A CLASS OF NONLINEAR INTEGER PROGRAMS
SOLVABLE BY A SINGLE LINEAR PROGRAM

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ABSTRACT

Although the addition of integrality constraints to the existing constraints of an optimization problem will, in general, make the determination of an optimal solution more difficult, we consider here a class of nonlinear programs in which the imposition of integrality constraints on the variables makes it possible to solve the problem by a single, easily-constructed linear program. The class of problems addressed has a separable convex objective function and a totally unimodular constraint matrix. Such problems arise in logistic and personnel assignment applications.

I. Introduction

Nonlinear integer programs of the form

$$\min_{x} \sum_{i=1}^{n} f_i\,(x_i)$$

$$\text{(1.1)} \qquad \text{s.t.} \sum_{i=1}^{n} x_i = r,$$

$$x = (x_1,\ldots,x_n)^T \geq 0,\ x_i \text{ integer } (i = 1,\ldots,n)$$

arise in logistic and personnel assignment applications and have been the subject of a number of studies [2, 7, 8, 9, 10]. Here, we consider the broader class of problems of the form

$$\min_{x} \sum_{i=1}^{n} f_i\,(x_i)$$

$$\text{(1.2)} \qquad Ax = b$$

$$x \geq 0,\ x \text{ integer},$$

where A is a totally unimodular (T.U.) $m \times n$ and b is integer*, (in the following, a vector is said to be integer if all its components are integer), and we will show that a solution to the problem (1.2)

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*Recall that a matrix is said to be totally unimodular if the determinant of each of its square submatrices has value 0 or ±1. Totally unimodular matrices typically arise in optimization problems defined on networks, but may also arise in other contexts such as bounds on sums or differences of subsets of variables. Although we assume here the equality constraints $Ax = b$, analogous results hold if $Ax = b$ is replaced by $Ax \leq b$ or by <u>any combination of equations and inequalities</u> whose aggregate coefficient matrix is totally unimodular, since the conversion of such constraints to a set of equations (by the addition of slack variables) yields a new coefficient matrix that will also be totally unimodular.

may be obtained by solving a <u>single</u> linear program provided that known bounds exist for the feasible set of (1.2) and each f_i is a <u>convex</u> function. This result thus also generalizes the well-known property [6] that, in the case that all the f_i are linear (so that (1.2) is a linear integer program), the solution of (1.2) may be obtained by solving a single linear program.

For the case in which bounds for the feasible set are <u>not</u> known, a column-generation procedure is developed, and it is shown that, if an optimal solution to (1.2) exists, this procedure will yield an optimal solution (and a proof of its optimality) by the solution of a finite number of linear programs. This column-generation procedure also has computational advantages in the bounded case if the bounds are very large and/or one or more of the f_i are "costly" to evaluate

2. An Equivalent Linear Program

In this section we will establish the equivalence of the nonlinear integer program (1.2) to a linear program ((2.5) below) under the following <u>hypotheses</u>:

(A) there exist non-negative integers ℓ_i, u_i $(i=1,\ldots,n)$ such that $F \equiv \{x|Ax = b, x \geq 0\} \leq \{x|\ell_i \leq x_i \leq u_i, i=1,\ldots,n\}$

(B) $F \neq \emptyset$

(C) the matrix A is an $m \times n$ totally unimodular matrix and b is integer

(D) (for $i = 1,\ldots,n$) f_i is a real-valued convex function on $[\ell_i,u_i]$

Note that under hypotheses (A) and (B), the nonlinear integer program (1.2) has an optimal solution since the number of feasible points is finite and non-zero. (The hypotheses (A) and (B) can, in fact, be deleted, as is shown in section 5, but the proofs for the more general case are straightforward extensions of the results of this section.) Let $\tilde{f}_i$ denote the continuous piecewise-linear function defined on $[\ell_i, u_i]$ that <u>coincides</u> with f_i at the integer points in $[\ell_i, u_i]$ and is linear between those points. It is easily seen that each $\tilde{f}_i$ is also convex. (In fact, it is convexity of the $\tilde{f}_i$ that is crucial rather than convexity of the f_i.) The problem (1.2)

is therefore equivalent* to

$$\min_x \sum_{i=1}^{n} \tilde{f}_i(x_i)$$
$$\text{s.t. } Ax = b \tag{2.1}$$
$$x \geq 0, \ x \text{ integer},$$

since the objective functions of (1.2) and (2.1) <u>coincide</u> over their common feasible set. (Put another way, the values of the objective function terms at non-integer points are completely irrelevant to the optimization problem (1.2), so we can take advantage of this fact by "simplifying" the form of the objective function terms between consecutive integers.) We will now exploit properties of a particular representation of the $\tilde{f}_i$ in order to get rid of the integrality constraints. (The overall strategy is thus to exploit the integrality constraints to modify the objective function, and then to exploit the modified objective function to get rid of the integrality constraints.)

It is a well-known result of separable programming (see [4]) that, for $x_i \in [\ell_i, u_i]$, we have the following representation for the $\tilde{f}_i$:

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*If integrality constraints were not present in (1.2), then the $\tilde{f}_i$ would merely be approximations to the f_i; but, given the integrality constraints, no "error" is incurred over the discrete domain $[\ell_i, u_i]$ by replacing f_i by $\tilde{f}_i$, so that in this context the $\tilde{f}_i$ should <u>not</u> be thought of as "approximations", as is the case when similar substitutions are done in the continuous variable case (see [1,3]).

$$f_i(x_i) = \min_{\delta_{i,j}} f_i(\ell_i) + \sum_{j \in R_i} \delta_{i,j} \, [f_i(j+1) - f_i(j)]$$

$$\text{(2.2)} \qquad \text{s.t.} \quad \ell_i + \sum_{j \in R_i} \delta_{i,j} = x_i$$

$$0 \leq \delta_{i,j} \leq 1, \; j \in R_i,$$

where $R_i = \{j | j \text{ integer}, \; j \in [\ell_i, u_i - 1]\}$. Thus, it is easily seen the problem (2.1) is <u>equivalent</u> to the problem

$$\min_{x,\delta} \sum_{i=1}^{n} f_i(\ell_i) + \sum_{i=1}^{n} d_i \, \delta_i$$

$$\text{(2.3)} \qquad \begin{array}{lll} \text{s.t. } Ax & = b & \\ \quad x_i - e_i^T \delta_i & = \ell_i \;, & i = 1,\ldots,n \\ \quad x & \geq 0 & \\ \quad \delta & \geq 0 & \\ \quad \delta_i & \leq e_i, & i = 1,\ldots,n \\ \quad x \;\; \text{integer} & & \end{array}$$

where $\delta_i = (\delta_{i,1}, \delta_{i,2}, \ldots)^T$, $\delta = (\delta_1, \ldots, \delta_n)$, $d_i = (f_i(\ell_i + 1) - f_i(\ell_i), \ldots, f_i(u_i) - f_i(u_{i-1}))$, and $e_i^T = (1, \ldots, 1)$ (a row vector of $u_i - \ell_i$ 1's).

