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ABSTRACT

Let A be a real $n \times n$ matrix. A is <u>TP</u> (<u>totally positive</u>) if all the minors of A are nonnegative. A has an LU-factorization if $A = LU$ where L is a lower triangular matrix and U is an upper triangular matrix.

The following results are proved:

<u>Theorem 1</u>: A is TP iff A has an LU-factorization such that L and U are TP.

<u>Theorem 2</u>: If A is TP then there exists a TP matrix S and a tridiagonal TP matrix T such that: (i) $TS = SA$; and (ii) the matrices A and T have the same eigenvalues. If A is nonsingular then S is also nonsingular.

<u>Theorem 3</u>: If A is an $n \times n$ matrix of rank m then A is TP iff every minor of A formed from any columns

$$\beta_1, \ldots, \beta_p \quad \text{satisfying} \quad \sum_{i=2}^{p} |\beta_i - \beta_{i-1}| \leq n - m$$

is nonnegative.

<u>Theorem 4</u>: If A is a nonsingular lower triangular matrix then A is TP iff every minor of A formed from consecutive initial columns is nonnegative.

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1. INTRODUCTION

Let A be a real $n \times n$ matix. A is TP (totally positive) if all the minors of A are non-negative. A has an LU-factorization if $A = LU$ where L is a lower triangular matrix and U is an upper triangular matrix.

The main results of the present paper are:

Theorem 1.1

A is TP iff A has an LU-factorization such that L and U are TP.

which is proved in section 4.

Theorem 1.2

If A is TP then there exists a TP matrix S and a tridiagonal TP matrix T such that: (i) $TS = SA$; and (ii) the matrices A and T have the same eigenvalues. If A is nonsingular then S is also nonsingular.

which is proved in section 5, and

Theorem 1.3

If A is an $n \times n$ matrix of rank m then A is TP iff every minor of A formed from any columns

$$\beta_1, \ldots, \beta_p \quad \text{satisfying} \quad \sum_{i=2}^{p} |\beta_i - \beta_{i-1}| \leq n - m$$

is nonnegative.

Theorem 1.4

If A is a nonsingular lower triangular matrix then A is TP iff every minor of A formed from consecutive initial columns is nonnegative.

which are proved in section 6 .

2. NOTATION

We use multi-subscripts (Marcus and Minc [8,p. 9]). If $1 \leq p \leq n$ then $Q^{(p,n)}$ denotes the set of strictly increasing sequences $\alpha = \{\alpha_1,\ldots,\alpha_p\}$ of p integers chosen from $1,\ldots,n$. If $\alpha \in Q^{(p,n)}$ we set

$$d(\alpha) = \sum_{k=1}^{p-1} (\alpha_{k+1}-\alpha_k-1) = \alpha_p - \alpha_1 - (p-1) ,$$

with the convention that $\sum_{k=1}^{0} = 0$.

In particular, $d(\alpha) = 0$ iff the integers α_1 through α_p are consecutive. If $\beta_1,\ldots,\beta_p$ is a sequence of distinct integers chosen from $1,\ldots,n$ then following Koteljanskii [7] we denote by

$$\{\beta_1,\ldots,\beta_p\}_N$$

the corresponding ordered sequence in $Q^{(p,n)}$. The null sequence is denoted by ϕ .

Let $A = (a_{ij})$ be an $n \times m$ matrix. The minor of A formed from rows $\alpha \in Q^{(p,n)}$ and columns $\beta \in Q^{(p,m)}$ will be denoted by $A(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_p)$ or $A(\alpha;\beta)$ or

$$A\begin{pmatrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_p\end{pmatrix} .$$

The matrix A will be said to be TP (totally positive) if all its minors are nonnegative, and will be said to be STP (strictly totally positive) if all its minors are strictly positive. We observe that Rainey and Habetler [16] call TP matrices CNN matrices (completely nonnegative matrices).

The submatrix of A formed from rows $\alpha \in Q^{(p,n)}$ and columns $\beta \in Q^{(q,m)}$ will be denoted by

$A[\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q]$ or $A[\alpha;\beta]$ or

$$A\begin{bmatrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{bmatrix} .$$

When considering submatrices the original numbering of the rows and columns will be used; thus, if $B = A[\alpha;\beta]$, then by row α_i of B will be meant the row of B corresponding to row α_i of A.

The absence of rows or columns of a matrix or elements of a sequence will be indicated by means of a hat. Thus

$$\{\beta_1,\ldots,\hat{\beta}_i,\ldots,\hat{\beta}_j,\ldots,\beta_p\}$$

denotes the sequence obtained by deleting β_i and β_j from β; the possibility that $i = j$ or $i > j$ is not excluded. Similarly

$$\hat{A}\begin{bmatrix} & i & \\ j & & k\end{bmatrix} \quad \text{or} \quad \hat{A}[i;j,k]$$

denotes the submatrix of A obtained by deleting row i and columns j and k, while

$$\hat{A}\begin{bmatrix}\phi\\ j\end{bmatrix} \quad \text{or} \quad \hat{A}[\phi;j]$$

denotes the submatrix of A obtained by deleting column j.

I_r will denote the $r \times r$ unit matrix. If $c \geq 0$,

$$F_r(c) = \begin{pmatrix} 1 & & c \\ & \bigcirc & \\ 0 & & 1\end{pmatrix}$$

will denote the $r \times r$ matrix with ones in the upper left and lower right corners, c in the upper right corner, and zeros elsewhere.

$$G_r = \begin{pmatrix} 0 & & 1 \\ & \bigcirc & \\ 0 & & 1 \end{pmatrix}$$

will denote the $r \times r$ matrix with ones in the upper right and lower right corners, and zeros elsewhere.

$$H_r = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & \bigcirc & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

will denote the $r \times r$ matrix with one's on the diagonal immediately above the main diagonal and zeros elsewhere.

