An Almost Surely Optimal Algorithm for the Euclidean Traveling Salesman Problem

by

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(†) That is, optimal with probability one.

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ABSTRACT

This paper presents an algorithm for the Euclidean Traveling Salesman Problem in any k-dimensional Lebesgue set $\underline{\underline{E}}$ of zero-volume boundary. For n points independently, uniformly distributed in $\underline{\underline{E}}$, we show that, in probability, the time taken by the algorithm can be made to be of order less than $n\sigma(n)$, as $n \to \infty$, for any choice of an increasing function σ(however slow its rate of increase.) The resulting tour-length will, with probability one (i.e. almost surely), be asymptotic, as $n \to \infty$, to the minimal tour length (which has previously been determined to be asymptotic to $\beta n^{1-1/k} v(\underline{\underline{E}})^{1/k}$, where β depends only on k and $v(\underline{\underline{E}})$ is the k-dimensional Lebesgue measure (volume) of the set $\underline{\underline{E}}$.) This result is stronger and the algorithm is faster than any other we have been able to find in the literature.

<u>EXTENDED ABSTRACT</u>

1. Introduction and Summary

Given an integer $k \geqslant 2$ the k-dimensional Euclidean Traveling Salesman Problem (k-TSP) can be defined as follows: given a set of n points distributed in the k-dimensional Euclidean space R^k, determine a tour, i.e., a closed path visiting each of the n points exactly once, so that the tour is the shortest possible one (we take the distance between two points to be the ordinary Euclidean distance).

For the particular case of $k = 2$, the fact that the TSP is NP-hard (Garey et al. [1976], Papadimitriou [1977]) is ~~an~~ evidence that there are no polynomial-time algorithms for obtaining an exact solution for this problem. delet

On the other hand, there has been some research on heuristic methods for the solution of the 2-TSP. For example, computer programs to find near-optimal solutions for 2-TSP instances of up to 300 points in an acceptable amount of time were described by Krolak et al. [1970] and by Lin and Kernighan [1973]. Their programs seem to give good results but no rigorous analyses of the algorithms are available.

Our point of departure is a paper on the asymptotic behavior of the length of the shorte~~r~~st tour in a uniformly and independently distributed k-TSP instance, by Beardwood, Halton, and Hammersley [1959].

In Section 2 of this abstract we present Algorithm A, a non-recursive, divide-and-conquer algorithm for the k-TSP, $k \geqslant 2$. (~~As we will see,~~) Algorithm A is a significant improvement upon a similar, but recursive, algorithm presented by Karp [1977] in terms of its simplicity, its dimensional generality and its running time (Theorem 1, below). Moreover, we will see also that the theorem on the asymptotic behavior of Algorithm A (Theorem 2, below) is stronger than the corresponding result claimed by Karp [1977] (his claim is for expected behavior only, while ours is true with probability one); and further, we have been unable to follow Karp's proof of his result. delete

We assume Condition D, that

[1] the points of a sequence $\underset{\sim}{P}$ are distributed uniformly and independently in a Lebesgue subset $\underline{\underline{E}}$ (of volume $v(\underline{\underline{E}})$) of $\lambda\underline{\underline{C}}$, the k-dimensional hypercube of side λ,

[2] the boundary of $\underline{\underline{E}}$ is of zero k-dimensional volume (Lebesgue measure.)

[3] we choose a non-zero function $\delta(n)$ such that

$\delta(n) \to \infty$ and $\delta(n)e^{\delta(n)}/n \to 0$, as $n \to \infty$.

The function $\delta(n)$ is used in defining Algorithm A.

Let $\underset{\sim}{P}^n$ denote the first n points of $\underset{\sim}{P}$. In the paper, we prove the following:

Theorem 1: Under Condition D, if Algorithm A is applied to a k-TSP instance $\underset{\sim}{P}^n$ in $\underline{\underline{E}}$, then the Algorithm runs in time

$R_n \sim A \;\; n\, \delta^*(n)\, e^{\delta^*(n)}$, in probability, where A is a constant and $\delta^*(n) = \delta(n)\,\lambda^k/v(\underline{\underline{E}})$.

We are thinking in particular of very slowly increasing functions $\delta(n)$, bearing in mind that Karp [1977] has an upper bound for the expected running time of his algorithm which cannot be less than $\underline{0}(n\,(\log n)^2)$. We notice that, for example, if we let $\delta(n) = \log\log\log n$ in Theorem 1, we would have

$R_n \sim \underline{0}\,(n \;\; \log\log\log n \;\; \log\log n)$, in probability.

Indeed, by choosing $\delta(n) = \alpha \log \sigma(n)$, for any $0 < \alpha < 1$, we obtain

Corollary 1: Under the hypotheses of Theorem 1, we can find a function $\delta(n)$, such that, for any arbitrarily slowly increasing function $\sigma(n)$, the running time of Algorithm A will be

$$R_n \sim \cancel{O}(n\, \sigma(n)), \quad \text{in probability.}$$

Let $T_0(n)$ denote the length of an optimal solution for a given k-TSP instance $\underset{\sim}{P}^n$, and let $T(n)$ denote the length of the closed path given by Algorithm A for $\underset{\sim}{P}^n$. In the paper, we characterize the asymptotic performance of Algorithm A by the following:

ASYMPTOTI

<u>Theorem 2:</u> Under the hypotheses of Theorem 1, we have

$T(n)/T_0(n) \to 1$, with probability one, as $n \to \infty$.

2. The Algorithm

Algorithm A computes a closed path which visits some of the points more than once. We will see later in this section that it is easy to transform such a closed path into a tour with a shorter length.

In specifying Algorithm A, we need the function $\delta(\cdot)$ defined in Condition D, and an integer m defined as the smallest even integer greater than or equal to

$$\left(\frac{n}{\delta(n)}\right)^{1/k}, \qquad \text{where } n \text{ is the number of points of a}$$

k-TSP instance $\underset{\sim}{J}$ in $\underline{\underline{C}}$.

Now we specify:

<u>Algorithm A:</u>

[1] Divide each size of $\lambda\underline{\underline{C}}$ into m equal parts, thus creating a cubic lattice of m^k cells (of side h) in $\lambda\underline{\underline{C}}$.

[2] Let $\underset{\sim}{B}$ be the set of cell-centers (mid-points of cells created in [1]). Form the union $\underset{\sim}{B} \cup \underset{\sim}{J}$.

[3] For each of the m^k cells, find the shortest tour through the points of $\underset{\sim}{B} \cup \underset{\sim}{J}$ in the cell by applying a dynamic programming algorithm (such as that by Bellman [1962] and by Held and Karp [1962]);) ⊛

[4] Construct a basic tour through the points of $\underset{\sim}{B}$ added in step [2] above, using Algorithm B below.

[5] The closed path consisting of all the subtours constructed in step [3] chain-connected by the basic tour built in step [4] is the result of the algorithm.

Algorithm $\underset{\sim}{B}$: To construct a basic tour.

For k=2, Figure 1 indicates the basic tour. Note that there are m^2 cells, with m even.

Iteratively (see Figure 2 below), suppose we have a basic tour for a given k=j. Number the cell-centers in the order of the tour, with $K_1 = ((2m-1)h/2, h/2, h/2, \ldots, h/2)$,

$$K_1 \; K_2 \; \ldots \; K_M \; K_1 \quad (M = m^j).$$

Consider the case of $k = j+1$. Take the section of the $(j+1)$-dimensional dissected hypercube defined by $x_{j+1} = h/2$. It forms a j-dimensional dissected hypercube. Form the above basic tour of the cell-centers in this section. Number the cell-centers of the chosen section so that the basic tour is:

$$K'_1 \quad K'_{2m} \quad K'_{2m+1} \quad K'_{4m} \quad K'_{4m+1} \quad K'_{6m} \quad K'_{6m+1} \quad \cdots$$
$$\cdots \quad K'_{M'} \quad K'_1 \qquad (M'=M \; m=m^{j+1}).$$

Each cell of the chosen j-dimensional section sits in line with a stack of m cells in the (j+1)-dimensional cube, and we construct the basic tour for the $k = j+1$ case by zig-zagging along the full length of these stacks. All m^{j+1} cell-centers are numbered as follows: if a stack of cell-centers is in line with a cell-center numbered K'_{2sm} the cell-centers are numbered

$$K'_{2sm-1} \quad K'_{2sm-2} \cdots K'_{(2s-1)m+1},$$

moving from the chosen section; while if a stack of cell-centers is in line with a cell-center numbered K'_{2sm+1} the cell-centers are numbered

$$K'_{2sm+2} \quad K'_{2sm+3} \cdots K'_{(2s+1)m},$$

moving from the chosen section. The new tour is then simply

$$K'_1 \quad K'_2 \quad \cdots \quad K'_{M'} \quad K'_1 \; .$$

This defines the algorithm iteratively on k.

While it is relatively easy to imagine, Algorithm B is rather laborous to implement and execute; so we have devised another, Algorithm B[3/], which is executed in time proprotional to m^k (by defining the successor-cell of each cell) and which we have proved to yield the same basic tour as Algorithm B.

Apostrophe

Now we want to show how the closed path constructed by Algorithm A can be transformed into a tour with a shorter length.

First, if any cell has no points of $\underset{\sim}{J}$, then the basic tour can be shortened by connecting the previous cell-center to the next one. This may be repeated until the basic tour contains only points of $\underset{\sim}{B}$ from cells

CONTENTS

1. Introduction and Summary

Given an integer $k \geqslant 2$ the k-dimensional Euclidean Traveling Salesman Problem (k-TSP) can be defined as follows: given a set of n points distributed in the k-dimensional Euclidean space R^k, determine a tour, i.e., a closed path visiting each of the n points exactly once, so that the tour is the shortest possible one (we take the distance between two points to be the ordinary Euclidean distance).

For the particular case of k=2, the fact that the TSP is NP-hard (Garey et al. [1976], Papadimitriou [1977]) is an evidence that there are no polynomial-time algorithms for obtaining an exact solution for this problem. Even when one wants only a solution of some guaranteed accuracy, the 2-TSP seems to be hard (Sahni and Gonzales [1976]).

On the other hand, there has been some research on heuristic methods for the solution of the 2-TSP. For example, computer programs to find near-optimal solutions for 2-TSP instances of up to 300 points in an acceptable amount of time were described by Krolak et al. [1970] and by Lin and Kernighan [1973]. Their programs seem to give good results but no rigorous analyses of the algorithms are available.

In Section 2 of this paper we present Algorithm A (defined below), a non-recursive, divide-and-conquer algorithm for the k-TSP, $k \geqslant 2$ As we will see, Algorithm A is a significant improvement upon a similar, but recursive, algorithm presented by Karp [1977] in terms of its simplicity, its dimensional generality and its running time (Theorem 1, below). Moreover, we will see also that the theorem on the asymptotic behavior of Algorithm A (Theorem 2, below) is stronger than the corresponding result claimed by Karp [1977] (his claim is for expected behavior only, while ours is true with probability one); and further, we have been unable to follow Karp's proof of his result. (See Appendix III.)

We assume Condition C, that

[1] the points of a sequence $\underset{\sim}{P}$ are distributed uniformly and independently in the k-dimensional unit hypercube $\underline{\underline{C}}$;

[2] we choose a non-zero function $\delta(n)$ such that

$$\delta(n) \to \infty \text{ and } \delta(n)e^{\delta(n)}/n \to 0\,, \quad \text{as } n \to \infty\,.$$

Remark E: We note from Condition C [2] that, as $n \to \infty$: (i) $n/\delta(n) \to \infty$; (ii) $[\delta(n)]^m = \underline{o}(n)$, for any power m, since $[\delta(n)]^m/n = [\delta(n)e^{\delta(n)}/n]/[e^{\delta(n)}/\delta(n)^{m-1}] \to 0$ because $e^{\delta(n)}$ increases faster than any power of $\delta(n)$; and (iii) while $\delta(n) \sim K \log n$ does not satisfy C[2], any $\delta(n) = \underline{o}(\log n)$ will satisfy C[2]; since, if $\delta(n) = \varepsilon(n) \log n$ and $\varepsilon(n) \to 0$, then $\delta(n)e^{\delta(n)}/n = \varepsilon(n)\ (\log n)\ n^{\varepsilon(n)-1} = \underline{o}[\varepsilon(n)\log n/n^{1/2}] \to 0$.

The function $\delta(n)$ is used in defining Algorithm A.

Let $\underset{\sim}{P}^n$ denote the first n points of $\underset{\sim}{P}$. In Section 3 of this paper we prove the following:

Theorem 1: Under Condition C, if Algorithm A is applied to a k-TSP instance $\underset{\sim}{P}^n$ then Algorithm A runs in time

$$R_n \sim A \quad n\ \delta(n)\ e^{\delta(n)}, \quad \text{in probability, where A is a constant.}$$

We are thinking in particular of very slowly increasing functions $\delta(n)$, bearing in mind that Karp [1977] has an upper bound for the expected running time of his algorithm which cannot be less than $\underline{O}(\ n\ (\log n)^2)$. We notice that, for example, if we let $\delta(n) = \log\log\log n$ in Theorem 1, we would have

$$R_n \sim \underline{O}(\ n \quad \log\log\log n \quad \log\log n), \text{ in probability.}$$

Indeed, by choosing $\delta(n) = \alpha \log \sigma(n)$, for any $0 < \alpha < 1$, we obtain

Corollary 1: Under the hypotheses of Theorem 1, we can find a function $\delta(n)$, such that, for any arbitrarily slowly increasing function $\sigma(n)$, the running time of Algorithm A will be

$$R_n \sim \underline{o}(\, n\, \sigma(n)), \text{ in probability.}$$

Let $T_0(n)$ denote the length of an optimal solution for a given k-TSP instance $\underset{\sim}{P}^n$, and let $T(n)$ denote the length of the closed path given by Algorithm A for $\underset{\sim}{P}^n$. In Section 4 of this paper we characterize the asymptotic performance of Algorithm A by the following:

Theorem 2: Under the hypotheses of Theorem 1, we have

$T(n)/T_0(n) \to 1$, with probability one, as $n \to \infty$.

Finally, in Section 5, we consider Condition D, that

[1] the points of a sequence $\underset{\sim}{P}$ are distributed uniformly and independently in a Lebesgue subset $\underline{\underline{E}}$ of $\lambda\underline{\underline{C}}$, the k-dimensional hypercube of side λ,

[2] the boundary of $\underline{\underline{E}}$ is of zero k-dimensional volume (Lebesgue measure.)

[3] we choose a non-zero function $\delta(n)$ as in Condition C.

In this case, we apply Algorithm A to $\lambda\underline{\underline{C}}$ (instead of $\underline{\underline{C}}$, as given in Section 2) and obtain

Theorem 3: Under Condition D;

(1) Theorem 1 holds, with $\delta(n)$ replaced by $\delta(n)\lambda^k/v(\underline{\underline{E}})$;

(2) Theorem 2 holds; and the important result (4.43) in the proof this theorem holds with β replaced by $\beta[v(\underline{\underline{E}})]^{1/k}$.

2. The Algorithm

As in Karp [1977], Algorithm A computes a closed path which visits some of the points more than once. We will see later in this section that it is easy to transform such a closed path into a tour with a shorter length.

In specifying Algorithm A, we need a function $\delta(\cdot)$ and an integer m defined as the smallest even integer greater than or equal to

$\left(\frac{n}{\delta(n)}\right)^{1/k}$, where n is the number of points of a k-TSP instance $\underset{\sim}{J}$ in $\underline{\underline{C}}$.

Now we are able to specify:

<u>Algorithm A</u>:

[1] Divide each side of $\underline{\underline{C}}$ into m equal parts, thus creating a cubic lattice of m^k cells (of side h) in $\underline{\underline{C}}$.

