interval containing x_0 on which

2.1) $$|H(x,\varepsilon_n)| \geq \eta \quad .$$

Then there are constants K_ν, K'_ν depending only on ν, η and C_0 such that,

2.2) $$\left|\left(\frac{d}{dx}\right)^\nu G(x,\varepsilon_n)\right| + \left|\left(\frac{d}{dx}\right)^\nu H(x,\varepsilon_n)\right| \leq K_\nu, \quad a + K'_\nu \ \varepsilon|\ell n\ \varepsilon| \leq x \leq b - K_\nu\ \varepsilon|\ell n\ \varepsilon| \quad .$$

Once these estimates have been proven, it is an easy matter to establish

Corollary: Let $G(x_0,\varepsilon_n) \to G_0$, $H(x_0,\varepsilon_n) \to H_0$, $H'(x_0,\varepsilon_n) \to H_1$, $H''(x_0,\varepsilon_n) \to H_2$. Let

$$\tau_0 = G_0/H_0 \quad .$$

Let $\langle \bar{H}(x), \bar{G}(x)\rangle$ be the solution of the reduced equation given by (1.5a), (1.5b), (1.5c). Then

2.3) $$\text{Max}\{|(\tfrac{d}{dx})^\nu[G(x,\varepsilon_n) - \bar{G}(x)]| + |(\tfrac{d}{dx})^\nu[H(x,\varepsilon_n) - \bar{H}(x)]|, \ a \leq x \leq b\} \to 0 \quad .$$

We require the following basic lemma which was proven in [3].

Lemma 2.1: Consider the differential equation

2.4) $$\varepsilon\ dy/dx + a(x)y = F(x), \quad \alpha \leq x \leq \beta$$

where a, F are continuous functions with Real $a > 0$ and $\varepsilon > 0$ is a (small) positive constant. The solutions of (2.4) satisfy the estimates

2.5) $$|y(x)| \leq \varepsilon^{-1}|x-\alpha| \max_{\alpha\leq\eta\leq x} |F(\eta)| + s(x-\alpha)|y(\alpha)|, \ x \geq \alpha$$

2.6) $$|y(x)| \leq \max_{\alpha \leq \eta \leq x} |F(\eta)/\text{Re } a(\eta)| + s(x-\alpha)|y(\alpha)|, \; x \geq \alpha$$

where

$$s(x-\alpha) = \exp\{-\frac{1}{\varepsilon}\int_\alpha^x \text{Re } a(\xi)d\xi\} \quad .$$

If Real $a < 0$ the corresponding estimates hold. We have to replace $s(x-\alpha)|y(\alpha)|$ by $s(\beta-x)|y(\beta)|$ and $\alpha \leq \eta \leq x$ by $x \leq \eta \leq \beta$.

Proof: See lemma 2.1 of [3].

If $H(x_0,\varepsilon_n) < 0$ we consider the functions

2.7) $$\tilde{H}(x,\varepsilon_n) = -H(1-x,\varepsilon_n), \quad \tilde{G}(x,\varepsilon_n) = G(1-x,\varepsilon_n) \quad .$$

These functions satisfy (1.1), (1.2) and H.2. Moreover

$$\tilde{H}(1-x_0,\varepsilon_n) = -H(x_0,\varepsilon_n) \geq \delta > 0 \quad .$$

Since estimates on $\tilde{H}(x,\varepsilon_n)$, $\tilde{G}(x,\varepsilon_n)$ are easily translated into estimates on $H(x,\varepsilon_n)$, $G(x,\varepsilon_n)$ we may assume

2.8) $$H(x_0,\varepsilon_n) \geq \delta > 0 \quad .$$

Lemma 2.2: Let $0 < \varepsilon \leq 1$. Let $\langle H(x,\varepsilon), G(x,\varepsilon)\rangle$ be a solution of (1.1), (1.2) which satisfies H.2. Then, for every positive integer p there is a constant C_p which depends only on p and C_0 such that

2.9a) $$|(\frac{d}{dx})^p G(x,\varepsilon)| \leq C_p \varepsilon^{-p} \quad , \quad p = 1,2,\cdots$$

2.9b) $$|(\frac{d}{dx})^p H(x,\varepsilon)| \leq C_p \varepsilon^{-p+1}, \quad p = 1,2,\cdots \quad .$$

Proof: Let

$$\tilde{x} = x/\varepsilon \quad .$$

Then equations (1.1), (1.2) become

$$\left(\frac{d}{d\tilde{x}}\right)^2 G + H \frac{d}{d\tilde{x}} G - \left(\frac{dH}{d\tilde{x}}\right) G = 0 \quad , \quad 0 \le \tilde{x} \le 1/\varepsilon \ .$$

$$\left(\frac{d}{d\tilde{x}}\right)^4 H + H\left(\frac{d}{d\tilde{x}}\right)^3 H + \varepsilon^2 G \frac{dG}{d\tilde{x}} = 0 \quad , \quad 0 \le \tilde{x} \le 1/\varepsilon \ .$$

Thus, the estimates (2.9a), (2.9b) follow as in lemma 2.2 of [4].

Let $[a,b]$ be the largest interval containing x_0 on which

$$H(x,\varepsilon_n) \ge \eta > 0 \ .$$

Let

$$\alpha = \max[a - \eta/(2C_0), \frac{\varepsilon|\ell n\ \varepsilon|}{\eta}] \ ,$$
$$a' = a + 2\frac{\varepsilon|\ell n\ \varepsilon|}{\eta} \ .$$

Then (see lemma A.1 of [4])

2.10) $$H(x,\varepsilon_n) \ge \eta/2, \quad \alpha \le x \le b \ .$$

For any function $f \in C[\alpha,b]$ and any β, x with $\alpha \le \beta \le x \le b$ let

$$\|f\|_{\beta,x} = \max\{|f(t)|;\ \beta \le t \le x\} \ .$$

Lemma 2.3: Suppose $\langle H(x,\varepsilon), G(x,\varepsilon)\rangle$ is a solution of (1.1), (1.2) which satisfies H.2. Suppose (2.8) and (2.10) hold. Then for every β, β', x with $\alpha \le \beta \le \beta' \le x \le b$ we have

2.11a) $$|G'(x,\varepsilon)| \le \|G/H\|_{\beta,x} \|H'\|_{\beta,x} + s(x-\beta)|G'(\beta,\varepsilon)| \ ,$$

2.11b) $$|H'''(x,\varepsilon)| \le \|G/H\|_{\beta',x} \|G'\|_{\beta',x} + s(x-\beta')|H'''(\beta',\varepsilon)| \ ,$$

2.11c) $$|G''(x,\varepsilon)| \le \|G/H\|_{\beta',x} \|H''\|_{\beta',x} + s(x-\beta')|H''(\beta',\varepsilon)| \ ,$$

Proof: We obtain (2.11a) from equation (1.2) and lemma 2.1. We obtain (2.11b) from equation (1.1) and lemma 2.1. Differentiating equation (1.2) we have

$$\varepsilon G''' + HG'' = H''G \ .$$

Thus, (2.11c) follows from lemma 2.1.

Proof of theorem 2.1: Differentiation of (1.1), (1.2) gives equations of the form

2.12a) $$\varepsilon H^{(3+k+1)} + HH^{(3+k)} = \mathcal{H}_k(H,H',\cdots H^{(3+k-1)}, G,G',\cdots G^{(k+1)})$$

2.12b) $$\varepsilon G^{(2+k+1)} + HG^{(2+k)} = \mathcal{G}_k(H,H',\cdots H^{(2+k)}, G,G',\cdots G^{(k)})$$

where $\mathcal{H}_k$, $\mathcal{G}_k$ are quadratic functions of their arguments.

Let

$$\xi = 2\ \varepsilon|\ln\ \varepsilon|/\eta\ .$$

Let $\beta = \alpha$, and $\beta' = \frac{1}{2}(\alpha+a'+4\xi)$. Applying (2.11a) of lemma 2.3 and lemma 2.2 we see that

$$\|G'\|_{\beta',b} \leq 2C_0^2/\eta + C_1\varepsilon^{-1}\exp\{-|\ln\ \varepsilon|\} = 2C_0^2/\eta + C_1\ .$$

Thus, G' is bounded on the interval $[\beta',b]$. Let $\beta'' = \frac{1}{2}(\beta'+a'+6\xi)$. Then (2.11b) of lemma 2.3 and lemma 2.2 imply that

$$\|H'''\|_{\beta'',b} \leq (2C_0/\eta)\|G'\|_{\beta',b} + C_3\varepsilon^{-3}\exp\{-3|\ln\ \varepsilon|\}\ .$$

Thus $\|H'''\|_{\beta'',b}$ is bounded. Since $0 < \eta \leq \delta/2$ we have

$$b - \beta'' \geq b - x_0 \geq \min(1 - x_0,\ x_0 + \eta/2C_0) \doteq L\ .$$

Applying Landau's Theorem [5] (lemma 2.1 of [4]) we have

$$\|H''\|_{\beta'',b} \leq \frac{L}{4}C_0 + \frac{4}{L}\|H'''\|_{\beta'',b}\ .$$

Thus, $\|H''\|_{\beta'',b}$ is bounded. The complete Theorem now follows from a straight forward induction based on lemma 2.1 and (2.12a), (2.12b).

