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COMPUTER SCIENCES DEPARTMENT
University of Wisconsin
1210 West Dayton Street
Madison, Wisconsin 53706

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SIMPLIFICATION AND IMPROVEMENT OF A NUMERICAL METHOD FOR NAVIER-STOKES PROBLEMS

by

David Schultz
&
Donald Greenspan

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Abstract

Previously, a viable numerical method for the Navier-Stokes equations was developed and applied to two-dimensional, steady state problems, to three-dimensional, axially symmetric, steady state problems, and to a class of nonsteady problems which had steady state solutions. The method applied for all Reynolds numbers. Among other things, it required the construction of a double sequence of stream and vorticity functions and an appropriate selection of smoothing parameters to assure convergence . Both these complexities are eliminated in the method of this paper. Moreover, illustrative examples show that the new method is faster than the previous one and more accurate for physically sensitive problems.

CONTENTS

1. Introduction

In this paper we will develop a new method for the numerical solution of a class of Navier-Stokes problems. This new method is a significant improvement over the one applied by D. Greenspan to a variety of temperature independent problems for arbitrary Reynolds number [3,4,5,6,7] and extended by D. Schultz [10] to problems which include temperature dependence. The new method has been found to be approximately ten times faster than the original method. In addition, the new method requires less computer storage, does not require smoothing, and has been found to be more accurate for physically sensitive problems.

2. General Description

The numerical method to be developed is a finite difference method and is applicable to nonlinear coupled systems of differential equations similar in structure to Navier-Stokes equations which have a "stream-vorticity" formulation.

In the previous method, we would have started with some initial numerical estimate of the solution of the system. Then the algebraic system of equations which approximated only the first differential equation would have been solved completely. Next, this new solution would have been smoothed (averaged) with the initial approximation. Using these results the next differential equation would have been solved and the new solution averaged with its initial approximation. This process would be repeated for all differential equations of the system, in order. After all equations were solved, one would then return to the first equation and repeat the process. This step-by-step iteration for each equation continued until the results converged to within some tolerance.

Analytically, the above numerical method was shown to be convergent for the biharmonic problem (Reynolds number zero) by J. Smith [11].

In the new method we have found that convergence can be increased by a factor of ten (or more) by solving all equations simultaneously. In this manner, we will eliminate the need for smoothing and reduce the amount of computer core required. For example, if we have n coupled differential equations to solve, the old method required $2n + 1$ storage arrays while the new method requires only $n + 1$ arrays. Since the step size h is limited by the amount of computer core available we can now work with a much smaller h and thus increase the accuracy of our results.

The general procedure described above will be illustrated next, in detail, by considering several problems of physical interest.

3. The Eddy Problem in a Rectangular Cavity

This problem is defined over a rectangle with interior R and boundary S. The vertices of the rectangle are taken to be (0,0), (a,0), (0,b) and (a,b). The equations of motion are

$$\Delta\psi = -\omega \tag{3.1}$$

$$\Delta\omega + \Re\left(\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}\right) = 0, \tag{3.2}$$

where ψ is the stream function, ω is the vorticity and $\Re$ is the Reynolds number. On S the boundary conditions are

$$\psi = 0, \quad \frac{\partial\psi}{\partial x} = 0; \qquad \text{on} \qquad x = 0 \tag{3.3}$$

$$\psi = 0, \quad \frac{\partial\psi}{\partial y} = 0; \qquad \text{on} \qquad y = 0 \tag{3.4}$$

$$\psi = 0, \quad \frac{\partial\psi}{\partial x} = 0; \qquad \text{on} \qquad x = a \tag{3.5}$$

$$\psi = 0, \quad \frac{\partial\psi}{\partial y} = -1; \qquad \text{on} \qquad y = b. \tag{3.6}$$

In general, the boundary value problem (3.1) to (3.6) cannot be solved by existing analytical techniques.

3.1 Numerical Method for the Eddy Problem

Consider the particular case $a = b = 1$ (other cases can be treated in a completely analogous fashion). For a fixed positive integer n, set $h = \frac{1}{n}$, and construct and number the set of interior grid points R_h and the set of boundary grid points S_h in the usual way. Initially, set

$$\psi^{(0)} = C_1 \quad \text{on} \quad R_h$$

$$\omega^{(0)} = C_2 \quad \text{on} \quad R_h + S_h.$$

Then, as in [6], at each point of R_h of the form (h, ih), $i = 2, \ldots, n-2$ approximate (3.3) by

$$\psi(h, ih) = \frac{\psi(2h, ih)}{4}. \tag{3.7}$$

At each point of R_h of the form (ih, h), $i = 1, 2, \ldots, n-1$ approximate (3.4) by

$$\psi(ih, h) = \frac{\psi(ih, 2h)}{4}. \tag{3.8}$$

At each point of R_h of the form $(1-h, ih)$, $i = 2, 3, \ldots, n-2$ approximate (3.5) by

$$\psi(1-h, ih) = \frac{\psi(1-2h, ih)}{4}. \tag{3.9}$$

At each point of R_h of the form $(ih,1-h)$, $i = 1,2,\ldots,n-1$ approximate (3.6) by

$$\psi(ih,1-h) = \frac{h}{2} + \frac{\psi(ih,1-2h)}{4}. \tag{3.10}$$

At each remaining point of R_h write down the following difference analogue of (3.1):

$$-4\psi(x,y) + \psi(x+h,y) + \psi(x,y+h) + \psi(x-h,y) + \psi(x,y-h) = -h^2\omega(x,y). \tag{3.11}$$