[2] Let $\underset{\sim}{B}$ be the set of <u>cell-centers</u> (mid-points of cells created in [1]). Form the union $\underset{\sim}{B} \cup \underset{\sim}{J}$.

[3] For each of the m^k cells, find the shortest tour through the points of $\underset{\sim}{B} \cup \underset{\sim}{J}$ in the cell by applying a dynamic programming algorithm (by Bellman [1962] and by Held and Karp [1962]);

[4] Construct a <u>basic tour</u> through the points of $\underset{\sim}{B}$ added in step [2] above, using Algorithm B below.

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Consider the case of $k = j+1$. Take the section of the $(j+1)$-dimensional dissected hypercube defined by $x_{j+1} = h/2$. It forms a j-dimensional dissected hypercube. Form the above basic tour of the cell-centers in this section. Number the cell-centers of the chosen section so that the basic tour is:

$$K'_1\ K'_{2m}\ K'_{2m+1}\ K'_{4m}\ K'_{4m+1}\ K'_{6m}\ K'_{6m+1}\ \ldots$$
$$\ldots\ K'_{M'}\ K'_1 \quad (M'=M\ m=m^{j+1}).$$

Each cell of the chosen j-dimensional section sits in line with a stack of m cells in the $(j+1)$-dimensional cube, and we construct the basic tour for the $k = j+1$ case by zig-zagging along the full length of these stacks. All m^{j+1} cell-centers are numbered as follows: if a stack of cell-centers is in line with a cell-center numbered K'_{2sm} the cell-centers are numbered

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moving from the chosen section; while if a stack of cell-centers is in line with a cell-center numbered K'_{2sm+1} the cell-centers are numbered

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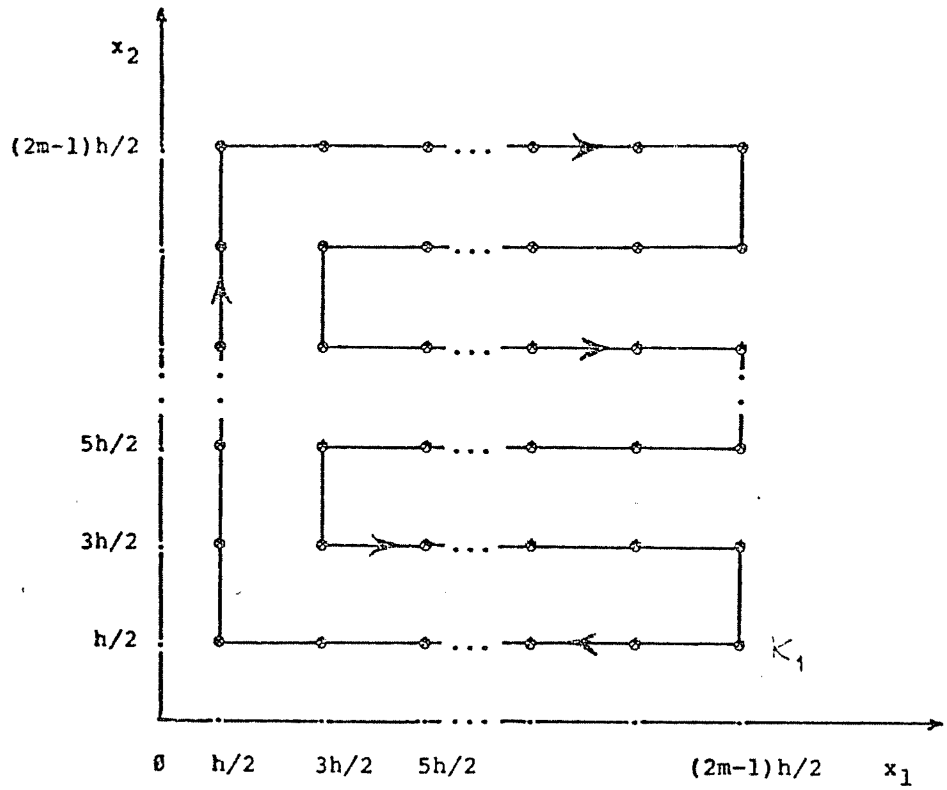
moving from the chosen section. The new tour is then simply

$$K'_1\ K'_2\ \ldots\ K'_{M'}\ K'_1\ .$$

This defines the algorithm iteratively on k.

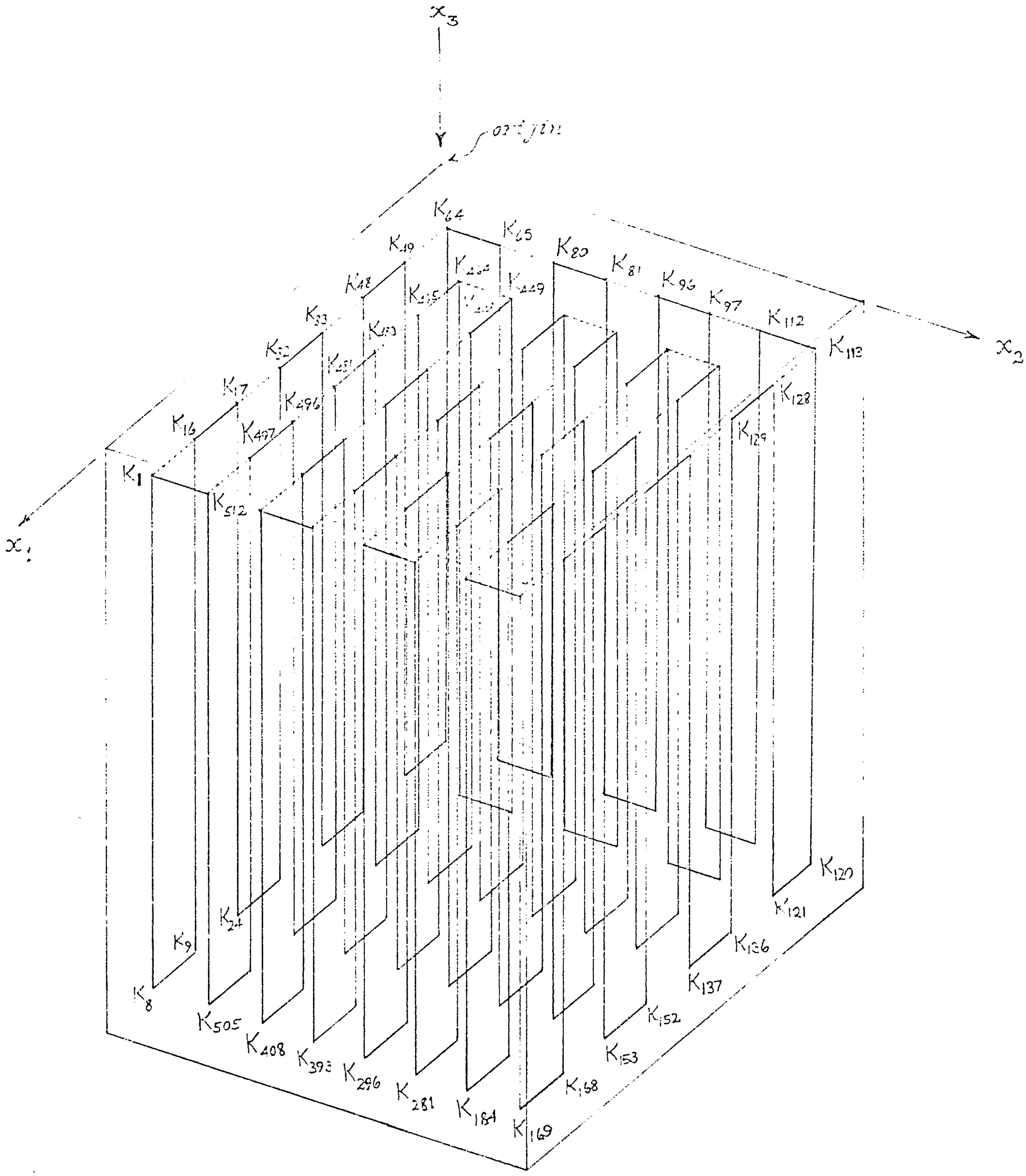
Now we want to show how the closed path constructed by Algorithm A can be transformed into a tour with a shorter length.

First, if any cell has no points of $\underset{\sim}{J}$, then the basic tour can be shortened by connecting the previous cell-center to the next one. This may be repeated until the basic tour contains only points of $\underset{\sim}{B}$ from cells



<u>Figure 1</u>:

The basic tour for the case of dimension k=2.



<u>Figure 2:</u>

The "basic tour" of cell-centers for the case of k=3 and m=8. The dotted segments indicate the basic tour for k=2, in the top layer of cells, illustrating Algorithm B.

containing points of $\underset{\sim}{J}$ (without changing the sequential order of cell-centers in the original basic tour). This does not affect steps [4] and [5] of Algorithm A. Moreover, this can clearly be done in time proportional to m^k, i.e., $\underline{O}(n/\delta(n))$.

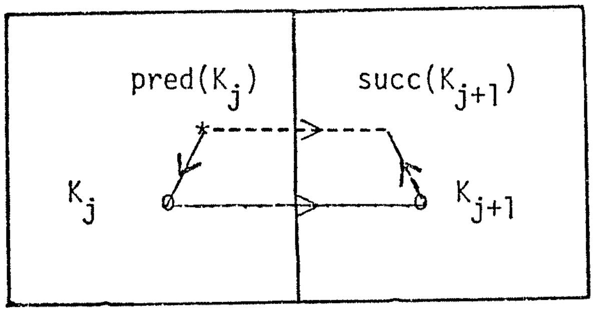


Figure 3

Secondly, let K_j and K_{j+1} be two consecutive cell-centers and let pred(K) and succ(K) denote the predecessor and the successor of a cell-center K, respectively, according to an order assigned to the closed path. Then, if $K_j \notin \underset{\sim}{J}$ and $K_{j+1} \notin \underset{\sim}{J}$, replace the edges

$(\text{pred}(K_j), K_j)$,

(K_j, K_{j+1}),

and $(K_{j+1}, \text{succ}(K_{j+1}))$,

by the edge $(\text{pred}(K_j), \text{succ}(K_{j+1}))$, as illustrated in Figure 3.

If $K_j \in \underset{\sim}{J}$ and $K_{j+1} \notin \underset{\sim}{J}$, replace the edges

(K_j, K_{j+1})

and $(K_{j+1}, \text{succ}(K_{j+1}))$,

by the edge $(K_j, \text{succ}(K_{j+1}))$. Proceed similarly if $K_j \notin \underset{\sim}{J}$ and $K_{j+1} \in \underset{\sim}{J}$.

After applying the procedure above to all pairs (K_j, K_{j+1}) of cell-centers, we get a tour which is shorter than the original closed path, since each replacement of edges always shortens the length of the closed path. Moreover, this shortening procedure can be clearly executed in time proportional to m^k, i.e., $\underline{O}(\ n/\delta(n)\ )$.

3. Asymptotic Execution Time

Before giving the proof of Theorem 1, we want to make some observations on the basic tour and prove two lemmas which will be useful in this section.

We gave Algorithm B to construct the basic tour because it is relatively simple to understand; but now we want to describe Algorithm B', below, which is more efficient than Algorithm B. Theorem 4 below will establish the equivalence of the two algorithms.

To construct the basic tour, we have a cubic lattice of cubic cells of side h, m in each coordinate direction, m^k in all, where m is a positive even integer. Suppose that, for $a_i \in L = \{0,1,2,\ldots,m-1\}$, $1 \leqslant i \leqslant k$, the cell containing the cell-center with coordinates

$$(\ (2a_1+1)h/2,\ (2a_2+1)h/2,\ \ldots\ ,\ (2a_k+1)h/2)$$

is identified by the vector

$$\underset{\sim}{a} = (\ a_1, a_2,\ \ldots\ ,\ a_k)\ .$$

Let $\underset{\sim}{e}_i$ denote the unit vector in the i-th coordinate direction and write

$$r_i = r_i(\underset{\sim}{a}) = (-1)^{1+a_1+a_2+\ldots+a_{i-1}} \qquad \text{for } 2 \leqslant i \leqslant k. \tag{3.1}$$

Algorithm B': Given cell $\underset{\sim}{a}$, finds its successor $\underset{\sim}{b}$ according to the basic tour.

[1] If there exists one value d such that

$$d \geqslant 3,\ a_d + r_d \in L \text{ and } a_i + r_i \notin L \text{ for } d+1 \leqslant i \leqslant k; \tag{3.2}$$

then the successor of $\underset{\sim}{a}$ is

$$\underset{\sim}{b} = \underset{\sim}{a} + r_d \underset{\sim}{e}_d \tag{3.3}$$

(i.e., for all $i \neq d$, $b_i = a_i$, and $b_d = a_d + r_d$).

[2] Otherwise, if (3.2) cannot be satisfied by any d, the successor is determined as follows:

$$\underset{\sim}{b} = \underset{\sim}{a} - \underset{\sim}{e}_1, \quad \text{if } a_1 = 1,\ a_2 = 0, \quad \text{or } a_1 > 1,\ a_2 \text{ even}; \tag{3.4}$$

$$\underset{\sim}{b} = \underset{\sim}{a} + \underset{\sim}{e}_1, \quad \text{if } a_1 = 0,\ a_2 = m - 1, \quad \text{or } 0 < a_1 < m - 1,\ a_2 \text{ odd}; \tag{3.5}$$

$$\underset{\sim}{b} = \underset{\sim}{a} - \underset{\sim}{e}_2, \quad \text{if } a_1 = 1,\ a_2 \text{ even},\ a_2 \neq 0, \quad \text{or } a_1 = m - 1,\ a_2 \text{ odd}; \tag{3.6}$$

$$\underset{\sim}{b} = \underset{\sim}{a} + \underset{\sim}{e}_2, \quad \text{if } a_1 = 0,\ a_2 < m - 1. \tag{3.7}$$

Having defined Algorithm B', we observe that the step [2] above is executed only when

$a_4 = a_5 = \ldots = a_k = 0$; and $a_1 + a_2$ is odd and $a_3 = m-1$,

or $a_1 + a_2$ is even and $a_3 = 0$.

This is so because step [2] is executed when

$$a_i + r_i \notin L \quad \text{for } 3 \leqslant i \leqslant k:$$

thus $a_3 + (-1)^{1+a_1+a_2} \notin L$;

whence $a_1 + a_2$ is odd and $a_3 = m - 1$ (odd),

or $a_1 + a_2$ is even and $a_3 = 0$ (even).

If $a_1 + a_2$ is odd, then

$$r_3 = (-1)^{1+a_1+a_2} = +1,$$

and $a_3 = m - 1$, whence $r_4 = (-1)^{a_3} r_3 = -1$;

$a_4 + r_4 \notin L$, whence $a_4 = 0$, so $r_5 = (-1)^{a_4} r_4 = r_4 = -1$;

$a_5 + r_5 \notin L$, whence $a_5 = 0$, so $r_6 = r_5 = -1$;

. . . .

$$a_k = 0.$$

Also, if $a_1 + a_2$ is even then

$$r_3 = (-1)^{1+a_1+a_2} = -1;$$

and $a_3 = 0$, whence $r_4 = (-1)^{a_3} r_3 = r_3 = -1$;

$a_4 = 0$, whence $r_5 = r_4 = -1$;

. . . .

$a_k = 0$.

We shall now establish the equivalence of Algorithms B and B' as follows:

Theorem 4: The orders of cell succession determined by Algorithm B and B' are the same.

Proof (by induction on k):

For $k = 2$, step [2] prevails and from the observation made above it is easy to verify that (3.4) - (3.7) prescribe the entire set of successors $\underset{\sim}{b}$ of possible vectors $\underset{\sim}{a}$ and is in accordance with the basic tour of Algorithm B for $k = 2$.

