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THE ARITHMETIC BASIS OF SPECIAL RELATIVITY - 1

by

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at time t_k, while it is at (x_k', y_k', z_k') in the rocket frame at time t_k', then the events (x_k, y_k, z_k, t_k) and (x_k', y_k', z_k', t_k') are related by the Lorentz transformation [8] :

(2.1) $x_k' = c(x_k - ut_k)/(c^2 - u^2)^{\frac{1}{2}}$, or, $x_k = c(x_k' + ut_k')/(c^2 - u^2)^{\frac{1}{2}}$

(2.2) $y_k' = y_k$

(2.3) $z_k' = z_k$

(2.4) $t_k' = (c^2 t_k - ux_k)/(c(c^2 - u^2)^{\frac{1}{2}})$, or, $t_k = (c^2 t_k' + ux_k')/(c(c^2 - u^2)^{\frac{1}{2}})$,

where c is the speed of light.

Formulas (2.1)-(2.4) are geometric formulas in the sense that they relate the coordinates of the two systems under consideration. Arithmetic formulas for the basic physical concepts of velocity and acceleration will be given next. The forward difference operator Δ ,where

(2.5) $$\Delta F(k) = F(k+1) - F(k)$$

will be used throughout.

In the lab frame, let particle P be in motion in the X-direction. Then at time t_k, P's velocity $v(t_k) = v_k$ and acceleration $a(t_k) = a_k$ are defined by

(2.6) $$v_k = \Delta x_k / \Delta t_k$$

(2.7) $$a_k = \Delta v_k / \Delta t_k \quad .$$

Similarly, in the rocket frame, at time t_k' one defines v_k' and a_k' by

(2.8) $$v_k' = \Delta x_k' / \Delta t_k'$$

(2.9) $$a_k' = \Delta v_k' / \Delta t_k' \quad .$$

The structures of (2.6) and (2.8) are the same, as is also the case with (2.7) and (2.9).

In order to find the relationships between v_k and v_k', and between a_k and a_k', note first that the linearity of (2.1)-(2.4) implies

(2.10) $$\Delta x_k' = c(\Delta x_k - u\,\Delta t_k)/(c^2-u^2)^{\frac{1}{2}}$$

(2.11) $$\Delta y_k' = \Delta y_k$$

(2.12) $$\Delta z_k' = \Delta z_k$$

(2.13) $$\Delta t_k' = (c^2\,\Delta t_k - u\,\Delta x_k)/(c(c^2-u^2)^{\frac{1}{2}}) \quad .$$

Thus, (2.8), (2.10) and (2.13) imply

(2.14) $$v_k' = (c^2(v_k-u))/(c^2-uv_k) \quad ,$$

while (2.9), (2.13) and (2.14) imply

(2.15) $$a_k' = \frac{c^3(c^2-u^2)^{3/2}}{(c^2-uv_k)^2(c^2-uv_{k+1})}\, a_k \quad .$$

Finally, for later convenience, we note that if the quantity τ is defined by

(2.16) $$\tau = (c^2 t_k{}^2 - x_k{}^2 - y_k{}^2 - z_k{}^2)^{\frac{1}{2}} \quad ,$$

then (2.1)-(2.4) imply

(2.17) $$\tau = (c^2 t_k'{}^2 - x_k'{}^2 - y_k'{}^2 - z_k'{}^2)^{\frac{1}{2}} \quad ,$$

so that τ is an invariant of the Lorentz transformation. In the case when

(2.18) $$c^2 t_k{}^2 - x_k{}^2 - y_k{}^2 - z_k{}^2 > 0,$$

τ is called the proper time of the event (x_k, y_k, z_k, t_k). In a similar fashion, it follows that $\Delta\tau$, defined by

(2.19) $$\Delta\tau = (c^2(\Delta t_k)^2 - (\Delta x_k)^2 - (\Delta y_k)^2 - (\Delta z_k)^2)^{\frac{1}{2}}$$

is also an invariant and it is called the proper time between successive events (x_k, y_k, z_k, t_k) and $(x_{k+1}, y_{k+1}, z_{k+1}, t_{k+1})$, provided that (2.18) is valid for both k and k+1.