As in [6], to obtain ω on the boundary S_h, set

$$\omega(ih,0) = -\frac{2\psi(ih,h)}{h^2} \qquad i = 0,1,2,\ldots,n \tag{3.12}$$

$$\omega(0,ih) = -\frac{2\psi(h,ih)}{h^2} \qquad i = 1,2,\ldots,n-1 \tag{3.13}$$

$$\omega(1,ih) = -\frac{2\psi(1-h,ih)}{h^2} \qquad i = 1,2,\ldots,n-1 \tag{3.14}$$

$$\omega(ih,1) = \frac{2}{h} - \frac{2\psi(ih,1-h)}{h^2} \qquad i = 0,1,\ldots,n. \tag{3.15}$$

Finally, to assure diagonal dominance of the difference equation, at each point (x,y) in R_h set

$$\alpha = \psi(x+h,y) - \psi(x-h,y)$$

$$\beta = \psi(x,y+h) - \psi(x,y-h)$$

and approximate (3.2) by

$$\frac{-4\omega(x,y) + \omega(x+h,y) + \omega(x,y+h) + \omega(x-h,y) + \omega(x,y-h)}{h^2}$$

$$+ R\left(\frac{\psi(x+h,y) - \psi(x-h,y)}{2h}F - \frac{\psi(x,y+h) - \psi(x,y-h)}{2h}G\right) = 0 \quad (3.16)$$

where

$$F = \frac{\omega(x,y+h) - \omega(x,y)}{h} \quad \text{if} \quad \alpha \geq 0 \quad (3.17)$$

$$F = \frac{\omega(x,y) - \omega(x,y-h)}{h} \quad \text{if} \quad \alpha < 0 \quad (3.18)$$

$$G = \frac{\omega(x,y) - \omega(x-h,y)}{h} \quad \text{if} \quad \alpha \geq 0 \quad (3.19)$$

$$G = \frac{\omega(x+h,y) - \omega(x,y)}{h} \quad \text{if} \quad \alpha < 0. \quad (3.20)$$

To start the method applied previously [6], make an initial guess $\psi^{(0)}$ on R_h and $\omega^{(0)}$ on $R_h + S_h$. From these initial guesses two sequences of discrete stream and vorticity functions, called outer iterates, are produced. The iterations performed in getting each one of the outer iterates are called inner iterations. The sequences

$$\begin{array}{ccccccc} \psi^{(0)}, & & \psi^{(1)}, & & \psi^{(2)}, & & \ldots \\ & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\ \omega^{(0)}, & & \omega^{(1)}, & & \omega^{(2)}, & & \ldots \end{array}$$

are calculated in the indicated order until the differences between the $k^{\underline{th}}$ and $(k-1)^{\underline{st}}$ outer iterates agree to within fixed, positive tolerances. Thus, to obtain $\psi^{(1)}$ solve equations (3.7) to (3.11) with $\omega = \omega^{(0)}$ by successive over-relaxation (SOR) and denote this solution by $\bar{\psi}^{(1)}$. Then $\psi^{(1)}$ would be defined by

$$\psi^{(1)} = \rho\psi^{(0)} + (1-\rho)\bar{\psi}^{(1)}$$

where

$$0 \leq \rho \leq 1.$$

Using $\psi = \psi^{(1)}$, one now finds $\bar{\omega}^{(1)}$ on the boundary from equations (3.12) to (3.15) and $\bar{\omega}^{(1)}$ on R_h by solving system (3.16) to (3.20) by SOR. Finally, $\omega^{(1)}$ is defined on $R_h + S_h$ by the smoothing formula

$$\omega^{(1)} = \mu\omega^{(0)} + (1-\mu)\overline{\omega}^{(1)}$$

where

$$0 \leq \mu \leq 1.$$

$\psi^{(2)}$ and then $\omega^{(2)}$ are now calculated in the same fashion as were $\psi^{(1)}$ and $\omega^{(1)}$. This process is continued until the outer iterates have converged, that is, when

$$|\psi^{(k)} - \psi^{(k+1)}| < \varepsilon_\psi \qquad \text{on} \quad R_h$$

and

$$|\omega^{(k)} - \omega^{(k+1)}| < \varepsilon_\omega \qquad \text{on} \quad R_h + S_h,$$

for fixed positive tolerances ε_ψ and ε_ω.

In the new method we solve the system of linear algebraic equations generated by (3.7) to (3.20) simultaneously by SOR with overrelaxation factors r_ψ used in the inner ψ iterations and r_ω used for the inner ω iterations. This procedure eliminates the need for smoothing and the necessary search for adequate smoothing parameters, since there are no longer any outer iterates.

3.2 Results

For $h = 0.1$, $R = 10$, $\varepsilon_\psi = \varepsilon_\omega = 0.001$, $r_\omega = 1.0$, $r_\psi = 1.8$, $\rho = 0.03$, $\mu = 0.95$, $\psi^{(0)} = 0$ on R_h, and $\omega^{(0)} = 0$ on $R_h + S_h$ the old method converged in 36 seconds. The new method with the same parameters converged in one second. For $h = 0.05$, $R = 10$, $\varepsilon_\psi = \varepsilon_\omega = 0.001$, $r_\omega = 1.0$, $r_\psi = 1.8$, $\rho = 0.03$, $\mu = 0.95$, $\psi^{(0)} = 0$ on R_h, and $\omega^{(0)} = 0$ on $R_h + S_h$ the old method did not converge in 2 minutes. The new method converged to $\varepsilon_\psi = \varepsilon_\omega = .0001$ in 6 seconds. For $h = 0.05$, $R = 100000$, $r_\omega = 1.0$, $r_\psi = 1.0$, $\varepsilon_\psi = \varepsilon_\omega = 0.005$, $\rho = 0.03$, $\mu = 0.70$, $\psi^{(0)} = 0$ on R_h, and $\omega^{(0)} = 0$ on $R_h + S_h$ the old method converged in one minute 30 seconds. The new method converged in 16 seconds. The results obtained by the new method are shown in Figures 1 and 2. Note that the r_ψ and r_ω used above were those which were relatively optimal for the old method, so that it is possible that the new method could have been made to converge even faster if a search had been made for new optimal r_ψ and r_ω.