The problem (2.3) is a <u>linear mixed-integer program</u>, but, as will be shown in Lemma 2.1 below, the constraint matrix of (2.3) is totally unimodular, so that <u>deletion</u> of the integrality constraints of (2.3) has <u>no effect</u> on the optimal value. (This result is somewhat surprising in view of the fact that if $(\bar{x}, \bar{\delta})$ is an extreme point of (2.3), then $\bar{x}$ need <u>not</u> be an extreme point of F. For, consider

the case in which the constraints $Ax = b$ are given by $x_1 + x_2 = 2$; it is easily seen that in the corresponding set of constraints of the form (2.3) that there is an extreme point with $x_1 = x_2 = 1$, whereas the extreme points of F are (0, 2) and (2, 0). Thus, the knowledge that the extreme points of F must be integer is not useful in this analysis.)

The proof of Lemma 2.2 requires the following elementary result:

<u>Lemma 2.1</u>: If B is a T. U. matrix and d is a unit vector (one component = 1, all other components = 0) of the appropriate size, then any composite matrix of the form [B d], [B -d], $\begin{bmatrix} B \\ d \end{bmatrix}$, or $\begin{bmatrix} B \\ -d \end{bmatrix}$ is also T. U.

<u>Proof</u>: Given a square submatrix of one of the composite matrices that contains a portion of a unit vector d or -d as a row or column, expand the determinant by that row or column and use the total unimodularity of B to show that the cofactors are 0 or ± 1. □

<u>Lemma 2.2</u>: If A is totally unimodular, then the constraint matrix of (2.3) is also totally unimodular.

<u>Proof</u>: If A is T. U., then, by repeated applications of Lemma 2.1, so is the composite matrix $\begin{bmatrix} A \\ I \end{bmatrix}$. Again applying Lemma 2.1, (column by column with $B = \begin{bmatrix} A \\ I \end{bmatrix}$) we conclude that the matrix

$$\text{(2.4)} \qquad \begin{bmatrix} A & 0 & 0 & \cdots & 0 \\ & -e_1 & 0 & \cdots & 0 \\ I & \vdots & \vdots & & \vdots \\ & 0 & \cdots & 0 & -e_n \end{bmatrix}$$

is also T. U. Finally, by taking the matrix in (2.4) as B and applying Lemma 2.1 to the remaining rows of (2.3), we conclude that (2.3) has a T. U. constraint matrix. □

Now consider the <u>linear programming relaxation:</u>

$$\min_{x,\delta} \sum_{i=1}^{n} f_i(\ell_i) + \sum_{i=1}^{n} d_i \, \delta_i$$

$$\begin{array}{lllll} & \text{s.t. } Ax & & & \\ (2.5) & x_i - e_i^T \delta_i & & = \ell_i & (i = 1,\ldots,n) \\ & x & & \geq 0 & \\ & & \delta & \geq 0 & \\ & & \delta_i & \leq e_i & (i = 1,\ldots,n) \end{array}$$

<u>Theorem 2.3</u>: The optimal value of the LP (2.5) is equal to the optimal value of the nonlinear integer program (1.2), and if (x^*, δ^*) is an optimal extreme point of (2.5), then x^* is an optimal solution of (1.2).

<u>Proof</u>: Since the constraint matrix of (2.5) is T. U., every extreme point of the feasible set of (2.5) is integer. Since (2.5) has an optimal solution (its feasible set is non-empty and bounded), it has an extreme point that is optimal, so the deletion of the integrality constraints had no effect on the optimal value of problem (2.3), and the theorem is proved. □

From Theorem 2.3 it follows that the simplex algorithm applied to the linear program (2.5) will yield the optimal solution of the nonlinear integer program (1.2), under hypotheses (A) - (D). In the

remaining sections of this paper, we will establish a number of theoretical and computational refinements of this basic result.

Before considering these refinements, however, we wish to point out an alternative derivation that is possible in some cases.

If the constraints $Ax = b$, $x \geq 0$ can be given a <u>network</u> interpretation in which each x_i represents the flow on a directed arc connecting two nodes y and z (see Figure 1), then the above result can be obtained from the theory of minimum cost network flows [5] by appropriately augmenting the original network. This is done by replacing the single arc connecting y and z by multiple arcs as shown in Figure 1, and replacing the convex cost function by the <u>unit costs</u> shown in Figure 1. (This transformation is equivalent to the algebraic transformation of (2.2).) When this is done for each arc, the resulting network has a linear cost function and therefore an integer optimal solution. (An analogous conversion procedure for a class of uncapacitated bipartite networks is suggested in Dantzig [4, p. 498].)

However, if the constraints $Ax = b$, $x \geq 0$ did not arise from a network formulation, it might be difficult to derive an "equivalent" directed network problem, and, indeed, the problem of conversion to such a network formulation does not appear to be addressed in the literature. In any case, Theorem 2.3 provides an algebraic approach that eliminates the need for a network formulation.

