CHARACTERIZATIONS OF BOUNDED SOLUTIONS OF LINEAR COMPLEMENTARITY PROBLEMS

by

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ABSTRACT

A number of equivalent characterizations for the existence and boundedness of solutions of the linear complementarity problem: $Mx + q \geqq 0$, $x \geqq 0$, $x^T(Mx+q) = 0$ where M is an $n\times n$ real matrix and q is an n-vector, are given for the case when M is copositive plus. The special case when M is skew-symmetric covers the linear programming case. One useful characterization of existence and boundedness of solutions is given by solving a simple linear program. Other important characterizations are the Slater constraint qualification and the stability condition that for all arbitrary but sufficiently small perturbations of the data M and q which maintain copositivity plus, the perturbed linear complementarity problem is solvable and its solutions are uniformly bounded. An interesting sufficient condition for boundedness of solutions is that the linear complementarity problem have a nondegenerate vertex solution. Another result is that the subclass $\mathcal{M}$ of copositive plus matrices for which the linear complementarity problem has a solution for each q in R^n, that is $\mathcal{M} \subset Q$, coincides with the subclass of copositive plus matrices for which the linear complementarity problem has a nonempty bounded solution set for each q in R^n.

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1. Introduction

The principal purpose of this paper is to give a number of characterizations for the existence and boundedness of solutions of the linear complementarity problem of finding an x in the n-dimensional real Euclidean space R^n such that

$$Mx + q \geqq 0,\ x \geqq 0,\ x^T(Mx+q) = 0 \tag{1}$$

where M is a given $n\times n$ real matrix, q is a given vector in R^n and T denotes the transpose. We shall refer to this problem as LCP(M,q). Existence of solutions for the linear complementarity problem has been investigated by many authors [7,8,3,15,5]. One well known existence result [5,2] is that if the feasible region of (1)

$$S(M,q) = \{x \mid Mx+q \geqq 0,\ x \geqq 0\} \tag{2}$$

is nonempty then the solution set of (1)

$$\bar{S}(M,q) = \{x \mid Mx+q \geqq 0,\ x \geqq 0,\ x^T(Mx+q)=0\} \tag{3}$$

is nonempty when the matrix M is copositive plus, that is

$$\begin{aligned} &\text{(a)}\quad x \geqq 0 \ \text{ implies } \ x^TMx \geqq 0 \ \text{ (copositive)} \\ &\text{(b)}\quad x \geqq 0,\ x^TMx = 0 \ \text{ imply } \ (M+M^T)x = 0 \ \text{ (plus)} \end{aligned} \tag{4}$$

Copositive plus matrices include positive semidefinite matrices and positive matrices. Other existence results are given in [5,11,12,17]. Global uniqueness of solution of the linear complementarity problem has been investigated in among others [15,1,14] and local uniqueness in [13]. Robinson [20] and Doverspike [4] have characterized nonemptiness and boundedness of $\bar{S}(M,q)$ by the nonemptiness of $\bar{S}(\tilde{M},\tilde{q})$ for all $(\tilde{M},\tilde{q})$ sufficiently close to (M,q) for the cases when M is positive semidefinite and copositive plus respectively. In addition, for positive semidefinite M Robinson has further characterized the nonemptiness and boundedness of $\bar{S}(M,q)$ by the nonemptiness and uniform boundedness of $\bar{S}(\tilde{M},\tilde{q})$ for positive semidefinite M and $\tilde{M}$. One of our characterizations, (xvi) of Theorem 2, extends this result of Robinson to the copositive plus M and $\tilde{M}$. Another characterization of the nonemptiness and boundedness of $\bar{S}(M,q)$, (x) of Theorem 2, extends the corresponding boundedness results of Williams [23] for a dual pair of linear programming problems to the linear complementarity problem with a copositive plus matrix. A useful feature of another characterization, Theorem 2 (vii), is that without knowing whether (1) has a solution and without knowing any of its solutions we can determine if its solution set $\bar{S}(M,q)$ is nonempty and bounded from the following equivalence

$$\left.\begin{array}{l}\bar{S}(M,q) \text{ is nonempty}\\ \text{and bounded}\end{array}\right\} \underset{\substack{\text{copositive}\\ \text{plus}}}{\overset{M}{\Longleftrightarrow}} \left\{ \max_{u} \{e^T u \mid M^T u \leqq 0,\ q^T u \leqq 0,\ u \geqq 0\} = 0 \right. \tag{5}$$

where e is a vector of ones in R^n. Note that the right hand side of the equivalence (5) can be easily checked by solving a simple linear programming problem. Another interesting characterization of the nonemptiness and boundedness of $\bar{S}(M,q)$ for copositive plus matrices, Theorem 2 (iii), is that the feasible region $S(M,q)$ be stable [19] or equivalently that it satisfies the Slater constraint qualification [9].