We observe that the matrices I_r, F_r, G_r, and H_r are TP matrices. If U is a block diagonal matrix with the matrices I_r, F_r, G_r, or H_r as diagonal elements then U is a TP matrix. Moreover, premultiplication (postmultiplication) by U is equivalent to performing elementary row (column) operations.

For example, if $c \geq 0$ then

$$U = \text{diag}\,(I_p, F_q(c), I_r)$$

is a TP matrix, and postmultiplication by U is equivalent to adding c times column $p + 1$ to column $p + q$ and then setting columns $p + 1$ through $p + q - 1$ equal to zero.

3. PRELIMINARY RESULTS

The first lemma is a generalization of a determinantal identity which is often used in the theory of TP matrices. We give two proofs: the first proof involves row and column interchanges and the use of the usual identity (Karlin [6, p.8]); the second proof, which is somewhat more elegant, involves the use of the quadratic relations between the subdeterminants of an array (Marcus and Minc [8, p. 15]).

Lemma 3.1

Let $B = (b_{ij})$ be a matrix with n rows and $n+1$ columns, where $n \geq 2$. Let i be any row of B and let j_1, j_2, j_3 be three distinct columns of B. Then

$$\hat{B}\binom{i}{j_1,j_2}\hat{B}\binom{\phi}{j_3}p(j_1,j_3)p(j_2,j_3) +$$

$$+ \hat{B}\binom{i}{j_1,j_3}\hat{B}\binom{\phi}{j_2}p(j_1,j_2)p(j_3,j_2) +$$

$$+ \hat{B}\binom{i}{j_2,j_3}\hat{B}\binom{\phi}{j_1}p(j_2,j_1)p(j_3,j_1) = 0 .$$

where

$$p(j_1,j_2) = \begin{cases} +1, & \text{if } j_2 > j_1 , \\ -1, & \text{if } j_2 < j_1 . \end{cases}$$

First Proof: For the special case $i = n$, $j_1 = 1$, $j_2 = n$, $j_3 = n+1$ the result to be proved takes the form

$$\hat{B}\binom{n}{1,n}\hat{B}\binom{\phi}{n+1} - \hat{B}\binom{n}{1,n+1}\hat{B}\binom{\phi}{n} + \hat{B}\binom{n}{n,n+1}\hat{B}\binom{\phi}{1} = 0 , \qquad (1)$$

which is proved by Karlin [6, p.8].

Now assume that $j_1 < j_2 < j_3$. Denote by C the array obtained from B by

(F1) moving row i to position n

(F2) moving column j_1 to position 1

(F3) moving column j_3 to position $n+1$

(F4) moving column j_2 to position n

Then it follows from (1) that

$$\hat{C}\binom{n}{1,n}\hat{C}\binom{\phi}{n+1} - \hat{C}\binom{n}{1,n+1}\hat{C}\binom{\phi}{n} + \hat{C}\binom{n}{n,n+1}\hat{C}\binom{\phi}{1} = 0. \qquad (2)$$

But,

$$\hat{C}\binom{\phi}{1} = (-1)^{n-i}(-1)^{n-j_2}(-1)^{n+1-j_3}\hat{B}\binom{\phi}{j_1},$$

$$\hat{C}\binom{\phi}{n+1} = (-1)^{n-1}(-1)^{j_1-1}(-1)^{n-j_2}\hat{B}\binom{\phi}{j_3},$$

$$\hat{C}\binom{\phi}{n} = (-1)^{n-i}(-1)^{j_1-1}(-1)^{n+1-j_3}\hat{B}\binom{\phi}{j_2},$$

$$\hat{C}\binom{n}{n,n+1} = (-1)^{j_1-1}\hat{B}\binom{i}{j_2,j_3},$$

$$\hat{C}\binom{n}{1,n+1} = (-1)^{n-j_2}\hat{B}\binom{i}{j_1,j_3},$$

$$\hat{C}\binom{n}{1,n} = (-1)^{n+1-j_3}\hat{B}\binom{i}{j_1,j_2}.$$

To illustrate how these relations are derived, we consider $\hat{C}(\phi;n)$. We can obtain $\hat{B}[\phi;j_2]$ from $\hat{C}[\phi;n]$ as follows:

(R1) moving row n to position i ($n-i$ interchanges),

(R2) moving column 1 to position j_1, (j_1-1 interchanges),

(R3) moving column $n+1$ to position j_3.

To count the number of interchanges in step F3 we observe that, in constructing C, steps F3 and F4 do not involve the first j_1 columns; thus, step F3 requires $n + 1 - j_3$ interchanges. To count the number of interchanges in step R3 we observe that $\hat{C}[\phi;n]$ is obtained from C by deleting column n so that the effect of step F4 is negated and the relative position of the last columns of $\hat{C}[\phi;n]$ is as though step F4 had never occurred; thus, step R3 requires $n + 1 - j_3$ interchanges.

Substituting the above expressions into (2) we find that

$$\hat{B}\binom{i}{j_1,j_2}\hat{B}\binom{\phi}{j_3} - \hat{B}\binom{i}{j_1,j_3}\hat{B}\binom{\phi}{j_2} + \hat{B}\binom{i}{j_2,j_3}\hat{B}\binom{\phi}{j_1} = 0 ,$$

or, equivalently,

$$\hat{B}\binom{i}{j_1,j_2}\hat{B}\binom{\phi}{j_3}p(j_1,j_3)p(j_2,j_3) +$$

$$+ \hat{B}\binom{i}{j_1,j_3}\hat{B}\binom{\phi}{j_2}p(j_1,j_2)p(j_3,j_2) +$$

$$+ \hat{B}\binom{i}{j_2,j_3}\hat{B}\binom{\phi}{j_1}p(j_2,j_1)p(j_3,j_1) = 0 . \qquad (3)$$

Since (3) is invariant to interchanges between the numbers j_1,j_2,j_3, the lemma holds for arbitrary j_1,j_2,j_3.