<u>Proof of the corollary</u>: Let $\bar{H}(x,\varepsilon_n)$, $\bar{G}(x,\varepsilon_n)$ be the solution of the reduced equations (1.3), (1.4) determined by $H(x_0,\varepsilon_n)$, $H'(x_0,\varepsilon_n)$, $H''(x_0,\varepsilon_n)$, $G(x_0,\varepsilon_n)$. Then, from (1.2) we have

$$\frac{G(x,\varepsilon_n)}{H(x,\varepsilon_n)} = \frac{G(x_0,\varepsilon_n)}{H(x_0,\varepsilon_n)} - \varepsilon_n\int_{x_0}^{x}\frac{G''(t,\varepsilon_n)}{H^2(t,\varepsilon_n)}dt\ .$$

Let

$$\tau = G(x_0,\varepsilon_n)/H(x_0,\varepsilon_n) \quad ,$$

then

2.13a) $$|G(x,\varepsilon_n) - \tau\, H(x,\varepsilon_n)| \le \frac{\varepsilon_n}{\eta^2} K_2 C_0 \quad ,$$

and

2.13b) $$G(x_0,\varepsilon_n) = \bar{G}(x_0,\varepsilon_n) \quad .$$

Moreover

$$G'(x,\varepsilon_n) = H'(x,\varepsilon_n)\left[\frac{G(x,\varepsilon_n)}{H(x,\varepsilon_n)}\right] - \varepsilon_n\left[\frac{G''(x,\varepsilon_n)}{H(x,\varepsilon_n)}\right] .$$

Thus

2.13c) $$|G'(x,\varepsilon_n) - \tau\, H'(x,\varepsilon_n)| \le \varepsilon_n K_2 C_0\left[\frac{1}{\eta^2} + \frac{1}{\eta^3}\right] \quad .$$

Substitution into (1.1) gives

$$|H''' + \tau^2 H'| \le \tau\varepsilon_n M$$

where M is a constant depending only on K_4, K_2, τ, C_0 and η. Hence, in view of the initial conditions we have

2.14a) $$H'(x,\varepsilon_n) = \bar{H}'(x,\varepsilon_n) + O(\tau\varepsilon_n) \quad , \quad a \le x \le b$$

2.14b) $$H(x,\varepsilon_n) = \bar{H}(x,\varepsilon_n) + O(\tau\varepsilon_n) \quad , \quad a \le x \le b \quad ,$$

2.14c) $$G(x,\varepsilon_n) = \bar{G}(x,\varepsilon_n) + O(\tau\varepsilon_n + \varepsilon_n) \quad , \quad a \le x \le b \quad .$$

Finally, the conclusion of the corollary follows from the continuous dependence of $\bar{H}(x,\varepsilon_n)$, $\bar{G}(x,\varepsilon_n)$ on the initial conditions. That is

$$\bar{H}(x,\varepsilon_n) \to \bar{H}(x) \quad \text{as} \quad \varepsilon_n \to 0+$$

$$\bar{G}(x,\varepsilon_n) \to \bar{G}(x) \quad \text{as} \quad \varepsilon_n \to 0+ \quad .$$

3. Behavior at a point β with $\bar{H}(\beta) = 0$, $\bar{H}'(\beta) \neq 0$.

The purpose of this section is to prove the following fact: Either $\beta = 1$ and $H(x,\varepsilon_n)$, $H'(x,\varepsilon_n)$, $H''(x,\varepsilon_n)$, $G'(x,\varepsilon_n)$ are bounded for $x_0 \leq x \leq 1$, or $\beta < 1$. If $\beta < 1$ the $H(x,\varepsilon_n)$ has a "nodal" zero near $x = \beta$, i.e. $H(x,\varepsilon_n)$ really changes sign about $x = \tilde{\beta}(\varepsilon)$ and $H'(\tilde{\beta}(\varepsilon_n),\varepsilon_n) < \frac{1}{2}\bar{H}'(\beta)$. Moreover, $H(x,\varepsilon_n)$, $H'(x,\varepsilon_n)$, $H''(x,\varepsilon_n)$, $G'(x,\varepsilon_n)$ are uniformly bounded in a fixed neighborhood of $x = \beta$.

The argument is carried out in three steps.

Step 1: There is a point $x_2 = x_2(\varepsilon_n)$ and a K, depending only on the K_ν of Theorem 2.1, such that

3.1) $$x_0 < x_2(\varepsilon_n)$$

3.2) $$H(x_2(\varepsilon_n),\varepsilon_n) = K\sqrt{\varepsilon_n}$$

3.3) $$H(x,\varepsilon_n) \geq K\sqrt{\varepsilon_n}, \quad x_0 \leq x \leq x_2$$

and $H(x,\varepsilon_n)$, $H'(x,\varepsilon_n)$, $H''(x,\varepsilon_n)$, $H'''(x,\varepsilon_n)$, $G(x,\varepsilon_n)$, $G'(x,\varepsilon_n)$, $G''(x,\varepsilon_n)$ are uniformly bounded on the interval $[x_0,x_2]$.

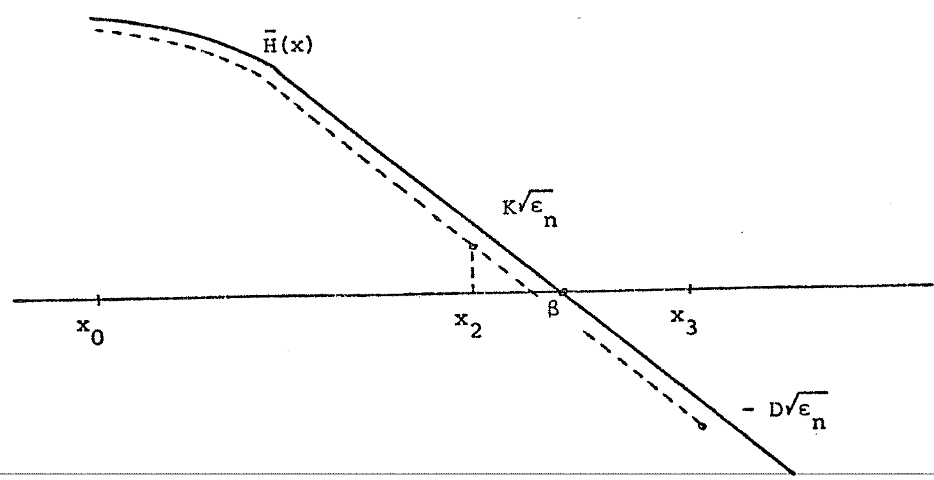

Figure 1

Step 2: We "shoot" through the zero of $H(x,\varepsilon_n)$ or we shoot until $x = 1$. To accomplish this we make the change of variables

3.4) $$\xi = (x-x_2)/\sqrt{\varepsilon_n}, h(\xi) = H(x,\varepsilon_n)/\sqrt{\varepsilon_n}, \; g(\xi) = G(x,\varepsilon_n) \quad .$$

The solution $\langle h(\xi), g(\xi)\rangle$ is now continued to the right until either $x = 1$ or $\xi = \alpha$ where α is any fixed constant. The corresponding functions $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$ are smooth for $x_2 \leq x \leq x_2 + \alpha\sqrt{\varepsilon_n}$. Indeed we may take α logrithmically large and still show that $H'(x,\varepsilon_n)$ and $H''(x,\varepsilon_n)$ are bounded.

Step 3: If $\beta \neq 1$ one can actually shoot through until reaching a value $x_4 > x_2$ at which

$$H(x_4,\varepsilon_n) \leq -\delta_3 < 0$$

where δ_3 is a constant independent of ε_n.

The complete smoothness now follows from "patching" the results for $H(x,\varepsilon_n) < -K\sqrt{\varepsilon_n}$ with the results already obtained in steps 1 and 2.

Before beginning this program we observe that - as long as $H(x,\varepsilon) \neq 0$ - (1.2) can be rewritten as

3.5) $$\frac{d}{dx}\left(\frac{G}{H}\right) = -\varepsilon \frac{G''}{H^2} \quad .$$

Lemma 3.1: Let $\eta = \delta/2$ and let K_ν be the constants $K_\nu(\eta)$ of Theorem 1.2. Let

3.6a) $$\tau = G(x_0,\varepsilon_n)/H(x_0,\varepsilon_n)$$

and

3.6b) $$M_0 = C_0/\delta \geq |\tau| \quad .$$

Assume that

3.7) $$\varepsilon_n \leq \frac{M_0\delta^2}{2K_2} \quad .$$

Let $[x_0,x_1] \subset [x_0,1]$ be the largest interval on which

3.8) $$|G(x,\varepsilon_n)/H(x,\varepsilon_n) - \tau| \leq 2M_0 \quad .$$

Then

3.9) $$x_1 - x_0 \geq \min(\delta/2C_0,\ 1-x_0) = L \quad ,$$

3.10a) $$|G'(x,\varepsilon_n)| \leq (2M_0 + |\tau|)C_0 + |G'(x_0,\varepsilon_n)|, \quad x_0 \leq x \leq x_1 \quad ,$$

3.10b) $$|H'''(x,\varepsilon_n)| \leq (2M_0 + |\tau|)^2 C_0 + |H'''(x_0,\varepsilon_n)|, \quad x_0 \leq x \leq x_1 \quad .$$

Moreover, there is a constant M_1 such that

3.11a) $$\|G'\|_{x_0,x_1} + \|H'''\|_{x_0,x_1} \leq M_1$$

3.11b) $$\|H''\|_{x_0,x_1} \leq \frac{L}{4} C_0 + \frac{4}{L} M_1 = M_2$$

3.11c) $$|G''(x,\varepsilon_n)| \leq (2M_0 + |\tau|)M_2 + |G''(x_0,\varepsilon_n)|, \quad x_0 \leq x \leq x_1$$

3.11d) $$\|G''\|_{x_0,x_1} \leq (2M_0 + |\tau|)M_2 + K_2 = M_3 \quad .$$

<u>Proof</u>: Let $[a,b]$ be the interval on which

$$H(x,\varepsilon_n) \geq \delta/2 = \eta \quad .$$

Then (3.5) implies that, if $x \in [x_0,b]$ we have

$$|G(x,\varepsilon_n)/H(x,\varepsilon_n) - \tau| \leq \varepsilon_n \int_{x_0}^{x} \frac{|G''(t,\varepsilon_n)|}{H^2(t,\varepsilon_n)}\, dt \leq \frac{4}{\delta^2}\varepsilon_n K_2 \quad .$$

Since (3.7) holds, we see that $b \leq x_1$. Thus (3.9) holds. The estimates (3.10a), (3.10b) follow at once from lemma 2.3. The estimate (3.11a) follow from Theorem 2.1. The estimate (3.11b) is Landau's Theorem. The estimates (3.11c), (3.11d) follow from lemma 2.3 and Theorem 2.1.