4. Biharmonic Problem

In the recent literature [1,2,9,11] there has been a renewed interest in numerical techniques for biharmonic problems. In our method [9], the biharmonic problem was treated as a system of second order elliptic equations in the spirit of [6]. We will show how to improve the speed of this method by a factor better than 10.

We will not make any attempt to rank the different types of fast new methods now available for the biharmonic problem. The differences between methods, computers, programs, languages, and the like make it difficult, if not impossible, to compare computer times. In addition any comparisons could not take into account the amount of work needed to implement the various procedures. However, it is worth noting that the existence of several, viable fast methods provides the computer user with some means for increasing the reliability of his numerical output against machine and program errors.

The problem to be considered is to find a function $\psi(x,y)$ which is a solution of the biharmonic equation over the region described in Section 3.1. The equation is

$$\Delta\Delta\psi = 0 \tag{4.1}$$

subject to the boundary conditions

$$\psi(x,0) = f_1(x), \quad \psi_y(x,0) = g_1(x); \qquad 0 \leq x \leq 1$$
$$\psi(1,y) = f_2(y), \quad \psi_x(1,y) = g_2(y); \qquad 0 \leq y \leq 1$$
$$\psi(x,1) = f_3(x), \quad \psi_y(x,1) = g_3(x); \qquad 0 \leq x \leq 1$$
$$\psi(0,y) = f_4(y), \quad \psi_x(0,y) = g_4(y); \qquad 0 \leq y \leq 1.$$

We replace equation (4.1), as in [9], by the system of coupled equations.

$$\Delta\psi = -\omega \tag{4.2}$$

$$\Delta\omega = 0 \tag{4.3}$$

Physically, one can interpret ψ and ω from, say, the fluid dynamics point of view, as stream and vorticity functions respectively. Indeed, for $\mathcal{R} = 0$, equations (3.1) and (3.2), reduce to (4.2) and (4.3). Therefore, the new method described in Section 3 can be used for the biharmonic problem also. The only difference will be that the biharmonic problem does not require the special inner boundary equations for the stream function, i.e., (3.7) - (3.10) are not used and (3.11) is applied at all interior grid points.

4.1 Example

Using the boundary conditions $f_1 = x^3$, $f_2 = 2y + 1 - 3y^2$, $f_3 = x^3 + 2x - 3$, $f_4 = -3y^2$, $g_1 = 2x$, $g_2 = 2y + 3$, $g_3 = 2x - 6$ and $g_4 = 2y$, the methods of Section 3 were executed with $\mathcal{R} = 0$, $h = 0.05$, $\rho = 0.2$, $\mu = 0.85$, $\varepsilon_\psi = 10^{-4}$, $\varepsilon_\omega = 10^{-3}$, $r_\psi = 1.8$ and $r_\omega = 1.0$, $\psi^{(0)} = 0$, $\omega^{(0)} = 0$. Using the original method [9], convergence resulted in approximately 6 minutes on the Univac 1108. Using the new method the problem converged in 24 seconds on the same computer. Both methods yielded results which agreed with the exact solution $u = x^3 - 3y^2 + 2xy$ to at least three decimal places.

5. Heated Cavity Flow

Consider now convective flow in a heated cavity [10]. Again, the region of interest will be the square cavity. The equations of motion are

$$\Delta\psi = -\omega \tag{5.1}$$

$$\Delta\theta + \left(\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}\right) = 0 \tag{5.2}$$

$$\Delta\omega + \frac{1}{\sigma}\left(\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}\right) + A\frac{\partial\theta}{\partial y} = 0 \tag{5.3}$$

where ψ is the stream function, ω is the vorticity, θ is a measure of temperature, σ is the Prandtl number, and A is the Rayleigh number. The boundary conditions are

$$\psi = 0, \quad \frac{\partial\psi}{\partial y} = 0, \quad \theta = 0; \qquad \text{on} \quad y = 0 \tag{5.4}$$

$$\psi = 0, \quad \frac{\partial\psi}{\partial x} = 0, \quad \theta = y; \qquad \text{on} \quad x = 1 \tag{5.5}$$

$$\psi = 0, \quad \frac{\partial\psi}{\partial y} = 0, \quad \theta = 1; \qquad \text{on} \quad y = 1 \tag{5.6}$$

$$\psi = 0, \quad \frac{\partial\psi}{\partial x} = 0, \quad \theta = y; \qquad \text{on} \quad x = 0. \tag{5.7}$$

5.1 Numerical Method

To solve the problem numerically approximate (5.1) by (3.11). Equation (5.2) is approximated by

$$\frac{-4\theta_0+\theta_1+\theta_2+\theta_3+\theta_4}{h^2} + \left(\frac{\psi_1-\psi_3}{2h}F - \frac{\psi_2-\psi_4}{2h}G\right) = 0 \tag{5.8}$$

where F and G are defined in (3.17) to (3.20) with ω_n replaced by θ_n. To approximate ω on the boundary use (3.12) to (3.14) with (3.15) replaced by

$$\omega(ih,1) = \frac{-2\psi(ih,1-h)}{h^2}. \tag{5.9}$$

Finally, the approximation to (5.3) is of the form

$$\frac{-4\omega_0+\omega_1+\omega_2+\omega_3+\omega_4}{h^2} + \frac{1}{\sigma}\left(\frac{\psi_1-\psi_3}{2h}F - \frac{\psi_2-\psi_4}{2h}G\right) + A\,\frac{\theta_2-\theta_4}{2h} = 0 \tag{5.10}$$

where F and G are defined as in (3.17) to (3.20). The subscripts in (5.8) and (5.10) refer to the same points as in (3.16).