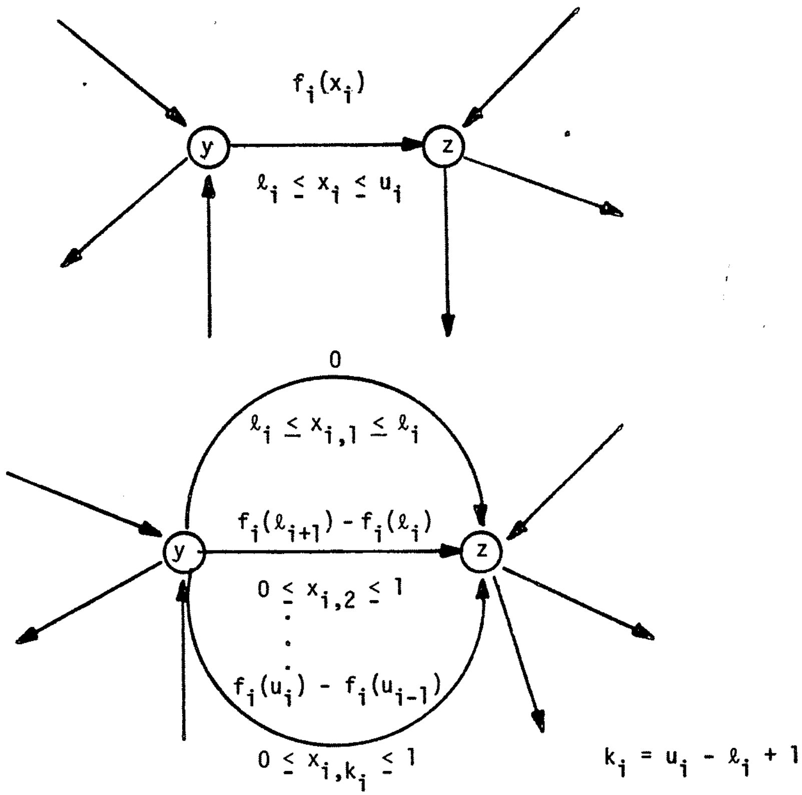

Figure 1. Equivalent Network Formulations

3. An Alternative Approach: the λ-formulation

The separable programming representation of $\tilde{f}_i$ in section 2 is sometimes referred to as the <u>δ-formulation</u>, and in this section we will consider the so-called λ-formulation of $\tilde{f}_i$. This alternative approach requires a slightly more complicated analysis, but has computational advantages that will be discussed below. We again assume that hypotheses (A) - (D) of section 2 hold.

Defining the index set

(3.1) $\quad R_i' = \{j \mid j \in [\ell_i, u_i],\ j \text{ integer}\}$,

by a well-known result of separable programming we have for $x_i \in [\ell_i, u_i]$:

$$f_i(x_i) = \min_{\lambda_{i,j}} \sum_{j \in R_i'} \lambda_{i,j} f_i(j)$$

(3.2)
$$\text{s.t.} \quad \sum_{j \in R_i'} j\lambda_{i,j} = x_i$$
$$\sum_{j \in R_i'} \lambda_{i,j} = 1;\ \lambda_{i,j} \ge 0.$$

Thus, the problem (2.1) is equivalent to the problem

$$\min_{\lambda, x} \sum_{i=1}^{n} c_i \lambda_i$$

(3.3)
$$\text{s.t. } Ax = b,\ x \ge 0,\ x \text{ integer},$$
$$D\lambda = x,\ E\lambda = e,\ \lambda \ge 0$$

where $\lambda_i = (\lambda_{i,1}, \lambda_{i,2}, \ldots)^T$, $\lambda = (\lambda_1, \ldots, \lambda_n)$, $c_i = (f_i(\ell_i), \ldots, f_i(u_i))$, $e = (1, \ldots, 1)^T$, and the constraints $D\lambda = x$, $E\lambda = e$, $\lambda \ge 0$ represent the constraints of (3.2) as i ranges from 1 to n. The problem (1.2) has thus been transformed into an equivalent <u>linear</u> mixed-integer program (3.3). Now if the constraint matrix of (3.3) were totally unimodular, then the integrality constraints of (3.3) could be deleted without affecting the optimal value. However, because D contains integer entries other than 0 or ±1, the constraint matrix of (3.3) is not totally unimodular, and if we consider the linear programming relaxation of (3.3)

$$\min_{\lambda,x} \sum_{i=1}^{n} c_i \lambda_i$$

(3.4) s.t. $Ax = b,\ x \geq 0$

$D\lambda = x,\ E\lambda = e,\ \lambda \geq 0,$

examples are easily constructed to show that the feasible set of (3.4) may have non-integer extreme points. However, we will show that if $(\hat{\lambda}, \hat{x})$ is an extreme point of (3.4), then the vector $\hat{x}$ must be integer, and thus this condition is sufficient to guarantee that the optimal value of (3.4) is equal to the optimal value of (1.2). For notational convenience, we denote the equality constraints of (3.4) as

(3.5) $Ax = b$

(3.6) $D\lambda = x$

(3.7) $E\lambda = e$

Theorem 3.1: If $(\hat{\lambda}, \hat{x})$ is an extreme point of (3.4), then $\hat{x}$ is integer.

Proof: Let λ_B and x_B be the basic variables corresponding to the extreme point $(\hat{\lambda}, \hat{x})$. It is easily seen from (3.6) and (3.7) that at least one and at most two variables from each λ_i must be in λ_B. Let

$x_B' \equiv \{x_i | x_i$ is basic and λ_B contains exactly one variable of $\lambda_i\}$ and $x_B'' \equiv \{x_i | x_i$ is basic and λ_B contains exactly two variables of $\lambda_i\}$, with corresponding definitions for λ_B' and λ_B''. If x_i is in x_B', let μ_i denote the basic variable in λ_i, so that (3.6) and (3.7) imply $\hat{\mu}_i = 1$ and $\hat{x}_i = d_i\mu_i = d_i$ for some integer d_i in R_i'.

Thus, the variables x_B' are all integer-valued, and we will now show that this is the case for x_B'' also. For each variable x_i in x_B'', we let μ_i be one of the corresponding basic variables in λ_i, so that the other basic variables in λ_i can be replaced by $1 - \mu_i$ because of (3.7). Denote the coefficient of the variable in (3.6) corresponding to $(1 - \mu_i)$ as d_i and the coefficient of the other basic variable μ_i as $d_i + h_i$ (note that h_i is a non-zero integer). Using the change of variable $x_i = d_i + x_i'$, we have from (3.6), $d_i + x_i' = d_i(1 - \mu_i) + (d_i + h_i)\mu_i$ or $x_i' = h_i\, \mu_i$, so that each such μ_i is uniquely determined by x_i'. We will now show that the columns of A corresponding to x_B'' are linearly independent. For, suppose that they were not, and set all variables other than x_B'' and λ_B'' to their valu-s in the solution $(\hat{\lambda}, \hat{x})$. If the columns of A corresponding to x_B'' were linearly dependent, there would be infinitely many sets of values of x_B'' for which (3.5) (with the other variables set to their values in $\hat{x}$) would be satisfied, and for each such set of values, values of μ_i could be determined so that (3.6) and (3.7) were also satisfied. This contradicts the fact that the system (3.5)-(3.7) must have a unique solution when the non-basics are set to 0. Thus having shown that the columns of A corresponding to x_B" are linearly independent, it follows from the T. U. of A that $\hat{x}_B$" is integer. □

Theorem 3.2: The optimal value of (3.4) is equal to the optimal value of (1.2), and if (λ^*, x^*) is an optimal extreme point of (3.4), then x^* solves (1.2).