We now complete Step 1.

<u>Lemma 3.2</u>: Let

3.12) $$K = (M_3/2M_0)^{1/2} + 1 \quad .$$

Then either $x_1 = 1$ or there is a point x_2, $x_0 \le x_2 \le x_1$ such that (3.2) and (3.3) hold. Moreover,

3.13) $$|G(x_2,\varepsilon_n)| \le (|\tau| + 2M_0)K\sqrt{\varepsilon_n} \quad .$$

Proof: Integration of (3.5) yields

$$|G(x,\varepsilon_n)/H(x,\varepsilon_n) - \tau| \le \varepsilon_n \int_{x_0}^{x} \frac{|G''(t,\varepsilon_n)|dt}{H^2(t,\varepsilon_n)} \quad .$$

Suppose that at all points $x \in [x_0,x_1]$ we have

$$H(x,\varepsilon_n) \ge K\sqrt{\varepsilon_n} \quad .$$

Then

$$|G(x_1,\varepsilon_n)/H(x_1,\varepsilon_n) - \tau| \le \frac{\varepsilon_n}{K^2\varepsilon_n} \int_{x_0}^{x_1} M_3 dt < 2M_0 \quad .$$

But, because of the strict inequality, either $x_1 = 1$ or $[x_0,x_1]$ can be enlarged to the right without violating (3.8). However $[x_0,x_1]$ was chosen as large as possible. Hence if $x_1 \neq 1$, there are points $x_2 \in [x_0,x_1]$ which satisfy (3.2). We choose x_2 as the first such point. Then (3.3) holds and (3.13) follows from the triangle inequality.

Having found $x_2 = x_2(\varepsilon_n)$ we now wish to "shoot" the right. In order to do this we must establish the following facts:

3.14a) $$x_2(\varepsilon_n) \to \beta \quad ,$$

and, for ε_n small enough

3.14b) $$H'(x_2(\varepsilon_n),\varepsilon_n) \le -\frac{1}{2}\bar{H}'(\beta) = -\Delta \quad .$$

To do this we use a slight variant of the Ascoli-Anzela lemma. Let

3.15a) $$Q(x,\varepsilon_n) = \begin{cases} H'''(x,\varepsilon_n) \ , & x_0 \le x \le x_1(\varepsilon_n) \\ H'''(x_1,\varepsilon_n), & x_1(\varepsilon_n) \le x \le 1 \end{cases} \quad .$$

Let

$$\text{3.15b)} \qquad \begin{cases} q(x,\varepsilon_n) = H(x_0,\varepsilon_n) + H'(x_0,\varepsilon_n)(x-x_0) + \frac{1}{2} H''(x_0,\varepsilon_n)(x-x_0)^2 + \\ \int_{x_0}^{x} \int_{x_0}^{t} \int_{x_0}^{s} Q(\lambda,\varepsilon_n)\,d\lambda\,ds\,dt \quad . \end{cases}$$

The functions $q(x,\varepsilon_n) \in C^3[x_0,1]$ and $q'''(x,\varepsilon_n)$ are uniformly bounded. Thus, a subsequence converges on $[x_0,1]$ in the $C^2[x_0,1]$ topology. Moreover, on any interval $[x_0,b]$ on which the limit function is strictly positive, that limit function is $\bar{H}(x)$. Thus, since we may assume that

$$x_2(\varepsilon_n) \to \bar{x}_2$$

and $H(x_2(\varepsilon_n),\varepsilon_n) \to 0+$ we see that

$$\text{3.16)} \qquad \bar{x}_2 = \beta \quad .$$

Moreover,

$$|H'(x_2,\varepsilon_n) - \bar{H}'(x_2)| = |q'(x_2,\varepsilon_n) - \bar{H}'(x_2)| \to 0 \quad .$$

Thus we obtain (3.14b).

Let us summarize our results at this point.

<u>Theorem 3.1</u>: Let $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$ be a sequence of solutions of (1.1), (1.2) which satisfy H.2 and (2.8). Suppose

$$H(x,\varepsilon_n) \to \bar{H}(x), \; G(x,\varepsilon_n) \to \bar{G}(x)$$

as in the corollary to Theorem 2.1. Suppose that there is a point $\beta > x_0$ such that

$$\text{3.17a)} \qquad \bar{H}(x) > 0, \quad x_0 \le x < \beta$$

$$\text{3.17b)} \qquad \bar{H}(\beta) = 0, \quad \bar{H}'(\beta) = -2\Delta < 0 \quad .$$

Then, for ε_n sufficiently small there is a point $x_2 = x_2(\varepsilon_n) > x_0$ such that (3.2), (3.3), hold. Moreover, there are constants M_1, M_2, M_3 such that

3.18a) $$\|G'\|_{x_0,x_2} + \|H'''\|_{x_0,x_2} \leq M_1$$

3.18b) $$\|H''\|_{x_0,x_2} \leq M_2$$

3.18c) $$\|G''\|_{x_0,x_2} \leq M_3 \quad .$$

Finally

3.19a) $$x_2(\varepsilon_n) \to \beta$$

3.19b) $$H'(x_2,\varepsilon_n) \leq -\Delta \quad .$$

We are now ready to shoot to the right. We make the change of variables (3.4) and note that we have the differential equation

3.20a) $$\ddot{g} + h\dot{g} - g\dot{h} = 0, \quad 0 \leq \xi \leq \bar{\xi} = (1-x_2)/\sqrt{\varepsilon_n} \quad ,$$

3.20b) $$\ddddot{h} + h\dddot{h} + g\dot{g} = 0, \quad 0 \leq \xi \leq \bar{\xi} \quad ,$$

together with the initial conditions

3.21a) $$h(0) = K, \ \dot{h}(0) < -\Delta, \ \ddot{h}(0) = O(\sqrt{\varepsilon_n}), \ \dddot{h}(0) = O(\varepsilon_n) \quad ,$$

3.21b) $$g(0) = O(\sqrt{\varepsilon_n}), \ \dot{g}(0) = O(\sqrt{\varepsilon_n}) \quad \ddot{g}(0) = O(\varepsilon_n) \quad .$$

Furthermore

3.21c) $$|\dot{h}(\xi)| + |g(\xi)| \leq C_0 \quad .$$

We require two basic lemmas.

Lemma 3.3: Consider a function $r(\xi)$ which satisfies

3.22) $$\ddot{r} + h\dot{r} = f, \quad 0 \leq \xi \quad .$$

Let

3.23) $$A(\xi) = \exp\{(K + C_0\xi)\xi\} \quad .$$

Then

3.24) $$|\dot{r}(\xi)| \le |\dot{r}(0)|A(\xi) + \xi A(\xi)\|f\|_{0,\xi} \quad .$$

Proof: Observe that

$$|h(\xi)| \le K + C_0\xi$$

and solve (3.22).

Lemma 3.4: Consider a function $r(\xi)$ which satisfies

3.25a) $$\ddot{r} + h\dot{r} + Ur = f \quad , \quad 0 \le \xi$$

where

3.25b) $$|U(\xi)| \le U_0 \quad .$$

Let

3.26) $$B(\xi) = \exp\{[1 + U_0 + K + C_0\xi]\xi\} \quad .$$

Then

3.27) $$|r(\xi)| + |\dot{r}(\xi)| \le (|r(0)| + |\dot{r}(0)| + \xi\|f\|_{0,\xi})B(\xi) \quad .$$

Proof: We write (3.25a) as the first order system

$$\frac{d}{d\xi}\begin{bmatrix} r \\ \dot{r} \end{bmatrix} = \tilde{A}\begin{bmatrix} r \\ \dot{r} \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}$$

where

$$\tilde{A} = \begin{bmatrix} 0 & 1 \\ U & h \end{bmatrix} \quad .$$

Thus

$$\|\tilde{A}\|_2 \le U_0 + 1 + K + C_0\xi$$

and (3.26) follows from well known estimates.

Theorem 3.2: For every $N > 0$ such that $x_2(\varepsilon_n) + N\sqrt{\varepsilon_n} \le 1$ there is a constant $B_1 = B_1(N)$ such that

3.28) $$|H''(x,\varepsilon_n)| + |H'''(x,\varepsilon_n)| + |G'(x,\varepsilon_n)| + |G''(x,\varepsilon_n)| \le B_1, \; x_2 \le x \le x_2 + N\sqrt{\varepsilon_n} \quad .$$

For every $D > 0$ such that

3.29) $$x_2(\varepsilon_n) + \frac{(1+D+K)}{\Delta}\sqrt{\varepsilon_n} \le 1$$

there is an $\bar{\varepsilon} = \bar{\varepsilon}(D)$ and an $x_3(\varepsilon_n)$ such that

3.30a) $$x_2(\varepsilon_n) < x_3(\varepsilon_n) \le x_2(\varepsilon_n) + \frac{(1+D+K)}{\Delta}\sqrt{\varepsilon_n}$$

and for $0 < \varepsilon_n \le \bar{\varepsilon}$ we have

3.30b) $$H(x_3,\varepsilon_n) = -D\sqrt{\varepsilon_n} \quad .$$

Moreover,

3.30c) $$H'(x_3,\varepsilon_n) \le -\Delta + B(N) \cdot N\sqrt{\varepsilon_n} \quad .$$

Proof: Consider the functions $g(\xi)$, $h(\xi)$ given by (3.4). Let $U(\xi) = \dot{h}(\xi)$. Let $r(\xi) = g(\xi)$. Applying lemma 3.4 obtain

3.31) $$|g(\xi)| + |\dot{g}(\xi)| = O(B(N)\sqrt{\varepsilon_n}) \quad .$$

Let $r(\xi) = \ddot{h}(\xi)$. Consider the equation (3.20b) together with the initial conditions (3.21a). Applying lemma 3.3 and the estimate (3.31) we have

3.32) $$|\dddot{h}(\xi)| \le O(B^2(N)A(N)\varepsilon_n) \quad .$$

An integration, together with the initial conditions (3.21a) gives

3.33) $$|\ddot{h}(\xi)| \le O(B^2(N)A(N)N\sqrt{\varepsilon_n}) \quad .$$

Differentiation of (3.20a) gives

$$\dddot{g} + h\ddot{g} = \ddot{h}g \quad .$$

Let $r(\xi) = \dot{g}(\xi)$. Lemma 3.3, together with (3.21a), gives

3.34) $$|\ddot{g}(\xi)| = O(B^2(N)A(N)N\varepsilon_n) \quad .$$

Returning to the variables $H(x,\varepsilon_n)$, $G(x,\varepsilon_n)$ we obtain (3.28).