The principal results of the paper are contained in Theorem 2 which gives a number of equivalent characterizations for the nonemptiness and boundedness of the solution set $\bar{S}(M,q)$. The first eight characterizations of Theorem 2 are stability or constraint qualification conditions for the feasible region $S(M,q)$ and they do not require any assumptions on the matrix M. The last eight characterizations of Theorem 2 however make essential use of the copositivity plus of the matrix M. The equivalence between (ix) and (xv) of Theorem 2 is due to Doverspike [4] and is stated separately as Theorem 1. Corollary 1 shows that whenever a linear complementarity problem with a copositive plus matrix has a nondegenerate vertex solution, its solution set must be bounded. Corollary 2 establishes the characterization (5) stated above. Corollary 3 characterizes the subclass of copositive plus matrices which is in Q, that is the class of matrices for which the linear complementarity problem has a solution for each q in R^n. Corollary 4 specializes Theorem 2 to the case of a symmetric M and uses the same condition as that of [10, Theorem 2.2] which ensures the boundedness of the iterates of the algorithms of [10].

We briefly describe now the notation of this paper. All matrices and vectors are real. For the $m \times n$ matrix A, row i is denoted by A_i and the element in row i and column j by A_{ij}. For x in the real n-dimensional Euclidean space R^n, element j is denoted by x_j. All vectors are column vectors unless transposed by the superscript T. For $I \subset \{1,\ldots,m\}$ and $J \subset \{1,\ldots,n\}$, A_I denotes the submatrix of A with rows A_i, $i \in I$, A_{IJ} denotes the submatrix of A with elements A_{ij}, $i \in I$, $j \in J$, and x_J denotes x_j, $i \in J$. Superscripts such as A^i, x^i, denote specific matrices and vectors and usually refer to elements of a sequence. For simplicity we shall write A^{i^T} for $(A^i)^T$. The Euclidean norm $(x^Tx)^{\frac{1}{2}}$ of a vector x in R^n will be denoted by $\|x\|$ and the corresponding induced matrix norm $\max\limits_{\|x\|=1} \|Ax\|$ will be denoted by $\|A\|$. The vector e will denote a vector of ones usually in R^n. A partition $\{I,J\}$ of the set of integers $\{1,\ldots.,n\}$ is defined as $I \subset \{1,\ldots.,n\}$, $J \subset \{1,\ldots.,n\}$, $I \cup J = \{1,\ldots.,n\}$ and $I \cap J = \phi$.

2. Principal Results

We begin with a theorem which was established by Robinson [20] for the case when M is positive semidefinite and extended by Doverspike [4] to Eaves' class L of matrices [5] which includes the copositive plus case. This theorem will be utilized in establishing one of the equivalences of Theorem 2.

Theorem 1 [4] Let M be an $n\times n$ copositive plus matrix and let q be in R^n. The following are equivalent:

(i) The solution set $\bar{S}(M,q)$ of the linear complementarity problem (1) is nonempty and bounded.

(ii) There exists an $\varepsilon > 0$ such that the solution set $\bar{S}(\tilde{M},\tilde{q})$ of the perturbed linear complementarity problem $LCP(\tilde{M},\tilde{q})$ is nonempty for $\max\{\|\tilde{M}-M\|, \|\tilde{q}-q\|\} \leqq \varepsilon$.

Our principal result which follows establishes the equivalence of a number of conditions for the nonemptiness and boundedness of the solution set $\bar{S}(M,q)$ of the linear complementarity problem (1).

Theorem 2 For any $n\times n$ matrix M and any vector q in R^n the statements (i) to (viii) below are equivalent. If in addition M is copositive plus then the statements (i) to (xvi) below are equivalent.

(i) The system

$$Mx + q\zeta > 0,\ x \geqq 0,\ \zeta \geqq 0$$

has a solution (x,ζ) in R^{n+1}.

(ii) The system

$$Mx + q > 0, \ x \geqq 0$$

has a solution x in R^n.

(iii) The system

$$Mx + q > 0, \ x > 0$$

has a solution x in R^n.

(iv) For each h in R^n the system

$$Mx + q + \gamma h \geqq 0, \ x \geqq 0, \ \gamma > 0$$

has a solution (x,γ) in R^{n+1}.

(v) There exists a $\delta > 0$ such that the system

$$Mx + \tilde{q} \geqq 0, \ x \geqq 0$$

has a solution x for each $\tilde{q}$ in R^n such that $\|\tilde{q}-q\| \leqq \delta$.

(vi) There exists an $\varepsilon > 0$ such that the system

$$\tilde{M}x + \tilde{q} \geqq 0, \ x \geqq 0$$

has a solution x for each $n \times n$ matrix $\tilde{M}$ and each $\tilde{q}$ in R^n such that

$$\max \{\|\tilde{M}-M\|, \|\tilde{q}-q\|\} \leqq \varepsilon.$$

(vii) The system

$$M^T u \leqq 0, \ q^T u \leqq 0, \ 0 \leqq u \neq 0$$

has no solution u in R^n.