Second Proof: Let A denote the $(n+1) \times (n+2)$ array obtained from B by adjoining on the right the column vector with zero elements except for a 1 in the i-th position.

Let $s = n$,

$$\alpha = \{1,\ldots,\hat{j}_1,\ldots,\hat{j}_3,\ldots n,n+1,n+2\},$$

$$\beta = \{1,\ldots.,\hat{j}_2,\ldots.,n,n+1\} .$$

Applying the quadratic relations given by Marcus and Minc [8,p.15] and ignoring determinants with two identical columns, we obtain

$$A\begin{pmatrix}1,\ldots\ldots,n\\ \alpha\end{pmatrix} A\begin{pmatrix}1,\ldots\ldots,n\\ \beta\end{pmatrix}$$

$$= A\begin{pmatrix}1,\ldots\ldots\ldots\ldots\ldots\ldots\ldots,n\\ 1,\ldots,\hat{j}_1,\ldots,\hat{j}_3,\ldots,n,n+1,j_1\end{pmatrix} .$$

$$.A\begin{pmatrix}1,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots n\\ 1,\ldots,j_1-1,n+2,j_1+1,\ldots,\hat{j}_2,\ldots,n,n+1\end{pmatrix} +$$

$$+ A\begin{pmatrix}1,\ldots\ldots\ldots\ldots\ldots\ldots\ldots,n\\ 1,\ldots,\hat{j}_1,\ldots,\hat{j}_3,\ldots,n,n+1,j_3\end{pmatrix} .$$

$$. A\begin{pmatrix}1,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,n\\ 1,\ldots,j_3-1,n+2,j_3+1,\ldots,\hat{j}_2,\ldots,n,n+1\end{pmatrix},$$

which, for convenience, we write in the form

$$D_1D_2 = D_3D_4 + D_5D_6 .$$

Now,

$$D_1 = \hat{B}\binom{i}{j_1,j_3}(-1)^{n-i} ,$$

$$D_2 = \hat{B}\binom{\phi}{j_2} ,$$

$$D_3 = \hat{B}\binom{\phi}{j_3}p(j_1,j_3)(-1)^{n-j_1} ,$$

$$D_4 = \hat{B}\binom{i}{j_1,j_2}p(j_1,j_2)(-1)^{n-j_1}(-1)^{n-i} ,$$

$$D_5 = \hat{B}\binom{\phi}{j_1}p(j_3,j_1)(-1)^{n-j_3} ,$$

$$D_6 = \hat{B}\binom{i}{j_3,j_2}p(j_3,j_2)(-1)^{n-j_3}(-1)^{n-i} .$$

Substituting these expressions for $D_1,\dots,D_6$, and cancelling out the common factor $(-1)^{n-i}$, we obtain

$$\hat{B}\binom{i}{j_1,j_3}\hat{B}\binom{\phi}{j_2}$$

$$= \hat{B}\binom{i}{j_1,j_2}\hat{B}\binom{\phi}{j_3}p(j_1,j_3)p(j_1,j_2)(-1)^{2n-2j_1} +$$

$$+ \hat{B}\binom{i}{j_3,j_2}\hat{B}\binom{\phi}{j_1}p(j_3,j_1)p(j_3,j_2)(-1)^{2n-2j_3} .$$

Multiplying by $p(j_1,j_2)p(j_3,j_2)$ and bringing all the terms onto the left hand side, we obtain

$$\hat{B}\binom{i}{j_1,j_3}\hat{B}\binom{\phi}{j_2}p(j_1,j_2)p(j_3,j_2) -$$

$$- \hat{B}\binom{i}{j_1,j_2}\hat{B}\binom{\phi}{j_3}p(j_1,j_3)p(j_3,j_2) -$$

$$- \hat{B}\binom{i}{j_3,j_2}\hat{B}\binom{\phi}{j_1}p(j_3,j_1)p(j_1,j_2) = 0 ,$$

which is equivalent to the desired result.

The next lemma is a generalization of Sylvester's identity (Gantmacher and Krein [5,p.16]) which is stated by Koteljanskii [7,p.7] and Karlin [6,p.3] . For completeness we give a proof.

<u>Lemma 3.2</u>

Let $B = (b_{ij})$ be an $m \times n$ matrix. Let $\alpha \in Q^{(k,m)}$ and $\beta \in Q^{(k,n)}$. Let $C = (c_{st})$ be the $(m-k) \times (n-k)$ matrix defined by

$$c_{st} = B\begin{pmatrix}\alpha_1,\dots,\alpha_k,s\\ \beta_1,\dots,\beta_k,t\end{pmatrix}_N$$

for $1 \le s \le m$, $s \notin \alpha$ and $1 \le t \le n$, $t \notin \beta$.