Assume now that the theorem is true for the $(k - 1)$ - dimensional case $(k \geqslant 3)$ and let us proceed inductively on k. Then, by the inductive step of Algorithm B and by the inductive hypothesis, the order of cell succession (not necessarily consecutive succession in k-dimensions) in coordinates 1,2,..., (k-1) is prescribed identically by Algorithms B and B'; hence, whenever $a_k = 0$ or $(m-1)$ and $a_k + r_k \notin L$, the successions determined by Algorithms B and B' are the same as for (k-1) - dimensions. When $a_k + r_k \in L$, step [1] tells us that $b_k = a_k + r_k$, $b_i = a_i$ for all $i \neq k$. Thus r_k is not changed (since r_k depends only on $a_1, a_2, \ldots, a_{k-1}$ according to (3.1)); so that the direction of the succession (i.e., a_k being increased or decreased) is not changed, as in Algorithm B. When finally we arrive at $a_k = 0$ or $(m-1)$ and $a_k + r_k \notin L$, we make a move according to the order of succession in (k-1) - dimensions (which is the same by B or B', as noted above) and this changes just one of $a_1, a_2 \ldots, a_{k-1}$ by +1 or -1, thus changing r_k to $-r_k$ and ensuring that $a_k + r_k \in L$ again; i.e., that the direction of succession is changed as in Algorithm B. Therefore, the cell successions determined by Algorithms B and B' are the same in k dimensions, and the induction is complete.

QED

Now we want to present two auxiliary lemmas. Their proofs will be given in Appendix I.

Let S_n denote the time needed to compute the $M = m^k$ shortest tours through the points in each of the cells $\underline{\underline{C}}_j$ constructed in Algorithm A, and let $(n)_i$ denote $n(n-1)(n-2) \ldots (n-i+1)$, as is customary.

Lemma 3.1: Under Condition C, if Algorithm A is applied to a k - TSP instance $\underset{\sim}{P}^n$ then there is a constant A, such that

$$\mathcal{E}\, S_n \sim A\, n\, \delta(n)\, e^{\delta(n)}\, (1 - 1/\delta(n)), \text{ as } n \to \infty .$$

Lemma 3.2: Under the same conditions as in Lemma 3.1, we have that

$$\operatorname{var} S_n \leqslant A^2 e^{2\delta(n)} \left\{ 16n\delta(n)^3 e^{\delta(n)} + \frac{n^2}{4\delta(n)^2} \right\} [1+\underline{O}(1/\delta(n))]$$

as $n \to \infty$, where A is the constant in Lemma 3.1.

We are now able to present the

Proof of Theorem 1:

We have three terms to consider for the execution time of Algorithm A:

(i) the time to determine which points are in each of the $M = m^k$ cells;

(ii) the time to compute the shortest tours through the points in each of the M cells (step [3] of Algorithm A)

(iii) the time to construct the basic tour (Algorithm B).

We assume that $\underline{O}(n)$ (on- or off-line) memory space is available and a hashing technique may be used to determine the points in each cell and term (i) is then $\underline{O}(n)$ (otherwise, a sorting requiring $\underline{O}(n \log n)$ would be needed).

We estimate term (ii) as follows.

Since, for any $\varepsilon > 0$ and for all sufficiently large n, by Lemma 3.1,

$$|\mathcal{E}S_n - An\delta(n)e^{\delta(n)}| < \frac{\varepsilon}{2} An\delta(n)e^{\delta(n)},$$

we see, by the Chebyshev inequality with Lemma 3.2, for any $\varepsilon > 0$ and all sufficiently large n, that

$$\Pr[An\delta(n)e^{\delta(n)}(1-\varepsilon) \leqslant S_n \leqslant An\delta(n)e^{\delta(n)}(1+\varepsilon)]$$

$$= \Pr[|S_n - An\delta(n)e^{\delta(n)}| \leqslant \varepsilon\, An\delta(n)e^{\delta(n)}]$$

$$\geqslant \Pr[|S_n - \mathcal{E}S_n| \leqslant \frac{\varepsilon}{2} An\delta(n)e^{\delta(n)}]$$

$$\geqslant 1 - \operatorname{var} S_n / \frac{\varepsilon^2}{4} A^2 n^2 \delta(n)^2 e^{2\delta(n)}$$

$$\geqslant 1 - \{64n^{-1}\delta(n)e^{\delta(n)} + \delta(n)^{-4}\}\,[1+\underline{O}(1/\delta(n))]/\varepsilon^2 \to 1 \quad \text{as} \quad n \to \infty .$$

Therefore, since $n^{-1}\delta(n)e^{\delta(n)} \to 0$ as $n \to \infty$ (for example, when $\delta(n) = \underline{O}\,(\log n/\log\log n)$)

$$S_n \sim An\delta(n)e^{\delta(n)}, \quad \text{in probability, as } n \to \infty .$$

Finally, the basic tour can be constructed by using Algorithm B' M times so that the term (iii) is clearly

$$\underline{O}(M) = \underline{O}(\, n/\delta(n)).$$

The proof is now complete, since the term (ii) dominates the others.

QED

4. Asymptotic Performance

Before proving Theorem 2, we need to prove three auxiliary lemmas. First, let us establish a notation for some concepts used in this section (following the notation in Beardwood, Halton, and Hammersley [1959]).

We have already stated that $\underset{\sim}{P}$ denotes a sequence of points, $\underset{\sim}{P}^n$ denotes the first n points of $\underset{\sim}{P}$ and $\underline{\underline{C}}$ denotes the unit hypercube. Let $\underline{\underline{E}}$ denote any bounded Lebesgue-measurable subset of R^k (we shall suppose that the boundary of $\underline{\underline{E}}$ has zero measure); $\underset{\sim}{P}^n\,\underline{\underline{E}}$ denote the subset of $\underset{\sim}{P}^n$ which lies in $\underline{\underline{E}}$; $N(\underset{\sim}{P}\underline{\underline{E}})$ denote the (possibly infinite) number of points of $\underset{\sim}{P}$ in $\underline{\underline{E}}$; $\ell\,(\underset{\sim}{P}\underline{\underline{E}})$ denote the length of the shortest tour through the points of $\underset{\sim}{P}\underline{\underline{E}}$; $\underline{\underline{C}}_1, \underline{\underline{C}}_2, \ldots$ denote semiclosed hypercubes (i.e., hypercubes open on their lower-left faces and closed on their upper-right faces) in different positions in R^k; and $v(\underline{\underline{E}})$ denote the volume (k-dimensional Lebesgue measure) of $\underline{\underline{E}}$. If ξ is a positive real number, we write $\xi\underline{\underline{E}}$ for the set of all points with coordinates $(\xi X_1, \xi X_2, \ldots, \xi X_k)$ such that the $(X_1, X_2, \ldots, X_k)$ are points of $\underline{\underline{E}}$. Thus $\xi\underline{\underline{E}}$ is a ξ-fold linear magnification of $\underline{\underline{E}}$, which leaves the origin of R^k invariant, and $v(\xi\underline{\underline{E}}) = \xi^k\, v(\underline{\underline{E}})$. We will use $\xi\underset{\sim}{P}\underline{\underline{E}}$ to denote the magnification of $\underset{\sim}{P}\underline{\underline{E}}$, whereas $\underset{\sim}{P}\,\xi\,\underline{\underline{E}}$ will denote the intersection of the unmagnified $\underset{\sim}{P}$ with the magnified $\underline{\underline{E}}$.

The phrase '$\underset{\sim}{P} \in u(\underline{\underline{E}})$', where $\underline{\underline{E}}$ is a Lebesgue set of strictly positive measure, means that $\underset{\sim}{P} = P_1, P_2, \ldots$ is a sample of random points independently distributed over $\underline{\underline{E}}$ with uniform probability density.

The phrase '$\underset{\sim}{P} \in W_\xi$' means that $\underset{\sim}{P} = P_1, P_2, \ldots$ is a sample from a Poisson process of density ξ over R^k; that is to say, for arbitrary disjoint Lebesgue sets $\underline{\underline{E}}_1, \underline{\underline{E}}_2, \ldots, \underline{\underline{E}}_m$,

$$\Pr\left\{ N(\underset{\sim}{P}\,\underline{\underline{E}}_j) = N_j \;;\; j = 1,2,\ldots,m \right\} = \prod_{j=1}^{m} \frac{\{\xi\, v(\underline{\underline{E}}_j)\}^{N_j}}{N_j!} \quad \exp\{-\xi\, v(\underline{\underline{E}}_j)\}$$

Finally, we adopt the abbreviation $q = 1 - 1/k$, where $k \geq 2$.

With these notational conventions in mind, we are now able to state and prove the following lemmas.

Lemma 4.1: Let $M = m^k$ (where m is a positive even integer) be an integer value (but not a function of n as in Algorithm A) and let $\underline{\underline{C}}_j$, $j = 1,2,\ldots,M$, be the cubic cells, congruent to $(1/m)\ \underline{\underline{C}}$, obtained by dissecting $\underline{\underline{C}}$, as in Algorithm A. If $\underset{\sim}{P} \in W_\xi$, then

$$\mathcal{E}\, \ell(\underset{\sim}{P}\ \underline{\underline{C}}_j) \sim \beta\xi^q/M \quad , \text{ as } \quad \xi/M \to \infty \quad , \tag{4.1}$$

where β is an absolute constant (independent of ξ,M and P; but depending on k, the dimension of the space).

Proof: If ζ is a positive real number, Lemma 5 of Beardwood, Halton, and Hammersley [1959] says that

$$\mathcal{E}\, \ell\ (\underset{\sim}{P}'\zeta\,\underline{\underline{E}}) \sim \beta\ \zeta^k\ v(\underline{\underline{E}}) \quad \text{as} \quad \zeta \to \infty\ , \text{ for } \underset{\sim}{P}' \in W_1 . \tag{4.2}$$

We notice that, by scaling, to each $\underset{\sim}{P}' \in W_1$ in $\zeta\underline{\underline{E}}$ corresponds a $\underset{\sim}{P} \in W_\xi$ in $\zeta\xi^{-1/k}\underline{\underline{E}}$ (and this correspondence is one-to-one). By the same scaling we have

$$\ell(\underset{\sim}{P}\ \zeta\ \xi^{-1/k}\ \underline{\underline{E}}) = \xi^{-1/k}\ \ell(\underset{\sim}{P}'\,\zeta\,\underline{\underline{E}}).$$

Thus, from (4.2) we have

$$\mathcal{E}\, \ell(\underset{\sim}{P}\ \zeta\ \xi^{-1/k}\ \underline{\underline{E}}) \ \sim\ \xi^{-1/k}\ \beta\ \zeta^k\ v(\underline{\underline{E}}) \text{ as } \zeta \to \infty . \tag{4.3}$$

Let us take $\zeta\xi^{-1/k} = 1/m$ and $\underline{\underline{E}} = \underline{\underline{C}}$, so that $\underline{\underline{C}}_j$ is a linear translation of $(1/m)\underline{\underline{C}} = \zeta\xi^{-1/k}\underline{\underline{E}}$. Then

$v(\underline{\underline{E}}) = v(\underline{\underline{C}}) = 1$, $\zeta^k = \xi/m^k = \xi/M$. Thus, as $\zeta \to \infty$, $\xi/M \to \infty$; and $\xi^{-1/k} \zeta^k = \xi^q/M$. Since W_ξ is homogeneous in R^k, so that translation of sets has no effect on the statistics, from (4.3) we get (4.1).

QED

Lemma 4.2: Under the same conditions as in Lemma 4.1, we have

$$\operatorname{var} \ell(\underset{\sim}{P}\underline{\underline{C}}_j) = \underline{o}(1)\, \xi^{-2/k} (\xi/M)^{2-2/k^2}, \quad \text{as } \xi/M \to \infty, \tag{4.4}$$

where $\underline{o}(1)$ depends only on k.

Proof: If ζ is a positive real number and if $\underline{\underline{E}} \subseteq \underline{\underline{C}}$, Lemma 6 of Beardwood, Halton, and Hammersley [1959] implies that

$$\operatorname{var} \ell(\underset{\sim}{P}'\zeta\underline{\underline{E}}) = \underline{O}\,(\zeta^{2k-2/(k-1)} \log^2 \zeta), \text{ as } \zeta \to \infty, \quad \text{for } \underset{\sim}{P}' \in W_1. \tag{4.5}$$

We notice that

$$\zeta^{2/k}\zeta^{-2/(k-1)} \log^2\zeta = \zeta^{-2/(k(k-1))} \log^2\zeta = \underline{o}(1), \quad \text{as } \zeta \to \infty, \text{ for all } k \geq 2.$$

Thus, from (4.5) we have that

$$\operatorname{var} \ell(\underset{\sim}{P}'\zeta\, \underline{\underline{E}}) = \underline{o}(1)\, \zeta^{2k-2/k}, \text{ as } \zeta \to \infty. \tag{4.6}$$

If $\underset{\sim}{P} \in W_\xi$ and we consider the set $\zeta\, \xi^{-1/k}\underline{\underline{E}}$; by scaling as in the proof of Lemma 4.1 above, we have from (4.6) that

$$\operatorname{var} \ell(\underset{\sim}{P}\, \zeta\xi^{-1/k}\underline{\underline{E}}) = \xi^{-2/k} \operatorname{var} \ell\, (\underset{\sim}{P}'\zeta\, \underline{\underline{E}}) =$$

$$= \underline{o}(1)\ \xi^{-2/k}\ \zeta^{2k-2/k}\ ,\ \text{as } \zeta \to \infty\ . \tag{4.7}$$

As before, if $\zeta\xi^{-1/k} = 1/m$ and $\underline{\underline{E}} = \underline{\underline{C}}$, then $\zeta = (\xi/M)^{1/k}$.

Thus, from (4.7) we have that

$$\operatorname{var}\ \ell(\underset{\sim}{P}(1/m)\underline{\underline{C}}) = \operatorname{var}\ \ell(\underset{\sim}{P}\ \underline{\underline{C}}_j) =$$

$$= \underline{o}(1)\ \xi^{-2/k}\ (\xi/M)^{2-2/k^2}\ ,$$

$$\text{as } \xi/M \to \infty\ .$$

QED

Let us now introduce $U_{\xi,M}$, a random variable conditional on ξ and M as parameters with M>1:

$$U_{\xi,M} = \sum_{j=1}^{M} \ell(\underset{\sim}{P}\ \underline{\underline{C}}_j) \qquad , \qquad \underset{\sim}{P} \in W_\xi\ ,\ \underline{\underline{C}}_j \text{ a translation of } (1/m)\underline{\underline{C}}.$$

Then, by the independence of $\underset{\sim}{P} \in W_\xi$ in the disjoint $\underline{\underline{C}}_j$'s we have from Lemmas 4.1 and 4.2 that

$$\mathcal{E}\, U_{\xi,M} = \sum_{j=1}^{M} \mathcal{E}\ell(\underset{\sim}{P}\ \underline{\underline{C}}_j) \sim \beta\ \xi^q\ ,\ \text{as } \xi/M \to \infty\ . \tag{4.8}$$

$$\text{var } U_{\xi,M} = \sum_{j=1}^{M} \text{var } \ell(\underset{\sim}{P}\ \underset{=}{C}_j) = \underline{o}(1)\ M\xi^{-2/k}(\xi/M)^{2-2/k^2}\ ,$$

$$\text{as} \quad \xi/M \to \infty\ . \tag{4.9}$$

Lemma 4.3: Given any set $\underset{\sim}{P}^n$ of n points in $\underset{=}{C}$, let $M_1 = m_1{}^k$ and $M_2 = m_2{}^k$, where $m_1 < m_2$ and m_1, m_2 are positive even integers. Consider the dissections of $\underset{=}{C}$, into M_1 cells $\underset{=}{C}_{1i}$ congruent to $m_1{}^{-1}\underset{=}{C}$, and into M_2 cells $\underset{=}{C}_{2j}$ congruent to $m_2{}^{-1}\underset{=}{C}$, as in Algorithm A. Then

$$\sum_{j=1}^{M_2} \ell(\underset{\sim}{P}^n\underset{=}{C}_{2j}) \leq \sum_{i=1}^{M_1} \ell(\underset{\sim}{P}^n\underset{=}{C}_{1i})$$

$$+\ \underline{O}\left[M_2{}^{1/k(k-1)}n^{1-1/(k-1)}\right] + \underline{O}\left[M_2{}^{1-1/k}\right]. \tag{4.10}$$

Proof: Since $M_1 < M_2$, the cells $\underset{=}{C}_{2j}$ are smaller than the cells $\underset{=}{C}_{1i}$ (sides are $m_2{}^{-1}$ and $m_1{}^{-1}$, respectively); thus any cell $\underset{=}{C}_{2j}$ can contain at most one corner of the dissection into cells $\underset{=}{C}_{1i}$. Therefore, $\underset{=}{C}_{2j}$ contains all or part of at most 2^k minimal cell-tours T_i (say) of $\underset{\sim}{P}^n\underset{=}{C}_{1i}$. We distinguish two cases: k=2 and k>2.