The old numerical method for generating the numerical solution is analogous to the old method described in Section 3.1. To apply the improved method, equations (3.7) - (3.11), (5.8), (3.12) to (3.14), (5.9) and (5.10) are, in the spirit of Section 3.1, simply solved simultaneously for ψ, θ, ω. Note, of course, that in equation (3.10) the term $\frac{h}{2}$ will not appear for this problem.

5.2 Results

For $h = 0.1$, $A = 500$, $\sigma = 0.73$, $\varepsilon_\psi = \varepsilon_\theta = 0.0002$, $\varepsilon_\omega = 0.001$ $\psi^{(0)} = 0$, $\theta^{(0)} = 0$, $\omega^{(0)} = 0$, the original method converged in 11 seconds. The new method converged in 1 second. For $h = 0.05$, $A = 500$, $\sigma = 0.73$, $\varepsilon_\omega = 0.001$, $\varepsilon_\psi = \varepsilon_\theta = 0.00001$, $\psi^{(0)} = 0$, $\theta^{(0)} = 0$, $\omega^{(0)} = 0$, the original method converged in 2 minutes. The new method converged in 13 seconds. For $h = 0.025$, $A = 10$, $\sigma = 0.73$, $\varepsilon_\omega = 0.002$, $\varepsilon_\psi = \varepsilon_\theta = 0.00003$, $\psi^{(0)} = 0$, $\theta^{(0)} = 0$, $\omega^{(0)} = 0$ the original method converged in 3 minutes. The new method converged in 43 seconds. For $h = 0.1$, $A = 10000$, $\sigma = 0.73$, $\varepsilon_\omega = 0.001$, $\varepsilon_\psi = \varepsilon_\theta = 0.00002$, $\psi^{(0)} = 0$, $\theta^{(0)} = 0$, $\omega^{(0)} = 0$ the original method converged in 25 seconds. The new method converges in 2.4 seconds. For $h = 0.05$, $A = 10000$, $\varepsilon_\omega = 0.001$, $\varepsilon_\psi = 0.00002$, $\psi^{(0)} = 0$, $\theta^{(0)} = 0$, $\omega^{(0)} = 0$, the original method converged in 10 minutes. The new method converged in 36 seconds. For $h = 0.1$, $A = 20000$,

$\sigma = 0.73$, $\varepsilon_\psi = \varepsilon_\theta = 0.000002$, $\varepsilon_\omega = 0.001$, $\psi^{(0)} = 0$, $\theta^{(0)} = 0$, $\omega^{(0)} = 0$, the original method converged in 25 seconds. The new method converged in 2.4 seconds.

The graphs of the case $A = 10000$ are the same for both the old and the new methods and are shown in Figures 3, 4, 5.

6. Rotating Coaxial Disks

The problem we consider now is the steady motion of a viscous, incompressible fluid between two rotating, infinite coaxial disks [8]. The first disk is, in (x,y,z) space, in the plane $z = 0$ with its center at $(0,0,0)$ and has an angular velocity Ω_1. The second disk is in the plane $z = 1$ with its center at $(0,0,1)$ and has an angular velocity Ω_2. If the cylinderical coordinates of (x,y,z) are (r,θ,z), and if the fluid at (x,y,z) has velocity components (u,v,w) then the substitutions

$$u = -\frac{1}{2}rH'(z), \qquad v = rG(z), \qquad w = H(z)$$

transform the dimensionless, steady state Navier-Stokes equations to

$$H'' = M \quad , \qquad 0 \le z \le 1 \tag{6.1}$$

$$G'' + R(GH' - G'H) = 0 \ , \qquad 0 \le z \le 1 \tag{6.2}$$

$$M'' - R(HM' + 4GG') = 0, \qquad 0 \le z \le 1, \tag{6.3}$$

where differentiation is with respect to z. The boundary conditions are

$$G(0) = \Omega_1, \qquad G(1) = \Omega_2 \tag{6.4}$$

$$H(0) = 0 \ , \qquad H(1) = 0 \tag{6.5}$$

$$H'(0) = 0 \ , \qquad H'(1) = 0. \tag{6.6}$$

To obtain the solution we subdivide $0 \le z \le 1$ into n equal parts of length $h = \Delta z = \frac{1}{n}$. Let the points of subdivision be $0 = z_0 < z_1 < z_2 < \dots < z_n = 1$. For convenience define $F_i = F(z_i)$. The original method [8] again approximates H, G, M by generating three sequences $H^{(k)}$, $G^{(k)}$, and $M^{(k)}$ of outer iterates and requires smoothing. The difference equations used were