Proof: Analogous to the proof of Theorem 2.3. □

Thus, the original nonlinear integer program (1.2) can be solved by computing the values of each f_i at the integer points in $[\ell_i, u_i]$ and solving the linear program (3.4) by the simplex method, which will generate an optimal extreme point.

It should also be noted that Theorem 3.2 also implies that (1.2) and (3.4) have the same optimal value as the problem obtained from (3.1) by deleting its integrality constraints, namely

$$\min \sum_{i=1}^{n} \tilde{f}_i(x_i) \qquad (3.8)$$

$$\text{s.t. } Ax = b, \quad x \geq 0 .$$

The next two results show that analogous conclusions can be obtained when each f_i is replaced by a piecewise - linear convex function that coincides with f_i at some rather than all of the integer points in the interval $[\ell_i, u_i]$. These more general results suggest the use of "column-generation" strategies in the event that

evaluation of the f_i at all integer points in the intervals $[\ell_i u_i]$ would be "costly". (Details of these "column-generation" procedures are given in section 4).

Corollary 3.3: Let the functions $f_i (i=1,\ldots,n)$ be convex piece-wise-linear functions of the form

$$\hat{f}_i(x_i) = \min_{\lambda_{i,j}} \sum_{j \in R_i"} f_i(j)\lambda_{i,j}$$

$$(3.9) \qquad \text{s.t. } \sum_{j \in R_i"} j\lambda_{i,j} = x_i$$

$$\sum_{j \in R_i"} \lambda_{i,j} = 1, \ \lambda_{i,j} \geq 0 ,$$

where each R_i" is a finite, non-empty subset of the integers. If the optimal value of the problem

$$\min \sum_{i=1}^{n} \hat{f}_i(x_i) \qquad (3.10)$$

$$\text{s.t. } Ax = b, \ x \geq 0, \ x \text{ integer}$$

exists, then it is equal to the optimal value of

$$\min \sum_{i=1}^{n} \hat{f}_i(x_i) \qquad (3.11)$$

$$\text{s.t. } Ax = b, \ x \geq 0 .$$

<u>Proof</u>: Since (3.10) is assumed to have an optimal solution, it is easily seen that (3.11) must also have an optimal solution, and by an argument analogous to the proof of Theorem 3.1, the LP equivalent to (3.11) must have an optimal solution with x integer-valued. ∎

Note that if the functions $\hat{f}_i$ are all <u>linear</u> or <u>affine</u> functions, then the equivalence of (3.10) and (3.11) is well-known (see [6]), so that Corollary 3.3 may be thought of as the generalization of this well-known (linear) integer programming result to <u>piecewise-linear convex functions</u> with breakpoints at integers. It should be recognized, however, that the conclusion of Corollary 3.3 need <u>not</u> hold if the $\hat{f}_i$ are general convex functions or if the $\hat{f}_i$ are even piecewise-linear convex functions with "breakpoints" at non-integer points. This is easily seen by letting the constraints $Ax = b$ be given by $x_1 + x_2 = 1$ ($n = 2$) and letting $\hat{f}_i(x_i) = (x_i - \frac{1}{2})^2$ or $|x_i - \frac{1}{2}|$. Such convex functions must be replaced by "equivalent" piecewise-linear functions with integral breakpoints before the integrality constraints may be deleted.

We will now show that the optimal value of the problem (3.11) conincides with the optimal value of the nonlinear problem (1.2) if the index sets R_i'' are sufficiently "fine" near an integer optimal solution of (3.11). This result is essentially equivalent to the fact that a local solution of (3.8) must also be a global solution of (3.8).

<u>Theorem 3.4</u>: If x^{**} is an integer optimal solution of (3.11) and if $R_i'' \supseteq \{x_i^{**} - 1, x_i^{**}, x_i^{**} + 1\} \cap [\ell_i, u_i]$ for $i = 1,\ldots,n$,

then x^{**} is an optimal solution of the nonlinear integer program (1.2).

<u>Proof</u>: Since the feasible sets of (3.8) and (3.11) coincide, x^{**} is feasible for (3.8). Moreover, $\hat{f}_i(y) = \tilde{f}_i(y)$ for $y \in [x_i^{**}-1, x_i^{**}+1] \cap [\ell_i, u_i]$, so x^{**} must be a local minimum of (3.8). Because of convexity, x^{**} is also a global minimum of (3.8), and the conclusion follows from Corollary 3.2 and the equivalence of (3.4) and (3.8). ∎

It should be noted that it is <u>not</u> sufficient for optimality to simply have $R_i'' \supseteq \{x_i^{**}\}$ for all i , as may be seen from following <u>example</u>: consider the following problem of the form (1.2)

$$\min \ (x_1-1)^2 + (x_2-1)^2$$

$$\text{s.t. } x_i + x_2 = 2$$

$$x_i \geq 0 \text{ and integer,}$$

and let $\ell_i = 0$, $u_i = 2$, $R_i'' = \{0,2\}$ for $i = 1,2,$; then $\hat{f}_i \equiv 1$ on $[J,2]$, so that optimal solutions for the corresponding approximating problem occur at $x_1 = 0$, $x_2 = 2$ and $x_1 = 2$, $x_1 = 0$, but the unique optimal solution of the original problem is $x_1 = 1$, $x_2 = 1$.

Finally, note that these results do <u>not</u> generalize to the case in which the x_i are <u>vector</u> variables rather than the scalar components of x . This may be seen from the following example.