Let $\mu \in Q^{(r,m)}$ and $\nu \in Q^{(r,n)}$ be such that $\mu \cap \alpha = \phi$ and $\nu \cap \beta = \phi$. Then

$$C(\mu;\nu) = [B(\alpha;\beta)]^{r-1} B\binom{\alpha,\mu}{\beta,\nu}_N .$$

Proof: Let $\tilde{B}$ be obtained from B by reordering rows and columns so that the first k rows and columns of $\tilde{B}$ consist of rows α and columns β of B . Let $\tilde{C} = (\tilde{c}_{st})$ be the $(m-k) \times (n-k)$ matrix defined by

$$\tilde{c}_{st} = B\begin{pmatrix}\alpha_1,\ldots,\alpha_k,s\\ \beta_1,\ldots,\beta_k,t\end{pmatrix} ,$$

for $1 \le s \le m$, $s \notin \alpha$, and $1 \le t \le n$, $t \notin \beta$
From Sylvester's identity applied to $\tilde{B}$,

$$\tilde{C}\begin{pmatrix}\mu_1,\ldots,\mu_r\\ \nu_1,\ldots,\nu_r\end{pmatrix} = [B(\alpha;\beta)]^{r-1} B\begin{pmatrix}\alpha_1,\ldots,\alpha_k,\mu_1,\ldots,\mu_r\\ \beta_1,\ldots,\beta_k,\nu_1,\ldots,\nu_r\end{pmatrix}$$

Let $I(s)$ denote the number of interchanges needed to reorder $\alpha_1,\ldots,\alpha_k,s$ in increasing order. Let $J(t)$ denote the number of interchanges needed to reorder $\beta_1,\ldots,\beta_k,t$ in increasing order. Then

$$\tilde{c}_{st} = (-1)^{I(s) + J(t)} c_{st} .$$

Thus,

$$\tilde{C}\begin{pmatrix}\mu_1,\ldots,\mu_r\\ \nu_1,\ldots,\nu_r\end{pmatrix} = (-1)^{\sum_{i=1}^{r} [I(\mu_i) + J(\nu_i)]} C(\mu;\nu) .$$

Since $\alpha,\beta,\mu,$ and ν are ordered sequences, the number of interchanges needed to reorder $\alpha_1,\ldots,\alpha_k,\mu_1,\ldots,\mu_r$ and

$\beta_1,\ldots,\beta_k,\nu_1,\ldots,\nu_r$ in increasing order are $\sum_{i=1}^{r} I(\nu_i)$ and $\sum_{i=1}^{r} J(\nu_i)$, respectively. Hence

$$B\begin{pmatrix}\alpha,\mu\\ \beta,\nu\end{pmatrix}_N = (-1)^{\sum_{i=1}^{r}[I(\mu_i)+J(\nu_i)]} B\begin{pmatrix}\alpha,\mu\\ \beta,\nu\end{pmatrix} .$$

The lemma follows.

Lemma 3.3

Let $B = (b_{ij})$ be an $m \times (n+1)$ matrix with $m \geq n \geq 2$. Assume that the first n columns of B form a TP matrix, and that the last n columns of B form a TP matrix. Also assume that for some $\alpha \in Q^{(n,m)}$ and some k such that $2 \leq k \leq n$,

$$B\begin{pmatrix}\alpha_1,\ldots\ldots\ldots,\alpha_n\\ 1,\ldots,\hat{k},\ldots,n+1\end{pmatrix} < 0 .$$

Then

(i) Columns $2,\ldots,\hat{k},\ldots,n$ of B have rank $n-2$ and column k of B depends linearly upon columns $2,\ldots,\hat{k},\ldots,n$ of B . (If $n = 2$ then column k of B is zero.)

(ii) All minors of B of order less than n are nonnegative.

Proof: Set

$$C = B\begin{bmatrix}\alpha_1,\ldots,\alpha_n\\ 1,\ldots,n+1\end{bmatrix}$$

so that C consists of rows α of B .

From Lemma 3.1,

$$\hat{C}\binom{\alpha_i}{1,k}\hat{C}\binom{\phi}{n+1} - \hat{C}\binom{\alpha_i}{1,n+1}\hat{C}\binom{\phi}{k} + \hat{C}\binom{\alpha_i}{k,n+1}\hat{C}\binom{\phi}{1} = 0,$$

for $1 \le i \le n$. Since the first n columns of B form a TP matrix the minors $\hat{C}(\alpha_i;1,n+1)$, $\hat{C}(\alpha_i;k,n+1)$, and $\hat{C}(\phi;n+1)$ are nonnegative. Similarly, the minors $\hat{C}(\alpha_i;1,k)$ and $\hat{C}(\phi;1)$ are nonnegative. Finally,

$$\hat{C}(\phi;k) = B\begin{pmatrix}\alpha_1, \ldots\ldots\ldots, \alpha_n \\ 1,\ldots,\hat{k},\ldots,n+1\end{pmatrix} < 0 .$$

We can thus conclude that $\hat{C}(\alpha_i;1,n+1) = 0$ for $1 \le i \le n$; that is, columns $2,\ldots,n$ are linearly dependent. Since $\hat{C}(\phi;k) < 0$ columns $2,\ldots,\hat{k},\ldots,n$ of C are linearly independent. Thus columns $2,\ldots,n$ of C have rank $n - 2$ and column k of C depends linearly upon columns $2,\ldots,\hat{k},\ldots,n$ of C .

We now assert that columns 2 through n of B have rank $n - 2$ and that column k of B depends linearly upon columns $2,\ldots,\hat{k},\ldots,n$, of B . We first consider the case $n = 2$. Then $k = 2$ and C is a 2×3 matrix with zero middle column. Since $B(\alpha_1,\alpha_2;1,3) < 0$ it follows that $b_{\alpha_1,3}$ and $b_{\alpha_2,1}$ are strictly positive. Thus B has the form

$$\begin{matrix} b_{11} & b_{12} & b_{13} \\ \cdot & \cdot & \cdot \\ b_{\alpha_1,1} & 0 & b_{\alpha_1,3} > 0 \\ \cdot & \cdot & \cdot \\ b_{\alpha_2,1} > 0 & 0 & b_{\alpha_2,3} \\ \cdot & \cdot & \cdot \\ b_{m1} & b_{m2} & b_{m3} \end{matrix}$$

Since the last two columns of B form a TP matrix,

$$B(\alpha_1,i;2,3) = -b_{\alpha_1,3}b_{i,2} \geq 0$$

for $i > \alpha_1$, from which it follows that $b_{i,2} = 0$ for $i > \alpha_1$. Since the first two columns of B form a TP matrix,

$$B(i,\alpha_2;1,2) = -b_{i,2}b_{\alpha_2,1} \geq 0$$

for $i < \alpha_2$, from which it follows that $b_{i,2} = 0$ if $i < \alpha_2$. Thus column $k = 2$ of B is zero and the assertion is true.