Case (i): k=2. We form a tour of $\underset{\sim}{P}^n\underset{=}{C}_{2j}$ as follows. Any pieces of T_i $(i=1,\ldots,M_1)$ intersecting $\underset{=}{C}_{2j}$ can be formed into a simple closed polygon by tracing parts of the perimeter of $\underset{=}{C}_{2j}$. (See Fig. 4) This perimeter is of length $4m_2{}^{-1} = 4M_2{}^{-1/2}$. Any T_i contained entirely in $\underset{=}{C}_{2j}$ can be connected to the above polygon by a double chord of length less than $m_2{}^{-1}$ (See Fig. 5) Such included tours cannot be more than 4 in number. Since each part of every T_i will lie in exactly one of the $\underset{=}{C}_{2j}$, the sum of the tours constructed above will not exceed

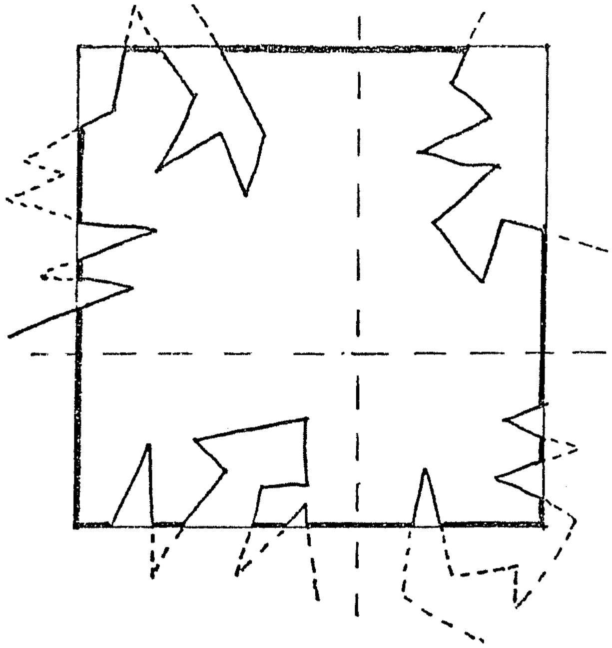


Figure 4

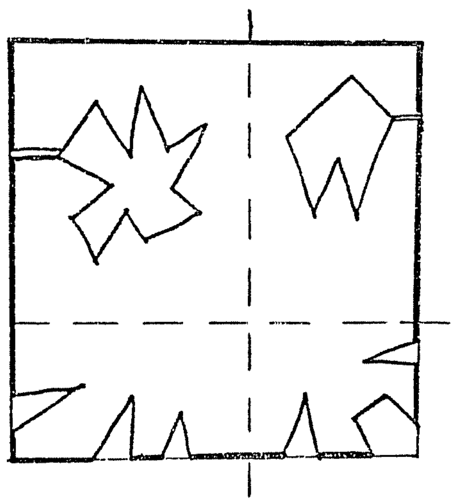


Figure 5

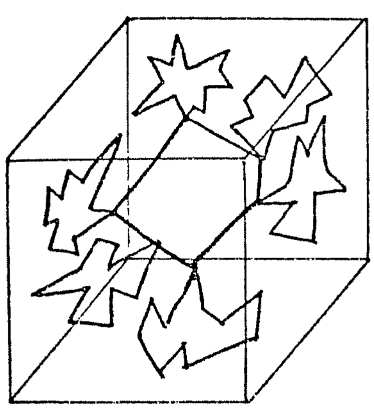


Figure 6

$\sum_{i=1}^{M_1} \ell(\underset{\sim}{P}^n \underline{\underline{C}}_{1i}) + 8\, M_2^{1/2}$, and will not be less than the minimal sum $\sum_{j=1}^{M_2} \ell(\underset{\sim}{P}^n \underline{\underline{C}}_{2j})$. This proves (4.10) for k=2.

Case (ii): $k \geqslant 3$. The cell $\underline{\underline{C}}_{2j}$ now has 2k faces (of k-1 dimensions), of (k-1)-dimensional volume $m_2^{-(k-1)} = M_2^{-(k-1)/k}$. Various tours T_i will cross a particular face (say) F times; and so, we may form a tour of these F intersections by a polygon L, of length not exceeding $\alpha'_{k-1}\, M_2^{-1/k}\, F^{1-1/(k-1)}$ (by Lemma 4 of Beardwood, Halton, and Hammersley [1959]; with $\alpha'_{k-1} \geqslant \alpha_{k-1}$ independent of M_2, F, or $\underset{\sim}{P}^n$); and therefore, all pieces of tours T_i entering into $\underline{\underline{C}}_{2j}$ by the given face may be connected into a simple closed polygon by parts of such a polygon L, rather as in Case (i). All 2k such paths belonging to $\underline{\underline{C}}_{2j}$ may then be joined into a single simple closed polygon by 2k segments of total length not exceeding $2k^{3/2} M_2^{-1/k}$ (Figure 6), since the diagonal of $\underline{\underline{C}}_{2j}$ is $k^{1/2} M_2^{-1/k}$. As in Case (i), we see that there are at most 2^k tours T_i entirely contained in $\underline{\underline{C}}_{2j}$, and these can be incorporated into our tour of $\underset{\sim}{P}^n \underline{\underline{C}}_{2j}$ by double chords of length less than $M_2^{-1/k}$. Again, each part of every T_i will lie in exactly one $\underline{\underline{C}}_{2j}$, and the sum of all the numbers F of intersections of faces with tours cannot exceed 4n, since each point of $\underset{\sim}{P}^n$ is connected to its successor, in its T_i, by just one chord, and this can only cross at most two faces of the finer dissection; and every such intersection is counted twice.

Thus the sum of the tours constructed above cannot exceed

$$\sum_{i=1}^{M_1} \ell(\underset{\sim}{P}^n \underline{\underline{C}}_{1i}) + \alpha'_{k-1}\, M_2^{-1/k} \sum_{\text{faces}} F^{1-1/(k-1)}$$

$$+ 2k^{3/2} M_2^{1-1/k} + 2^k M_2^{1-1/k} . \qquad (4.11)$$

By Hölder's inequality, since every face intersected at all will be counted twice, and there are at most $(M_2 - M_2^{1-1/k})2k$ such faces,

$$\sum_{\text{faces}} (1)^{1/(k-1)} F^{1-1/(k-1)} \leqslant \left[\sum_{\text{faces}} (1) \right]^{1/(k-1)} \left[\sum_{\text{faces}} F \right]^{1-1/(k-1)}$$

$$= \left[4k(M_2 - M_2^{1-1/k}) \right]^{1/(k-1)} (4n)^{1-1/(k-1)}$$

$$= \underline{O} \left[M_2^{1/(k-1)} n^{1-1/(k-1)} \right]$$

Thus the upper bound given by (4.11) is

$$\sum_{i=1}^{M_1} \ell(\underset{\sim}{P}^n \underset{=}{C}_{1i}) + \underline{O} \left[M_2^{1/(k-1)-1/k} n^{1-1/(k-1)} \right]$$

$$+ \underline{O} \left[M_2^{1-1/k} \right] .$$

Since the sum of the tours constructed above cannot be less than $\sum_{j=1}^{M_2} \ell(\underset{\sim}{P}^n \underset{=}{C}_{2j})$, we obtain (4.10) for $k \geqslant 3$.

Q.E.D.

Finally, we are now able to proceed to:

Proof of Theorem 2:

First, assume the conditions of Lemmas 4.1 and 4.2.

From (4.8) we know that for all sufficiently large ξ/M and for any arbitrary $\varepsilon > 0$ we have

$$| \mathcal{E}\, U_{\xi,M} - \beta \xi^q | < \frac{1}{2} \varepsilon \beta \xi^q ;$$

and then by Chebyshev's inequality, much as in the proof of Theorem 1,

$$\Pr\{\ |U_{\xi,M} - \beta\,\xi^q\ | \leqslant \varepsilon\ \beta\ \xi^q\ \} \geqslant \Pr\{\ |U_{\xi,M} - \mathcal{E}\ \ U_{\xi,M}\ | \leqslant \frac{1}{2}\ \varepsilon\ \beta\ \xi^q\ \}$$

$$\geqslant 1 - \underline{o}(1)\ M\ \xi^{-2/k}(\xi/M)^{2-2/k^2} / \left(\frac{1}{2}\ \varepsilon\beta\xi^q\right)^2 \qquad \text{(by (4.9))}$$

$$= 1 - \frac{\underline{o}(1)}{\varepsilon^2} \qquad \frac{1}{\xi^{2/k^2}\ M^{1-2/k^2}}\ , \ \text{as}\ \frac{\xi}{M} \to \infty\ . \qquad (4.12)$$

Also, if $N(\underset{\sim}{P}\ \underline{\underline{C}}) = n_\xi$, then by Chebyshev's inequality,

$$\Pr\ \{\ \ |\ n_\xi - \xi\ | \leqslant \varepsilon\ \xi\ \} \quad \geqslant 1 - \frac{1}{\varepsilon^2 \xi}\ , \qquad (4.13)$$

since $\mathcal{E}\ n_\xi \ = \text{var}\ n_\xi = \xi$.

Thus, from (4.12) and (4.13) we have for all sufficiently large ξ/M that

$$\Pr\left[\beta\ \frac{1-\varepsilon}{(1+\varepsilon)^q}\ \leqslant\ \frac{U_{\xi,M}}{n_\xi^{\ q}}\ \ \leqslant\ \beta\ \frac{1+\varepsilon}{(1-\varepsilon)^q}\right] \geqslant 1 - \left[\frac{1}{\xi^{2/k^2} M^{1-2/k^2}} + \frac{1}{\xi}\right] \varepsilon^{-2}\ . \qquad (4.14)$$

Now, let $V_{n,M} = \sum_{j=1}^{M} \ell(\underset{\sim}{P}^n \underline{\underline{C}}_j)$ where n is a positive integer value and $P \in u(\underline{\underline{C}})$.

Next, define $f(n,M)$ by

$$1 - f(n,M) = \text{Prob}\left[\beta\ \frac{1-\varepsilon}{(1+\varepsilon)^q}\ \leqslant\ \frac{V_{n,M}}{n^q}\ \leqslant\ \beta\ \frac{1+\varepsilon}{(1-\varepsilon)^q}\right]. \qquad (4.15)$$

Since the conditional probability distribution of $U_{\xi,M}$ given $n_\xi = n$ is the unconditional probability distribution of $V_{n,M}$, we have

$$\sum_{n=0}^{\infty} e^{-\xi} \frac{\xi^n}{n!} [1-f(n,M)] = \Pr\left[\beta \frac{1-\varepsilon}{(1+\varepsilon)^q} \leqslant \frac{U_{\xi,M}}{n_\xi q} \leqslant \beta \frac{1+\varepsilon}{(1-\varepsilon)^q}\right]$$

$$\geqslant 1-\left[\frac{1}{\xi^{2/k^2}M^{1-2/k^2}} + \frac{1}{\xi}\right]\varepsilon^{-2}, \qquad (4.16)$$

for all sufficiently large ξ/M.

Since $0 \leqslant 1-f(n,M) \leqslant 1$, (4.16) gives us that

$$\sup_{|t-\xi|\leqslant \varepsilon\xi} [1-f(t,M)] \sum_{|n-\xi|\leqslant \varepsilon\xi} e^{-\xi} \frac{\xi^n}{n!} + \sum_{|n-\xi|>\varepsilon\xi} e^{-\xi} \frac{\xi^n}{n!}$$

$$\geqslant 1-\left[\frac{1}{\xi^{2/k}M^{1-2/k^2}} + \frac{1}{\xi}\right]\varepsilon^{-2}. \qquad (4.17)$$

By observing that the first summation above is less than 1 and the second summation is less than $1/(\varepsilon^2\xi)$, by (4.13), we have that

$$\sup_{|t-\xi|\leqslant \varepsilon\xi} [1-f(t,M)] \geqslant 1-\left[\frac{1}{\xi^{2/k^2}M^{1-2/k^2}} + \frac{2}{\xi}\right]\varepsilon^{-2},$$

for all sufficiently large ξ/M. (4.18)

Since by hypothesis $M > 1$, for all sufficiently large ξ/M we have

$$\sup_{|t-\xi|\leqslant \varepsilon\xi} [1-f(t,M)] \geqslant 1-\left[\frac{1}{\xi^{2/k^2}} + \frac{2}{\xi}\right]\varepsilon^{-2}$$

$$\geqslant 1- C\xi^{-2/k^2}, \qquad (4.19)$$

for all sufficiently large ξ/M, where C is a constant (depending upon ε and k but not on M.)

The supremum in (4.19) is taken over the range:

$$(1-\varepsilon)\ \xi \leqslant t \leqslant (1+\varepsilon)\ \xi$$

If $\xi = \dfrac{(1+\varepsilon)^m}{(1-\varepsilon)^{m+1}}$, and if J_m is the set of integers t satisfying: $\left(\dfrac{1+\varepsilon}{1-\varepsilon}\right)^m \leqslant t \leqslant \left(\dfrac{1+\varepsilon}{1-\varepsilon}\right)^{m+1}$, $m = 1,2,\ldots\ldots$, then J_m becomes the range of the supremum in (4.19).

