SIMPLIFIED CHARACTERIZATIONS OF LINEAR COMPLEMENTARITY PROBLEMS SOLVABLE AS LINEAR PROGRAMS

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SIMPLIFIED CHARACTERIZATIONS OF LINEAR COMPLEMENTARITY PROBLEMS SOLVABLE AS LINEAR PROGRAMS [1)]

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New and simplified characterizations are given for solving, as a linear program, the linear complementarity problem of finding an x in R^n such that $Mx + q \geqq 0$, $x \geqq 0$ and $x^T(Mx+q) = 0$. The simplest such characterization given here is that if there exist n-dimensional vectors c,r,s which are nonnegative, and n-by-n matrices Z_1,Z_2, with nonpositive off-diagonal elements such that (i) $MZ_1 = Z_2 + qc^T$ and (ii) $r^TZ_1 + s^TZ_2 > 0$, then each solution of the linear program: $\underset{x}{\text{Minimize}}\ (r^T+s^TM)x$ subject to $Mx + q \geqq 0$, $x \geqq 0$, solves the linear complementarity problem. Conversely if the linear complementarity problem has at least one vertex solution x which is nondegenerate, that is, $x + Mx + q > 0$, then there exist nonnegative vectors c,r,s and matrices Z_1,Z_2, with nonpositive off-diagonal elements, such that (i)-(ii) are satisfied.

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In [5, Theorem 1] it was shown that the linear complementarity problem of finding an x in R^n such that

$$\text{LCP} \qquad Mx + q \geqq 0, \quad x \geqq 0, \quad x^T(Mx+q) = 0 \qquad (1)$$

is solvable, if and only, certain conditions were satisfied. When these conditions are satisfied the linear program

$$\text{LP} \qquad \underset{x}{\text{Minimize}}\ (r^T+s^TM)x \quad \text{subject to} \quad Mx + q \geqq 0, \quad x \geqq 0 \qquad (2)$$

is solvable and each of its solutions solves the LCP. Here, r and s are nonnegative vectors which appear in the characterization Theorem 1 of [5]. In this paper we give a number of considerably simpler characterizations for the solvability of the linear complementarity problem as a linear program. We begin by showing in Theorem 1 below that the simple conditions of Lemma 1 of [3] can be made necessary as well as sufficient for the solvability of the LCP as an LP. By using these conditions we establish in Theorem 2 what is probably the simplest characterization so far of the LCP solvability as an LP. In fact the conditions of this theorem, conditions (7), are a simplification of the sufficient conditions for the LCP solvability as an LP of [4, Theorem 1]. In order to make these simplified sufficient conditions necessary as well, we have required here that the LCP have at least one nondegenerate vertex solution. Examples show for this particular characterization that the nondegenerate vertex assumption is a sharp assumption. Finally, we show in Theorem 3 that another set of conditions, equivalent to an earlier set of sufficient conditions for the LCP solvability as a linear program [5, Theorem 2] also characterize an LCP that is solvable as a linear program. Recently Pang [6] has shown that the conditions of [5, Theorem 2] are satisfied by any solvable LCP.

We shall need the following dual problem associated with the LP

$$\text{DP} \qquad \underset{y}{\text{Maximize}}\ -q^Ty \quad \text{subject to} \quad -M^Ty + r + M^Ts \geqq 0, \quad y \geqq 0 \qquad (3)$$

We shall use the following notation. All matrices and vectors considered here are real. If A is an n-by-m matrix then this is denoted as $A \in R^{n \times m}$, and A_i denotes its i-th row and A_{ij} denotes its ij-th element. We shall also use the notation Z_1, Z_2 to denote specific matrices. This will be made clear from the context. A matrix $D \in R^{n \times n}$ is a diagonal matrix if $D_{ij} = 0$ for $i \neq j$. All vectors are column vectors unless transposed by a superscript T to a row vector. If x is an n-dimensional vector then this is denoted as $x \in R^n$, and x_i denotes its i-th element. The vector e will denote the vector of ones in R^n. The matrix I will denote the diagonal matrix of ones in $R^{n \times n}$. A square matrix with nonpositive off-diagonal elements is said to be a Z-matrix or it is said to belong to Z. The LCP is said to be feasible if the set $\{x \mid Mx+q \geqq 0,\ x \geqq 0\}$ is nonempty.