We now consider the case $n > 2$. It has been shown that columns $2,\ldots,\hat{k},\ldots,n$ of C have rank $n - 2$. Thus there exist α_u and α_v with $\alpha_v > \alpha_u$ such that

$$B\begin{pmatrix}\alpha_1,\ldots,\hat{\alpha}_u,\ldots,\hat{\alpha}_v,\ldots,\alpha_n\\ 2,\ldots\ldots,\hat{k},\ldots\ldots\ldots\ldots,n\end{pmatrix} = g$$

with $g \neq 0$. Since the first n columns of B form a TP matrix, $g > 0$. Let $D = (d_{st})$ be the $[m - (n-2)] \times 3$ matrix defined by

$$d_{st} = B\begin{pmatrix}\alpha_1,\ldots,\hat{\alpha}_u,\ldots,\hat{\alpha}_v,\ldots,\alpha_n,s\\ 2,\ldots\ldots\ldots,\hat{k},\ldots\ldots\ldots,n,t\end{pmatrix}_N$$

for $1 \leq s \leq m$, $s \notin \{\alpha_1,\ldots,\hat{\alpha}_u,\ldots,\hat{\alpha}_v,\ldots,\alpha_n\}$ and $t = 1,k,n+1$. From Lemma 3.2 we have that

$$D\begin{pmatrix}s_1,\ldots,s_r\\ t_1,\ldots,t_r\end{pmatrix} = g^{r-1}\,B\begin{pmatrix}\alpha_1,\ldots,\hat{\alpha}_u,\ldots,\hat{\alpha}_v,\ldots,\alpha_n,s_1,\ldots,s_r\\ 2,\ldots,\hat{k},\ldots\ldots\ldots\ldots\ldots n,t_1,\ldots,t_r\end{pmatrix}_N$$

for $r = 1,2,3$. Since the first n columns of B form a TP matrix, the first two columns of D form a TP matrix. Similarly, the last two columns of D form a TP matrix. Finally,

$$D\begin{pmatrix}\alpha_u,\alpha_v\\ 1,n+1\end{pmatrix} = g\ B\begin{pmatrix}\alpha_1,\ldots\ldots\ldots\ldots,\alpha_n\\ 1,\ldots,\hat{k},\ldots,n+1\end{pmatrix} < 0 .$$

Thus D satisfies the conditions of the lemma, and since part (i) of the lemma has already been proved for $n = 2$ we can conclude that the middle column of D is zero.

Now let B_0 denote the matrix consisting of columns 2 through n of B . Since

$$g = B\begin{pmatrix}\alpha_1,\ldots,\hat{\alpha}_u,\ldots,\hat{\alpha}_v,\ldots\ldots,\alpha_n\\ 2,\ldots\ldots\ldots\ldots\hat{k}\ldots\ldots\ldots,n\end{pmatrix} \neq 0$$

rows $\alpha_1,\ldots,\hat{\alpha}_u,\ldots,\hat{\alpha}_v,\ldots,\alpha_n$ of B_0 are linearly independent. On the other hand,

$$d_{sk} = B\begin{pmatrix}\alpha_1,\ldots,\hat{\alpha}_u,\ldots,\hat{\alpha}_v,\ldots,\alpha_n,s\\ 2,\ldots\ldots\ldots\ldots\ldots\ldots\ldots,n\end{pmatrix}_N = 0$$

for $1 \leq s \leq m$ and $s \notin \{\alpha_1,\ldots,\hat{\alpha}_u,\ldots,\hat{\alpha}_v,\ldots,\alpha_n\}$, so that rows $\alpha_1,\ldots,\hat{\alpha}_u,\ldots,\hat{\alpha}_v,\ldots,\alpha_n$, s of B_0 are linearly dependent for $1 \leq s \leq m$. We conclude that B_0 has row rank $n - 2$.

Using the well-known equivalence between row rank and column rank (Mirsky [13,p.139]) it follows that B_0 has column rank $n - 2$; that is, columns 2 through n of B have rank $n - 2$. Since columns $2,\ldots,\hat{k},\ldots,n$ of C are linearly independent, columns $2,\ldots,\hat{k},\ldots,n$ of B must be linearly independent. The only possibility therefore is that column k of B depends linearly upon columns $2,\ldots,\hat{k},\ldots,n$ of B . The proof of part (i) of the lemma is therefore complete.