<u>Example</u>: Let $f(x_1,x_2,x_3,x_4) = f_1(x_1,x_2) + f_2(x_3,x_4)$, where f_1 is any convex function such that $f_1(0,0) = f_1(1,1) = 1$ and $f_1(0,1) = f_1(1,0) = -1$ (for example, f_1 could be taken as $2(x_1+x_2-1)^2-1$ or as a convex piecewise-linear function with those values), and f_2 is any convex function such that $f_2(0,0) = f_2(1,1) = -1$ and $f_2(0,1) = f_2(1,0) = 1$ (for example, f_2 could be taken as $2(x_3-x_4)^2-1$ or as a convex piecewise-linear function with those values). Consider the problem:

$$\begin{array}{lll} \min & f_1(x_1,x_2) + f_2(x_3,x_4) & \\ \text{s.t.} & x_1 \quad -x_3 & = 0 \\ & x_2 \quad -x_4 & = 0 \\ & 0 \le x_i \le 1 & \\ & x_i \text{ integer} \quad (i = 1,\ldots,4) . & \end{array}$$

It is easily seen that the constraint matrix is totally unimodular. The four feasible points (0,0,0,0), (1,0,1,0), (0,1,0,1), (1,1,1,1) all have objective function values of 0 , and hence are all <u>optimal solutions</u>. Suppose, however, we replace the f_i by piecewise linear convex functions $\bar{f}_i$ that agree with the f_i at the points at which they are defined, and delete the integrality constraints to obtain the problem

$$\begin{array}{lll} \min & \bar{f}_1(x_1,x_2) + \bar{f}_2(x_3,x_4) & \\ \text{s.t.} & x_1 \quad -x_3 & = 0 \\ & x_2 \quad -x_4 & = 0 \\ & 0 \le x_i \le 1 \quad (i = 1,\ldots,4) & \end{array}$$

It is easily seen that the <u>optimal solution</u> of this <u>new</u> problem occurs at the point (1/2, 1/2, 1/2, 1/2), where the objective value is -2 , so that the deletion of the integrality constraints results in a decrease of the optimal value. □

4. Computational Considerations

Theorem 3.4 establishes the validity of the following column-generation procedure for solving the nonlinear integer program (1.2) under the hypotheses (A)-(D) of section 2:*

(4.1) Set the iteration index $k = 0$, and select initial set of breakpoints $R_i^0 \supseteq \{\ell_i, u_i\}$ $(i = 1,\ldots,n)$.

(4.2) Solve the LP (3.11) with $R_i'' = R_i^k$.

(4.3) If the optimal solution obtained for (3.11) satisfies the optimality conditions of Theorem 3.4, then it also solves (1.2), and the algorithm terminates; otherwise, increase k by 1 and add the breakpoints that would have been required to satisfy the breakpoint hypotheses of Theorem 3.4 for the solution obtained in (4.2) (thereby obtaining "finer" index sets R_i^k) and return to (4.2).

Since the maximum possible number of breakpoints is finite, and at least one new breakpoint is added at each iteration, this procedure must terminate in a finite number of iterations with the optimal solution of (1.2). As with other column-generation procedures, each succeeding iteration can be started with the optimal basis from the previous iteration. If function evaluations are

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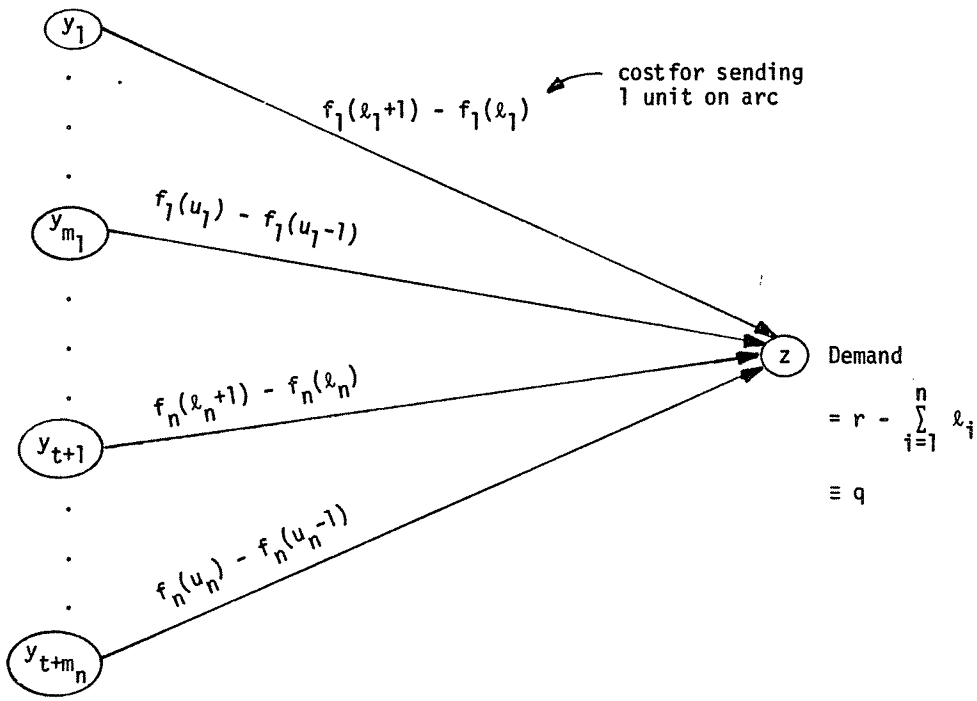
*This procedure may also be used if hypothesis (B) is violated (i.e., $F = \phi$), since $F = \phi$ if and only if the feasible set of (1.2) is empty, and the first attempt to analyze an LP in (4.2) will establish $F = \phi$ if this is the case. The case in which hypothesis (A) is violated (i.e., F is unbounded) is dealt with in section 5, where it is shown that an analogous procedure will converge in a finite number of iterations if (1.2) has an optimal solution.

much more "expensive" than pivot operations, the procedure could be modified by selecting the initial breakpoints close to an estimate of the optimal solution and adding only some of the "missing" breakpoints in step (4.3).

In the case of the problem (1.1), it should be noted that the equivalent LP has such a simple structure that its solution is obvious. In fact, rather than dealing with (1.1), we will consider the more general class of problems of the form

$$
\begin{aligned}
&\min \sum_{i=1}^{n} f_i(x_i) \\
(4.4)\quad &\text{s.t.} \sum_{i=1}^{n} x_i = r \\
&\qquad \ell_i \le x_i \le u_i \quad (i = 1,\ldots,n) \\
&\qquad\quad x_i \text{ integer } (i = 1,\ldots,n)
\end{aligned}
$$

(Note that (1.1) is equivalent to a problem of the form (4.4) with $\ell_i = 0$ and $u_i = r$ $(i = 1,\ldots,n)$.) Figure 2 shows the network that results after the arcs corresponding to flows fixed at lower bounds (see Figure 1) have been accounted for by reducing the "demand". (We assume that the condition $\sum_{i=1}^{n} \ell_i \le r \le \sum_{i=1}^{n} u_i$, which is necessary and sufficient for feasibility, has been verified.) This network has $t + m_n$ supply points, each of which can ship at most 1 unit to the demand point z, at which the demand is $q \equiv r - \sum_{i=1}^{n} \ell_i$. If $q = 0$, the optimal solution is obtained by setting $x_i = \ell_i$ for all i; otherwise, the optimal solution is obtained by sending one unit along each of the q "least expensive" arcs. Because of convexity, these arcs and the optimal values of the x_i may be