We begin by establishing a simple theorem which characterizes the solvability of the LCP as an LP. This theorem is an improvement of Lemma 1 of [3] and shows that the conditions of that lemma can be made necessary as well as sufficient for the solvability of the LCP as an LP.

Theorem 1 The LCP has a solution if and only if the LP is solvable for some $r,s \in R^n$, $r,s \geqq 0$, such that an associated dual optimal variable y satisfies

$$(I-M)^T y + r + M^T s > 0 \tag{4}$$

Furthermore, each solution of the LP solves the LCP.

Proof (Sufficiency) If x solves the LP and y is a corresponding optimal dual variable satisfying (4) then by the equality of the optimal primal and dual objective functions it follows that

$$0 = (r^T+s^TM)x + q^Ty = x^T(-M^Ty+r+M^Ts) + y^T(Mx+q)$$

From the nonnegativity of x,y and of the last two terms in parentheses and from (4) it follows that $x^T(Mx+q) = 0$. Hence x solves the LCP. If $\hat{x}$ is another optimal solution of the LP with an associated dual optimal $\hat{y}$ which does not satisfy (4) then because $(r^T+s^TM)x = (r^T+s^TM)\hat{x} = -q^T\hat{y} = -q^Ty$ it follows that

$$0 = (r^T+s^TM)\hat{x} + q^Ty = \hat{x}^T(-M^Ty+r+M^Ts) + y^T(M\hat{x}+q)$$

and hence as before $\hat{x}^T(M\hat{x}+q) = 0$ and $\hat{x}$ solves the LCP.

(Necessity) Let x solve the LCP. Define

$$r_i = \begin{cases} 1 & \text{for } x_i = 0 \\ 0 & \text{for } x_i > 0 \end{cases}, \quad s_i = \begin{cases} 1 & \text{for } M_i x + q_i = 0 \\ 0 & \text{for } M_i x + q_i > 0 \end{cases} \tag{5}$$

Hence $r^T x + s^T(Mx+q) = 0$ and x is a solution to the LP with the above r and s. In addition s is dual feasible because $-M^T s + r + M^T s = r \geqq 0$, and in fact s is also dual optimal because

$$(r^T + s^T M)x + q^T s = r^T x + s^T(Mx+q) = 0 \tag{6}$$

Finally condition (4) is satisfied by $y = s$ because

$$(I - M^T)s + r + M^T s = r + s > 0$$

where the inequality follows from the definition (5) of r and s. □

We obtain now a characterization of the LCP solvability as an LP by simplifying the sufficient conditions of [4, Theorem 1] and by showing that under appropriate assumptions these simplified conditions are also necessary.

<u>Theorem 2</u> If there exist $c, r, s \in R^n$ and $Z_1, Z_2\ R^{n \times n}$ such that

$$\begin{aligned} &MZ_1 = Z_2 + qc^T \\ &r^T Z_1 + s^T Z_2 > 0 \\ &c, r, s \geqq 0, \quad Z_1, Z_2 \in Z \end{aligned} \tag{7}$$

then the feasible LCP has a solution which can be obtained by solving the LP. Conversely if the LCP has at least one solution x which is a vertex of $\{x|Mx+q\geqq 0,\ x\geqq 0\}$ which is also nondegenerate, that is $x + Mx + q > 0$, then conditions (7) are satisfied and each solution of the LP solves the LCP.

<u>Proof</u> We begin by showing the equivalence of conditions (7) and (7') below. Observe first that conditions (7) are equivalent to

$$-MR + U + (-I+M)D - qc^T = 0$$

$$(r^T+s^TM)R-(r^T+s^TM)D + d + q^Tsc^T = 0$$

$$d > 0,\quad c,r,s \geqq 0,\quad R,\ U,\ D \geqq 0,\quad D = \text{Diagonal}$$

where the substitutions $Z_1 = D-R$ and $Z_2 = D-U$ have been made. By Motzkin's theorem [2] this is equivalent to the system

$$\begin{array}{rcl} -M^Ty & +(r+M^Ts)\zeta & \geqq 0 \\ y & & \geqq \\ (-I+M^T)_iy & -(r+M^Ts)_i\zeta & \geqq 0 \\ & \zeta & > 0 \\ -q^Ty & + \quad q^Ts\zeta & \geqq 0 \end{array}$$

having no solution $y\epsilon R^n$, $\zeta\epsilon R$, for each $i=1,\text{---},n$. This in turn is equivalent to

$$\left.\begin{array}{rl} -M^Ty +(r+M^Ts) & \geqq 0 \\ y & \geqq 0 \\ -q^Ty & \geqq -q^Ts \end{array}\right\rangle \Rightarrow \quad (I-M^T)y +(r+M^Ts) > 0 \qquad (7')$$

Thus conditions (7) and (7') are equivalent.