Suppose (3.29a) holds. Let

$$\tilde{x} = x_2(\varepsilon_n) + \left(\frac{1+D+K}{\Delta}\right)\sqrt{\varepsilon_n} \quad .$$

Then

$$H(\tilde{x},\varepsilon_n) \leq K\sqrt{\varepsilon_n} - \Delta(\tilde{x}-x_2) + \frac{1}{2}H''(\xi,\varepsilon_n)(\tilde{x}-x_2)^2 \quad .$$

That is

$$H(\tilde{x},\varepsilon_n) \leq -(1+D)\sqrt{\varepsilon_n} + O(\varepsilon_n) \quad .$$

Thus, for ε_n small enough, $H(\tilde{x},\varepsilon_n) \leq -D\sqrt{\varepsilon_n}$. Since $H(x_2,\varepsilon_n) = K\sqrt{\varepsilon_n}$ and $H(x,\varepsilon_n)$ is a continuous function there exists an appropriate $x_3(\varepsilon_n)$.

If there are no D's so that (3.29) holds, then $\beta = 1$ and we have established smoothness on the entire interval $[x_0,1]$. In any case, we have now completed step 2.

<u>Lemma 3.5</u>: Assume that there is a value $D > 0$ such that (3.29) holds for all sufficiently small ε_n. If $\beta = 1$ let $H(x,\varepsilon_n)$ satisfy the boundary condition

3.35) $$H(1,\varepsilon_n) = 0 \quad .$$

Then there is an N_1 and an $\bar{\bar{\varepsilon}} > 0$ such that

3.36a) $$1 \leq x_2(\varepsilon_n) + N_1\sqrt{\varepsilon_n}, \qquad 0 \leq \varepsilon_n \leq \bar{\bar{\varepsilon}} \quad .$$

In that case Theorem 3.1 asserts that $H(x,\varepsilon_n) \in C^3[x_0,1]$, $G(x,\varepsilon_n) \in C^2[x_0,1]$ uniformly. That is; $H, H', H'', H''', G, G', G''$ are all uniformly bounded for $x_0 \leq x \leq 1$.

If $\beta < 1$ there is a point $x_4 > x_3$ and a positive constant δ_3, independent of ε_n, such that

3.36b) $$H(x_4,\varepsilon_n) \leq -\delta_3 < 0 \quad .$$

<u>Proof</u>: Suppose the lemma is false. In either case we have the following situation; given an $N > 0$ there is an $\hat{\varepsilon} = \hat{\varepsilon}(N)$ such that

3.37) $$x_2(\varepsilon_n) + N\sqrt{\varepsilon_n} < 1, \qquad 0 < \varepsilon_n \leq \hat{\varepsilon}(N) \quad .$$

Let N_0 be fixed and let $x_3 = x_3(\varepsilon_n)$ be chosen so that

3.38a) $$H(x_3,\varepsilon_n) = -N_0\sqrt{\varepsilon_n}$$

3.38b) $$H'(x_3,\varepsilon_n) < -\frac{1}{2}\Delta \quad .$$

Since the lemma is false there must be a point $x_5 = x_5(\varepsilon_n)$ such that

3.39a) $$H(x_5,\varepsilon_n) \to 0- \quad \text{as} \quad \varepsilon_n \to 0+ \quad ,$$

3.39b) $$H'(x_5,\varepsilon_n) = \frac{1}{2} H'(x_3,\varepsilon_n) < -\frac{1}{4}\Delta$$

3.39c) $$H'(x,\varepsilon_n) < \frac{1}{2} H'(x_3,\varepsilon_n), \; x_3 \leq x < x_5 \quad .$$

That is, take x_5 as the first point after x_3 at which (3.39b) holds. If there is no such point then either $H(1,\varepsilon_n) \neq 0$ or (3.36b) follows from an integration of $H'(t,\varepsilon_n)$. If (3.39a) does not hold then (3.36b) holds and furthermore H.2 implies that $x_4 < 1 - \delta_3/C_0$ which implies that $\beta \neq 1$ (because of (3.14a)). Thus, (3.39a) must hold.

Moreover,

3.39d) $$x_5(\varepsilon_n) - x_3(\varepsilon_n) \to 0 \quad .$$

If not, an integration of $H'(t,\varepsilon_n)$ from x_3 to x_5 would once more imply (3.36b).

Finally, (3.37) and the fact (see theorem 3.2) that H'' is bounded on $[x_2, x_2+N\sqrt{\varepsilon_n}]$ together with (3.39b) implies that

3.39e) $$\lim_{\varepsilon_n=0} \frac{|H(x_5(\varepsilon_n),\varepsilon_n)|}{\sqrt{\varepsilon_n}} = +\infty \quad .$$

Consider the change of variables

3.40) $$\tau = (x-x_3)/|H(x_5,\varepsilon_n)|, \; u(\tau) = G(x,\varepsilon_n), \; v(\tau) = \frac{H(x,\varepsilon_n)}{|H(x_5,\varepsilon_n)|} \quad .$$

Substitution into (1.1), (1.2) gives

3.41a) $$\tilde{\varepsilon}_n \ddddot{v} + v\dddot{v} + u\dot{u} = 0$$

3.41b) $$\tilde{\varepsilon}_n \ddot{u} + v\dot{u} - \dot{v}u = 0$$

where

3.41c) $$\tilde{\varepsilon}_n = \varepsilon_n/|H^2(x_5,\varepsilon_n)| \to 0 \quad \text{as} \quad \varepsilon_n \to 0 \quad .$$

The initial conditions are

3.42a) $$v(0) = -N_0\sqrt{\tilde{\varepsilon}_n}, \quad \dot{v}(0) = H'(x_3,\varepsilon_n), \quad \ddot{v}(0) = H''(x_3,\varepsilon_n)|H(x_5,\varepsilon_n)|$$

3.42b) $$\dddot{v}(0) = H'''(x_3,\varepsilon_n)|H(x_5,\varepsilon_n)|^2, \; u(0) = O(\sqrt{\tilde{\varepsilon}_n}|H(x_5,\varepsilon_n)|)$$

3.42c) $$\dot{u}(0) = G'(x_3,\varepsilon_n)|H(x_5,\varepsilon_n)| \quad .$$

At $\tau_5 = (x_5-x_3)/|H(x_5,\varepsilon_n)|$ we have

3.43) $$v(\tau_5) = -1, \; \dot{v}(\tau_5) = H'(x_5,\varepsilon_n) = \frac{1}{2}H'(x_3,\varepsilon_n) \quad .$$

Moreover

$$H(x_5,\varepsilon_n) = H(x_3,\varepsilon_n) + \int_{x_3}^{x_5} H'(t,\varepsilon_n)dt \quad .$$

Since $H(x_5,\varepsilon_n)$, $H(x_3,\varepsilon_n)$ are both negative and (3.39c) holds we have

$$|H(x_5,\varepsilon_n)| \geq |H(x_3,\varepsilon_n)| + |H'(x_5,\varepsilon_n)|(x_5-x_3) \quad .$$

Using (3.39b) we have

$$|H(x_5,\varepsilon_n)| \geq \frac{1}{2}|H'(x_3,\varepsilon_n)|(x_5-x_3) \quad .$$

Therefore

$$\frac{|H(x_5,\varepsilon_n)|-N_0\sqrt{\varepsilon_n}}{C_0} \leq x_5 - x_3 \leq \frac{2|H(x_5,\varepsilon_n)|}{|H'(x_3,\varepsilon_n)|}$$

and

3.44) $$\frac{1-N_0\sqrt{\varepsilon_n}/|H(x_5,\varepsilon_n)|}{C_0} \leq \tau_5 \leq \frac{2}{|H'(x_3,\varepsilon_n)|} \leq \frac{4}{\Delta} \quad .$$

Thus, $\langle v(\tau,\tilde{\varepsilon}_n), u(\tau,\tilde{\varepsilon}_n)\rangle$ are solutions of (1.1), (1.2) on an interval of finite length $[0,\tau_5]$ satisfying the conditions (3.42a)-(3.42c) and (3.43).