We now prove part (ii) of the lemma. If $n = 2$, part (ii) follows immediately from part (i), so that it suffices to consider the case $n > 2$. Suppose that $B(\mu;\nu) < 0$ for some $\mu \in Q^{(q,m)}$, $\nu \in Q^{(q,n+1)}$, with $1 \le q < n$. Since the first n columns of B form a TP matrix, and the last n columns of B form a TP matrix, we conclude that $q \ge 2$ and that $\nu_1 = 1$, $\nu_q = n + 1$. Assume that $2 \le \ell \le n$ and that $\ell \notin \nu$. Set $\tau = \{\nu_1,\ldots,\nu_q,\ell\}_N$; then $\tau_1 = 1 < \ell < \tau_{q+1} = n + 1$. Let C be the $m \times (q+1)$ matrix consisting of columns τ of B. The first q columns of C are a subset of the first n columns of B and thus form a TP matrix. Similarly, the last q columns of C form a TP matrix. Finally,

$$C\begin{pmatrix}\mu_1, \ldots\ldots\ldots\ldots, \mu_q \\ \tau_1, \ldots, \hat{\ell}, \ldots, \tau_{q+1}\end{pmatrix} = B(\mu;\nu) < 0 .$$

Applying part (i) of the lemma to C we conclude that column ℓ of B depends linearly upon columns $\tau_2,\ldots,\hat{\ell},\ldots,\tau_q$ of B. Since this is true for each ℓ such that $2 \le \ell \le n$ and $\ell \notin \nu$, it follows that columns 2 through n of B have rank at most $q - 2 < n - 2$ in contradiction to the fact that, from part (i), columns 2 through n of B have rank $n - 2$. The truth of part (ii) has therefore been established.

<u>Lemma 3.4</u>

Let $B = (b_{ij})$ be an $m \times (n+1)$ matrix with $m \ge n \ge 2$. Assume that all the assumptions of Lemma 3.3 hold. Assume furthermore that

$$B\begin{pmatrix}\alpha_1,\ldots\ldots\ldots,\alpha_n\\ 1,\ldots,\hat{j},\ldots\ldots,n+1\end{pmatrix} \geq 0$$

if $2 \leq j \leq n$ and $j \neq k$.

Then column k of B is zero.

Proof:

If $n = 2$ then the lemma is true by virtue of Lemma 3.3. We therefore assume that $n \geq 3$.

Applying Lemma 3.3 we observe that columns 2 through n of B have rank $n - 2$. Also, column k of B depends linearly upon columns $2,\ldots,\hat{k},\ldots,n$ of B . Finally, all minors of B of order less than n are nonnegative.

Denoting the j-th column of B by $\underline{b}_j$ we have that

$$\underline{b}_k = \sum_{\substack{\ell=2\\ \ell\neq k}}^{n} u_\ell \underline{b}_\ell ,$$

where the u_ℓ are constants.

If all the u_ℓ are zero, then column k of B is zero and the lemma is true. We thus assume that $u_j \neq 0$ for some j satisfying $2 \leq j \leq n$, $j \neq k$.

Then

$$B\begin{pmatrix}\alpha_1,\ldots\ldots\ldots,\alpha_n\\ 1,\ldots,\hat{j},\ldots,n+1\end{pmatrix} = (-1)^{k+1-j} u_j \, B\begin{pmatrix}\alpha_1,\ldots\ldots\ldots,\alpha_n\\ 1,\ldots,\hat{k},\ldots,n+1\end{pmatrix} .$$

The minor on the left of this equation is nonnegative, and the minor on the right is strictly negative so that

$$(-1)^{k-j} u_j > 0 .$$

Since columns $2,\ldots,\hat{k},\ldots,n$ of B have rank $n-2$, there exists $\beta \in Q^{(n-2,m)}$ such that

$$B\begin{pmatrix} \beta_1,\ldots\ldots,\beta_{n-2} \\ 2,\ldots,\hat{k},\ldots,n \end{pmatrix} = g \neq 0 .$$

Since the minors of B of order less than n are nonnegative, $g > 0$.

Now,

$$B\begin{pmatrix} \beta_1,\ldots\ldots\ldots,\beta_n \\ 2,\ldots,\hat{j},\ldots,n \end{pmatrix} = (-1)^{k+1-j} u_j \; B\begin{pmatrix} \beta_1,\ldots\ldots\ldots,\beta_n \\ 2,\ldots,\hat{k},\ldots,n \end{pmatrix}$$

$$= (-1)^{k+1-j} u_j \; g .$$

The minor on the left of this expression is nonnegative and $g > 0$ so that

$$(-1)^{k-j} u_j < 0 .$$

We have thus arrived at a contradiction, and the proof of the lemma is complete.

Remark 3.1

The example

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

with $k = 2$ satisfies the conditions of Lemma 3.3 but column k is not zero.

Remark 3.2

Gantmacher and Krein [5,p.108] prove a related result namely that if B is an $n \times (n+1)$ TP matrix and $\hat{B}(\phi;n) = 0$ then either column $n + 1$ is zero or the first n columns are linearly dependent.

4. LU-FACTORIZATION OF TP MATRICES

The purpose of the section is to prove Theorem 1.1, namely that an $n \times n$ matrix A is TP iff it has an LU-factorization such that L and U are TP matrices. The proof is constructive: the off-diagonal elements of A are successively reduced to zero by row operations (premultiplication by upper triangular matrices) and by column operations (postmultiplication by lower triangular matrices). This approach is a modification of methods used by Rainey and Habetler [16] and Metelmann [11,12].

We begin by considering an $n \times n$ TP matrix A which is of the form

$$\begin{pmatrix} a_{11} & 0 & 0 \ \cdots & 0 & 0 \cdots 0 & 0 & 0 \cdots 0 \\ a_{21} & a_{22} & 0 \ \cdots & 0 & 0 \cdots 0 & 0 & 0 \cdots 0 \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\ a_{k1} & a_{k2} & \cdots a_{kk} & a_{k\ell} & 0 \cdots 0 & a_{km} & 0 \cdots 0 \\ a_{k+1,1} & a_{k+1,2} & \cdots a_{k+1,k} \cdots & a_{k+1,\ell} & \cdots & a_{k+1,m} & \cdots a_{k+1,n} \\ \cdot & \cdot & \cdots \cdot \cdots & \cdot & & & \cdots \cdot \\ a_{n1} & a_{n2} & \cdots a_{nk} \cdots & a_{n\ell} & \cdots & a_{nm} & \cdots a_{nn} \end{pmatrix} \tag{4.1}$$

where

$$1 \le k \le \ell < m \le n .$$

That is,

$$a_{ij} = 0 \quad \text{if} \quad \begin{cases} \quad 1 \le i < k & \text{and} \quad i < j \le n , \\ \text{or} \quad i = k & \text{and} \quad m < j \le n , \\ \text{or} \quad k \le i < n & \text{and} \quad \ell < j < m . \end{cases}$$

In words: the first $k - 1$ rows of A are in lower triangular form, in row k the elements in columns $m + 1$ through n are zero, and columns $\ell + 1$ through $m - 1$ are zero.