We observe that, for any M and for sufficiently large m, ξ/M can be made as large as we like, and so can ensure that (4.19) above holds true. In particular, if we let n be any member of J_m, for fixed ε, and let $M = M(n) = n/\delta(n)$. We have

$$\frac{\xi}{M(n)} = \frac{(1+\varepsilon)^m}{(1-\varepsilon)^{m+1}}\ \frac{\delta(n)}{n} \geqslant \frac{(1+\varepsilon)^m}{(1-\varepsilon)^{m+1}}\ \frac{(1-\varepsilon)^{m+1}}{(1+\varepsilon)^{m+1}}\ \delta(n) = \frac{\delta(n)}{(1+\varepsilon)}$$

Since $\delta(\cdot)$ is an increasing function; for fixed ε and for sufficiently large m, $\xi/M(n)$ can be made as large as we like; so that from (4.19) we have

$$\sup_{t\in J_m}\ [1-f(t,M(n))] \geqslant 1 - C'\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{-2m/k^2}, \qquad (4.20)$$

for all sufficiently large m,

where $C' = C(1-\varepsilon)^{2/k^2}$ is a constant (depending only on ε and k). That is, there is an integer m_0 (depending on ε and k) such that (4.20) holds for all $m \geq m_0$. Further, since J_m contains only a finite number of integers, it contains an integer n_m (depending on ε, k and n) such that

$$1-f(n_m, M(n)) = \sup_{t\in J_m} [1-f(t,M(n)]; \quad \text{whence}$$

$$\sum_{m=0}^{\infty} \left\{ 1-\Pr \left[\beta \frac{1-\varepsilon}{(1+\varepsilon)^q} \le \frac{V_{n_m,M(n)}}{n_m^{\ q}} \le \beta \frac{1+\varepsilon}{(1-\varepsilon)^q} \right] \right\}$$

$$= \sum_{m=0}^{m_0-1} f(n_m, M(n)) + \sum_{m=m_0}^{\infty} f(n_m, M(n)) \le m_0 + \sum_{m=m_0}^{\infty} C' \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{-2m/k^2} < \infty . \tag{4.21}$$

By the Borel-Cantelli lemma, (4.21) implies that, with probability one, for any choices of n (and consequent values of M and n_m) in each J_m,

$$\beta \frac{1-\varepsilon}{(1+\varepsilon)^q} \leqslant \lim_{m\to\infty} \inf \frac{V_{n_m,M(n)}}{n_m^{\ q}} \leqslant \lim_{m\to\infty} \sup \frac{V_{n_m,M(n)}}{n_m^{\ q}} \leqslant \beta \frac{1+\varepsilon}{(1-\varepsilon)^q} . \tag{4.22}$$

Next, for choices n', n, and n'' in J_{m-1}, J_m, and J_{m+1}, respectively, write $\mu_n = n_{m-1}$, $V_n = n_{m+1}$. From the definition of J_m and n_m we have

$$\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{m-1} \leqslant \mu_n \leqslant \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{m} \leqslant n \leqslant \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{m+1} \leqslant \nu_n \leqslant \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{m+2} . \tag{4.23}$$

From (4.23),
$$\begin{aligned} 0 \leqslant n-\mu_n &\leqslant n - \frac{n}{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{m+1}} \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{m-1} \\ &= n \left[1-\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2\right] \\ &= n \frac{4\varepsilon}{(1+\varepsilon)^2} \leqslant 4\,\varepsilon\, n . \end{aligned} \tag{4.24}$$

Similarly, from (4.23), $0 \leqslant \nu_n - n \leqslant \dfrac{n}{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^m} \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{m+2} - n$

$$= n\left[\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2 - 1\right]$$

$$= n \frac{4\varepsilon}{(1-\varepsilon)^2} \leqslant 5 \varepsilon n \qquad (4.25)$$

for sufficiently small ε $(< \frac{1}{5+\sqrt{20}})$.

Thus $\left(\underset{\sim}{P}^{\mu_n}\right)^c \underset{\sim}{P}^n$ consists of a set of not more than $4 \varepsilon n$ points in $\underline{\underline{C}}$ and $\underset{\sim}{P}^{\nu_n} (\underset{\sim}{P}^n)^c$ consists of a set of not more than $5 \varepsilon n$ points in $\underline{\underline{C}}$.

Now, if $\bar{\underline{\underline{E}}}$ denotes the closure of $\underline{\underline{E}}$, by Lemma 4 of Beardwood, Halton and Hammersley [1959] there is an α such that $\limsup_{n\to\infty} n^{-q} \ell(\underset{\sim}{P}^n \underline{\underline{E}}) \leq \alpha \; v^{1/k}(\bar{\underline{\underline{E}}})$ i.e. there is an α' such that

$$(\forall n) \; n^{-q} \ell(\underset{\sim}{P}^n \underline{\underline{E}}) \leqslant \alpha' \; v^{1/k}(\bar{\underline{\underline{E}}}) \; , \qquad (4.26)$$

where α and α' are absolute constants (depending on k). If $a_j = N(\underset{\sim}{P}^{\nu_n} (\underset{\sim}{P}^n)^c \underline{\underline{C}}_j)$, by applying (4.26) to $\underline{\underline{C}}_j$ we have

$$(\forall a_j) \;\; a_j^{-q} \ell(P^{\nu_n} (\underset{\sim}{P}^n)^c \underline{\underline{C}}_j) \leqslant \alpha' M(n)^{-1/k} \; , \qquad (4.27)$$

since $v(\underline{\underline{C}}_j) = M(n)^{-1}$.

From (4.25) we have

$$V_{\nu_n, M(n)} \leqslant V_{n, M(n)} + \sum_{j=1}^{M(n)} [\ell(\underset{\sim}{P}^{\nu_n} (\underset{\sim}{P}^n)^c \underline{\underline{C}}_j) + 2\sqrt{k} \, M(n)^{-1/k}] \qquad (4.28)$$

($\sqrt{k} \, M(n)^{-1/k}$ is the diameter of $\underline{\underline{C}}_j$).

Since $\sum_{j=1}^{M(n)} a_j^q \leqslant M(n)^{1/k}\left(\sum_{j=1}^{M(n)} a_j\right)^q$ (Hölder's inequality), from (4.27) we have

$$\sum_{j=1}^{M(n)} [\ell(\underset{\sim}{P}^{\nu_n}(\underset{\sim}{P}^n)^c \underline{\underline{C}}_j)] \leqslant \sum_{j=1}^{M(n)} a_j^q \, \alpha' M(n)^{-1/k}$$

$$\leqslant \alpha' M(n)^{-1/k} \, M(n)^{1/k} \left[\sum_{j=1}^{M(n)} a_j\right]^q$$

$$\leqslant \alpha' \; (5\varepsilon n)^q \, . \tag{4.29}$$

From (4.28) and (4.29), we have

$$n^{-q} \, V_{\nu_n, M(n)} \leqslant n^{-q} \, V_{n,M(n)} + n^{-q}[\alpha'(5\varepsilon n)^q + 2\sqrt{k}\, M(n)^q]$$

$$= n^{-q} V_{n,M(n)} + \alpha'(5\varepsilon)^q + 2\sqrt{k}\left(\frac{M(n)}{n}\right)^q \tag{4.30}$$

and the last term in (4.30) is $\underline{o}(1)$, as $n \to \infty$. On the other hand, since $M(n) < M(n'')$ and $\nu_n^{-q} \leqslant n^{-q}$, by Lemma 4.3 we have that

$$\nu_n^{-q} \, V_{\nu_n, M(n'')} \leqslant n^{-q} V_{\nu_n, M(n)} + n^{-q}\left[\underline{O}\left(M(n'')^{\frac{1}{k(k-1)}} \, \nu_n^{1-\frac{1}{k-1}}\right) + \underline{O}(M(n'')^q)\right] . \tag{4.31}$$

Since, by (4.25), $\nu_n \leqslant (1+5\varepsilon)n$ and similarly $n'' \leqslant (1+5\varepsilon)n$, we have that

$$n^{-q}\underline{O}\left(M(n'')^{\frac{1}{k(k-1)}}\, \nu_n^{\,1-\frac{1}{k-1}}\right) = \underline{O}\left[\left(\frac{(1+5\varepsilon)n}{\delta(n'')}\right)^{\frac{1}{k(k-1)}} \frac{((1+5\varepsilon)n)^{1-\frac{1}{k-1}}}{n^{1-\frac{1}{k}}}\right]$$

$$= \underline{O}\ (\delta(n'')^{\frac{-1}{k(k-1)}}) = \underline{o}(1), \text{ as } n \to \infty, \tag{4.32}$$

and also

$$n^{-q}\underline{O}(M(n'')^q) = \underline{O}\left[\left(\frac{(1+5\varepsilon)n}{n\ \delta(n'')}\right)^q\right] = \underline{o}(1), \text{ as } n \to \infty . \tag{4.33}$$

We have from (4.30), (4.31), (4.32), and (4.33)

$$\nu_n^{-q}\ V_{\nu_n, M(n'')} \leqslant n^{-q}\ V_{n,M(n)} + \alpha'(5\varepsilon)^q + \underline{o}(1), \text{ as } n \to \infty .$$

We see that, in this inequality, the independent variables are ε, n (in J_m, which determines m), and n'' (chosen in J_{m+1}, which determines $\nu_n = n_{m+1}$). Applying (4.22), we thus get that

$$\frac{\beta(1-\varepsilon)}{(1+\varepsilon)^q} \leqslant \liminf_{n\to\infty}\ n^{-q}\ V_{n,M(n)} + \alpha'(5\varepsilon)^q, \text{ with probability one.} \tag{4.34}$$

Similarly, if $b_j = N(\mathcal{P}^n(\mathcal{P}^{\mu_n})^c \underline{\underline{C}}_m)$, by applying (4.26) to $\underline{\underline{C}}_j$, we get

$$(\forall b_j)\ \ b_j^{-q}\ \ell(\mathcal{P}^n(\mathcal{P}^{\mu_n})^c\underline{\underline{C}}_j) \leqslant \alpha' M(n)^{-1/k} . \tag{4.35}$$

From (4.24) we have

$$V_{\mu_n,M(n)} \geqslant V_{n,M(n)} - \sum_{j=1}^{M(n)} [\ell(\mathcal{P}^n(\mathcal{P}^{\mu_n})^c \underline{\underline{C}}_j) + 2\sqrt{k}\, M(n)^{-1/k}] \,. \quad (4.36)$$

From (4.35) we have

$$\sum_{j=1}^{M(n)} [\ell(\mathcal{P}^n(\mathcal{P}^{\mu_n})^c \underline{\underline{C}}_j)] \leqslant \sum_{j=1}^{M(n)} b_j^q\, \alpha' M(n)^{-1/k}$$

$$\leqslant \alpha'\, M(n)^{-1/k}\, M(n)^{1/k} \left[\sum_{j=1}^{M(n)} b_j\right]^q$$

$$\leqslant \alpha'(4\varepsilon n)^q \quad (\text{by } (4.24)) \,. \quad (4.37)$$

From (4.36) and (4.37) we have

$$n^{-q}\, V_{\mu_n,M(n)} \geqslant n^{-q}\, V_{n,M(n)} - n^{-q}[\alpha'(4\varepsilon n)^q + 2\sqrt{k}\, M(n)^q]$$

$$= n^{-q}\, V_{n,M(n)} - \alpha'(4\varepsilon)^q + \underline{o}(1) \,, \quad (4.38)$$

$$\text{as} \quad n \to \infty \,.$$

On the other hand, since $M(n') < M(n)$ and $n^{-q} \leqslant \mu_n^{-q}$, by Lemma 4.3 we have

$$\mu_n^{-q}\, V_{\mu_n,M(n')} + \mu_n^{-q}\left[\underline{O}\left(M(n)^{\overline{k(k-1)}^{\,1}}\, \mu_n^{\,1-\frac{1}{k-1}}\right) + \underline{O}(M(n)^q)\right]$$

$$\geqslant n^{-q}\, V_{\mu_n,M(n)} \,. \quad (4.39)$$

Since, by (4.24), $\mu_n \geqslant (1-4\varepsilon)n$ we have

$$\mu_n^{-q}\, \underline{O}\left(M(n)^{\frac{1}{k(k-1)}}\ \mu_n^{1-\frac{1}{k-1}}\right) = \underline{O}\ \left[\left(\frac{(1-4\varepsilon)^{-1}\mu_n}{\delta(n)}\right)^{\frac{1}{k(k-1)}} \frac{\mu_n^{1-\frac{1}{k-1}}}{\mu_n^{1-\frac{1}{k}}}\right]$$

$$= \underline{O}\ \left(\delta(n)^{\frac{-1}{k(k-1)}}\right) = \underline{o}(1), \quad \text{as } n \to \infty, \tag{4.40}$$

and also

$$\mu_n^{-q}\, \underline{O}(M(n)^q) = \underline{O}\ \left[\left(\frac{(1-4\varepsilon)^{-1}\mu_n}{\delta(n)\mu_n}\right)^q\right] = \underline{o}(1), \quad \text{as } n \to \infty . \tag{4.41}$$

We have from (4.38), (4.39), (4.40), and (4.41) that

$$\mu_n^{-q}\, V_{\mu_n, M(n')} \geqslant n^{-q}\, V_{n,M(n)} - \alpha'(4\varepsilon)^q + \underline{o}(1), \quad \text{as } n \to \infty .$$

As in obtaining (4.34), we note that the independent variables in this inequality are ε, n (which determines m), and n' (which determines $\mu_n = n_{m-1}$). Applying (4.22) again, we get

$$\frac{\beta(1+\varepsilon)}{(1-\varepsilon)^q} \geqslant \limsup_{n \to \infty}\ n^{-q}\, V_{n,M(n)} - \alpha'(4\varepsilon)^q , \tag{4.42}$$

with probability one.

Since ε is arbitrary and $n^{-q}\, V_{n,M(n)}$ does not depend on ε, (4.34) and (4.42) imply that

$$\lim_{n \to \infty} n^{-q}\, V_{n,M(n)} = \beta, \quad \text{with probability one.} \tag{4.43}$$

Now let $X_{n,M(n)} = \sum_{j=1}^{M(n)} \ell(\underset{=}{P}^n \underset{=}{C}_j \cup K_j)$ where K_j is the singleton containing the cell-center of $\underset{=}{C}_j$. Then we have

$$X_{n,M(n)} \leqslant V_{n,M(n)} + M(n) \quad [2\sqrt{k}\, M(n)^{-1/k}]$$

$$= V_{n,M(n)} + \underline{O}[M(n)^q]$$

$$\sim \beta \quad n^q + \underline{O}[n^q / \delta(n)^q] \qquad \text{(by (4.43))}$$

$$= \beta \quad n^q + \underline{o}(n^q) \,.$$

Thus: $X_{n,M(n)} \leqslant \beta \quad n^q + \underline{o}(n^q)$, as $n \to \infty$, with probability one.

(4.44)

Since the basic tour has length $M(n)\ h$ (there are $M(n)$ cell centers being connected by edges of length h), where $h = 1/m = 1/M(n)^{1/k}$, we have that the length of the closed path given by Algorithm A is, by (4.44)

$$T(n) = X_{n,M(n)} + \underline{O}[M(n)^q]$$

$$= X_{n,M(n)} + \underline{o}(n^q)$$

$$\leqslant \beta\, n^q + \underline{o}(n^q)\,, \quad \text{as } n \to \infty, \quad \text{with probability one.} \qquad (4.45)$$

On the other hand, by Lemma 7 of Beardwood, Halton, and Hammersley [1959], the length $T_0(n)$ of the optimal tour is such that

$$T_0(n) \sim \beta\, n^q \,, \quad \text{as} \quad n \to \infty \,, \quad \text{with probability one.} \qquad (4.46)$$

From (4.45) and (4.46) we have

$$1 \leqslant \frac{T(n)}{T_0(n)} \leqslant \frac{\beta n^q + \underline{o}(n^q)}{\beta n^q}$$

Thus $\quad \dfrac{T(n)}{T_0(n)} \sim 1$ as $n \to \infty$, with probability one.

QED

In constructing the proof of Theorem 2, we had occasion to review Beardwood, Halton, and Hammersley [1959]; and, in particular, to check the proofs of the lemmas there. We found a small inaccuracy in the proof of Lemma 7, which is corrected in Appendix II.

5. A Generalization of the Results

As we mentioned in Section 1, Algorithm A can be applied to $\lambda\underline{\underline{C}}$, the k-dimensional hypercube of side λ, instead of $\underline{\underline{C}}$. In this section, we want to show how Theorems 1 and 2 can be modified so that, under Condition D, Theorem 3 holds true.

Under Condition D, we partition $\lambda\underline{\underline{C}}$ into $M = n/\delta(n)$ cubic cells, $\underline{\underline{C}}_j$ say. Let us define index sets H_0, H_1, H_2 as follows, for $1 \leq j \leq M$,

$$
\begin{aligned}
j \in H_0 , &\quad \text{iff} \quad \underline{\underline{C}}_j \subseteq \underline{\underline{E}}^c ; \\
j \in H_1 , &\quad \text{iff} \quad C_j \subseteq \underline{\underline{E}} ; \\
j \in H_2 , &\quad \text{iff} \quad j \notin H_0 \text{ and } j \notin H_1 .
\end{aligned}
$$

Let $N(H_0) = N_0$, $N(H_1) = N_1$, $N(H_2) = N_2$, and $M' = v(\underline{\underline{E}})M/\lambda^k$.