We wish to apply Theorem 2.1 and Theorem 3.2. Consider the change of variables

3.45) $$\tilde{v}(\tau,\tilde{\varepsilon}_n) = -v(\tau_5-\tau,\tilde{\varepsilon}_n),\ \tilde{u}(\tau,\tilde{\varepsilon}_n) = \tilde{u}(\tau_5-\tau,\tilde{\varepsilon}_n) \quad .$$

These functions now satisfy all the hypotheses of Theorem 2.1 and Theorem 3.2. Expressing the results directly in terms of the original functions $v(\tau,\tilde{\varepsilon}_n)$, $u(\tau,\tilde{\varepsilon}_n)$ we may assert the following. There exists a subsequence which converges uniformly on $[0,\overline{\tau}_5]$ where $\overline{\tau}_5$ is the limit of $\tau_5(\varepsilon_n)$. Moreover, if $\overline{u}(\tau)$ and $\overline{v}(\tau)$ are the limit functions, then

$$\dot{v}\ (\tau,\varepsilon_n) \to \dot{\overline{v}}\ (\tau) \quad , \quad 0 \le \tau \le \overline{\tau}_5$$

$$\ddot{v}\ (\tau,\varepsilon_n) \to \ddot{\overline{v}}\ (\tau) \quad , \quad 0 \le \tau \le \overline{\tau}_5$$

$$\dddot{v}\ (\tau,\varepsilon_n) \to \dddot{\overline{v}}\ (\tau) \quad , \quad 0 \le \tau \le \overline{\tau}_5$$

$$\dot{u}\ (\tau,\varepsilon_n) \to \dot{\overline{u}}\ (\tau) \quad , \quad 0 \le \tau \le \overline{\tau}_5$$

$$\ddot{u}\ (\tau,\varepsilon_n) \to \ddot{\overline{u}}\ (\tau) \quad , \quad 0 \le \tau \le \overline{\tau}_5 \quad .$$

These results, together with the initial condition (3.42a), (3.42b), (3.42c) imply that

3.46) $$\overline{v}(\tau) = \overline{v}(0) + v_1\tau, \quad 0 \le \tau \le \overline{\tau}_5 \quad ,$$

for some constant v_1.

However (3.43a) and (3.42a) imply that

3.47) $$\dot{\overline{v}}(0) = \frac{1}{2}\,\dot{\overline{v}}(\overline{\tau}_5) \quad .$$

Since (3.46) and (3.47) are in contradiction, the lemma is true.

We have now completed step 3.

We conclude this chapter with the following summary and extension of these results.

Theorem 3.3: Let $\langle H(x,\varepsilon_n),\ G(x,\varepsilon_n)\rangle$ be a sequence of solutions of (1.1), (1.2) which satisfy H.2 and (2.8). Suppose

$$H(x,\varepsilon_n) \to \overline{H}(x),\ G(x,\varepsilon_n) \to \overline{G}(x)$$

in an interval $[x_0,\ x_0+\alpha]$.

Suppose there is a point $\beta \geq \alpha$ such that (3.17a), (3.17b) hold. The functions $(\bar{H}(x), \bar{G}(x))$ are the solutions of the reduced equations (1.3), (1.4) and are given by (1.5a), (1.5b), (1.5c), depending on the limit values of $G(x_0,\varepsilon_n)$, $H(x_0,\varepsilon_n)$, $H'(x_0,\varepsilon_n)$, $H''(x_0,\varepsilon_n)$.

If $\beta = 1$ assume that

3.48) $$H(1,\varepsilon_n) = 0 \quad .$$

In this case there is a constant $\bar{B}$ such that

3.49a) $$|H(x,\varepsilon_n)| + |H'(x,\varepsilon_n)| + |H''(x,\varepsilon_n)| + |H'''(x,\varepsilon_n)| \leq \bar{B}, \quad x_0 \leq x \leq 1$$

3.49b) $$|G(x,\varepsilon_n)| + |G'(x,\varepsilon_n)| + |G''(x,\varepsilon_n)| \leq \bar{B}, \quad x_0 \leq x \leq 1 \quad .$$

If $\beta < 1$, let β_1 be the next zero (if one exists) of $\bar{H}(x)$ and let

3.50) $$\beta_2 = \min(\beta_1, 1) \quad .$$

Then, for any constant q, $0 < q < \frac{1}{2}(\beta_2 - \beta)$ we have the uniform convergence

3.51a) $$(\frac{d}{dx})^k H(x,\varepsilon_n) \to (\frac{d}{dx})^k \bar{H}(x), \quad k = 0,1,2, \qquad x_0 \leq x \leq \beta_2 - q \quad ,$$

3.51b) $$(\frac{d}{dx})^j G(x,\varepsilon_n) \to (\frac{d}{dx})^j \bar{G}(x), \quad j = 0,1, \qquad x_0 \leq x \leq \beta_2 - q \quad .$$

Proof: It is only necessary to establish (3.51a), (3.51b). However, once $H(x_5,\varepsilon_n) \leq -\delta_3 < 0$ we may apply Theorem 1.2 on the interval on which $\bar{H}(x,\varepsilon_n) < 0$. Theorem 3.2 assures us that $H'''(x,\varepsilon_n)$ and $G''(x,\varepsilon_n)$ are bounded in a transition layer in which $H(x,\varepsilon_n)$ goes from $K\sqrt{\varepsilon_n}$ to $-\tilde{K}\sqrt{\varepsilon_n}$ for any $\tilde{K}$. Arguing as in lemma 3.5 we consider the change of variables (2.7). The region where $H(x,\varepsilon) < 0$ now becomes a region where $\tilde{H}(x,\varepsilon) > 0$, and β, which was to the left of the negative values, goes over to $\beta' = 1 - \beta$ which is to the right of the positive values. Therefore, we may apply Theorem 3.2 and match bounds in the overlapping regions to see that $H'''(x,\varepsilon_n)$ and $G''(x,\varepsilon_n)$ are bounded on $[x_0, \beta_2 - q]$. Thus, (3.51a), (3.51b) follow at once.

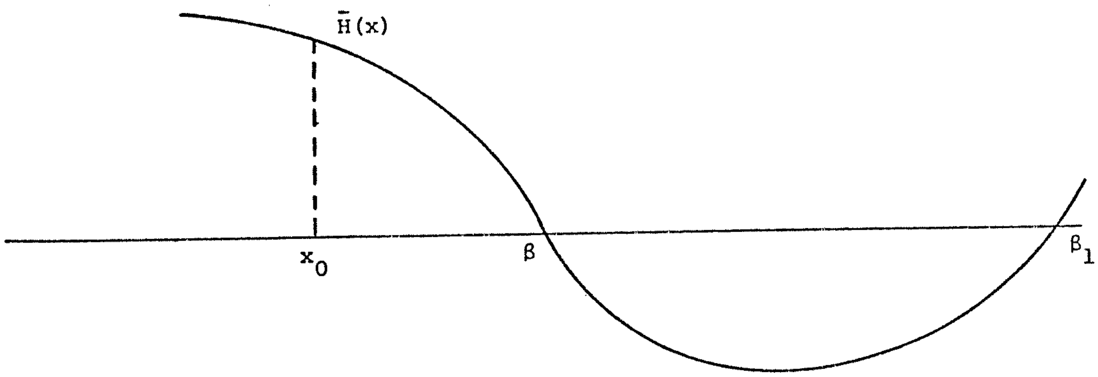

Figure 2

4. Non-Existence Theorems

In this section we consider solutions $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$ which satisfy the boundary conditions (1.6a), (1.6b) and prove a basic non-existence result: there do not exist limit solutions $\bar{H}(x)$ which satisfy (3.17a), (3.17b). Our first result is an immediate consequence of theorem 3.3.

Theorem 4.1: Let $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$ be a sequence of solutions of (1.1), (1.2), (1.6a), (1.6b) which satisfy H.2 and (2.8). Suppose that

4.1) $$H(x,\varepsilon_n) \to \bar{H}(x)$$

uniformly on $[0,1]$. Suppose that $\beta = 1$ is the first zero of $\bar{H}(x)$ with $\beta > x_0$. Then

4.2) $$\bar{H}'(1) = 0 \quad .$$

Proof: From the form of $\bar{H}(x)$ given by (1.5a) or (1.5b) we see that either (4.2) holds or (3.17b) holds. Suppose (3.17b) holds. Then we may apply Theorem 3.3. However, (3.49) of Theorem 3.3 and (1.6b) imply that we may extract a subsequence which will converge to a limit $\tilde{H}(x)$ and this convergence will be $C^1[x_0,1]$ convergence. Hence $\tilde{H}'(1) = 0$. But, of course, $\tilde{H}(x) = \bar{H}(x)$ and the theorem is proven.

We now turn our attention to the case where β the first zero of $\bar{H}(x)$ greater than x_0, satisfies

$$x_0 < \beta < 1 \quad .$$

In this case we make use of the properties of the function

4.3) $$\Phi(x,\varepsilon) = [G'(x,\varepsilon)]^2 + [H''(x,\varepsilon)]^2 \quad .$$

The basic result is due to McLeod [7], [8].

Lemma 4.1: The function $\Phi(x,\varepsilon)$ satisfies the differential equation

4.4) $$\varepsilon\Phi'' + H\Phi' = 2\varepsilon[(G'')^2 + (H''')^2] \quad ,$$

and the function

$$\Phi'(x,\varepsilon)\ \exp\{\frac{1}{\varepsilon}\int_{x_0}^{x} H(t,\varepsilon)dt\}$$

has at most one zero. Thus the behavior of the function $\Phi(x,\varepsilon)$ is described in one of the following three ways

(α) Φ is monotone decreasing on its interval of definition,

(β) Φ is monotone increasing on its interval of definition,

(γ) there is an interior point γ such that $\Phi' < 0$ for $x < \gamma$ and $\Phi' > 0$ for $x > \gamma$.

Lemma 4.2: Let $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$ be a sequence of solutions of (1.1), (1.2) which satisfy the hypotheses of theorem 3.3. Let β, the first zero of $\bar{H}(x)$ greater than x_0, satisfy

$$x_0 < \beta < 1 \quad .$$

As in theorem 3.3 let β_1 be the next zero of $\bar{H}(x)$ and let β_2 be given by (3.50). Let a the first zero of $\bar{H}(x)$ to the left of x_0. Let

4.5) $$a' = \max(a,0) \quad .$$

Then, $\bar{H}(x)$ is a quadratic of the form (1.5c) on the open interval $a' < x < \beta_2$.