Lemma 4.1

Assume that A is a TP matrix of the form (4.1) and that $a_{k\ell} > 0$. Let $\tilde{A}$ be obtained from A by subtracting $a_{km} / a_{k\ell}$ times column ℓ from column m. Thus $A = \tilde{A}U$ where U is the TP upper triangular block diagonal matrix

$$U = \mathrm{diag}(I_{\ell-1}, F_{m-\ell+1}(a_{km}/a_{k\ell}), I_{n-m}) .$$

Then

(i) $\tilde{A}$ is of the form (4.1) and $\tilde{a}_{km} = 0$,

(ii) If A is upper triangular then so is $\tilde{A}$. If A is upper Hessenberg then so is $\tilde{A}$,

(iii) $\tilde{A}$ is TP ,

Proof: Parts (i) and (ii) of the lemma follow immediately from the definition of $\tilde{A}$.

We now prove the hard part of the lemma, part (iii). This is a generalization of a result due to Rainey and Habetler [16,p.125] . Presumably the proof of Rainey and Habetler could be generalized, but instead we prove (iii) by induction upon the following hypothesis:

H(s): $\tilde{A}(\alpha;\beta) \geq 0$ for all $\alpha,\beta \in Q^{(r,n)}$ where $1 \leq r \leq s$.

Hypothesis H(1) is certainly true since $\tilde{a}_{ij} = a_{ij}$ if $j \neq m$, $\tilde{a}_{i,m} = 0$ if $i \leq k$, and

$$\begin{aligned}\tilde{a}_{im} &= a_{im} - a_{i\ell}a_{km}/a_{k\ell} \\ &= \frac{1}{a_{k\ell}} A\begin{pmatrix}k,i\\ \ell,m\end{pmatrix} \geq 0 ,\end{aligned}$$

if $i > k$.

Now assume that $H(s)$ is true for $s \le t-1$ but that $H(t)$ is not true. Then there exist $\alpha,\beta \in Q^{(t,n)}$ such that $\tilde{A}(\alpha;\beta) < 0$. Clearly $m \in \beta$ and $\ell \notin \beta$ since otherwise $\tilde{A}(\alpha;\beta) = A(\alpha;\beta) \ge 0$. Set

$$\nu = \{\beta_1,\ldots,\beta_t,\ell\}_N$$

$$= \{\beta_1,\ldots,\beta_{q-1},\ell,\beta_q,\ldots,\beta_t\} \in Q^{(t+1,n)} .$$

Since $m \in \nu$ and $\ell < m$ we know that $1 \le q \le t$; the cases $q = 1$ and $1 < q < t+1$ will be considered separately.

Case 1: $q = 1$

That is, $\beta_1 > \ell$. Then $\alpha_1 > k$ since otherwise the first row of $\tilde{A}[\alpha;\beta]$ is zero in contradiction to the assumption that $\tilde{A}(\alpha;\beta) < 0$. Set

$$\mu = \{k,\alpha_1,\ldots,\alpha_t\} \in Q^{(t+1,n)} .$$

Expanding $\tilde{A}(\mu;\nu)$ by its first row we obtain that

$$\tilde{A}(\mu;\nu) = a_{k\ell}\,\tilde{A}(\alpha;\beta) .$$

But

$$\tilde{A}(\mu;\nu) = A(\mu;\nu)$$

since $\tilde{A}$ is obtained by subracting a multiple of column $\ell \in \nu$ from column $m \in \nu$. Thus

$$\tilde{A}(\alpha;\beta) = A(\mu;\nu)/a_{k\ell} \ge 0$$

contrary to assumption.

Case 2: $1 < q < t+1$

Let B be the $t \times (t+1)$ array $B = \tilde{A}[\alpha;\nu]$. Then

$$\hat{B}\binom{\phi}{\nu_q} = \tilde{A}(\alpha;\beta) < 0 .$$

By hypothesis $H(t-1)$ all the subdeterminants of B of order less than t are nonnegative. Furthermore, if $j \ne q$ then

$$\hat{B}\binom{\phi}{\nu_j} = \tilde{A}\binom{\alpha_1, \quad \cdots \quad , \alpha_t}{\nu_1,\ldots,\hat{\nu}_j,\ldots,\nu_{t+1}}$$

$$= A\binom{\alpha_1, \quad \cdots \quad , \alpha_t}{\nu_1,\ldots,\hat{\nu}_j,\ldots,\nu_{t+1}} \geq 0 \ .$$

Applying Lemma 3.4 we conclude that column $\nu_q = \ell$ of B is zero; that is, $a_{i\ell} = 0$ if $i \in \alpha$. Hence

$$\tilde{a}_{im} = a_{im} - a_{km}a_{i\ell}/a_{k\ell} = a_{im} \ , \quad \text{if} \quad i \in \alpha,$$

so that

$$\tilde{A}(\alpha;\beta) = A(\alpha;\beta) \geq 0$$

contrary to assumption.

The proof of the lemma is therefore complete.