(viii) For each (a,α) in R^{n+1} the set

$$\{u \mid M^T u \leqq a,\ q^T u \leqq \alpha,\ u \geqq 0\}$$

is empty or bounded.

(ix) The solution set $\bar{S}(M,q)$ of the linear complementarity problem (1) is nonempty and bounded.

(x) $\{x \mid Mx \geqq 0,\ q^T x \leqq 0,\ (M+M^T)x=0,\ 0 \leqq x \neq 0\} = \phi$.

(xi) For some or each solution $\bar{x}$ of the linear complementarity problem (1) the set

$$\{x \mid Mx+q \geqq 0,\ x \geqq 0,\ q^T x \leqq 0,\ (M+M^T)(x-\bar{x})=0\}$$

is bounded.

(xii) $\{(x,u) \mid Mx+M^T u < q,\ x \leqq u\} \neq \phi$.

(xiii) For each h in R^n there exists a positive γ such that LCP$(M,q+\gamma h)$ is solvable.

(xiv) There exists a $\delta > 0$ such that LCP$(M,\tilde{q})$ is solvable for each $\tilde{q}$ in R^n such that $\|\tilde{q}-q\| \leqq \delta$.

(xv) There exists an $\varepsilon > 0$ such that LCP$(\tilde{M},\tilde{q})$ is solvable for each $n \times n$ matrix $\tilde{M}$ and each $\tilde{q}$ in R^n such that

$$\max\ \{\|\tilde{M}-M\|,\ \|\tilde{q}-q\|\} \leqq \varepsilon.$$

(xvi) There exist $\varepsilon > 0$ and $\alpha > 0$ such that $LCP(\tilde{M},\tilde{q})$ is solvable for each $n \times n$ copositive plus matrix $\tilde{M}$ and each $\tilde{q}$ in R^n satisfying

$$\max \{\|\tilde{M}-M\|, \|\tilde{q}-q\|\} \leqq \varepsilon$$

and furthermore $\|x\| \leqq \alpha$ for all x in $\bar{S}(\tilde{M},\tilde{q})$.

<u>Proof</u> (i) $\Leftarrow$ (ii): If x solves the system of (ii) then x and $\zeta = 1$ solve the system of (i).

(i) $\Rightarrow$ (ii): Let $(\hat{x},\hat{\zeta})$ satisfy $M\hat{x} + q\hat{\zeta} > 0$, $\hat{x} \geqq 0$, $\hat{\zeta} \geqq 0$. Because the open set $\{(x,\zeta) | Mx+q\zeta>0\}$ contains $(\hat{x},\hat{\zeta})$ it must also contain $(\hat{x},\hat{\zeta}+\delta)$ for some sufficiently small positive δ and hence $M\hat{x} + q(\hat{\zeta}+\delta) > 0$. The point $\hat{x}/(\hat{\zeta}+\delta)$ solves the system of (ii).

(ii) $\Leftarrow$ (iii): Obvious

(ii) $\Rightarrow$ (iii): Let $\hat{x}$ satisfy $M\hat{x} + q > 0$, $\hat{x} \geqq 0$. Because the open set $\{x | Mx+q>0\}$ contains $\hat{x}$ it must also contain $\hat{x} + \delta e$ for some sufficiently small positive δ. The point $\hat{x} + \delta e$ solves the system of (iii).

(i) $\Longleftrightarrow$ (vii): This follows from Motzkin's theorem of the alternative [9].

(iv) $\Longleftrightarrow$ (vii): Condition (vii) is equivalent to the system

$$M^T u \leqq 0,\ q^T u \leqq 0,\ u \geqq 0,\ g^T u < 0$$

not having a solution u for each g in R^n. This in turn is equivalent, by Motzkin's theorem of the alternative, to the system

$$My + q\zeta + g \geqq 0,\ y \geqq 0,\ \zeta \geqq 0$$

having a solution (y,ζ) in R^{n+1} for each g in R^n. By defining $g = q + h$, $x = \frac{y}{1+\zeta}$, $\gamma = \frac{1}{1+\zeta}$ we have that (iv) follows from this last result, while the converse follows by defining $h = g$, $y = \frac{x}{\gamma}$, and $\zeta = \frac{1}{\gamma}$.

(iv) $\Leftarrow$ (v): Take $\gamma \leqq \delta/\|h\|$ if $h \neq 0$, otherwise take $\gamma = 1$.

(v) $\Leftarrow$ (vi): Set $\delta = \varepsilon$ and $\tilde{M} = M$.