Since each cell is of volume λ^k/M and $\underline{\underline{E}}$ has volume $v(\underline{\underline{E}})$; under Condition D, the probability of s points falling into $\underline{\underline{C}}_j$ is

$$\binom{n}{s} \left\{\frac{\lambda^k/M}{v(\underline{\underline{E}})}\right\}^s \left\{1 - \frac{\lambda^k/M}{v(\underline{\underline{E}})}\right\}^{n-s} = \binom{n}{s} \; (1/M')^s \; (1-1/M')^{n-s} \tag{5.1}$$

if $j \in H_1$; while if $j \in H_0$, the probability is zero; and if $j \in H_2$, the probability will have λ^k/M replaced by $v(\underline{\underline{C}}_j\underline{\underline{E}}) \leqslant \lambda^k/M$.

Since the boundary of $\underline{\underline{E}}$ has zero k-dimensional Lebesgue measure, asymptotically, only $M\, v(\underline{\underline{E}})/\lambda^k$ cells contain some of the n points. More precisely, we have

$$N_1/M' \to 1 \quad \text{and} \quad N_2/N_1 \to 0 , \quad \text{as} \quad n \to \infty . \tag{5.2}$$

In the proof of Lemma 3.1 (in Appendix I) we have, under Condition D,

$$\mathcal{E}\, S_n = \left(\sum_{j \in H_0} + \sum_{j \in H_1} + \sum_{j \in H_2} \right) \sum_{s=0}^{n} \Pr\,[s \text{ points in } \underline{\underline{C}}_j]\; t_s$$

$$= N_1 \sum_{s=0}^{n} \binom{n}{s} (1/M')^s (1-1/M')^{n-s}\, t_s$$

$$+ \sum_{j \in H_2} \sum_{s=0}^{n} \binom{n}{s} (1/M_j)^s (1-1/M_j)^{n-s}\, t_s$$

$$= N_1\, \psi(M',n) + \sum_{j \in H_2} \psi(M_j,n)\,,$$

where, for each $j \in H_2$, $M_j = v(\underline{\underline{E}})/v(\underline{\underline{C}}_j\underline{\underline{E}}) \geq M'$. (5.3)

Then the proof of Lemma (3.1) shows that

$$\psi(M,n) \sim A(n/M)^2\, e^{(n/M)}\, (1-M/n)\,, \quad \text{as} \quad n \to \infty\,. \tag{5.4}$$

Hence, if $M_j \geq M'$, $\psi(M_j,n) = \underline{O}[\psi(M',n)]$. (5.5)

Applying (5.2) and (5.5) to (5.3) we have

$$\mathcal{E}\, S_n = N_1\, \psi(M',n) + \sum_{j \in H_2} \psi(M_j,n)$$

$$\sim M'\psi(M',n) + M' \left(\frac{N_1}{M'}\right) \left(\frac{N_2}{N_1}\right) \underline{O}[\psi(M',n)]$$

$$\sim M'\psi(M',n)\,, \quad \text{as} \quad n \to \infty\,; \tag{5.6}$$

so that, by (5.4), we have, if $\delta^*(n) = n/M'$,

$$\mathcal{E}\, S_n \sim An\delta^*(n)\, e^{\delta^*(n)}[1-1/\delta^*(n)]$$

$$= An[\delta(n)\lambda^k/v(\underline{\underline{E}})]e^{[\delta(n)\lambda^k/v(\underline{\underline{E}})]}[1-v(\underline{\underline{E}})/\delta(n)\lambda^k]$$

$$\text{as} \quad n \to \infty\,; \qquad (5.7)$$

i.e., Lemma 3.1 holds true under Condition D, after replacing each $\delta(n)$ by $\delta^*(n) = n/M'$.

Similarly, if we denote by $\psi_1(n/\delta(n),n) = \psi_1(M,n)$ the right-hand side of (I.9) (in Appendix I), and we denote by $\psi_2(M,n)$ the expression for $\mathcal{E}(t_{n_i}t_{n_j}|i \neq j)$ in (I.11), and we denote by $\psi_3(M,n)$ the expression for $(\mathcal{E}\, S_n)^2/M^2$ in (I.12); then we have, under Condition D,

$$\begin{aligned}
\text{var}\ S_n \leqslant\ & N_1\psi_1(M',n) + \underline{o}(N_1)\psi_1(M',n) \\
& + N_1^{\,2}\psi_2(M',n) + \underline{o}(N_1^{\,2})\ \underline{O}[\psi_2(M',n)] \\
& - \left\{N_1^{\,2}\psi_3(M',n) + \underline{o}(N_1^{\,2})\ \underline{O}[\psi_3(M',n)]\right\} \text{ by (I.9), (I.11) and (I.12)).} \\
\sim\ & M'\psi_1(M',n) \\
& + M'^2\psi_2(M',n) \\
& - M'^2\psi_3(M',n) \qquad \text{(by (5.2))}
\end{aligned}$$

$$= M'\psi_1(M',n) + M'^2[\psi_2(M',n) - \psi_3(M',n)]$$

$$\leq A^2 \left\{16n\delta^*(n)^3 e^{3\delta^*(n)} + \frac{n^2}{4\delta^*(n)^2} e^{2\delta^*(n)}\right\} \{1+\underline{O}[1/\delta^*(n)]\} \quad (5.8)$$

(by the paragraph following (I.13)); i.e. Lemma 3.2 holds true under Condition D, when $\delta(n)$ is replaced by $\delta^*(n)$.

By (5.7) and (5.8), the proof of Theorem 1 holds under Condition D if we replace each $\delta(n)$ by $\delta^*(n)$; so that the first part of Theorem 3 is proved.

Now we want to show that the second part of Theorem 3 is true.

As in the proof of Lemma 4.1, we have from Lemma 5 of Beardwood, Halton, and Hammersley [1959] that, for $\underset{\sim}{P} \in W_\xi$,

$$\mathcal{E}\, \ell(\underset{\sim}{P}\zeta\xi^{-1/k}\underline{\underline{E}}') \sim \xi^{-1/k}\, \beta\zeta^k v(\underline{\underline{E}}'), \quad \text{as } \zeta \to \infty, \qquad (5.9)/(4.3)^*$$

for any bounded Lebesgue-measureable subset $\underline{\underline{E}}'$ of R^k, with boundary of zero measure.

Now, under Condition D, take $\zeta = \xi^{1/k}$ and $\underline{\underline{E}}' = \underline{\underline{C}}_j\, \underline{\underline{E}} \subseteq \underline{\underline{C}}_j$ (note that $\underline{\underline{C}}_j$ is congruent to $\lambda M^{-1/k}\underline{\underline{C}}$). Then, as $\zeta \to \infty$, $\xi \to \infty$, and from (5.9), we have

$$\mathcal{E}\, \ell(\underset{\sim}{P}\,\underline{\underline{C}}_j\underline{\underline{E}}) \sim \beta\xi^q v(\underline{\underline{C}}_j\underline{\underline{E}}), \quad \text{as } \xi \to \infty\ . \qquad (5.10)/(4.1)^*$$

As in the proof of Lemma 4.2, we have from Lemma 6 of Beardwood, Halton, and Hammersley [1959] that, for $\underset{\sim}{P} \in W_\xi$,

$$\operatorname{var} \ell(\underset{\sim}{P}\,\zeta\xi^{-1/k}\underline{\underline{E}}') = \underline{o}(1)\xi^{-2/k}\, \zeta^{2k-2/k}, \quad \text{as } \zeta \to \infty, \qquad (5.11)/(4.7)^*$$

uniformly in $\underline{\underline{E}}' \subseteq \{$any set congruent to $\underline{\underline{C}}\}$.

Now, under Condition D, take $\zeta\xi^{-1/k} = \lambda/M^{-1/k}$ and $\underline{\underline{E}}' = \lambda^{-1}M^{1/k}\,\underline{\underline{C}}_j\,\underline{\underline{E}} \subseteq \lambda^{-1}M^{1/k}\,\underline{\underline{C}}_j$ (note that $\lambda^{-1}M^{1/k}\,\underline{\underline{C}}_j$ is congruent to to $\underline{\underline{C}}$). Then $\zeta^k = \lambda\xi/M$ and thus, as $\zeta \to \infty$, $\xi/M \to \infty$, and from (5.11), we have

$$\operatorname{var} \ell(\underset{\sim}{P}\,\underline{\underline{C}}_j\,\underline{\underline{E}}) = \underline{o}(1)\xi^{-2/k}(\xi/M)^{2-2/k^2}, \quad \text{as} \quad \xi/M \to \infty, \qquad (5.12)/(4.4)^*$$

uniformly in j.

Under Condition D, $U_{\xi,M}$ is defined as follows:

$$U_{\xi,M} = \sum_{j=1}^{M} \ell(\underset{\sim}{P}\,\underline{\underline{C}}_j\,\underline{\underline{E}}) \quad .$$

Then we have, as in Section 4, from (5.10) and (5.12), that

$$\mathcal{E}\, U_{\xi,M} = \sum_{j=1}^{M} \mathcal{E}\, \ell(\underset{\sim}{P}\,\underline{\underline{C}}_j\,\underline{\underline{E}}) \sim \beta\xi^q \sum_{j=1}^{M} v(\underline{\underline{C}}_j\underline{\underline{E}})$$

$$= \beta\xi^q\, v(\underline{\underline{E}}), \quad \text{as } \xi \to \infty; \qquad (5.13)/(4.8)^*$$

and, by the uniformity of (5.12) over $j = 1,2,\ldots,M$,

$$\operatorname{var} \ \ell(\underset{\sim}{P}\,\underline{\underline{C}}_j\,\underline{\underline{E}}) = \sum_{j=1}^{M} \operatorname{var} \ (\underset{\sim}{P}\,\underline{\underline{C}}_j\,\underline{\underline{E}})$$

$$= M\,\underline{o}(1)\ \xi^{2q-2/k^2}/M^{2-2/k^2}$$

$$= \underline{o}(1)\ \xi^{2q-2/k^2}/M^{1-2/k^2}, \quad \text{as} \quad \xi/M \to \infty. \qquad (5.14)/(4.9)^*$$

As in the proof of Theorem 2, we have from (5.13) that, for sufficiently large ξ and for any $\varepsilon > 0$,

$$|\mathcal{E}\, U_{\xi,M} - \beta\xi^q\, v(\underline{\underline{E}})| \leqslant \tfrac{1}{2}\,\varepsilon\beta\xi^q\, v(\underline{\underline{E}})\,;$$

and then, by Chebyshev's inequality,

$$\Pr\{|U_{\xi,M} - \beta\xi^q\, v(\underline{\underline{E}})| \leqslant \varepsilon\, \beta\xi^q\, v(\underline{\underline{E}})\}$$

$$\geqslant \Pr\{|U_{\xi,M} - \mathcal{E}\, U_{\xi,M}| \leqslant \tfrac{1}{2}\,\varepsilon\beta\xi^q\, v(\underline{\underline{E}})\}$$

$$\geqslant 1 - \underline{o}(1)[\xi^{2q-2/k^2}/M^{1-2/k^2}]/(\tfrac{1}{2}\,\varepsilon\beta\xi^q\, v(\underline{\underline{E}}))^2 \qquad \text{(by 5.14))}$$

$$= 1 - \frac{\underline{o}(1)}{\varepsilon^2}\left[\frac{1}{\xi^{2/k^2}M^{1-2/k^2}}\right], \quad \text{as} \quad \xi/M \to \infty \quad \text{and} \quad \xi \to \infty. \qquad (5.15)/(4.12)^*$$

Also, as in the proof of Theorem 2, if $N(\underset{\sim}{P}\,\underline{\underline{E}}) = n_\xi$ then

$$\Pr\{|n_\xi - \xi v(\underline{\underline{E}})| \leqslant \varepsilon\xi v(\underline{\underline{E}})\} \;\geqslant\; 1 - \frac{1}{\varepsilon^2 \xi v(\underline{\underline{E}})}\,, \qquad (5.16)/(4.13)^*$$

since $\mathcal{E}\, n_\xi = \text{var}\; n_\xi = \xi v(\underline{\underline{E}})$.

From (5.15) and (5.16), for sufficiently large ξ/M and ξ, we have

$$\Pr\left[\beta\,\frac{\xi^q v(\underline{\underline{E}})}{\xi^q v(E)^q}\,\frac{(1-\varepsilon)}{(1+\varepsilon)^q} \leqslant \frac{U_{\xi,M}}{n_\xi{}^q} \leqslant \frac{\beta\xi^q v(\underline{\underline{E}})}{\xi^q v(\underline{\underline{E}})^q}\,\frac{(1+\varepsilon)}{(1-\varepsilon)^q}\right]$$

$$= \Pr\left[\beta v(\underline{\underline{E}})^{1/k}\,\frac{(1-\varepsilon)}{(1+\varepsilon)^q} \leqslant \frac{U_{\xi,M}}{n_\xi{}^q} \leqslant \beta v(\underline{\underline{E}})^{1/k}\,\frac{(1+\varepsilon)}{(1-\varepsilon)^q}\right]$$

$$\geqslant 1 - \left[\frac{1}{\xi^{2/k^2}M^{1-2/k^2}} + \frac{1}{\xi v(\underline{\underline{E}})}\right]\varepsilon^{-2} \qquad (5.17)/(4.14)^*$$

Under Condition D, $V_{n,M}$ is defined as follows:

$$V_{n,M} = \sum_{j=1}^{M} \ell(\underset{\sim}{P}^n C_j \underline{\underline{E}})$$

where n is a positive integer value and $\underset{\sim}{P} \in u(\underline{\underline{C}})$.

The remaining part of the proof of Theorem 2 holds true here if we replace each occurrence of β by $\beta v(\underline{\underline{E}})^{1/k}$ (as (5.17) above suggests), and if we replace each occurrence of the condition "sufficiently large ξ/M" by "sufficiently large ξ/M and ξ".

The only additional point to observe is that, since we take $\xi = \frac{(1+\varepsilon)^m}{(1-\varepsilon)^{m+1}}$ just before the definition of the sets J_m in the proof of Theorem 2, the condition "sufficiently large ξ" is satisfied for sufficiently large m.

Appendix I

First we want to prove a remark which will be used in the proofs of Lemmas 3.1 and 3.2.

Remark 1 (A)

If $x, q \geqslant 0$ are fixed and $M \to \infty$ in such a way that $M/n \to 0$, then $0 \leqslant e^{xn/M} - (1+x/M)^{n-q} \leqslant e^{xn/M}\left[\frac{xq}{M} + \frac{x^2 n}{2M^2}\right]$.

Proof: $e^{xn/M} = 1 + \frac{xn}{M} + \frac{1}{2}\frac{x^2n^2}{M^2} + \ldots + \frac{1}{m!}\frac{x^m n^m}{M^m} + \ldots$, $(1+\frac{x}{M})^{n-q} = 1 + \frac{x(n-q)}{M} + \frac{1}{2}\frac{x^2(n-q)_2}{M^2} + \ldots + \frac{1}{m!}\frac{x^m(n-q)_m}{M^m} + \ldots$. So $e^{xn/M} - (1+x/M)^{n-q}$

$= \sum_{m=1}^{\infty} \frac{1}{m!}\frac{x^m}{M^m}[n^m - (n-q)_m]$. Now $n^m \geqslant (n-q)_m \geqslant n^m - n^{m-1}\sum_{i=0}^{m-1}(i+q)$

$= n^m - n^{m-1}m[\frac{1}{2}(m-1)+q)]$.