3. CONSERVATION OF LINEAR MOMENTUM. Assume now that particle P, which is in motion in the X-direction in the lab frame, has mass m. Then the linear momentum p_k of P at time t_k is defined by

(3.1) $$p_k=mv_k \quad .$$

Similarly, in the rocket frame, let p_k' be defined by

(3.2) $$p_k'=m'v_k' \quad .$$

Now, instead of proving the conservation of linear momentum, we proceed by assuming it is valid in each frame of reference. This assumption is valid if and only if [8, pp 101-110] ,at time t_k in the lab frame,

(3.3) $$m=cm_0/(c^2-v_k{}^2)^{\frac{1}{2}} \quad ,$$

and, at corresponding time t_k' in the rocket frame,

(3.4) $$m'=cm_0/(c^2-v_k'^2)^{\frac{1}{2}} \quad ,$$

where m_0 is a constant called the rest mass of P.

Equations (3.3) and (3.4) are, of course, the classical relativistic formulas for the variation of mass with speed, and we continue under the assumption that they are valid.

4. SYMMETRY. From the dynamical and computational points of view, the actual motion of a particle in, say, the lab frame can be determined from (2.6) and (2.7) once an equation which relates force and acceleration is given. We assume now that this equation is

(4.1) $$F_k = \frac{c^2m}{((c^2-v_k{}^2)(c^2-v_{k+1}{}^2))^{\frac{1}{2}}} \frac{\Delta v_k}{\Delta t_k} \quad .$$

We will have the property of symmetry [9, pp 11-1, 52-1] ,then, if in the rocket frame the relationship between force and acceleration has exactly the same structure as (4.1), that is,

$$(4.2) \qquad F'_k = \frac{c^2 m'}{((c^2-v'^2_k)(c^2-v'^2_{k+1}))^{\frac{1}{2}}} \frac{\Delta v'_k}{\Delta t'_k} .$$

To show that we do, in fact, have symmetry under the Lorentz transformation, we need only show that (4.1) and (4.2) imply that $F_k=F'_k$. To do this, from (4.2) note that (2.14), (2.15), (3.3) and (3.4) yield

$$F'_k = \frac{c^3 m_0}{(c^2-v'^2_k)(c^2-v'^2_{k+1})^{\frac{1}{2}}} a'_k$$

$$= \frac{c^3 m_0}{(c^2-v_k^2)(c^2-v_{k+1}^2)^{\frac{1}{2}}} a_k$$

$$= \frac{c^2 m}{((c^2-v_k^2)(c^2-v_{k+1}^2))^{\frac{1}{2}}} \frac{\Delta v_k}{\Delta t_k} ,$$

and the proof is complete.

Note that taking limits in (4.1) yields the particular form

$$(4.3) \qquad F= \frac{c^2 m}{c^2 - v^2} \frac{dv}{dt}$$

of the classical relativistic dynamical equation

$$(4.4) \qquad F = \frac{d}{dt} (mv) ,$$

in which m is defined by (3.3).

5. ENERGY. The total energy E of particle P of mass m is defined by

(5.1) $$E = mc^2.$$

Experimental motivation for this definition follows, for example [9, p 15-11] , from experiments in which matter is annihilated, that is, converted totally to energy. Thus, when an electron and a positron come together at rest, each with rest mass m_0, they disintegrate and the two emerging gamma rays each has m_0c^2 measured energy.

From (3.3) and (5.1), then,

(5.2) $$E = \frac{c^2 m_0}{(1-v_k{}^2/c^2)^{\frac{1}{2}}} ,$$

so that for $v_k{}^2/c^2 < 1$

(5.3) $$E = c^2 m_0 (1 + \tfrac{1}{2} v_k{}^2/c^2 + \ldots\ldots) ,$$

or,

(5.4) $$E = c^2 m_0 + \tfrac{1}{2} m_0 v_k{}^2 + \ldots .$$

The quantity $\frac{1}{2}m_0 v_k{}^2$ is, of course, the classical Newtonian kinetic energy. For the special case $v_k=0$, (5.4) reduces to

(5.5) $$E_0 = c^2 m_0$$

which is called the rest energy of P.