(vi) $\Leftarrow$ (vii): Suppose not, then there exists a sequence of $n\times n$ matrices $\{M^i\}$ and a sequence of vectors $\{q^i\}$ in R^n, $i=1,2,\ldots$, converging to M and q respectively such that $M^i x + q^i \geqq 0$, $x \geqq 0$, has no solution for $i=1,2\ldots$. By Motzkin's theorem of the alternative this is equivalent to $M^{i^T} u \leqq 0$, $q^{i^T} u < 0$, $u \geqq 0$ having a solution u^i for $i=1,2,\ldots$. Since $u^i \neq 0$ it follows that

$$M^{i^T} \frac{u^i}{\|u^i\|} \leqq 0,\ q^{i^T} \frac{u^i}{\|u^i\|} < 0,\ \frac{u^i}{\|u^i\|} \geqq 0,\ i=1,2,\ldots\ .$$

Hence there exists an accumulation $\bar{u}$ such that

$$M^T \bar{u} \leqq 0,\ q^T \bar{u} \leqq 0,\ 0 \neq \bar{u} \geqq 0$$

which contradicts (vii).

(vii) $\Rightarrow$ (viii): Suppose not, then there exists an (a,α) in R^{n+1} such that the set

$$\{u \mid M^T u \leqq a,\ q^T u \leqq \alpha,\ u \geqq 0\}$$

is nonempty and unbounded. Hence there exists a sequence of $\{u^i\}$, $i=1,2,\ldots,$ such that $u^i \neq 0$, $\|u^i\| \to \infty$ and

$$M^T u^i/\|u^i\| \leqq a/\|u^i\|,\ q^T u^i/\|u^i\| \leqq \alpha/\|u^i\|,\ u^i/\|u^i\| \geqq 0,\ i=1,2,\ldots\ .$$

Consequently there exists an accumulation $\bar{u}$ such that

$$M^T\bar{u} \leqq 0,\ q^T\bar{u} \leqq 0,\ 0 \neq \bar{u} \geqq 0$$

which contradicts (vii).

(vii) $\Leftarrow$ (viii): We shall prove the contrapositive implication. Let $M^T\bar{u} \leqq 0$, $q^T\bar{u} \leqq 0$, $0 \leqq \bar{u} \neq 0$, then for $\lambda > 0$, $\lambda\bar{u}$ is unbounded as $\lambda \to \infty$ and

$$M^T(\lambda\bar{u}) \leqq 0 = a,\ q^T(\lambda\bar{u}) \leqq 0 = \alpha,\ \lambda\bar{u} \geqq 0.$$

(iv) $\Leftarrow$ (xiii): Obvious

(iv) $\Rightarrow$ (xiii): Because M is copositive plus and the feasible region $S(M, q+\gamma h)$ is nonempty it follows by Lemke's algorithm [8,3] that LCP$(M, q+\gamma h)$ has a solution.

(v) $\Leftarrow$ (xiv): Obvious

(v) ⇒ (xiv): Because M is copositive plus and the feasible region $S(M,\tilde{q})$ is nonempty it follows by Lemke's algorithm that $LCP(M,\tilde{q})$ has a solution.

(ix) ⟺ (xv): This follows from Theorem 1.

(vii) ⟺ (x): By recalling that M is copositive plus the contrapositive implications follow from the following

$$\left|\begin{array}{r} M^T u \leqq 0 \\ q^T u \leqq 0 \\ 0 \neq u \geqq 0 \end{array}\right| \Longleftrightarrow \left|\begin{array}{r} M^T u \leqq 0 \\ 0 \leqq u^T M^T u \leqq 0 \\ q^T u \leqq 0 \\ 0 \neq u \geqq 0 \end{array}\right| \Longleftrightarrow \left|\begin{array}{r} (M+M^T)u = 0 \\ q^T u \leqq 0 \\ Mu = -M^T u \geqq 0 \\ 0 \neq u \geqq 0 \end{array}\right.$$

(ix) ⇐ (x): We shall prove the contrapositive implication. Suppose that $\bar{S}(M,q)$ is empty or unbounded. If it is empty then because M is copositive plus it follows by Lemke's algorithm that the feasible region $S(M,q)$ is empty and consequently the system

$$Mx + q\zeta \geqq 0, \; x \geqq 0, \; \zeta > 0$$

has no solution (x,ζ) in R^{n+1}. By Motzkin's theorem of the alternative it follows that there exists a u satisfying

$$M^T u \leqq 0, \; u \geqq 0, \; q^T u < 0.$$

This however contradicts (vii) which, as established above, is equivalent to (x). Suppose now that $\bar{S}(M,q)$ is unbounded. Hence there exists a sequence of points $\{x^i\}$, $i=1,2,\ldots$, in R^n, $x^i \neq 0$, $\|x^i\| \to \infty$ such that

$$Mx^i/\|x^i\|+q/\|x^i\| \geqq 0,\ x^i/\|x^i\| \geqq 0, ({x^i}^T/\|x^i\|)(Mx^i/\|x^i\|+q/\|x^i\|) = 0,\ i=1,2,\ldots.$$

Hence there exists an accumulation point $\bar{x}$ such that $M\bar{x} \geqq 0$, $0 \neq \bar{x} \geqq 0$, $\bar{x}^T M\bar{x} = 0$. Since $q^T x^i/\|x^i\| = -{x^i}^T Mx^i/\|x^i\| \leqq 0$, it follows that $q^T\bar{x} \leqq 0$. Because M is copositive plus $(M+M^T)\bar{x} = 0$ follows from $\bar{x}^T M\bar{x} = 0$ and $\bar{x} \geqq 0$. The existence of this $\bar{x}$ satisfying the conditions $M\bar{x} \geqq 0$, $q^T\bar{x} \leqq 0$, $(M+M^T)\bar{x} = 0$, $0 \leqq \bar{x} \neq 0$ contradicts (x).