$$H_{i-1} - 2H_i + H_{i+1} = h^2 M_i^{(k)}, \quad i = 2,3,\dots,n-2 \tag{6.7}$$

$$4H_1 = H_2 \tag{6.8}$$

$$4H_{n-1} = H_{n-2}. \tag{6.9}$$

$$G_{i-1} + [-2 + \Re hH_i^{(k+1)}]G_i + [1 - \Re hH_i^{(k+1)}]G_{i+1}$$
$$= -\tfrac{1}{2}\Re hG_i^{(k)}[H_{i+1}^{(k+1)} - H_{i-1}^{(k+1)}]; \quad \text{if } H_i^{(k+1)} < 0, \ i = 1,2,\ldots,n-1 \tag{6.10}$$

$$[1 + \Re hH_i^{(k+1)}]G_{i-1} + [-2 - \Re hH_i^{(k+1)}]G_i + G_{i+1}$$
$$= -\tfrac{1}{2}\Re hG_i^{(k)}[H_{i+1}^{(k+1)} - H_{i-1}^{(k+1)}]; \quad \text{if } H_i^{(k+1)} \geq 0, \ i = 1,2,\ldots,n-1 \tag{6.11}$$

$$M_0^{(k+1)} = \frac{2H_i^{(k+1)}}{h^2} \tag{6.12}$$

$$M_n^{(k+1)} = \frac{2H_{n-1}^{(k+1)}}{h^2} \tag{6.13}$$

$$M_{i-1} + [-2 + \Re hH_i^{(k+1)}]M_i + [1 - \Re hH_i^{(k+1)}]M_{i+1}$$
$$= 2\Re hG_i^{(k+1)}[G_{i+1}^{(k+1)} - G_{i-1}^{(k+1)}], \quad \text{if } H_i^{(k+1)} < 0, \quad i = 1,2,\ldots,n-1 \tag{6.14}$$

$$[1 + \Re hH_i^{(k+1)}]M_{i-1} + [-2 - \Re hH_i^{(k+1)}]M_i + M_{i+1}$$
$$= 2\Re hG_i^{(k+1)}[G_{i+1}^{(k+1)} - G_{i-1}^{(k+1)}], \quad \text{if } H_i^{(k+1)} \geq 0, \quad i = 1,2,\ldots,n-1. \tag{6.15}$$

For the new method, we simply solve (6.7) and (6.10) – (6.15), simultaneously, with (6.7) extended to the range $i = 1,2,\ldots,n-1$.

6.1 Results

For each of the following cases the original method required approximately 30 seconds of computer time on the Univac 1108. The parameters for the first case are $\Re = 10$, $r_H = 1.8$, $r_G = 1.0$, $r_M = 1.0$, $\varepsilon_H = \varepsilon_G = \varepsilon_M = 0.005$, $H^{(0)} = 0$, $G^{(0)} = 0$, $M^{(0)} = 0$. The parameters for the second case are $\Re = 100$, $r_H = 1.8$, $r_G = 1.0$, $r_M = 1.5$, $\varepsilon_H = \varepsilon_G = 0.001$, $\varepsilon_M = 0.05$, $H^{(0)} = 0$, $G^{(0)} = 0$, $M^{(0)} = 0$. The parameters for the third case are $\Re = 1000$, $r_H = 1.8$, $r_G = 1.1$, $r_M = 1.5$, $\varepsilon_H = 0.005$, $\varepsilon_G = 0.03$, $\varepsilon_M = 0.3$, $H^{(0)} = 0$, $G^{(0)} = 0$, $M^{(0)} = 0$. For all three cases $h = \frac{1}{50}$, $\Omega_1 = 1$, $\Omega_2 = 0$. The new method with $r_G = r_M = 1.0$, $r_H = 1.8$, $\varepsilon_G = \varepsilon_H = \varepsilon_M = 0.001$ converged in a <u>total</u> of 7 seconds for all 3 cases.

For $h = \frac{1}{50}$, $\Omega_1 = 1$, $\Omega_2 = -1$ the old method converged in a maximum of 30 seconds and yielded spurious results for each of the cases $\Re = 10, 100, 1000$, (see [8] for the parameters). In the new method with $\Re = 10$, $h = \frac{1}{50}$, $\Omega_1 = 1$, $\Omega_2 = -1$, $r_G = r_M = 1.5$, $r_H = 1.8$ and $\varepsilon_H = \varepsilon_G = \varepsilon_M = 0.0002$ the problem converged in fewer than 5 seconds. For the case $\Re = 100$ with $r_G = r_M = 1.0$, $r_H = 1.8$, $\varepsilon = 0.0002$ for all parameters the new method converged in approximately 5 seconds. For the case $\Re = 1000$ with $r_G = r_M = 0.8$, $r_H = 1.8$, $\varepsilon = 0.001$ for all variables the method converged in 2.7 seconds. In addition the new method gave correct results (see Fig. [6]).

The fact that the computer results by the old method for the sensitive class of problems with $\Omega_1 = -\Omega_2$ were incorrect has now been established analytically (personal communication from S. V. Parter).

7. Conclusions

The new method has been found to work on all problems tested. It has been found to be faster, and thus more economical, to require less storage space and to free the researcher from the problem of trying to find the correct smoothing parameters necessary for convergence. In addition, because it requires less space one can go to much smaller grid sizes and thereby improve the results. Finally, and perhaps most importantly, it has been found to give more accurate results in at least one area of sensitive problems, that is, the case of counter rotating disks with $\Omega_1 = -\Omega_2$.

It is worth noting, in addition, that other computer centers also are experimenting with the new method described in the paper and have noticed the increased speed of convergence (personal communication from M. Friedman), but no published results are as yet available.