To prove the first part of the theorem we observe that the LP is solvable because it is feasible and s is dual feasible. Let x be any optimal solution of the LP and let y be a corresponding dual optimal solution. Hence $-q^Ty \geqq -q^Ts$ and by (7) or its equivalent (7') it follows that (4) holds and hence by Theorem 1, x solves the LCP.

To prove the second part of the theorem, suppose now that the vertex x solves the LCP and that x is nondegenerate. Define r and s as in (5). Then as before, x solves the LP and s is dual feasible and optimal. Because x is a nondegenerate vertex and it is a solution of the LP, it follows that its associated optimal dual variable is unique and must be equal to s. Hence

$$\left.\begin{array}{ll} -M^Ty+(r+M^Ts) & \geqq 0 \\ y & \geqq 0 \\ -q^Ty & \geqq -q^Ts \end{array}\right\rangle \Rightarrow \left\langle\, y = s \,\right\rangle \Rightarrow \left\langle \begin{array}{l} (I-M^T)y+(r+M^Ts) \\ = r+s>0 \end{array}\right.$$

Hence (7') holds. □

<u>Remark 1</u> The nondegenerate vertex assumption in Theorem 2 is a sharp assumption in the sense that relaxing it could invalidate the converse part of the theorem. Thus, Example 1 below shows that conditions (7) may not hold when the nondegeneracy assumption is dropped, while Example 3 shows that conditions (7) may not hold if the vertex assumption is dropped.

<u>Example 1</u> $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $q = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

This is a slight variation of Example 2.2 of [1]. It has a unique degenerate vertex solution $x = (2 \;\; 0)$. If we let

$$Z_1 = \begin{pmatrix} a_1 & -u_1 \\ -u_2 & a_2 \end{pmatrix} , \qquad Z_2 = \begin{pmatrix} a_3 & -u_3 \\ -u_4 & a_4 \end{pmatrix}$$

with $u_i \geqq 0$, $i=1,\dots,4$, then in order to satisfy the first condition of (7) we get that

$$Z_1 = \begin{pmatrix} a_1 & -(a_2+u_3+2c_2) \\ a_1+u_4+2c_1 & a_2 \end{pmatrix} , \quad Z_2 = \begin{pmatrix} 2a_1+u_4+4c_1 & -u_3 \\ -u_4 & -2a_2-u_3 \end{pmatrix}$$

$$u_i \geqq 0 , \quad i=1,\dots,4, \quad a_2 + u_3 + 2c_2 \geqq 0 , \quad a_1 + u_4 + 2c_1 \leqq 0$$

and hence

$$\begin{aligned}(r^T Z_1 + s^T Z_2)_1 &= r_1 a_1 + r_2(a_1+u_4+2c_1) + s_1(2a_1+u_4+4c_1) + s_2(-u_4) \\ &\leqq r_1 a_1 + r_2(a_1+u_4+2c_1) + s_1(a_1+u_4+2c_1) + s_1(-u_4) + s_2(-u_4) \\ &\leqq 0\end{aligned}$$

Thus the second condition of (7) cannot be satisfied.

Example 2 If we let q of Example 1 revert to $q = \begin{pmatrix}-2\\-1\end{pmatrix}$ as originally given in Example 2.2 of [1] then the resulting LCP has two nondegenerate vertex solutions $x = \begin{pmatrix}2\\0\end{pmatrix}$ and $x = \begin{pmatrix}3/2\\1/2\end{pmatrix}$. Conditions (7) can then be satisfied by

$$Z_1 = \begin{pmatrix}-1 & -1\\0 & 1\end{pmatrix} \quad Z_2 = \begin{pmatrix}1 & 0\\0 & -2\end{pmatrix} \quad c = \begin{pmatrix}1\\0\end{pmatrix} \quad r = \begin{pmatrix}0\\1\end{pmatrix} \quad s = \begin{pmatrix}1\\0\end{pmatrix}$$

or by

$$Z_1 = \begin{pmatrix}-3 & 0\\0 & -2\end{pmatrix} \quad Z_2 = \begin{pmatrix}3 & 0\\0 & 3\end{pmatrix} \quad c = \begin{pmatrix}3\\1\end{pmatrix} \quad r = \begin{pmatrix}0\\0\end{pmatrix} \quad s = \begin{pmatrix}1\\1\end{pmatrix}$$

However, as indicated in [1], conditions (7) cannot be satisfied with $c = 0$ for this example. In other words for this example the sufficient conditions of [3, Theorem 1] do not hold, but the new conditions (7) hold.