The first inequality holds because each factor of $(n-q)_m$ is less than or equal to n, and there are m factors on each side; the second is seen by induction: $n-q = n^1 - n^0 q$ $(m=1)$. If true for $m = h-1$, then $(n-q)_h = (n-q)_{h-1}(n-q-h+1) \geqslant \{n^{h-1} - n^{h-2}(h-1)[\frac{1}{2}(h-2)+q]\}(n-q-h+1)$

$\geqslant n^h - n^{h-1}(h-1)[\frac{1}{2}(h-2)+q] - (q+h-1)n^{h-1} = n^h - n^{h-1}h[\frac{1}{2}(h-1)+q]$ and induction is complete. Thus $0 \leqslant n^m - (n-q)_m \leqslant n^{m-1}m[\frac{1}{2}(m-1)+q]$, whence

$0 \leqslant e^{xn/M} - (1+x/M)^{n-q} \leqslant \sum_{m=1}^{\infty}\frac{1}{m!}\frac{x^m}{M^m}n^{m-1}m[\frac{1}{2}(m-1)+q]$

$$\leqslant \sum_{m=1}^{\infty}\frac{1}{(m-1)!}\frac{x^{m-1}n^{m-1}}{M^{m-1}}\frac{xq}{M}$$

$$+ \sum_{m=2}^{\infty}\frac{1}{(m-2)!}\frac{x^{m-2}n^{m-2}}{M^{m-2}}\frac{x^2 n}{2M^2}$$

$$= e^{xn/M}\left[\frac{xq}{M} + \frac{x^2 n}{2M^2}\right].$$

QED

Remark 1 (B)

If $x < 0$, $q \geqslant 0$ are fixed and $M \to \infty$ in such a way that $M/n \to 0$, then

$$\frac{x^2 n}{4M^2}(u-v) + \frac{xq}{2M}(u+v) \leqslant e^{xn/M} - (1+x/M)^{n-q} \leqslant \frac{x^2 n}{4M^2}(u+v) + \frac{xq}{2M}(u-v),$$

where $u = e^{xn/M}$, $v = 1/u = e^{-xn/M}$.

Proof: As in Remark 1(A), we see that

$$\Delta(x) = e^{xn/M} - (1+x/M)^{n-q} = \sum_{m=1}^{\infty} \frac{1}{m!} \frac{x^m}{M^m} [n^m - (n-q)_m],$$

and that

$$0 \leqslant n^m - (n-q)_m \leqslant n^{m-1} m [\tfrac{1}{2}(m-1)+q].$$

The series for $\Delta(x)$ now alternates. By collecting positive terms only, we obtain that

$$\Delta(x) \leqslant \sum_{\substack{m=2 \\ (m\text{ even})}}^{\infty} \frac{1}{(m-1)!} \left(\frac{xn}{M}\right)^m \frac{1}{n} \left[\frac{1}{2}(m-1)+q\right]$$

$$= \frac{x^2 n}{2M^2} \sum_{\substack{m=2 \\ (m\text{ even})}}^{\infty} \frac{1}{(m-2)!} \left(\frac{xn}{M}\right)^{m-2} + \frac{xq}{M} \sum_{\substack{m=2 \\ (m\text{ even})}}^{\infty} \frac{1}{(m-1)!} \left(\frac{xn}{M}\right)^{m-1}.$$

Now $\displaystyle\sum_{\substack{m=0 \\ (m\text{ even})}}^{\infty} \frac{1}{m!} \left(\frac{xn}{M}\right)^m = \frac{1}{2}(u+v)$ and $\displaystyle\sum_{\substack{m=1 \\ (m\text{ odd})}}^{\infty} \frac{1}{m!} \left(\frac{xn}{M}\right)^m = \frac{1}{2}(u-v)$.

Thus $\Delta(x) \leqslant \frac{x^2 n}{4M^2}(u+v) + \frac{xq}{2M}(u-v).$

Similarly, by collecting negative terms, we get

$$\Delta(x) \geqslant \sum_{\substack{m=1 \\ (m\ \text{odd})}}^{\infty} \frac{1}{(m-1)!}\left(\frac{xn}{M}\right)^m \frac{1}{n}\left[\tfrac{1}{2}(m-1)+q\right]$$

$$= \frac{x^2 n}{2M^2} \sum_{\substack{m=3 \\ (m\ \text{odd})}}^{\infty} \frac{1}{(m-2)!}\left(\frac{xn}{M}\right)^{m-2} + \frac{xq}{M} \sum_{\substack{m=1 \\ (m\ \text{odd})}}^{\infty} \frac{1}{(m-1)!}\left(\frac{xn}{M}\right)^{m-1}$$

$$= \frac{x^2 n}{4M^2}(u-v) + \frac{xq}{2M}(u+v).$$

QED

Lemma 3.1: Under Condition C, if Algorithm A is applied to a k-TSP instance $\mathcal{P}^n$ then there is a constant A, such that

$$\mathcal{E}\, S_n \sim An\delta(n)e^{\delta(n)}[1-1/\delta(n)], \quad \text{as} \quad n \to \infty .$$

Proof: Let t_s denote the time needed to compute a shortest tour through s points. From Bellman [1962] and Held and Karp [1962], we know there is a constant A (roughly, half the time needed for one addition), such that

$$\begin{aligned} t_s &= 2A(s-1)\ [2^{s-3}(s-2)+1] \\ &= A[2^{s-2}(s)_2-2^{s-1}s+2s+2^{s-1}-2] = t^*(s), \text{ for } s \geq 1, \text{ and } \quad t_0 = 0. \quad \text{(I.1)} \end{aligned}$$

If $k,p,$ and $q \geq 0$ are fixed, and $n \to \infty$, we see that

$$\begin{aligned} f(n;k,p,q) &= \sum_{s=0}^{n} \binom{n}{s} (1/M)^s(1-k/M)^{n-s}p^{s-q}(s)_q \\ &= (n)_q(1/M)^q \sum_{s=q}^{n} \binom{n-q}{s-q} (p/M)^{s-q}(1-k/M)^{n-s} \\ &= (n)_q(1/M)^q\ [1+(p-k)/M]^{n-q} . \qquad \text{(I.2)} \end{aligned}$$

By Remark 1 (A) we have that, if $p \geq k$, $q \geq 0$, then

$$\begin{aligned} \delta(n)^q e^{(p-k)\delta(n)}\Big(1- \frac{(p-k)q}{M} - \frac{(p-k)^2 n}{2M^2}\Big)(1+\underline{O}(1/n)) & \\ \leq f(n;k,p,q) & \\ \leq \delta(n)^q\, e^{(p-k)\delta(n)} . & \qquad \text{(I.3)} \end{aligned}$$

So $f(n;k,p,q) \sim \delta(n)^q e^{(p-k)\delta(n)}[1+\underline{O}(\delta(n)^2/n)]$. (I.4)

Now, if n_j denotes the number of points in cell $\underline{\underline{C}}_j$, we have

$$\mathcal{E}\, S_n = \mathcal{E} \sum_{j=1}^{M} t_{n_j} = M\, \mathcal{E}\, t_{n_j} \;; \tag{I.5}$$

and, since n_j has (binomial) probability $\binom{n}{s}$ $(1/M)^s(1-1/M)^{n-s}$ of taking the value s,

$$\begin{aligned}\mathcal{E}\, S_n &= M \sum_{s=0}^{n} \binom{n}{s} (1/M)^s(1-1/M)^{n-s}\, A[2^{s-2}(s)_2 \\ &\qquad -2^{s-1}s + 2s + 2^{s-1}-2] + M(1-1/M)^n(3/2)A \\ &= AM[f(n;1,2,2) - f(n;1,2,1) + 2f(n;1,1,1) \\ &\qquad + (1/2)\, f(n;1,2,0) - 2f(n;1,1,0) + (3/2)(1-1/M)^n]\end{aligned} \tag{I.6}$$

[Note that we use the general formula $t^*(s)$ in (I.1) for t_s even when $s = 0$. This incorrectly yields $t^*(0) = -(3/2)A$; forcing us to make the corresponding correction, $+M(1-1/M)^n(3/2)A$, above.] Thus

$$\begin{aligned}\mathcal{E}\, S_n &\sim A\,\frac{n}{\delta(n)}\,[\delta(n)^2e^{\delta(n)} - \delta(n)e^{\delta(n)} + 2\delta(n) \\ &\qquad + (1/2)e^{\delta(n)}-2+(3/2)e^{-\delta(n)}][1+\underline{O}(\delta(n)^2/n)] \quad \text{(by Remark 1 (A) \& (B))} \\ &\sim An\delta(n)e^{\delta(n)}[1-1/\delta(n) + 1/2\delta(n)^2 + \underline{O}(\delta(n)^2/n)],\end{aligned}$$

as $n \to \infty$, and the lemma follows from Remark E (ii), since

$[\delta(n)^2/n]/[1/\delta(n)] \to 0$, as $n \to \infty$.

QED

Now we want to prove a second remark which will be used in the proof of Lemma 3.2.

Remark 2

$$0 \leqslant (1+1/M)^{2n-k} - (1+2/M)^{n-k} \leqslant e^{2n/M} \frac{n}{M^2}\left(1+\frac{kM}{n}\right) \quad \text{if} \quad k \geqslant 0 \quad \text{is fixed and}$$

$M \to \infty$ in such a way that $M/n \to 0$.

Proof: The first inequality above is true since clearly

$$
\begin{aligned}
&(1+1/M)^{2n-k} - (1+2/M)^{n-k} \\
&= 1 + \frac{2n-k}{M} + \frac{(2n-k)(2n-k-1)}{2M^2} + \dots + \frac{(2n-k)\dots(2n-k-j+1)}{j!M^j} + \dots \\
&- 1 - \frac{2n-2k}{M} - \frac{(2n-2k)(2n-2k-2)}{2M^2} - \dots - \frac{(2n-2k)\dots(2n-2k-2j+2)}{j!M^j} - \dots \\
&\geqslant 0 .
\end{aligned}
$$

Now, the j-th term in the difference above is:

$$
\begin{aligned}
T_j &= \frac{(2n-k)\dots(2n-k-j+1) - (2n-2k)\dots(2n-2k-2j+2)}{j!M^j} \\
&= \frac{1}{j!M^j}\left\{\sum_{i=k}^{k+j-1}(2n)^{j-1}i - 3\sum_{\text{pairs}}(2n)^{j-2}ii' + 7\sum_{\text{triplets}}(2n)^{j-3}ii'i'' - \dots\right\}.
\end{aligned}
$$

By induction on j, we want to show that

$$T_j \leqslant \frac{1}{j!M^j}(2n)^{j-1}\sum_{i=k}^{k+j-1} i . \tag{I.7}$$

For $j=1$, $T_1 = \frac{k}{M} = \frac{1}{M}(2n)^0 \sum_{i=k}^{k} i = \frac{k}{M}$. Assume the inequality above is true for $j = h-1$.

Then $T_h h! M^h = [(2n-k)\dots(2n-k-h+2)](2n-k-h+1)$

$$- [(2n-2k)\dots(2n-2k-2h+4)]\,(2n-2k-2h+2)$$

$$= T_{h-1}(h-1)!M^{h-1}(2n-k-h+1) + (2n-2k)\dots(2n-2k-2h+4)(k+h-1)\,.$$

$$\leqslant (2n)^{h-2}\left(\sum_{i=k}^{k+h-2} i\right)\ (2n-k-h+1) + (2n-2k)\dots(2n-2k-2h+4)(k+h-1)$$

$$\leqslant (2n)^{h-1} \sum_{i=k}^{k+h-2} i + (2n)^{h-1}(k+h-1)$$

$$= (2n)^{h-1} \sum_{i=k}^{k+h-1} i \quad ,$$

and induction is complete.

From (I.7) we have that

$$T_j \leqslant \frac{1}{j!M^j} (2n)^{j-1}[\tfrac{1}{2}\,(k+j-1)(k+j) - \tfrac{1}{2}\,(k-1)k]$$

$$= \frac{1}{j!M^j} (2n)^{j-1}\, \frac{1}{2}\,[2jk+j(j-1)]\ ;$$

so that

$$\sum_{j=1}^{\infty} T_j \leqslant \frac{k}{M} \sum_{j=1}^{\infty} \frac{M^{-(j-1)}}{(j-1)!} (2n)^{j-1} + \frac{2n}{2M^2} \sum_{j=2}^{\infty} \frac{M^{-(j-2)}}{(j-2)!} (2n)^{j-2}$$

$$= \frac{k}{M}\, e^{2n/M} + \frac{n}{M^2}\, e^{2n/M}\,.$$

QED

Lemma 3.2: Under the same conditions as in Lemma 3.1, we have that

$$\operatorname{var} S_n \leqslant A^2 e^{2\delta(n)} \left\{ 16n\delta(n)^3 e^{\delta(n)} + \frac{n^2}{4\delta(n)^2} \right\} [1+\underline{O}(1/\delta(n))]$$

as $n \to \infty$, where A is the constant in Lemma 3.1.

Proof: $\mathcal{E}\, S_n^{\ 2} = \mathcal{E} \sum_{i=1}^{M} \sum_{j=1}^{M} t_{n_i} t_{n_j} = \mathcal{E} \left\{ \sum_{i=1}^{M} t_{n_i}^{\ 2} + 2 \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} t_{n_i} t_{n_j} \right\}$

$$= M\mathcal{E}t_{n_i}^{\ 2} + M(M-1)\,\mathcal{E}\,(t_{n_i} t_{n_j} | i \neq j) \qquad \text{(I.8)}$$

Since $[(s)_2]^2 = (s)_4 + 4(s)_3 + 2(s)_2$, $s(s)_2 = (s)_3 + 2(s)_2$, and $s^2 = (s)_2 + s$, using (I.1) and (I.3), we see that

$$\begin{aligned} \mathcal{E}\, t_{n_i}^{\ 2} = \sum_{s=0}^{n} \binom{n}{s} (1/M)^s (1-1/M)^{n-s} A^2 \{&[4^{s-2}(s)_4 + 4^{s-1}(s)_3 \\ &+ 2.4^{s-2}(s)_2] - [4^{s-1}(s)_3 + 2.4^{s-1}(s)_2] + [2^s(s)_3 \\ &+ 2^{s+1}(s)_2] + 4^{s-1}(s)_2 - 2^s(s)_2 + [4^{s-1}(s)_2 + 4^{s-1}s] \\ &- [2^{s+1}(s)_2 + 2^{s+1}s] - 2.4^{s-1}s + 2^{s+1}s + [4(s)_2 + 4s] \\ &+ 2^{s+1}s - 8s + 4^{s-1} - 2^{s+1} + 4\} - (9/4)A^2(1-1/M)^n . \end{aligned}$$