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ψ Contours

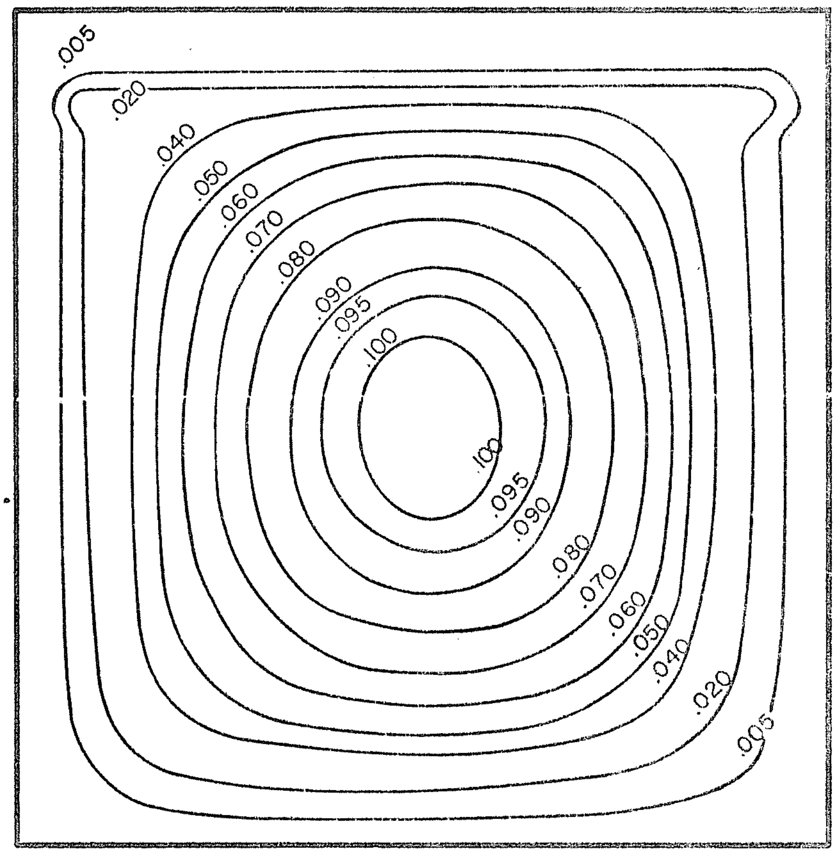

Fig. 1 Eddy Problem, $R = 10^5$

ω Contours

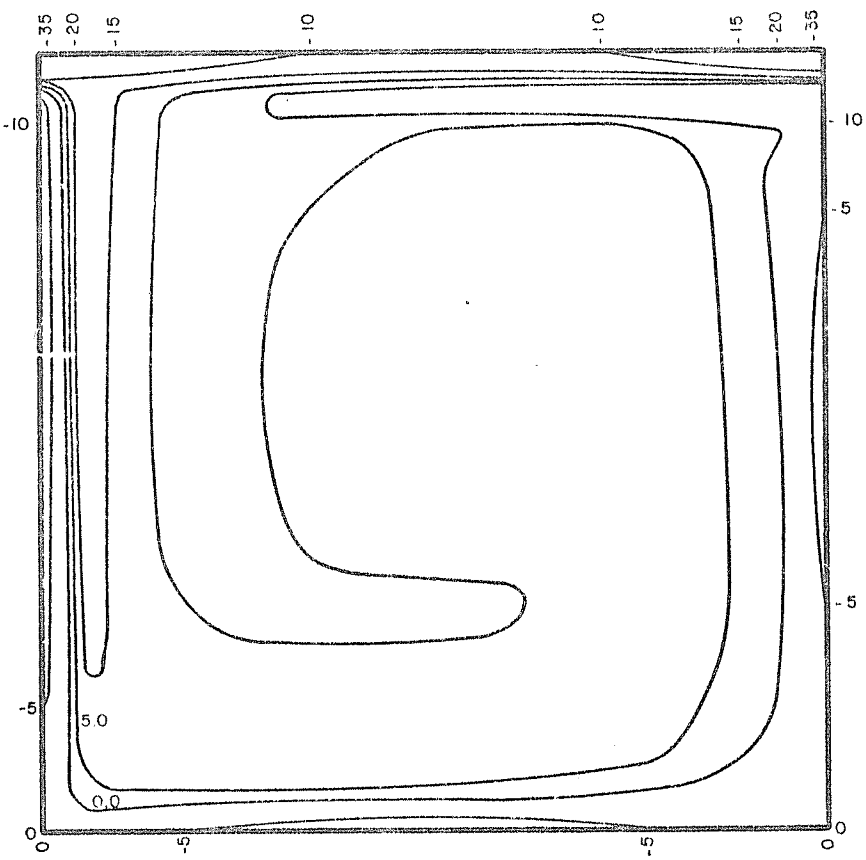

Fig. 2 Eddy Problem, $R = 10^5$

ψ **Contours**

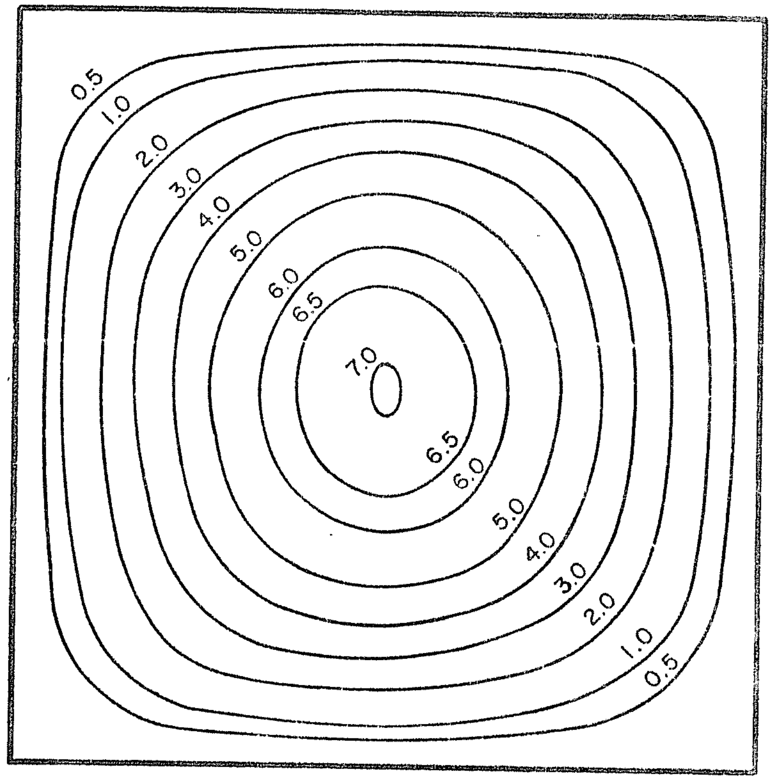

Fig. 3 Heated Cavity, $A = 10^4$

Temperature Contours

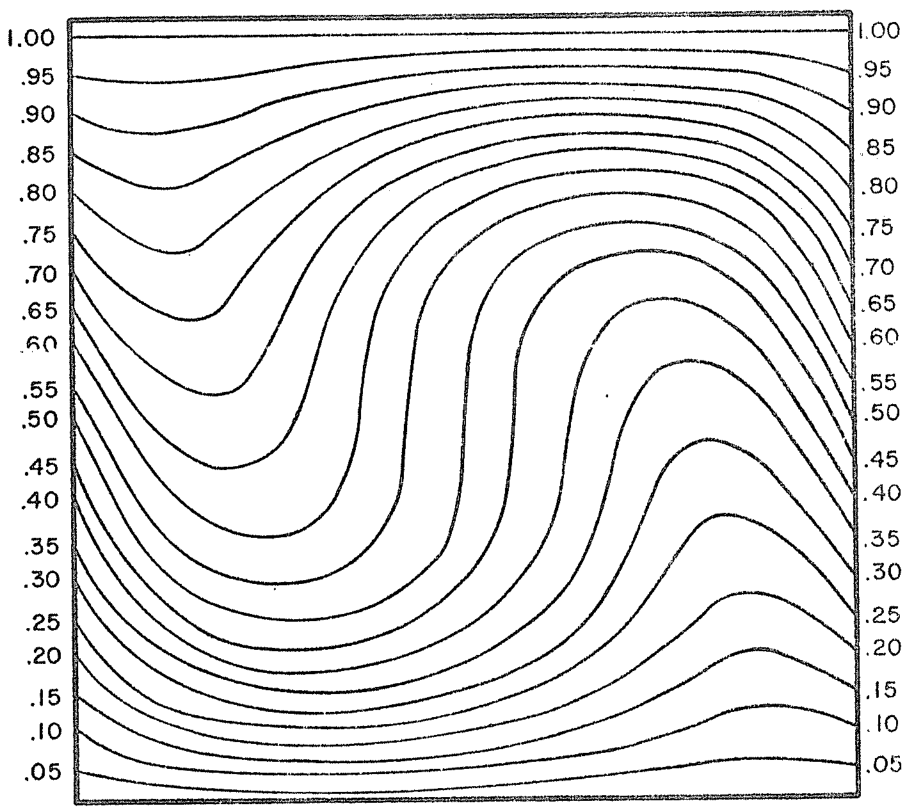

Fig. 4 Heated Cavity, $A = 10^4$

ω Contours

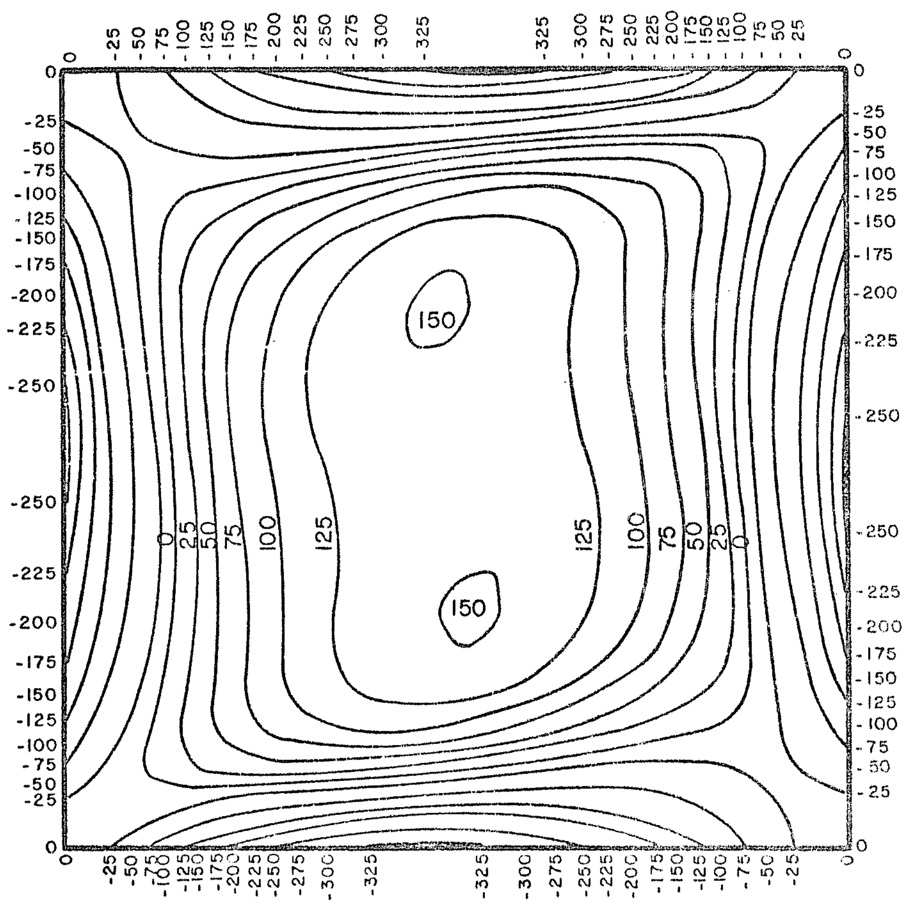

Fig. 5 Heated Cavity, $A = 10^4$