Proof: From theorem 3.3 and theorem 2.1 and its corollary we see that it is sufficient to show that $\bar{H}(x)$ is a quadratic on a subinterval of (a,β_2). We focus our attention on an interval $[x_0,x_0+\rho]$ on which

$$\bar{H}(x,\varepsilon_n) > \delta/2 \quad .$$

Suppose the lemma is false and $\bar{H}(x)$ is given by (1.5b) with $\tau_0 \neq 0$. We claim that, if ε_n is sufficiently small,

4.6) $$\Phi'(x,\varepsilon_n) > 0, \quad x_0 \leq x \leq x_0+\rho \quad .$$

To see this we observe that theorem 2.1 and the form of $\bar{H}(x)$ together with the differential equation (4.4) imply that

$$\Phi'(x,\varepsilon_n) = O(\varepsilon_n), \quad x_0 \leq x \leq x_0+\rho \quad .$$

Thus, since Φ''' is bounded on that interval, Landau's theorem implies that

$$\Phi''(x,\varepsilon_n) = O(\varepsilon_n^{1/2})$$

and

$$H\ \Phi' = 2\ \varepsilon_n[(G'')^2 + (H''')^2] + O(\varepsilon_n^{3/2}) \quad .$$

Thus, we have (4.6). From lemma 4.1 we see that

$$\Phi'(x,\varepsilon_n) > 0, \quad x_0 \le x \le 1 \quad .$$

However, let $[\tilde{a}\ ,b'] \subset (\beta,\beta_2)$ be an interval on which $H(x,\varepsilon_n) < 0$. Applying the argument above, we see that

$$\Phi'(x,\varepsilon_n) < 0, \quad \tilde{a} < x < b' \quad .$$

Thus the lemma is proven.

Remark: As we shall see, the results of section 5 show that the quantity

$$|H'''(x,\varepsilon_n)| + |G(x,\varepsilon_n)| + |G'(x,\varepsilon_n)|$$

is exponentially small (in ε_n) on $(a'+\delta',\beta_2-\delta')$.

Theorem 4.2: Let $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$ be a sequence of solutions of (1.1), (1.2), (1.6a) which satisfy H.2 and (2.8). Suppose that (4.1) holds uniformly on $[0,1]$. Let β, with $0 < \beta < 1$ be the first zero of $\bar{H}(x)$ with $\beta > x_0$. Then

4.7) $$\bar{H}(\beta) = 0 \quad .$$

Proof: Suppose the theorem is false. Then we may apply lemma 4.2 to see that $\bar{H}(x)$ is a quadratic in the interval $[a',\beta_2]$. Since $\bar{H}(\beta) = 0$, $\bar{H}(x)$ can have only one other zero. Thus, either $\bar{H}(x) > 0$ for all $x < \beta$ or $\bar{H}(x) < 0$ for all $x > \beta$. Whichever case occurs, the boundary condition (1.6a) is violated either at $x = 0$ or at $x = 1$.

5. Oscillating Solutions

In this section we consider the situation described in the introduction by (1.7a)-(1.7c), (1.8), (1.9a), (1.9b). Notice that we specifically give up the hypothesis H.2. The problem is illustrated in figure 3.

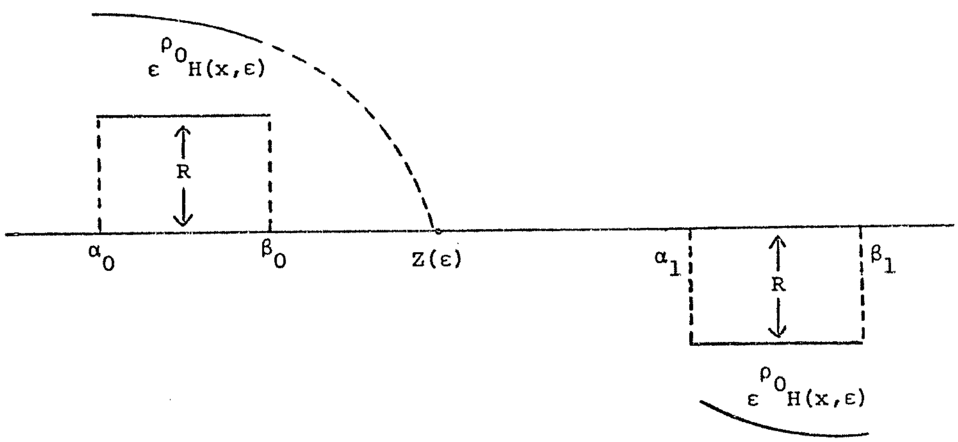

Figure 3

Our first goal is to establish the fact that $\varepsilon_n^{\rho_0}H(x,\varepsilon_n)$, $\varepsilon_n^{\rho_0}H'(x,\varepsilon_n)$, $\varepsilon_n^{\rho_0}H''(x,\varepsilon_n)$, $\varepsilon_n^{\rho_0}G(x,\varepsilon_n)$, $\varepsilon_n^{\rho_0}G'(x,\varepsilon_n)$ are uniformly bounded on $[\alpha_0+\delta', \beta_1-\delta']$ for any $\delta' > 0$. Thus, after extracting a subsequence, we may assume that

5.1) $$\varepsilon_n^{\rho_0}(\tfrac{d}{dx})^{\nu}H(x,\varepsilon_n) \to (\tfrac{d}{dx})^{\nu}h(x), \ \alpha_0 + \delta' \le x \le \beta_1 - \delta', \ \nu = 0,1,$$

5.2) $$\varepsilon_n^{\rho_0}G(x,\varepsilon_n) \to g(x), \ \alpha_0 + \delta' \le x \le \beta_1 - \delta' \quad .$$

Our major result is that $h(x)$ is a piecewise quadratic with at most two pieces. Moreover

$$\varepsilon^{\rho_0}(|H'''(x,\varepsilon_n)| + |G(x,\varepsilon_n)|)$$

is exponentially small in any sub interval on which $h(x)$ is a quadratic.

Since (1.7c) holds, i.e., $\rho_0 > -1$ we may normalize the problem so that for appropriate constants $C_0 > 0$, $\sigma \geq 0$ we have

5.3) $$|H(x,\varepsilon)| \leq C_0, \qquad \alpha_0 \leq x \leq \beta_1 \quad ,$$

5.4) $$|(\frac{d}{dx})^{\nu+1}H(x,\varepsilon)| + |(\frac{d}{dx})^{\nu}G(x,\varepsilon)| \leq C_0\varepsilon^{-\sigma}, \ \nu = 0,1,2,3,4, \ \alpha_0 \leq x \leq \beta_1 \quad .$$

To see this we observe that if

$$\tilde{H}(x,\varepsilon) = \varepsilon^{\rho_0}H(x,\varepsilon)$$

$$\tilde{G}(x,\varepsilon) = \varepsilon^{\rho_0}G(x,\varepsilon)$$

$$\tilde{\varepsilon} = \varepsilon^{1+\rho_0}$$

then multiplication of (1.1), (1.2) by $\varepsilon^{2\rho_0}$ shows that $\langle\tilde{H}(x,\varepsilon), \tilde{G}(x,\varepsilon)\rangle$ satisfies (1.1), (1.2) with ε replaced by $\tilde{\varepsilon}$. Moreover, $\tilde{\varepsilon} \to 0$ as $\varepsilon \to 0$.

Let $\gamma = \gamma(\varepsilon)$ be the point at which $\Phi(x,\varepsilon)$ assumes its minimum. Let $Z = Z(\varepsilon)$ be the first zero of $H(x,\varepsilon)$ greater than β_0. Since we may always apply the transformation (2.7), we may assume that

5.5) $$\beta_0 < Z(\varepsilon) \leq \gamma(\varepsilon) \quad .$$

Throughout this section we will assume $0 < \varepsilon < 1$ and that (5.3), (5.4) and (5.5) hold. In particular, the situation depicted in figure 3 holds with $\rho_0 = 1$.

In order to obtain the desired uniform bounds and (5.1), (5.2), (5.3) we make use of the function $\Phi(x,\varepsilon)$ given by (4.1). The basic result is: if $H(x,\varepsilon)$ is bounded away from zero on an interval and

$$\Phi'(x,\varepsilon)H(x,\varepsilon) < 0$$

then

$$|H'''(x,\varepsilon)| + |G(x,\varepsilon)| + |G'(x,\varepsilon)|$$

is exponentially small.

Lemma 5.1: Let $(H(x,\varepsilon),\ G(x,\varepsilon))$ be a solution of (1.1), (1.2) on $[\alpha_0,\beta_1]$. Let δ_1 be a fixed constant with

$$0 < \delta_1 \leq \frac{1}{2}(\beta_0-\alpha_0)$$

and let y_1 be the first point greater than β_0 - if such a point exists - at which

5.6) $$\int_{\alpha_0}^{y_1} H(t,\varepsilon)dt = R\delta_1 \quad .$$

If no such point exists, then $y_1 = \beta_1$. Let

5.7) $$y = \min(y_1,\gamma) \quad .$$

Then there is a constant K, depending only on C_0, such that, on the interval $[\alpha_0+\delta_1,y]$ we have the estimates

5.8a) $$|\Phi'(x,\varepsilon)| \leq K\varepsilon^{-2\sigma}\exp\{-R\delta_1/\varepsilon\}$$

5.8b) $$|H'''(x,\varepsilon)| + |G''(x,\varepsilon)| \leq K\varepsilon^{-\sigma}\exp\{-R\delta_1/4\varepsilon\}$$

5.8c) $$|G'(x,\varepsilon)| \leq K\varepsilon^{-\sigma}\exp\{-R\delta_1/16\varepsilon\}$$

5.8d) $$|H'(x,\varepsilon)| + |H''(x,\varepsilon)| \leq K(1 + \varepsilon^{-\sigma}\exp\{-R\delta_1/4\varepsilon\}) \quad .$$

Proof: From (4.3) we see that

$$|\Phi'| = 2|H''H''' + G'G''| \leq 4C_0^2\varepsilon^{-2\sigma} \quad .$$

Applying lemma 4.1 we have

$$0 \geq \Phi'(x,\varepsilon)\exp\{\frac{1}{\varepsilon}\int_{x_0}^{x} H(t,\varepsilon)dt\} \geq \Phi'(x_0,\varepsilon) \quad .$$

Since

$$\int_{\alpha_0}^{x} H(t,\varepsilon)dt \geq R\delta_1, \quad \alpha_0 + \delta_1 \leq x \leq y$$

we obtain (5.8a). Thus Landau's theorem implies that

$$|\Phi''| \leq C\varepsilon^{-2\sigma}\exp\{-R\delta_1/2\varepsilon\} .$$

The differential equation (4.4) now yields (5.8b). In order to prove (5.8c) we consider two cases.