Still another form for expressing E, which is equivalent to both (5.1) and (5.3) is

(5.6) $$E = m_0 c^3 \, \Delta t_k / \Delta \tau ,$$

where $\Delta\tau$ is defined by (2.19). This form follows from (2.6), (2.19), (3.3) and (5.1) since

$$(5.7)\quad E=mc^2=\frac{m_0c^3}{(c^2-v_k^{\,2})^{\frac{1}{2}}}=m_0c^3\frac{\Delta t_k}{(c^2(\Delta t_k)^2-(\Delta x_k)^2)^{\frac{1}{2}}}\quad.$$

Finally, we note that various relationships can be derived which relate energy and momentum. Thus, for example, (3.1) and (5.1) imply

$$(5.8)\qquad p_kc^2 = Ev_k\quad,$$

while (3.2), (3.3) and (5.1) imply

$$(5.9)\qquad E^2 = p_k^{\,2}c^2 + m_0^{\,2}c^4\quad.$$

Identity (5.9) implies immediately that conservation of momentum yields conservation of energy, which is why no special attention is directed toward the question of energy conservation. In relativistic mechanics, energy conservation is a direct consequence of momentum conservation.

6. THE MOMENTUM-ENERGY VECTOR. For all practical purposes, (2.2), (2.3) and the restricted type of motion under consideration require analysis only of x_k and t_k whenever one studies the 4-vector (x_k,y_k,z_k,t_k). Thus, we restrict attention now to the 2-vector (x_k,t_k) which maps under the Lorentz transformation into (x_k',t_k'). Also, thus far we have not placed any emphasis on a particular set of measurement units. In this connection, then, we will now be relatively specific as follows. Let

$$(6.1)\qquad E^* = E/c^2$$

be a normalized energy in the sense that the units of E^*, by (5.1), are units of mass. Attention will be directed to E^*, rather than to E.

Our present purpose is to show that the number couple (p_k,E^*), where p_k is given by (3.1) and E^* is given by (6.1), is, indeed, a vector, called the momentum-energy vector, in the sense that

(p_k, E^*) maps under the Lorentz transformation like (x_k, t_k). Specifically, from (2.1) and (2.4), we wish to show that

(6.2) $$p_k' = c(p_k - uE^*)/(c^2-u^2)^{\frac{1}{2}}$$

and

(6.3) $$E^{*\prime} = (c^2E^* - up_k)/(c(c^2-u^2)^{\frac{1}{2}}) \quad .$$

Now, from (2.14), (3.2)-(3.4), and (6.1),

$$\begin{aligned} p_k' &= m'v_k' \\ &= \frac{cm_0}{(c^2-{v_k}^2)^{\frac{1}{2}}} \; \frac{c^2-uv_k}{c(c^2-u^2)^{\frac{1}{2}}} \; \frac{c^2(v_k-u)}{c^2-uv_k} \\ &= mc(v_k-u)/(c^2-u^2)^{\frac{1}{2}} \\ &= c(mv_k-E^*u)/(c^2-u^2)^{\frac{1}{2}} \quad , \end{aligned}$$

which establishes (6.2). Then, from (2.14), (3.4), (3.5), (5.1) and (6.1),

$$\begin{aligned} E^{*\prime} &= m' \\ &= cm_0/(c^2-{v_k'}^2)^{\frac{1}{2}} \\ &= cm_0(c^2-uv_k)/(c(c^2-u^2)^{\frac{1}{2}}(c^2-{v_k}^2)^{\frac{1}{2}}) \\ &= m(c^2-uv_k)/(c(c^2-u^2)^{\frac{1}{2}}) \\ &= (c^2E^*-up_k)/(c(c^2-u^2)^{\frac{1}{2}}) \quad , \end{aligned}$$

which establishes (6.3).

7. CONCLUSIONS. We have shown in Sections 2-6 how to formulate the basic physical concepts of special relativity using only arithmetic processes. In particular, differences and difference quotients played a major role. Attention was restricted, for

simplicity, to a very special class of particle and rocket frame motions, but, even so, all the basic consequences related to linear momentum, energy, symmetry, and momentum-energy vectors were shown to be deducible within this arithmetic framework. The major implications are that continuity and limit concepts are shown to be unnecessary for the development of special relativity, while the resulting arithmetic formulation is already in the form necessary for dynamical problems to be solved by high-speed digital computers.

As indicated in the introduction, subsequent papers will deal with more general particle and rocket frame motions.

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