$m_i = u_i - \ell_i$

$t = \sum_{i=1}^{n-1} m_i$

Figure 2. A Network Equivalent of (4.4)

determined as follows:

(4.4) Set $\bar{x}_i = \ell_i$ and denote by V the set of differences

$$\{f_1(\ell_1+1) - f_1(\ell_1),\ldots,f_n(\ell_n+1) - f_n(\ell_n)\}$$

(4.5) Choose k such that $f_k(j) - f_k(j-1)$ is a minimum element of V, and increase $\bar{x}_k$ by 1.

(4.6) If $\sum_{i=1}^{n} \bar{x}_i = r$, terminate with the optimal values of the x_i being given by $\bar{x}_i$ $(i=1,\ldots,n)$; otherwise remove $f_k(\bar{x}_k) - f_k(\bar{x}_k - 1)$ from V, replacing it by $f_k(\bar{x}_k+1) - f_k(\bar{x}_k)$ if $\bar{x}_k < u_k$, and return to (4.5).

Note that this procedure requires at most* $2n + (r - \sum_{i=1}^{n} \ell_i) - 1$ function evaluations to generate an optimal solution. (If $(\sum_{i=1}^{n} u_i - r) < (r - \sum_{i=1}^{n} \ell_i)$, then by the transformation of variables $x_i' = u_i - x_i$, we obtain an equivalent problem requiring at most $2n + (\sum_{i=1}^{n} u_i - r) - 1$ function evaluations.) Using the convexity of the f_i, it may be shown that a necessary and sufficient optimality condition for a feasible solution $(x_1',\ldots,x_n')$ of the problem (1.1) with the additional constraints $\ell_i \le x_i \le u_i$ $(i = 1,\ldots,n)$ is that

---

*If the value μ^* of the minimum element of V in (4.5) at some iteration is not unique, then it is not necessary to compute the next difference needed to replace an element of value μ^* removed in (4.6) until all such elements of V have been removed from V, which may not occur before the termination test $\sum\bar{x}_i = r$ is satisfied.

$$\min_{i \in R_u} [f_i(x_i'+1) - f(x_i')] \geq \max_{i \in R_\ell} [f_i(x_i'-1) - f_i(x_i')] , \tag{4.7}$$

where $R_u \equiv \{i | x_i' < u_i\}$ and $R_\ell \equiv \{i | x_i' > \ell_i\}$. These conditions, as might be expected, are equivalent to the optimality conditions for the equivalent LP and network problems, and reduce to the optimality conditions of Gross [7] (see also Saaty [8]) for the problem without upper bounds by taking $\ell_i = 0$ and $R_u = \{1,\ldots,n\}$.

5. The Unbounded Case

The results of this section show that the hypotheses regarding boundedness of the feasible set and finiteness of the number of breakpoints can be deleted, provided that the conclusions are appropriately generalized.

<u>Theorem 5.1</u>: Let the function $\bar{f}_i$ $(i=1,\ldots,n)$ be continuous piecewise-linear convex functions on $[0,+\infty)$ whose derivatives are also continuous except possibly on subsets of the positive <u>integers</u>. Let A be an $m \times n$ totally unimodular matrix, and b be an integer vector. Then the problems

$$\begin{aligned} & \inf \quad \sum_{i=1}^{n} \bar{f}_i(x_i) \\ & \text{s.t.} \quad Ax = b,\ x \geq 0 ,\quad x \text{ integer} \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} & \inf \quad \sum_{i=1}^{n} \bar{f}_i(x_i) \\ & \text{s.t.} \quad Ax = b,\ x \geq 0 \end{aligned} \tag{5.2}$$

are equivalent in the sense that (5.1) is feasible if and only if (5.2) is feasible, (5.1) has an optimal solution with optimal value v^* if and only if (5.2) has an optimal solution with optimal value

v^*, and (5.1) is unbounded if and only if (5.2) is unbounded.

Proof: If (5.2) has a feasible solution, then the feasible set of (5.2) has an extreme point, which is thus integer and therefore a feasible solution of (5.1).

If (5.2) has an optimal solution, then there exist integer lower and upper bounds ℓ_i, u_i for each variable x_i that may be imposed in (5.2) without affecting its optimal value. The resulting problem then has an integer optimal solution because of Corollary 3.3 and this integer solution must be optimal for (5.1). If (5.1) has an optimal solution x^*, then define lower and upper bounds ℓ_i, u_i by $\ell_i \equiv \max\{0, x_i^*-1\}$, $u_i \equiv x_i^* + 1$, and add these bounds to (5.2). By Corollary 3.3, the resulting problem will also have x^* as its optimal solution, so that x^* is a local minimum of (5.2). Since (5.2) is a convex problem, x^* must also be the global solution of (5.2)

If (5.2) is unbounded, we will show that for any real number θ, there exists a feasible solution of (5.1) whose value is less than θ. Let $x(\theta)$ be a feasible solution of (5.2) whose objective value is less than θ. Now choose integer lower and upper bounds ℓ_i, u_i on x_i in (5.2) such that $x(\theta)$ satisfies these bounds. The resulting problem will have an optimal value less than θ, and thus the problem obtained by adding these bounds to (5.1) will have the same optimal value and hence a feasible solution with objective value less than θ. □

It should be noted that it is possible that (5.1) and (5.2) may have finite infima that are never attained. (If this is the case for one of these problems, it must be true for the other also.) In this situation, it is easily shown by an argument similar to that used in the unbounded case that these infima must coincide.

We now state the analog of Theorem 3.4 in the absence of upper bounds.

Theorem 5.2: If the hypotheses of Theorem 5.1 are satisfied, and x^{**} is an integer optimal solution of (5.2), then x^{**} is an optimal solution of

$$\inf \sum_{i=1}^{n} f_i(x_i)$$

(5.3)

$$\text{s.t.} \quad Ax = b,\ x \geq 0,\ \ x \text{ integer}$$

where, for $i = 1,\ldots,n$, f_i is any convex function that agrees with $\bar{f}_i$ on the set $\{x_i^{**} - 1, x_i^{**}, x_i^{**} + 1\} \cap \{y \mid y \geq 0\}$.