Case 1: There is a point, say $a \in [\alpha_0+\delta_1, y]$ at which (5.8c) holds. In this case (5.8c) holds on the entire interval $[\alpha_0+\delta_1, y]$ by virtue of (5.8b) and an integration.

Case 2: There is a constant, say E, so that: at every point $x \in [\alpha_0+\delta_1, y]$ we have

5.9) $$|G'(x,\varepsilon)| \geq E\varepsilon^{-\sigma}\exp\{-R\delta_1/16\varepsilon\} .$$

However, (5.8b) and (5.4) together with Landau's theorem imply that

$$|H^{IV}(x,\varepsilon)| \leq C\varepsilon^{-\sigma}\exp\{-R\delta_1/8\varepsilon\} .$$

Substitution into (1.1) now yields

$$|G(x,\varepsilon)\ G'(x,\varepsilon)| \leq C\varepsilon^{-\sigma}\exp\{-R\delta_1/8\varepsilon\} .$$

Thus, (5.9) implies that

5.10) $$|G(x,\varepsilon)| \leq \frac{C}{E}\exp\{-R\delta_1/16\varepsilon\} .$$

However, if (5.10) holds, (5.8c) follows from (5.8b) and Landau's theorem.

Finally (5.8d) follows from (5.8b) and Landau's theorem.

Having obtained these estimates we are able to establish the basic bounds.

Theorem 5.1: Let $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$ be a sequence of solutions of (1.1), (1.2) on the interval $[\alpha_0, \beta_1]$. Suppose that (5.3), (5.4) and (1.9a), (1.9b) hold (the problem has been normalized so that $\rho_0 = 1$). Let

5.11) $$\delta_1 \leq \frac{1}{2}\min(\beta_0-\alpha_0,\ \beta_1-\alpha_1) .$$

Then there is a constant $M > 0$, depending on δ_1, such that, on the interval $[\alpha_0+\delta_1,\ \beta_1-\delta_1]$ we have

5.12) $$\left|\left(\frac{d}{dx}\right)^{\nu} H(x,\varepsilon_n)\right| + \left|\left(\frac{d}{dx}\right)^{k} G(x,\varepsilon_n)\right| \le M, \quad \nu = 0,1,2, \quad k = 0,1 \quad .$$

Moreover, on the interval $[\alpha_0+\delta_1, y]$

5.13) $$|G(x,\varepsilon_n)| \le M\varepsilon_n^{-\sigma}\exp\{-R\delta_1/16\varepsilon_n\}$$

provided that ε_n is small enough.

Proof: We consider two cases.

Case 1: $$\gamma \le \beta_1-\delta' \quad .$$

In this case, after the change of variables (2.7) we may apply lemma 5.1 to find that

$$\Phi(x,\varepsilon_n) \le K_1, \quad \beta_1 - \delta' \le x \le \beta_1 - \frac{1}{2}\delta' \quad .$$

That is, using Landau's theorem,

5.14) $$|G'(x,\varepsilon_n)| + |H''(x,\varepsilon_n)| + |H'(x,\varepsilon_n)| \le K_2, \quad \beta_1 - \delta' \le x \le \beta_1 - \frac{1}{2}\delta' \quad .$$

Applying lemma 4.1, i.e., the fact that $\Phi(x,\varepsilon_n)$ assumes its maximum at the end points, we have - using Landau's theorem -

5.15) $$|G''(x,\varepsilon_n)| + |H''(x,\varepsilon_n)| + |H'(x,\varepsilon_n)| \le M_1, \quad \alpha_0 + \delta' \le x \le \beta_1 - \delta' \quad .$$

The function $H(x,\varepsilon_n)$ is converging to a quadratic function on $[\alpha_0+\delta', y]$ which has at least one zero in the interval $[\beta_0, y]$. Thus, there are points in the interval $[\alpha_0+\delta', y]$ at which $|H'(x,\varepsilon_n)| \ge R$. The estimate (5.13) follows from (1.2) and (5.8b), (5.8c).

Finally, the complete estimate (5.12) follows from (5.15), (5.13) and an integration.

Having established this basic result, we may apply the theory developed in sections 2, 3, and 4.

We now analyze the limit functions $\bar{H}(x)$, $\bar{G}(x)$. Let

5.16a) $$H(x,\varepsilon_n) \to \bar{H}(x) \quad , \quad \alpha_0 + \delta' \le x \le \beta_1 - \delta' \quad ,$$

5.16b) $$G(x,\varepsilon_n) \to \bar{G}(x) \quad , \quad \alpha_0 + \delta' \le x \le \beta_1 - \delta' \quad .$$

Let

5.17a) $$y \to \bar{y}$$

5.17b) $$Z(\varepsilon_n) \to Z_0 \quad ,$$

5.17c) $$\gamma(\varepsilon_n) \to \bar{\gamma} \quad .$$

Finally, let $\bar{Z} \geq Z_0$ be the first "crossing" zero of $\bar{H}(x)$. That is, there is a positive constant $p > 0$ such that

5.18a) $$\bar{H}(x) \geq 0, \quad \alpha_0 \leq x < \bar{Z}$$

5.18b) $$\bar{H}(\bar{Z}) = 0,$$

5.18c) $$\bar{H}(x) \leq 0, \quad \bar{Z} \leq x \leq \bar{Z} + p$$

5.18d) $$\bar{H}(\bar{Z}+p) < 0 \quad .$$

<u>Theorem 5.2</u>: Let the hypothesis of theorem 5.1 hold. Let (5.16a) - (5.18d) hold. Suppose

5.19) $$\bar{Z} < \bar{\gamma} \quad .$$

Then

5.20) $$H'(\bar{Z}) < 0 \quad ,$$

and $\bar{H}(x)$ is a piecewise quadratic with at most two pieces. Furthermore

5.21) $$|H'''(x,\varepsilon_n)| + |G(x,\varepsilon_n)| + |G'(x,\varepsilon_n)|$$

is exponentially small in ε_n on a proper subinterval of the interval on which $\bar{H}(x)$ is a quadratic.

<u>Proof</u>: From the definition of y in the construction of lemma 5.1 we see that

$$\bar{Z} < \bar{y} \leq \bar{\gamma} \quad .$$

Thus, the estimates of lemma 5.1 hold on the entire interval $[\alpha_0+\delta', \frac{1}{2}(\bar{Z}+\bar{y})]$. Therefore, $\bar{H}(x)$ is a quadratic on this interval. If (5.20) did not hold, $\bar{Z}$ would not be a crossing zero. Hence, (5.20) holds. Moreover

$$\bar{H}(x) > 0, \quad \alpha_0 < x < \bar{Z}$$

$$\bar{H}(x) < 0, \quad \bar{Z} < x < \frac{1}{2}(\bar{Z}+\bar{y}) \quad .$$

Thus we may apply lemma 4.2 to see that $\bar{H}(x)$ is a quadratic until the "next" zero (after $\bar{Z}$) of $\bar{H}(x)$. Moreover, from the estimates of lemma 5.1 and (5.13) we see that (5.21) is exponentially small on the interval $[\alpha_0+\delta', \frac{1}{2}(\bar{Z}+\bar{y})]$. A standard singular perturbation argument now shows that, in fact, this quantity is exponentially small in an interval $[\alpha_0+\delta', \beta_2-\delta_2]$ where β_2 is the next zero of $\bar{H}(x)$ (if it exists) and δ_2 is any positive constant.

Case 1: $\bar{Z}$ is the only zero of $\bar{H}(x)$ on the interval $[\alpha_0,\beta_1]$. In this case $\bar{H}(x)$ is a quadratic on the entire interval $[\alpha_0,\beta_1]$ and the theorem is proven.

Case 2: There is a β_2 with

5.22a) $$\bar{Z} < \beta_2 < \alpha_1$$

and β_2 is the next zero of $\bar{H}(x)$, i.e.,

5.22b) $$\bar{H}(\beta_2) = 0 \quad .$$

In this case, since $\bar{H}(x)$ is a quadratic on the interval $(\bar{Z},\beta_2)$ and $\bar{H}'(x)$ is continuous,

$$\bar{H}'(\beta_2) = -\bar{H}'(\bar{Z}) \neq 0 \quad .$$

Furthermore, there must be a third zero, say β_3, with

$$\beta_2 < \beta_3 < \alpha_1$$

and

$$\bar{H}(x) > 0, \quad \beta_2 < x < \beta_3 \quad .$$

However, from theorem 2.1 and its corollary we see that

$$\bar{H}'(\beta_3) = -\bar{H}'(\beta_2) = \bar{H}'(Z) \neq 0 \quad .$$

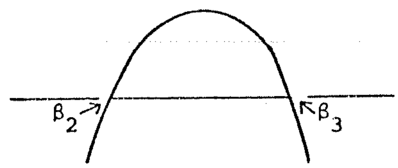

Figure 4

Thus we may apply lemma 4.2 to see that $\bar{H}(x)$ is a quadratic on the interval

$$\beta_2 < x < \tilde{\beta}$$

where $\tilde{\beta}$ is the next zero of $\bar{H}(x)$ beyond β_3. But, of course, $\bar{H}(x)$ is a quadratic which vanishes at β_2 and β_3. Hence there is no $\tilde{\beta}$. Thus, $\bar{H}(x)$ is a piecewise quadratic with a break in $\bar{H}''(x)$ at β_2. Moreover, the quantity (5.21) is exponentially small in any interval $[\alpha_0+\delta', \beta_2-\delta_2]$.