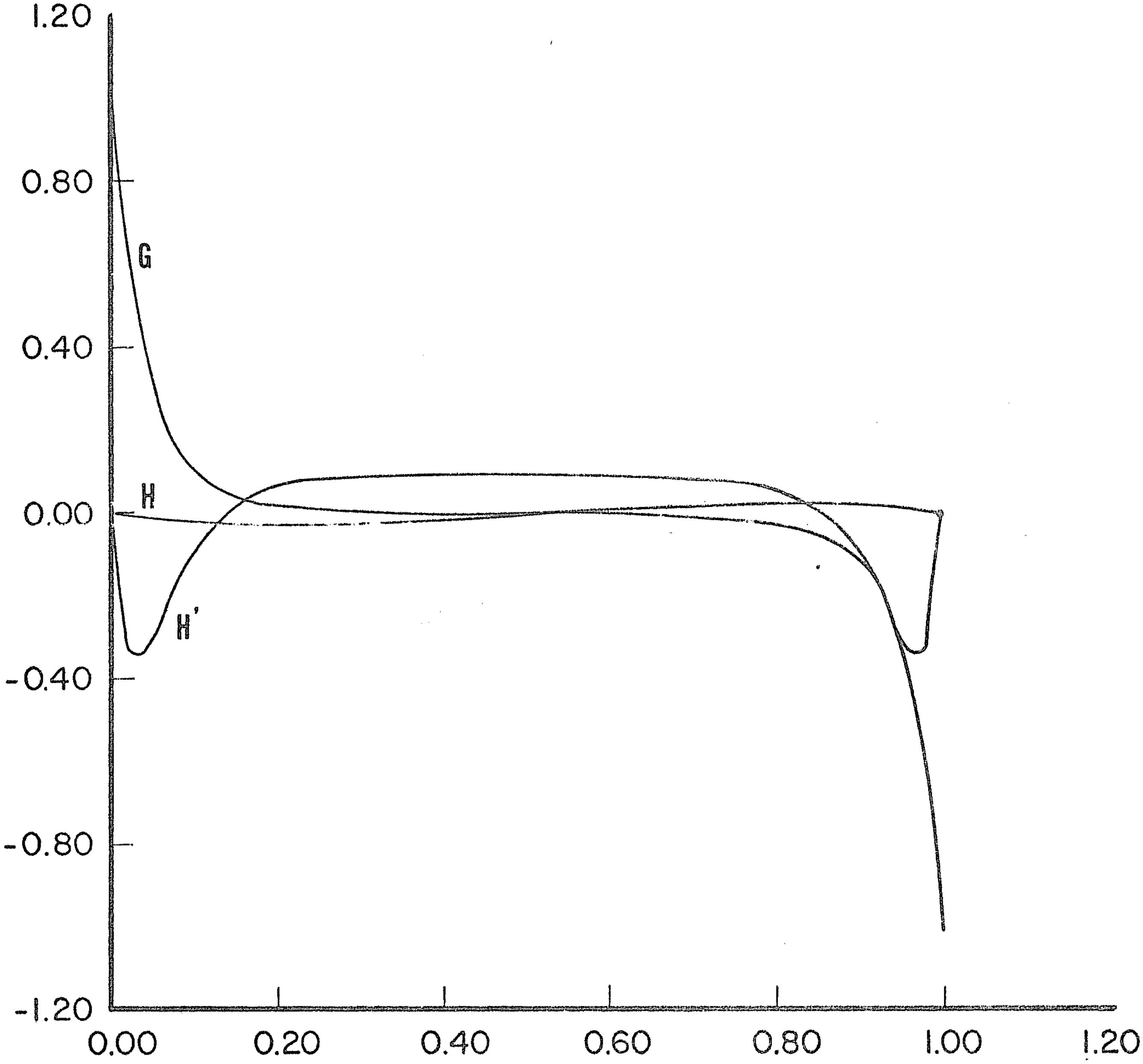

Fig. 6 Disk Problem $\Omega_1 = 1$, $\Omega_0 = -1$, $R = 10^3$

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15. Supplementary Notes

16. Abstracts

Previously, a viable numerical method for the Navier-Stokes equations was developed and applied to two-dimensional, steady state problems, to three-dimensional, axially symmetric, steady state problems, and to a class of nonsteady problems which had steady state solutions. The method applied for all Reynolds numbers. Among other things, it required the construction of a double sequence of stream and vorticity functions and an appropriate selection of smoothing parameters to assure convergence. Both these complexities are eliminated in the method of this paper. Moreover, illustrative examples show that the new method is faster than the previous one and more accurate for physically sensitive problems.

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