Proof: Suppose that there exists an $\bar{x}$ feasible for (5.3) such that $\sum_{i=1}^{n} f_i(\bar{x}_i) < \sum_{i=1}^{n} f_i(x_i^{**})$. Generate bounded variants of (5.2) and (5.3) by adding the constraints $\{\min \bar{x}_i, x_i^{**}\} \leq x_i \leq \max\{\bar{x}_i, x_i^{**}\}$ $(i=1,\ldots,n)$. The bounded variant of (5.3) must have an optimal solution x^* such that $\sum_{i=1}^{n} f_i(x_i^*) < \sum_{i=1}^{n} f_i(x^{**})$. However, by applying Theorem 3.4 to the bounded variants of (5.2) and (5.3); we obtain a contradiction. $\square$

In order to prove finite convergence of an extension to the unbounded case of the column-generation procedure of section 4, we will first establish a theorem that implies that <u>any</u> "non-degenerate", "primal" method for the problem (1.2) will converge in a finite number of steps. Note that the following theorem is actually valid for a more general class of problems than those considered previously, since total unimodularity of the matrix A is <u>not</u> assumed. (For notational convenience, we define $f(x) \equiv \sum_{i=1}^{n} f_i(x_i)$, and $F_I \equiv \{x | Ax = b, x \geq 0, x \text{ integer}\}$.)

<u>Theorem 5.3</u>: If f_i $(i = 1,\ldots,n)$ are convex functions on $[0,+\infty)$ and if the problem

$$\min \quad \sum_{i=1}^{n} f_i(x_i)$$

(5.4)

$$\text{s.t.} \quad Ax = b,\ x \geq 0,\ x \text{ integer}$$

has an optimal solution, then, for each real number M , the set $\{f(x) | x \in F_I\}$ contains a finite number (possibly 0) of distinct values in the range $(-\infty, M]$.

<u>Proof</u>: Suppose the result is false, and let $\{x^{(k)}\}$ be a sequence contained in F_I with the property that $\{f(x^{(k)})\}$ is monotone decreasing. We first show that there exist index sets J' and J'' (with $J' \cup J'' = \{1,\ldots,n\}$) and an increasing subsequence I_n' of the integers such that the sequences $\{x_i^{(k)}\}$ are constant for $i \in J'$ and have certain useful properties for $i \in J''$. Consider the sequence $\{x_1^{(k)}\}$, and if for each N there exists a $k(N)$ such that $k \geq k(N)$ implies $x_1^{(k)} \geq N$, set $I_1' \equiv \{1,2,3,\ldots\}$ and put 1 in the index set J'' . Otherwise, for some N , there exists an increasing subsequence of integers I_1 such that $k \in I_1$ implies $x_1^{(k)} < N$, and thus there exists a subsequence I_1' of I_1 such that $x_1^{(k)} = \bar{x}_1$, a constant, for all $k \in I_1'$; put 1 in the index set J' . Now carry out the analogous procedure for the sequence $\{x_2^{(k)} | k \in I_1'\}$, thereby obtaining a subsequence I_2' of I_1' and assigning 2 to J' or J'' . In general, given the index set I_i' we similarly construct I_{i+1} and place $i+1$ in J' or J'' until $i+1 = n$. Note that those indices $i \in J'$ have the property stated above, and those indi es $i \in J''$ have the property that, for every N , there exists a $k(N)$ such that $k \geq k(N)$ implies $x_i^{(k)} \geq N$. For notational convenience, partition the set of variables x into sets of variables y and z (one of which may vacuous) such that y is composed of the x_i with $i \in J'$ and z is composed of the x_i with $i \in J''$.

If z is vacuous, then clearly the sequence $\{f(x^{(k)}) | k \in I_n'\}$ is constant, a contradiction, and the theorem is proved, so suppose z is not vacuous. Let x^* be an optimal solution of (5.4) and let the corresponding values of y and z be y^* and z^*. Choose a $k \in I_n'$ such that $z^{(k)} \geq z^*$ and choose a $k' \in I_n'$ such that $z^{(k')} > z^{(k)}$, and let $\Delta \equiv x^{(k')} - x^{(k)}$. Note that for $i \in J'$, $\Delta_i = 0$, and for $i \in J''$, $\Delta_i > 0$, and that $(x^* + \Delta) \in F_I$. Using the convexity of the f_i and the non-negativity of Δ, we have

$$f(x^*) - f(x^* + \Delta) = \sum_{i=1}^{n} [f_i(x_i^*) - f_i(x_i^* + \Delta_i)]$$

$$= \sum_{i \in J''} [f_i(x_i^*) - f_i(x_i^* + \Delta_i)]$$

$$\geq \sum_{i \in J''} [f_i(x_i^{(k)}) - f_i(x_i^{(k)} + \Delta_i)] = f(x^{(k)}) - f(x^{(k')}) > 0 .$$

But this implies $f(x^*) > f(x^* + \Delta)$, contradicting the optimality of x^*. □

Theorem 5.3 will now be used to prove finite convergence of the following column-generation procedure in the case that (5.4) has an optimal solution:

(5.5) Use the simplex algorithm to determine an extreme point of the constraints $Ax = b$, $x \geq 0$. If no extreme point exists, then the problem (5.4) is infeasible. Otherwise, the extreme point is a feasible solution of (5.4); label this feasible solution $x^{(0)}$, set $R_i^{(0)} = \{\ell_i, u_i\}$, where ℓ_i and u_i are estimates (which need not be satisfied at an optimal solution) for lower and upper bounds on an optimal solution component x_i^*, set the iteration index $k=1$, and go to (5.6)

(5.6) Let $R_i^{(k)} \equiv R_i^{(k-1)} \cup [\{x_i^{(k-1)} - 1, x_i^{(k-1)}, x_i^{(k-1)} + 1\} \cap \{y | y \geq 0\}]$ and solve the linear program corresponding to (5.2)*, taking as $\bar{f}_i$ the piecewise-linear convex function defined by

$$\bar{f}_i(x_i) = \min_{\lambda_{i,j}} \sum_{j \in R_i^{(k)}} f_i(j) \lambda_{i,j}$$

$$\text{s.t.} \sum_{j \in R_i^{(k)}} j \, \lambda_{i,j} = x_i$$

$$\sum_{j \in R_i^{(k)}} \lambda_{i,j} = 1, \ \lambda_{i,j} \geq 0$$

(5.7) If the optimality conditions of Theorem 5.2 are satisfied, then terminate because the optimal solution of (5.4) has been determined. Otherwise denote the optimal solution of the LP as $x^{(k)}$, increase k by 1, and return to (5.6).