It remains to show that the quantity (5.21) is exponentially small on $[\beta_2+\delta_2, \beta_1-\delta']$. We sketch the argument.

<u>Case 2.1</u>:

$$\bar{\gamma} < \beta_3 \quad .$$

In this case we apply the change of variables (2.7) and repeat the above arguments.

<u>Case 2.2</u>: $$\beta_3 < \bar{\gamma} \quad .$$

In this case we apply lemma 5.1 on the interval $[\beta_2 + \frac{1}{2}\delta_2, \beta_3 - \frac{1}{2}\delta_2]$ to obtain the initial exponential bounds on the quantity (5.21). Then, we merely repeat the above discussion.

<u>Case 2.3</u>: $$\beta_3 = \bar{\gamma} \quad .$$

In this case we must match the exponential bounds to the right and left of β_3. To complete the proof "at β_3" we use a "shooting" argument as in section 3.

<u>Theorem 5.3</u>: Let the hypotheses of Theorem 5.1 hold. Let (5.16a) - (5.18d) hold. Suppose

5.23) $$\bar{z} = \bar{\gamma}$$

and

5.24) $$\bar{H}'(\bar{Z}) < 0 \quad .$$

Then the conclusions of Theorem 5.2 hold.

Proof: In the proof of Theorem 5.2 the condition (5.19) is used only for two purposes, to prove (5.20) - which we have explicitly assumed in (5.24) and to prove that (5.21) is exponentially small on a non vanishing interval beyond $\bar{Z}$. Thus, we need only prove that (5.21) is exponentially small. However, (5.21) is exponentially small whenever $H\Phi' < 0$. Thus, if $\bar{H}(x)$ has only one zero, $\bar{Z}$, we match the exponential decay on either side of $\bar{Z}$. If there are at least two other zeros, $\beta_2 < \beta_3 < \alpha_1$ we apply the transformation (2.7) and apply the above argument.

Theorem 5.4: Let the hypotheses of Theorem 5.1 hold. Let (5.16a) - (5.18d) hold. Suppose

5.25) $$\bar{Z} = \bar{\gamma} \quad ,$$

and

5.26) $$\bar{H}'(\bar{Z}) = 0 \quad .$$

Then $\bar{H}(x)$ is a quadratic on $[\alpha_0+\delta', \bar{Z}]$ and $\bar{H}(x)$ is a quadratic on $[\bar{Z}, \beta_1-\delta']$. Moreover (5.21) is exponentially small on every interval $[\alpha_0+\delta', \bar{Z}-\delta_2]$, $[\bar{Z}+\delta_2, \beta_1-\delta']$.

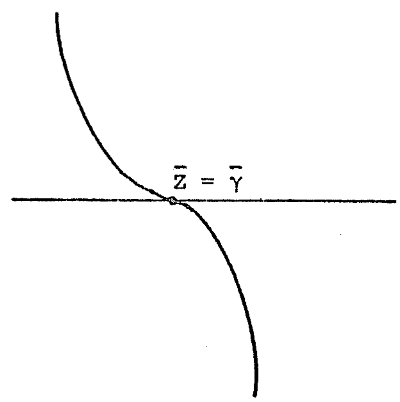

Figure 5

Proof: Apply lemma 5.1 on every interval $[\alpha_0+\delta', \bar{Z}-\delta_2]$. Thus, $\bar{H}(x)$ is quadratic on $[\alpha_0+\delta', \bar{Z}]$ and (5.21) is expenentially small on $[\alpha_0+\delta', \bar{Z}-\delta_2]$. To complete the proof we apply the change of variables (2.7) and argue on the interval

$$1 - (\bar{Z}+a) \leq x \leq 1 - \bar{Z} .$$

Thus, to the right of $\bar{Z}$, $\bar{H}(x)$ is a quadratic as long as it is negative! However, since

$$\bar{H}(\bar{Z}) = \bar{H}'(\bar{Z}) = 0$$

and $\bar{H}(x)$ is negative a bit to the right of $\bar{Z}$, it is always negative.

A simple argument shows that the case

$$\bar{Y} < \bar{Z}$$

is impossible. Thus, we have established the major result of this section: _On the interval_ $[\alpha_0, \beta_1]$ _the function_ $\bar{H}(x)$ _is a piecewise quadratic with at most two pieces. Moreover, on any proper subinterval of an interval on which_ $\bar{H}(x)$ _is quadratic_, (5.21) _is exponentially small in_ ε_n.

Now, let us turn to "oscillating cells".

Definition: Let $\langle H(x,\varepsilon_n), G(x,\varepsilon_n)\rangle$ be a sequence of solutions of (1.1), (1.2) which satisfy (1.7a), (1.7b). Suppose that

$$\varepsilon_n^{\rho_0} H(x,\varepsilon_n) \to h(x), \quad 0 < \delta' \leq x \leq 1 - \delta' < 1$$

for every δ', $0 < \delta' < \frac{1}{4}$. A "cell" is an interval (α,β) with $0 \leq \alpha < \beta \leq 1$ such that;

5.27a) either $\alpha = 0$ or $h(\alpha) = 0$, and

5.27b) either $\beta = 1$ or $h(\beta) = 0$, and

5.27c) $|h(x)| > 0, \quad \alpha < x < \beta$.

Note: As an example, the solutions obtained in [8] satisfy (1.7a), (1.7b) with $\rho_0 = -\frac{1}{2}$. The results of [8] show that those solutions converged to a function $h(x)$ with two cells.

In this context, theorems 5.2, 5.3, 5.4 assert that if $h(x)$ has two cells, (α_0,β_0), (α_1,β_1) and

5.28a) $h(x) > 0, \quad \alpha_0 < x < \beta_0$

5.28b) $h(x) < 0, \quad \alpha_1 < x < \beta_1$.

Then $h(x)$ has at most four cells. To see this, let (α_0,β_0) be the "first" interval on which $h(x) > 0$ and (α_1,β_1) be the "last" interval on which $h(x) < 0$. Applying Theorems 5.2, 5.3, 5.4 we see that $h(x)$ is a piecewise quadratic with at most two pieces.

Case 1: $h(x)$ is a quadratic on the entire interval (α_0,β_1). Then since $h(x)$ has an odd number of zeros in (α_0,β_1), $h(x)$ has exactly one zero, say β, in $\alpha_0 < x < \beta_1$. We note that $h(x)$ remains a quadratic for $x < \alpha_0$ as long as $h(x) > 0$ and $h(x)$ remains a quadratic for $x > \beta_1$ as long as $h(x) < 0$.

Case 1.1: $$h''(x) > 0, \quad \alpha_0 < x < \beta_1 \quad .$$

In this case $h(x) > 0$ for $0 < x < \beta$ and we have two or three cells depending on whether or not $h(x)$ becomes positive to the right of β_1.

Case 1.2: $$h''(x) < 0, \quad \alpha_0 < x < \beta_1 \quad .$$

In this case $h(x) < 0$ for $\beta_1 < x < 1$ and we have two or three cells depending on whether or not $h(x)$ becomes negative to the left of α_0.

Case 2: There is a point $\tilde{\beta} \in (\alpha_0,\beta_1)$ and $h''(x)$ jumps at $\tilde{\beta}$. That is

$$h(x) = \begin{cases} h_1(x), & \alpha_0 < x < \tilde{\beta} \\ h_2(x), & \tilde{\beta}_2 < x < \beta_1 \end{cases}$$

and $h_1(x)$, $h_2(x)$ are quadratic polynomials.

Case 2.1: $$h'(\tilde{\beta}) = 0 \quad .$$

In this case we have

$$h_1''(x) > 0$$

$$h_2''(x) < 0$$

and

$$h(x) = \begin{cases} h_1(x), & 0 < x < \tilde{\beta} \\ h_2(x), & \tilde{\beta} < x < 1 \end{cases}$$

and we have a two cell solution.

Case 2.2: $h'(\tilde{\beta}) \neq 0$.

Applying lemma 4.3 we see that β is the second zero of $h(x)$ for $\alpha_0 < x < \beta_1$. Thus

$$h_1'' > 0 \quad ,$$

and $h(x) > 0$ for $0 < x < \alpha_0$. Also, there must be a third zero, say β_3 and

$$h(x) > 0, \quad \beta < x < \beta_3 \quad .$$

Hence

$$h_2'' < 0$$

and $h(x) < 0$ for $\beta_1 < x < 1$. In this case we have a four cell solution.

REFERENCES

[1] G. K. Batchelor. Note on a class of solutions of the Navier-Stokes equations representing steady rotationally-symmetric flow. Quart. J. Mech. Appl. Math. 4 (1951), 29-41.

[2] T. von Kármán. Über laminare und turbulente Reibung. Z. Angew. Math. Mech. 1 (1921), 232-252.

[3] H. O. Kreiss. Difference methods for stiff ordinary differential equations. SIAM J. Num. Anal. 15 (1978), 21-58.

[4] H. O. Kreiss and S. V. Parter. On the swirling flow between rotating coaxial disks, asymptotic behavior I. MRC Report #1941 (1979).

[5] E. Landau. Einige Ungleichung für zweimal differenzbare funktionen. Proc. London Math. Soc. (2) 13 (1913), 43-49.

[6] J. B. McLeod. The asymptotic form of solutions of von Kármán's swirling flow problem. Quart. J. Math. Oxford Ser. 20 (1969), 483-496.

[7] J. B. McLeod. A note on rotationally symmetric flow above an infinite rotating disc. Mathematika 17 (1970), 243-249.

[8] J. B. McLeod and S. V. Parter. On the flow between two counter-rotating infinite plane disks. Arch. Rational Mech. Anal. 54 (1974), 301-327.

[9] G. L. Mellor, P. J. Chapple and V. K. Stokes. On the flow between a rotating and a stationary disk. J. Fluid Mech. 31 (1968), 95-112.

[10] S. M. Roberts and J. S. Shipman. Computation of the flow between a rotating and a stationary disk. J. Fluid Mech. 73 (1976), 53-63.

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