(ix) $\Rightarrow$ (x): We shall prove the contrapositive implication. Let x satisfy

$$Mx \geqq 0,\ q^Tx \leqq 0,\ (M+M^T)x = 0,\ 0 \leqq x \neq 0$$

and let $\bar{x}$ be a solution of LCP(M,q). Then for any $\lambda > 0$,

$$M(\bar{x}+\lambda x) + q \geqq 0,\ \bar{x} + \lambda x \geqq 0$$

and

$$\begin{aligned} 0 \leqq (\bar{x}+\lambda x)^T(M(\bar{x}+\lambda x)+q) &= \bar{x}^T(M\bar{x}+q) + \lambda\bar{x}^T(M+M^T)x \\ &\quad + \lambda^2 x^T Mx + \lambda q^T x \\ &= \lambda q^T x \leqq 0 \end{aligned}$$

Hence $\bar{x} + \lambda x$ solves LCP(M,q) and is unbounded as $\lambda \to \infty$. Hence $\bar{S}(M,q)$ is unbounded whenever it is nonempty.

(x) $\Longleftrightarrow$ (xii): By Tucker's theorem of the alternative [9] condition (x) is equivalent to the system

$$M^T v - q\zeta + (M+M^T)z < 0, \; v \geqq 0, \; \zeta \geqq 0$$

having a solution (v,z,ζ) in R^{2n+1}, and because the set $\{(v,z,\zeta) | M^T v - q\zeta + (M+M^T)z < 0\}$ is open this is equivalent to

$$Mz + M^T(v+z) < q, \; v \geqq 0$$

having a solution (v,z) in R^{2n}. Defining $x = z$ and $u = v + z$ gives (xii).

(x) ⇒ (xi): We shall establish (xi) for <u>each</u> solution of LCP(M,q) by proving the contrapositive implication. So let $\bar{x}$ be <u>some</u> solution of LCP(M,q). Then the set defined in (xi) is nonempty because M is copositive plus, $q^T\bar{x} = -\bar{x}^T M\bar{x} \leqq 0$ and hence $\bar{x}$ is in the set. Let this set be unbounded. Then there exists a sequence of points in R^n, $\{x^i\}$, $i=1,2,\ldots.,$ such that $x^i \neq 0$, $\|x^i\| \to \infty$ and

$$Mx^i/\|x^i\| + q/\|x^i\| \geqq 0, \; x^i/\|x^i\| \geqq 0, \; q^T x^i/\|x^i\| \leqq 0$$

$$(M+M^T)(x^i/\|x^i\| - \bar{x}/\|x^i\|) = 0, \; i=1,2,\ldots. \; .$$

Hence there exists an accumulation point $\tilde{x}$ satisfying

$$M\tilde{x} \geqq 0, \; 0 \neq \tilde{x} \geqq 0, \; q^T\tilde{x} \leqq 0, \; (M+M^T)\tilde{x} = 0.$$

This is a negation of (x).

(x) ⇐ (xi): We shall establish (x), with (xi) holding for <u>some</u> solution of LCP(M,q), by proving the contrapositive implication. So let

$\bar{x}$ be <u>any</u> solution of LCP(M,q). Let x satisfy

$$Mx \geqq 0,\ q^Tx \leqq 0,\ (M+M^T)x = 0,\ 0 \leqq x \neq 0.$$

Then for any $\lambda > 0$, $\bar{x} + \lambda x$ is unbounded as $\lambda \to \infty$, $M(\bar{x}+\lambda x) + q \geqq 0$, $\bar{x} + \lambda x \geqq 0$, $q^T(\bar{x}+\lambda x) \leqq -\bar{x}^T M\bar{x} \leqq 0$ and $(M+M^T)(\bar{x}+\lambda x-\bar{x}) = 0$. Hence the set defined in (xi) is unbounded. This is a negation of (xi) with $\bar{x}$ taken as some solution of LCP(M,q).

(xiv) $\Leftarrow$ (xvi): Set $\delta = \varepsilon$ and $\tilde{M} = M$.