[The last terms above is a correction similar to that in (I.6)]. So

$$\begin{aligned} \mathcal{E}\, t_{n_i}^{\ 2} = A^2[&16f(n;1,4,4) + 8f(n;1,2,3) + 2f(n;1,4,2) \\ &-4f(n;1,2,2) + 4f(n;1,1,2) - f(n;1,4,1) \\ &+4f(n;1,2,1) - 4f(n;1,1,1) + (1/4)f(n;1,4,0) \\ &-2f(n;1,2,0) + 4f(n;1,1,0) - (9/4)(1-1/M)^n] \end{aligned}$$

$$\leqslant A^2\{16\delta(n)^4 e^{3\delta(n)} + 8\delta(n)^3 e^{\delta(n)}$$

$$+ 2\delta(n)^2 e^{3\delta(n)} + 4\delta(n)^2 + 4\delta(n) e^{\delta(n)}$$

$$+ (1/4) e^{3\delta(n)+4}\} . \tag{I.9}$$

Just as in (I.2), we note that

$$\sum_{s=0}^{v} \binom{v}{s} p^{s-q}(s)_q(v-s)_r = (v)_{q+r} \sum_{s=q}^{v-r} \binom{v-q-r}{s-q} p^{s-q}$$

$$= (v)_{q+r}(p+1)^{v-q-r} = \sum_{s=0}^{v} \binom{v}{s} p^{v-s-r} (s)_q(v-s)_r ; \tag{I.10}$$

whence, by putting $v = s+u$, we get that

$$\mathcal{E}(t_{n_i}t_{n_j}|i\neq j) = \sum_{s=0}^{n} \sum_{u=0}^{n-s} \binom{n}{s+u}\binom{s+u}{s} (1/M)^{s+u}(1-2/M)^{n-s-u} t_s t_u$$

$$= A^2 \sum_{v=0}^{n} \binom{n}{v} (1/M)^v(1-2/M)^{n-v} \Bigg\{\sum_{s=0}^{v} \binom{v}{s} [2^{v-4}(s)_2(v-s)_2$$

$$-2^{v-3}(s)_2(v-s) + 2^{s-1}(s)_2(v-s) + 2^{v-3}(s)_2 - 2^{s-1}(s)_2$$

$$-2^{v-3}s(v-s)_2 + 2^{v-2}s(v-s) - 2^s s(v-s) - 2^{v-2}s$$

$$+2^s s + 2^{v-s-1} s(v-s)_2 - 2^{v-s}s(v-s) + 4s(v-s)$$

$$+2^{v-s}s - 4s + 2^{v-3}(v-s)_2 -2^{v-2}(v-s) + 2^s(v-s)$$

$$+2^{v-2} - 2^s - 2^{v-s-1}(v-s)_2 + 2^{v-s}(v-s) -4(v-s)$$

$$-2^{v-s}+4] + 3(2^{v-2}(v)_2 -2^{v-1}v + 2v + 2^{v-1}-2)\Bigg\}$$

$$+(9/4)A^2(1-2/M)^n$$

[The terms $3(2^{v-2}(v)_2-2^{v-1}v+2v+2^{v-1}+2)$ and $(9/4)A^2(1-2/M)^n$ at the end, above, are corrections for the use of $t^*(0)$ instead of t_0, similar to those in (I.6) and (I.9) above.]

$$
\begin{aligned}
= A^2 \sum_{v=0}^{n} \binom{n}{v} (1/M)^v(1-2/M)^{n-v}[&2^{v-4}(v)_4\, 2^{v-4} \\
&-2^{v-3}(v)_3 2^{v-3} + 2(v)_3 3^{v-3} + 2^{v-3}(v)_2 2^{v-2} \\
&-2(v)_2 3^{v-2} - 2^{v-3}(v)_3 2^{v-3} + 2^{v-2}(v)_2 2^{v-2} \\
&-2(v)_2 3^{v-2} - 2^{v-2}v2^{v-1} + 2v3^{v-1} + 2(v)_3 3^{v-3} \\
&-2(v)_2 3^{v-2} + 4(v)_2 2^{v-2} + v3^{v-1} - 4v2^{v-1} \\
&+2^{v-3}(v)_2 2^{v-2} - 2^{v-2}v2^{v-1} + v3^{v-1} + 2^{v-2}2^v \\
&-3^v - 2(v)_2 3^{v-2} + 2v3^{v-1} - 4v2^{v-1} - 3^v + 4.2^v \\
&+3(2^{v-2}(v)_2-2^{v-1}v+2v+2^{v-1}-2)] + (9/4)A^2(1-2/M)^n
\end{aligned}
$$

$$
\begin{aligned}
= A^2[&f(n;2,4,4) - 2f(n;2,4,3) + 2f(n;2,4,2) \\
&-f(n;2,4,1) + (1/4)\, f(n;2,4,0) + 4f(n;2,3,3) \\
&-8f(n;2,3,2) + 6f(n;2,3,1) - 2f(n;2,3,0) \\
&+7f(n;2,2,2) - 11f(n;2,2,1) + (11/2)f(n;2,2,0) \\
&+6f(n;2,1,1) - 6f(n;2,1,0) + (9/4)(1-2/M)^n]
\end{aligned}
$$

Then,

$$M^2\mathcal{E}(t_{n_i}t_{n_j}|i\neq j) = A^2M^2\{((n)_4/M^4)(1+2/M)^{n-4} - 2((n)_3/M^3)(1+2/M)^{n-3}$$

$$+ 2((n)_2/M^2)(1+2/M)^{n-2} - (n/M)(1+2/M)^{n-1} + (1/4)(1+2/M)^n$$

$$+ 4((n)_3/M^3)(1+1/M)^{n-3} - 8((n)_2/M^2)(1+1/M)^{n-2}$$

$$+ 6(n/M)(1+1/M)^{n-1} - 2(1+1/M)^n + 7(n)_2/M^2 - 11(n/M) + 11/2$$

$$+ 6(n/M)(1-1/M)^{n-1} - 6(1-1/M)^n + (9/4)(1-2/M)^n\}\ , \qquad \text{(I.11)}$$

On the other hand, from (I.6) we have that

$$\mathcal{E}\ S_n = AM[((n)_2/M^2)(1+1/M)^{n-2} - (n/M)(1+1/M)^{n-1}$$

$$+ 2n/M + (1/2)(1+1/M)^n - 2 + (3/2)(1-1/M)^n].$$

So

$$(\mathcal{E}\ S_n)^2 = A^2M^2\{[(n)_4+4(n)_3+2(n)_2](1/M)^4(1+1/M)^{2n-4}$$

$$-2(n)_2\ n(1/M)^3(1+1/M)^{2n-3}$$

$$+ [(n)_2+n^2](1/M)^2(1+1/M)^{2n-2}$$

$$- (n/M)(1+1/M)^{2n-1} + (1/4)(1+1/M)^{2n}$$

$$+ 4(\tfrac{n}{M} - 1 + (3/4)(1-1/M)^n)[((n)_2/M^2)(1+1/M)^{n-2} - (n/M)(1+1/M)^{n-1}$$

$$+ (1/2)(1+1/M)^n] + 4(\tfrac{n}{M} - 1)^2$$

$$+ 6(\tfrac{n}{M} - 1)(1-1/M)^n + (9/4)(1-1/M)^{2n}\}\ . \qquad \text{(I.12)}$$

Observing that, in (I.12), $(n)_2 n = (n)_3 + 2(n)_2$ and $n^2 = (n)_2 + n$, from (I.11) and (I.12), using Remarks 1 (A) and 2 we have that

$$M^2 \mathcal{E}(t_{n_i} t_{n_j} | i \neq j) - (\mathcal{E}\, S_n)^2$$

$$= A^2M^2\Big\{((n)_4/M^4)[(1+2/M)^{n-4} - (1+1/M)^{2n-4}]$$

$$+ ((n)_3/M^3)[-2(1+2/M)^{n-3} - (4/M)(1+1/M)^{2n-4} + 2(1+1/M)^{2n-3}]$$

$$+ ((n)_2/M^2)[2(1+2/M)^{n-2} - (2/M^2)(1+1/M)^{2n-4}$$

$$+ (4/M)(1+1/M)^{2n-3} - 2(1+1/M)^{2n-2}]$$

$$+ (n/M)[-(1+2/M)^{n-1} - (1/M)(1+1/M)^{2n-2} + (1+1/M)^{2n-1}]$$

$$+ (1/4)(1+2/M)^n - (1/4)(1+1/M)^{2n}$$

$$- 4((n)_3/M^4)(1+1/M)^{n-3} - 4((n)_2/M^3)(1+1/M)^{n-2}$$

$$+ 2(n/M^2)(1+1/M)^{n-1} - 4n/M^2 + 3(n)_2/M^2 - 3n/M + 3/2 + 6(n/M^2)(1-1/M)^n$$

$$+ (9/4)(1-2/M)^n - 3((n)_2/M^2)(1-1/M)^n(1+1/M)^{n-2}$$

$$+ 3(n/M)(1-1/M)^n(1+1/M)^{n-1} - (3/2)(1-1/M)^n(1+1/M)^n$$

$$- (9/4)(1-1/M)^{2n}\Big\}$$

$$\leq A^2M^2 \Big\{\delta(n)^3\, 2\, \frac{\delta(n)}{M}\, e^{2\delta(n)}[1 + \frac{3}{\delta(n)}] + \delta(n)^2 \frac{4}{M} e^{2\delta(n)}$$

$$+ \delta(n) \frac{\delta(n)}{M}\, e^{2\delta(n)}\, [1 + \frac{1}{\delta(n)}] + \frac{1}{4} e^{2\delta(n)}$$ [continued....]

$$+ 2\,\frac{\delta(n)}{M}\,e^{\delta(n)} + 3\delta(n)^2 + 3/2 + (6\delta(n)/M)\ \ [e^{-\delta(n)}(1+1/2M)$$

$$+ e^{\delta(n)}(\delta(n)+2)/4M] + (9/4)\ [e^{-2\delta(n)} + \delta(n)e^{2\delta(n)}/M]$$

$$+ 3\delta(n)[1+\delta(n)e^{2\delta(n)}/4M] \Big\}$$

$$= 2A^2 n\delta(n)^3\, e^{2\delta(n)} \Bigg\{ 1 + \frac{3}{\delta(n)} + \frac{5}{2\delta(n)^2} + \frac{1}{2\delta(n)^3} + \frac{n}{8\delta(n)^5}$$

$$+ \frac{1}{\delta(n)^3}\, e^{-\delta(n)} + \frac{ne^{-2\delta(n)}}{2} \Bigg[\frac{3}{\delta(n)^3} + \frac{3}{2\delta(n)^5}$$

$$+ \frac{6}{n\delta(n)^3}\ (e^{-\delta(n)}(1+1/2M) + e^{\delta(n)}(\delta(n)+2)/4M)$$

$$+ \frac{9}{4\delta(n)^5}\ (e^{-2\delta(n)} + \delta(n)e^{2\delta(n)}/M)$$

$$+ \frac{3}{\delta(n)^4}\ (1+\delta(n)e^{2\delta(n)}/4M) \Bigg] \Bigg\} \qquad \text{(I.13)}$$

From (I.8), (I.9) and (I.13) we have that

$$\text{var } S_n = \mathcal{E}\, S_n^{\ 2} - (\mathcal{E}\, S_n)^2$$

$$= M\mathcal{E}t_{n_i}^{\ 2} + M(M-1)\ \mathcal{E}\ (t_{n_i} t_{n_j} | i \neq j) - (\mathcal{E}\ S_n)^2$$

$$\leqslant A^2\ \Big\{ 16n\delta(n)^3\, e^{3\delta(n)} + 2n\delta(n)\, e^{3\delta(n)} + (n/4\delta(n))\, e^{3\delta(n)}$$

$$+ 8n\delta(n)^2\, e^{\delta(n)} + 4n\, e^{\delta(n)} + 4n\delta(n) + 4n/\delta(n)$$

$$+ 2n\delta(n)^3\, e^{2\delta(n)} + 6n\delta(n)^2\, e^{2\delta(n)} + 5n\delta(n)\, e^{2\delta(n)}$$

$$+ n\, e^{2\delta(n)} + \frac{n^2}{4\delta(n)^2}\, e^{2\delta(n)} + 2\,n\, e^{\delta(n)} + n^2 \Big[3 + \frac{3}{2\delta(n)^2}$$

[continued...]

$$+ \frac{6}{n} (e^{-\delta(n)}(1+1/2M) + e^{\delta(n)}(\delta(n)+2)/4M)$$

$$+ \frac{9}{4\delta(n)^2} (e^{-2\delta(n)} + \delta(n)e^{2\delta(n)}/M) + \frac{3}{\delta(n)}(1+\delta(n)e^{2\delta(n)}/4M)\Big]\Big\}$$

$$= A^2 \Big\{16n\delta(n)^3\, e^{3\delta(n)} + \frac{n^2}{4\delta(n)^2}\, e^{2\delta(n)} + 3n^2\Big[1+\delta(n)^2 e^{\delta(n)}/12n + 3e^{2\delta(n)}/4n$$

$$+\delta(n)e^{2\delta(n)}/4n\Big]\Big\}\,(1+\underline{O}(1/\delta(n))); \qquad (I.14)$$

but each of the terms in $3n^2[1+\delta(n)^2e^{\delta(n)}/12n + 3e^{2\delta(n)}/4n + \delta(n)e^{2\delta(n)}/4n]$ is $\underline{o}(\delta(n)^{-3}e^{2\delta(n)})$; since $\delta(n)^3\, e^{-2\delta(n)} \to 0$ ($e^{\delta(n)}$ increases faster than any power of $\delta(n)$), $\delta(n)^5\, e^{-\delta(n)}/n \to 0$ (similarly, $\delta(n)^3/n \to 0$ (by Remark E (ii)), and $\delta(n)^4/n \to 0$ (similarly). The lemma follows.

QED

Appendix II

In the proof of Lemma 7 of Beardwood, Halton, and Hammersley [1959], equation (7.15) is not valid; because the n_m depend on the corresponding intervals J_m, and these depend on the value of ε.

However, the argument via (7.16)-(7.19) is valid, except that, in each of (7.18) and (7.19), a factor of α should be inserted before $(5\varepsilon n)^q$, coming from Lemma 4 of that paper.

(7.20) now follows from (7.14) in the modified form

$$\frac{\beta-\varepsilon}{(1+\varepsilon)^q} v(\underline{\underline{E}})^{1/k} \leqslant \liminf_{n\to\infty} n^{-q} \ell(\underset{\sim}{P}^n) + \alpha(5\varepsilon)^q[v(\underline{\underline{\bar{E}}})+\varepsilon]^{1-q}, \tag{7.20*}$$

holding with probability one.

Similarly (7.21) and the next-following inequality (unnumbered) should have a factor of α inserted before $(5\varepsilon n)^q$; and, again directly from (7.14), we get a modified form of (7.22), holding with probability one:

$$\frac{\beta+\varepsilon}{(1-\varepsilon)^q} v(\underline{\underline{E}})^{1/k} \geqslant \limsup_{n\to\infty} n^{-q} \ell(\underset{\sim}{P}^n) - \alpha(5\varepsilon)^q[v(\underline{\underline{\bar{E}}})+\varepsilon]^{1-q}. \tag{7.22*}$$

Now we observe that ε is arbitrary and conclude that

$$\lim_{n\to\infty} n^{-q} \ell(\underset{\sim}{P}^n) = \beta\, v(\underline{\underline{E}})^{1/k},$$

establishing Lemma 7 of Beardwood, Halton, and Hammersley [1959].

Incidentally, equation (5.1) of the same paper should read

$$\mathcal{E}\ \ell(P\xi\underline{\underline{E}}) \sim \beta_k\ k^{1/2}\xi^k v(\underline{\underline{E}}) = \beta\xi^k v(\underline{\underline{E}}) \quad \text{as} \quad \xi \to \infty . \qquad (5.1)^*$$

Appendix III

In Karp [1977] the points of a 2-TSP instance are assumed to be distributed in a region A according to a two dimensional Poisson distribution with density n. As is noted in Karp [1977], it is then known that the expected number of points in A is $n\,v(A)$, where $v(A)$ denotes the area of A.

But the algorithms in Karp [1977], when applied to a region A with $v(A) = 1$, are analyzed as if the observed number of points in a 2-TSP instance were n, rather than considering n as the expected number of points. We conjecture that one possible way to rescue this part of the analysis in Karp [1977] is to prove that the observed number of points in A is asymptotic to the expected number of points in A, with probability one.

Furthermore, we note that Karp [1977] quotes a result (as Theorem 5 in Section 4 of his paper) from Beardwood, Halton, and Hammersley [1959] as if it held under the assumption of the Poisson distribution of the points with density n. But, actually that theorem is proved in Beardwood, Halton, and Hammersley [1959] (as Lemma 7) only for the uniform distribution of n points. The length of the proof of Theorem 2 in our paper indicates that the connection between the two is far from trivial; and, in fact, we do not believe that the results hold for the Poisson distribution more strongly than in probability.

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