---

*We assume that an integer optimal solution is generated and that the optimal solution is set to $x^{(k-1)}$ if $x^{(k-1)}$ satisfies the optimality test for the LP.

<u>Theorem 5.4</u>: If A is totally unimodular, and the f_i $(i=1,\ldots,n)$ are convex on $[0,+\infty)$, and the problem (5.4) has an optimal solution, then the algorithm (5.5)-(5.7) determines an optimal solution in a finite number of iterations.

<u>Proof</u>: Suppose that for some $k \geq 1$, $x^{(k)}$ does not satisfy the optimality test of Theorem 5.2. On the next iteration, $x^{(k)}$ will be a feasible point of the new optimization problem derived by refining the breakpoint sets. If $x^{(k)}$ is optimal for this new problem, then it satisfies the optimality test of Theorem 5.2 and the algorithm terminates. Otherwise, $x^{(k)}$ will not be optimal for the new problem, and thus $f(x^{(k)}) = \sum_{i=1}^{n} \bar{f}_i(x_i^{(k)}) > \sum_{i=1}^{n} \bar{f}_i(x_i^{(k+1)}) \geq f(x^{(k+1)})$. By Theorem 5.3, improvement in the objective function f can occur only a finite number of times, so the algorithm must terminate with proof of optimality in a finite number of iterations. □

There are two alternative strategies for updating the breakpoint sets in (5.6) that could prove helpful for troublesome problems. If good estimates for ℓ_i and u_i are not available, and if a solution $x^{(k)}$ is obtained in which $x_i^{(k)} > u_i$, then more rapid increases in x_i will be made possible by adding $2x_i^{(k)}$ as an additional breakpoint and replacing the upper bound estimate by $2x_i^{(k)}$. On the other hand, if storage becomes a problem, then an update of the breakpoint sets can be done in which most previous breakpoints are discarded, as long as the points $\{x_i^{(k-1)}-1, x_i^{(k-1)}, x_i^{(k-1)}+1\} \cap \{y \mid y \geq 0\}$ are retained, since only those points

are needed to guarantee convergence.

Linear programming can also be used to establish lower bounds on the optimal value of (1.2). If application of the algorithm (5.5)-(5.7) is terminated before satisfying the optimality conditions of Theorem 5.2, then a lower bound $\underline{z}$ on the optimal value of (1.2) will provide an estimate of the "quality" of the solution obtained. To get such a lower bound, the f_i are replaced by convex, piecewise-linear functions f_i^* whose values on the integers are no greater than the corresponding values of f_i. Specifically, we may define

$$
\begin{aligned}
f_i^*(x_i) \equiv \min_{z_i,\lambda_{i,j}} \quad & z_i \\
\text{s.t.} \quad & z_i \geq f_i(j) + \lambda_{i,j}(f_i(j+1) - f_i(j)) \\
& x_i = j + \lambda_{i,j} \quad (j \in R_i^{(k)}),
\end{aligned}
\tag{5.8}
$$

where $R_i^{(k)}$ is the "final" index set generated in (5.6) (in theory, any index set may be used, but finer index sets lead to better lower bounds).

Because of convexity, $f_i(x_i) \geq f_i^*(x_i)$ for integer $x_i \geq 0$, so that the optimal value of (1.2) is no greater than the optimal value of the problem

$$\min_{x} \sum_{i=1}^{n} f_i^*(x_i)$$

$$\text{s.t.} \quad Ax = b,\ x \geq 0,\ \ x \text{ int.} \tag{5.9}$$

Furthermore, the optimal value of (5.9) will be no greater than the optimal value of its relaxation

$$\min_{x} \sum_{i=1}^{n} f_i^*(x_i)$$

$$\text{s.t.} \quad Ax = b,\ x \geq 0\ . \tag{5.10}$$

By using (5.8), (5.10) may be converted into an equivalent LP, whose optimal value $\underline{z}$ is then a lower bound on the optimal value of (1.2).

An analogous procedure may clearly be used to provide lower bounds in conjunction with the algorithm (4.1)-(4.3), if, as a result of slow convergence, that algorithm is terminated prior to satisfaction of the optimality conditions.

6. Conclusions

We have shown how optimal solutions to bounded nonlinear integer programs of the form (1.2) (with f_i convex, A totally unimodular, and b integer) may be obtained by solving an easily-generated linear programming problem. These results generalize certain results in (linear) integer programming dealing with totally unimodular constraint matrices as well as results for nonlinear integer programs of the form (1.1), and provide a rigorous and efficient approach for obtaining optimal solutions. Furthermore, in the case that known bounds are not available for (1.2), it is shown than an appropriate linear programming "column-generation" algorithm will yield an optimal solution in a finite number of iterations if (1.2) actually has an optimal solution.

References

1. Claude Berge and A. Ghouila-Houri, Programming, Games and Transportation Networks, John Wiley, New York, 1965.

2. L. B. Boza, "The Interactive Flow Simulator: A System for Studying Personnel Flows", paper presented at the ORSA/TIMS Joint National Meeting, San Juan, Puerto Rico, October 1974.

3. A Charnes and W. W. Cooper, "Nonlinear Network Flows and Convex Programming over Incidence Matrices", Nav. Res. Log. Q., 5 (1958), pp. 231-240.

4. George B. Dantzig, Linear Programming and Extensions, Princeton University Press, Princeton, 1963.

5. L. R. Ford Jr. and D. R. Fulkerson, Flows in Networks, Princeton University Press, Princeton 1962.

6. Robert S. Garfinkel and George L. Nemhauser, Integer Programming, John Wiley, New York, 1972.

7. O. Gross, "Class of Discrete Type Minimization Problems, RM-1644, Rand Corp., Santa Monica, 1956.

8. Thomas L. Saaty, Optimization in Integers and Related Extremal Problems, McGraw-Hill, New York, 1970.

9. R. E. Schwartz and C. L. Dym, "An Integer Maximization Problem", Operations Res., 19 (1971), pp. 548-550.

10. Iram J. Weinstein and Oliver S. Yu, "Comment on an Integer Maximization Problem", Operations Res., 21 (1973), pp. 648-650.