(x) $\Rightarrow$ (xvi): Since we have already established the equivalence of (x) and (xv) we have from (xv) that for some $\varepsilon > 0$, $\bar{S}(\tilde{M},\tilde{q})$ is nonempty for $\max\ \{\|\tilde{M}-M\|, \|\tilde{q}-q\|\} \leqq \varepsilon$. Suppose now that (xvi) does not hold. Then there exist sequences $\{\|x^i\|\} \to \infty$, $\{M^i,q^i\} \to (M,q)$ such that $x^i \neq 0$ and M^i are copositive plus and

$$M^ix^i + q^i \geqq 0,\ x^i \geqq 0,\ x^{i^T}(M^ix^i+q^i) = 0$$

Consequently

$$M^i\frac{x^i}{\|x^i\|} + \frac{q^i}{\|x^i\|} \geqq 0,\ \frac{x^i}{\|x^i\|} \geqq 0,\ q^{i^T}\frac{x^i}{\|x^i\|} = -x^{i^T}\frac{M^ix^i}{\|x^i\|} \leqq 0$$

$$\frac{x^{i^T}}{\|x^i\|}M^i\frac{x^i}{\|x^i\|} + \frac{1}{\|x^i\|}q^{i^T}\frac{x^i}{\|x^i\|} = 0$$

Since $\{\|x^i\|\} \to \infty$, it follows that there exists an accumulation point z of the bounded sequence $\left\{\frac{x^i}{\|x^i\|}\right\}$ such that

$$Mz \geqq 0,\ 0 \neq z \geqq 0,\ q^T z \leqq 0,\ z^T Mz = 0$$

which contradicts (x). □

We note that the implication (vii) ⇒ (vi) also follows from [19, Part I] and the equivalence (vii) ⟺ (viii) follows from [22, Chapter 8]. We have included simple proofs of these relations here to make this paper more self-contained and accessible.

The following corollary gives among other things a simple sufficient condition for the boundedness of the solution set of the linear complementarity problem (1) when M is copositive plus. The first part of this corollary which is established from elementary arguments of perturbation theory of linear equations can also be established by using much more general perturbation results for complementarity problems [21].

Corollary 1 Let the linear complementarity problem (1) have a nondegenerate vertex solution $\bar{x}$, that is $\bar{x} + M\bar{x} + q > 0$ and $\bar{x}$ is a vertex of the feasible region $S(M,q)$, then assertion (xv) of Theorem 2 holds. If in addition M is copositive plus, assertions (i) to (xvi) of Theorem 2 hold as well.

Proof Since $\bar{x}$ is a nondegenerate vertex solution of (1) there exists a partition $\{I,J\}$ of $\{1,\ldots,n\}$ such that M_{JJ} is nonsingular and

$$M_{JJ}\,\bar{x}_J + q_J = 0$$

$$M_{IJ}\,\bar{x}_J + q_I > 0$$

$$\bar{x}_J \qquad > 0$$

By the Banach perturbation lemma [16, p. 45] for any $n\times n$ matrix $\tilde{M}$ satisfying $\|\tilde{M}_{JJ}-M_{JJ}\| < \|M_{JJ}^{-1}\|^{-1}$, $\tilde{M}_{JJ}$ is nonsingular and

$$\|\tilde{M}_{JJ}^{-1}\| \leqq \frac{\|M_{JJ}^{-1}\|}{1-\|M_{JJ}^{-1}\|\,\|\tilde{M}_{JJ}-M_{JJ}\|}$$

$$\|\tilde{M}_{JJ}^{-1}-M_{JJ}^{-1}\| \leqq \frac{\|M_{JJ}^{-1}\|^2\|\tilde{M}_{JJ}-M_{JJ}\|}{1-\|M_{JJ}^{-1}\|\,\|\tilde{M}_{JJ}-M_{JJ}\|}.$$

Hence for a sufficiently small perturbation of M and q, say $\max\{\|\tilde{M}-M\|, \|\tilde{q}-q\|\} \leqq \varepsilon$ for some $\varepsilon > 0$, we have

$$\tilde{x}_J = -\tilde{M}_{JJ}^{-1}\tilde{q}_J = \bar{x}_J - M_{JJ}^{-1}(\tilde{q}_J-q_J) - (\tilde{M}_{JJ}^{-1}-M_{JJ}^{-1})\tilde{q}_J > 0$$

and

$$\begin{aligned}\tilde{M}_{IJ}\tilde{x}_J + \tilde{q}_I &= -\tilde{M}_{IJ}\tilde{M}_{JJ}^{-1}\tilde{q}_J + \tilde{q}_I \\ &= (M_{IJ}\bar{x}_J + q_I) + M_{IJ}(\tilde{x}_J - \bar{x}_J) \\ &\quad + (\tilde{M}_{IJ} - M_{IJ})(-\tilde{M}_{JJ}^{-1}\tilde{q}_J) + (\tilde{q}_I - q_I) > 0.\end{aligned}$$

Hence $(\tilde{x}_J, 0_I)$ is a nondegenerate vertex solution of LCP$(\tilde{M},\tilde{q})$ when $\max\{\|\tilde{M}-M\|, \|\tilde{q}-q\|\} \leqq \varepsilon$, and consequently (xv) of Theorem 2 holds. By Theorem 2, the other assertions (i)-(xvi) hold also when M is copositive plus. □

The following example shows that the nondegenerate vertex condition is not necessary for the boundedness of the solution set.