Example 3 The following example due to Pang [7] shows that the requirement that the nondegenerate solution be a vertex cannot be dispensed with in Theorem 2.

$$M = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \qquad q = \begin{pmatrix}0\\0\end{pmatrix}$$

This problem has only one vertex solution, the origin, which however is degenerate. The positive parts of the x_1 and x_2 axes are nondegenerate nonvertex solutions of the LCP. It can be easily verified that conditions (7) of Theorem 2 cannot be satisfied for this problem.

Finally, we give a characterization which is equivalent to the conditions of Theorem 2 of [5]. Pang [6] has shown that these latter conditions hold whenever the LCP is solvable.

Theorem 3 The LCP has a solution if and only if the LP is solvable for some $r,s \in R^n$ which must satisfy the following conditions

$$\begin{aligned} &(a) \quad MZ_1 = Z_2 + qc^T \\ &(b) \quad r^T Z_1 + s^T Z_2 \geqq 0 \\ &(c) \quad r^T Z_1 + s^T Z_2 + c^T > 0 \\ &(d) \qquad\qquad r + s > 0 \\ &\qquad c,r,s \geqq 0 \ , \quad Z_1, Z_2 \in Z \end{aligned} \tag{8}$$

for some vector $c \in R^n$ and some matrices $Z_1, Z_2 \in R^{n \times n}$. Furthermore, each solution of the LP solves the LCP.

Proof (Necessity) Let x solve the LCP and let r and s be as defined in (5). It follows than that x solves the LP. If we further define, as in [6], $c=e$, $Z_1 = -xe^T$ and $Z_2 = -(Mx+q)e^T$ then condition (8) are satisfied.

(Sufficiency) We give a new proof here which is more transparent than that of Theorem 2 of [5]. We have that s is dual feasible. If s is dual optimal then

$$0 = s^T(Mx+q) + x^T(-M^Ts+r+M^Ts) = s^T(Mx+q) + r^Tx \ .$$

Because $r + s > 0$ we have that $x^T(Mx+q) = 0$ and x solves the LCP. Suppose now that s is <u>not</u> dual optimal. Then for any $x \in R^n$ satisfying $Mx+q \geqq 0$, $x \geqq 0$ we have that

$$(r^T+s^TM)x + q^Ts > 0$$

and hence from (8b) and (8c) we conclude that

$$r^TZ_1 + s^TZ_2 + ((r^T+s^TM)x + s^Tq)\ c^T > 0 \qquad (9)$$

Let $Z_1 = D-V$, $Z_2 = D-U$, $D,U,V \geqq 0$ and D is a positive diagonal matrix. Let x solve the LP and let y solve the DP. Then

$$r^TZ_1 + s^TZ_2 = r^TZ_1 + s^TMZ_1 - s^Tqc^T \qquad \text{(By (8a))}$$

$$= (r^T+s^TM)Z_1 + y^T(-MD+MV+D-U+qc^T)-s^Tqc^T \qquad \text{(By (8a))}$$

$$= (-y^TM+r^T+s^TM)(D-V) + y^T(D-U) + (y-s)^Tqc^T$$

$$\leqq (y^T(I-M) + (r^T+s^TM))D-(r^T+s^TM)xc^T - s^Tqc^T$$

(By dual feasibility and optimality of y)

By using (9) the above gives

$$0 < r^TZ_1 + s^TZ_2 +((r^T+s^TM)x+s^Tq)c^T \leqq (y^T(I-M)+(r^T+s^TM))D$$

Hence

$$y^T(I-M) + (r^T+s^TM) > 0$$

But by linear programming duality

$$x^T(-M^Ty+r+M^Ts) + y^T(Mx+q) = 0$$

Hence $x^T(Mx+q) = 0$ and x solves the LCP. □

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