Example 1

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Here M is positive and hence copositive plus. In fact M is also positive semidefinite. The solution set $\bar{S}(M,q) = \{(x_1,x_2) | x_1+x_2=1,\ x_1 \geqq 0,\ x_2 \geqq 0\}$ is bounded but does not contain a nondegenerate vertex.

The following two examples show that when M is not copositive plus the existence of a nondegenerate vertex solution to the linear complementarity may or may not imply boundedness of the solution set.

Example 2

$$M = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Here M is not copositive plus and the unbounded solution set $\bar{S}(M,q) = \{(1,0)\} \cup \{(x_1,x_2) | x_1=0,\ x_2 \geqq 0\}$ contains a nondegenerate vertex, $(1,0)$.

Example 3

$$M = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Again M is not copositive plus, but the bounded solution set $\bar{S}(M,q) = \{(1,0)\} \cup \{(x_1,x_2) | x_1=0,\ 0 \leqq x_2 \leqq 1\}$ contains a nondegenerate vertex, $(1,0)$.

The following example shows that the copositivity plus assumption on $\tilde{M}$ in (xvi) of Theorem 2 cannot be dropped.

Example 4

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} 0 & 1 \\ -1 & -\varepsilon \end{pmatrix}, \quad \tilde{q} = \begin{pmatrix} -1 \\ 1+\varepsilon \end{pmatrix}$$

Here M is skew-symmetric and hence is positive semidefinite but $\tilde{M}$ is not copositive plus. For this problem $\bar{S}(M,q) = \{(1,1)\}$ and $\bar{S}(\tilde{M},\tilde{q}) = \{(1,1), (0,1+\frac{1}{\varepsilon})\}$ which is not bounded for $\varepsilon \in (0,\bar{\varepsilon})$ for any $\bar{\varepsilon} > 0$. However if we take the copositive plus perturbation

$$\tilde{M} = \begin{pmatrix} \frac{\varepsilon}{4} & 1 \\ -1+\varepsilon & \frac{\varepsilon}{4} \end{pmatrix}, \quad \tilde{q} = q = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad 0 < \varepsilon \leqq \frac{4}{5}$$

which however is not positive semidefinite for positive ε we have that

$$\bar{S}(\tilde{M},\tilde{q}) = \left\{ \left(\frac{16+4\varepsilon}{16-16\varepsilon+\varepsilon^2}, \frac{16-20\varepsilon}{16-16\varepsilon+\varepsilon^2} \right) \right\}$$

which is bounded by the Euclidean ball of radius 7 around the origin.

The following corollary, which establishes the equivalence given by (5) in the Introduction, follows by paraphrasing (vii) of Theorem 2 as a linear programming condition (6).

Corollary 2 Let M be copositive plus. The linear complementarity problem (1) has a nonempty bounded solution set if and only if

$$\max_u \{e^T u \mid M^T u \leqq 0,\ q^T u \leqq 0,\ u \geqq 0\} = 0 \tag{6}$$

The following corollary characterizes the subclass of copositive plus matrices for which the linear complementarity problem has a solution for each q in R^n, that is those copositive plus matrices which are in the class Q [6]. The equivalence between (ii) and (iii) below has been given by Pang [18, Theorem 11] for a copositive plus M.

Corollary 3 Let M be an $n \times n$ matrix. Statements (i) and (ii) below are equivalent. Statements (i) to (iv) below are equivalent when M is copositive plus.

(i) For each q in R^n the system

$$Mx + q \geqq 0,\ x \geqq 0$$

has a solution x in R^n.

(ii) The system

$$Mx > 0,\ x \geqq 0$$

has a solution x in R^n.

(iii) For each q in R^n the solution set $\bar{S}(M,q)$ of the linear complementarity problem (1) is nonempty, that is $M \in Q$.

(iv) For each q in R^n the solution set $\bar{S}(M,q)$ of the linear complementarity problem (1) is nonempty and bounded.

Proof (i) $\Rightarrow$ (ii): Take $q = -e$.

(i) $\Leftarrow$ (ii) $\Rightarrow$ (iv): If $M\hat{x} > 0$ and $\hat{x} \geqq 0$ then obviously for each q in R^n there exists a $\lambda > 0$ such that $M(\lambda\hat{x}) + q > 0$ and $\lambda\hat{x} \geqq 0$. By the equivalence of (ii) and (ix) of Theorem 2 it follows that the solution set $\bar{S}(M,q)$ is nonempty and bounded for each q in R^n.

(iii) $\Leftarrow$ (iv): Obvious.

(i) $\Leftarrow$ (iii): Obvious. □

The following result complements Theorem 2.2 of [10] for symmetric copositive plus matrices. In a related result [18, Corollary 10] Pang has shown that when M is in Q, symmetric and copositive plus, then it is strictly copositive, that is $x^TMx > 0$ for $0 \neq x \geqq 0$.

Corollary 4 Let M be a symmetric copositive plus matrix. Assertions (i) to (xvi) of Theorem (2) hold **if and** only if $Mx + q > 0$ has a solution x in R^n.

Proof When M is symmetric condition (x) of Theorem 2 is equivalent to

$$M^Tu = 0,\ q^Tu \leqq 0,\ 0 \leqq u \neq 0 \text{ has no solution } u \in R^n.$$

By Tucker's theorem of the alternative [9] this is equivalent to

$$Mx + q\zeta > 0,\ \zeta \geqq 0 \text{ has a solution } (x,\zeta) \in R^{n+1}$$

Because the set $\{(x,\zeta) | Mx+q\zeta>0,\ (x,\zeta) \in R^{n+1}\}$ is open, the last condition is equivalent to the set $\{(x,\zeta) | Mx+q\zeta>0,\ \zeta>0,\ (x,\zeta) \in R^{n+1}\}$ being nonempty, which in turn is equivalent to $Mx + q > 0$ having a solution x in R^n. Hence condition (x) of Theorem 2 holds if and only if $Mx + q > 0$ has a solution x in R^n. The corollary now follows from Theorem 2. □

Remark 1 The equivalence between (ix) and (x) for the case when M is a skew-symmetric and hence copositive plus matrix degenerates to Williams' characterization [23, Theorem 3] of bounded solutions to a pair of dual programs. For the pair of dual linear programs

$$\begin{aligned}&(a)\quad \min_y \{c^Ty \mid Ay \geqq b,\ y \geqq 0\}\\&(b)\quad \max_z \{b^Tz \mid A^Tz \leqq c,\ z \geqq 0\}\end{aligned} \tag{7}$$

Williams' boundedness characterization for <u>both</u> the primal and dual solutions sets is that

(a) $Ay \geqq 0,\ c^Ty \leqq 0,\ 0 \neq y \geqq 0$ has no solution y

<u>and</u> (8)

(b) $A^Tz \leqq 0,\ b^Tz \geqq 0,\ 0 \neq z \geqq 0$ has no solution z.

We establish now that the negation of (8) is equivalent to the negation of (x) of Theorem 2 with $M = \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix}$, $q = \begin{pmatrix} c \\ -b \end{pmatrix}$ and $x = \begin{pmatrix} y \\ z \end{pmatrix}$.

The negation of (8) is

(a) $Ay \geqq 0,\ c^Ty \leqq 0,\ 0 \neq y \geqq 0$ has a solution y

<u>or</u> (9)

(b) $A^Tz \leqq 0,\ b^Tz \geqq 0,\ 0 \neq z \geqq 0$ has a solution z

We show now that (9) is equivalent to

$$Ay \geqq 0,\ A^Tz \leqq 0,\ -c^Ty + b^Tz \geqq 0,\ 0 \neq (y,z) \geqq 0 \text{ has a solution } (y,z) \tag{10}$$

which is the negation of (x) of Theorem 2 for the linear programming case. If (9a) holds take $z = 0$ in (10), if (9b) holds take $y = 0$

in (10), and hence (9) implies (10). Suppose now (10) holds. If either y or z is zero then (9) holds. If both y and z are nonzero then again (9a) holds when $c^T y \leqq 0$, and (9b) holds when $c^T y > 0$ because $b^T z \geqq c^T y > 0$. Hence (9) and (10) are equivalent and condition (x) of Theorem 2 degenerates to Williams' condition (8) when M and q are specialized to the linear programming case.

It is worthwhile to point out an interesting subtlety in connection with conditions (8). Taken together conditions (8a) and (8b) are equivalent to condition (x) of Theorem 2 and hence guarantee the existence and boundedness of the solution sets to <u>both</u> of the dual linear programs of (7). However, taking (8a) or (8b) one at a time merely guarantees the boundedness but not the existence of a solution set to the corresponding linear program. For example, for $A = 0$, $c = -1$, $b = -1$, condition (8a) is satisfied by $y = 1$, but the linear program (7a) has no solution because its objective is unbounded below.

<u>Remark 2</u> From the proof that (ix) $\Rightarrow$ (x) in Theorem 2 we conclude that if the linear complementarity problem (1) with a copositive plus M has an unbounded solution set then each of its solutions $\bar{x}$ lies on a ray of solutions $\bar{x} + \lambda x$ where $\lambda > 0$ and x is any nonzero element of the convex cone

$$\{x | Mx \geqq 0,\ q^T x \leqq 0,\ (M+M^T)x = 0,\ x \geqq 0\}.$$

Remark 3 It can be shown that Cottle's Theorem 3.1 [2] relating a solution ray of LCP(M,q) and the complementary cones of M follows from the equivalence of (x) and (xiii) of Theorem 2.

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