Lemma 4.2

Assume that A is a TP matrix of the form (4.1) with $\ell = k$, that column k of A is zero, and that A is upper triangular. Let $\tilde{A}$ be obtained from A by interchanging elements $a_{k,k}$ and a_{km} . Let U be the upper triangular TP block diagonal matrix.

$$U = \text{diag}(I_{k-1}, G_{m-k+1}, I_{n-m}) \ .$$

Then $A = \tilde{A}U$ and

(i) $\tilde{A}$ is of the form (4.1) and $\tilde{a}_{km} = 0$,

(ii) $\tilde{A}$ is upper triangular,

(iii) $\tilde{A}$ is TP .

Proof: Assertions (i) and (ii) are obvious, so that we need only prove that $\tilde{A}$ is TP . That is, if $\alpha,\beta \in Q^{(p,n)}$ then we must show that $\tilde{A}(\alpha;\beta) \geq 0$.

If $k \notin \alpha$ then $\tilde{A}[\alpha;\beta] = A[\alpha;\beta]$ and hence $\tilde{A}(\alpha;\beta) = A(\alpha;\beta) \geq 0$.

Since A is upper triangular, of the form (4.1), and has columns k through $m-1$ equal to zero, the only nonzero element in row k of A is a_{km} , and the only nonzero element of row k in $\tilde{A}$ is $\tilde{a}_{k,k} = a_{km}$. Hence, if $k \in \alpha$ but $k \notin \beta$ then $\tilde{A}(\alpha;\beta) = 0$.

Finally, let $k \in \alpha$ and $k \in \beta$ so that $k = \alpha_s = \beta_t$ for appropriate s and t . Set

$$v = \min(s,t), \qquad \sigma = \{1,\ldots,k\} ,$$
$$\alpha^{(1)} = \{\alpha_1,\ldots,\alpha_v\}, \qquad \alpha^{(2)} = \{\alpha_{v+1},\ldots,\alpha_p\},$$
$$\beta^{(1)} = \{\beta_1,\ldots,\beta_v\}, \qquad \beta^{(2)} = \{\beta_{v+1},\ldots,\beta_p\}.$$

Since the first k rows of $\tilde{A}$ are in diagonal form, we have that

$$\tilde{A}(\alpha;\beta) = \tilde{A}(\alpha^{(1)};\beta^{(1)})\ \tilde{A}(\alpha^{(2)};\beta^{(2)}),$$

with the convention that $\tilde{A}(\phi;\phi) = 1$.

Two cases arise:

Case (a): $s \neq t$

Since $\alpha_s = \beta_t$ it follows that $\alpha_v \neq \beta_v$. Remembering that $\tilde{A}[\sigma;\sigma]$ is diagonal we can conclude that $\tilde{A}(\alpha^{(1)};\beta^{(1)}) = 0$ which implies that $\tilde{A}(\alpha;\beta) = 0$.

Case (b): $s = t = v$

Then either $\alpha^{(2)} = \phi$ or $\alpha_{v+1} > \alpha_v = k$; in either case, $\tilde{A}[\alpha^{(2)};\beta^{(2)}] = A[\alpha^{(2)};\beta^{(2)}]$ so that $\tilde{A}(\alpha^{(2)};\beta^{(2)}) = A(\alpha^{(2)};\beta^{(2)}) \geq 0$. Moreover,

$$\tilde{A}(\alpha^{(1)};\beta^{(1)}) = \begin{cases} \tilde{a}_{kk} = a_{km} \geq 0 \ , \text{ if } \ v = 1 \ , \\ \tilde{a}_{kk} \ \ A\begin{pmatrix}\alpha_1,\ldots,\alpha_{v-1} \\ \beta_1,\ldots,\beta_{v-1}\end{pmatrix} \geq 0, \text{ if } \ v > 1 \ . \end{cases}$$

Thus,

$$\tilde{A}(\alpha;\beta) = \tilde{A}(\alpha^{(1)};\beta^{(1)}) \ \tilde{A}(\alpha^{(2)};\beta^{(2)}) \geq 0 \ ,$$

and the proof of the lemma is complete.

We now consider an $n \times n$ TP matrix A of the form

$$\begin{pmatrix} a_{11} & 0 & 0 \ldots 0 & \ldots & 0 & 0 & \ldots & 0 \\ a_{21} & a_{22} & 0 \ldots 0 & \ldots & 0 & 0 & \ldots & 0 \\ . & . & . \ldots . & \ldots & . & . & \ldots & 0 \\ a_{k1} & a_{k2} & \ldots \quad a_{kk} & \ldots & a_{km} & 0 & \ldots & 0 \\ a_{k+1,1} & a_{k+1,2} & \ldots \quad a_{k+1,k} & \ldots & a_{k+1,m} & a_{k+1,m+1} & \ldots & a_{k+1,n} \\ . & . & \ldots \quad . & \ldots & . & . & \ldots & . \\ a_{n1} & a_{n2} & \ldots \quad a_{nk} & \ldots & a_{nm} & . & \ldots & a_{nn} \end{pmatrix}, \quad (4.2)$$

where

$$1 \leq k < m \leq n \ .$$

That is,

$$a_{ij} = 0 \ \text{ if } \begin{cases} 1 \leq i < k \ \text{ and } \ i < j \leq n \ , \\ \text{or } \ i = k \ \text{ and } \ m < j \leq n \ . \end{cases}$$

In words: the first $k - 1$ rows of A are in lower triangular form, and in row k the elements in columns $m + 1$ through n are zero.

<u>Lemma 4.3</u>

If A is a TP matrix of the form (4.2) then one of the following three possibilities must hold:

1. $a_{km} = 0$,
2. $a_{km} > 0$ and A is of the form (4.1) for some ℓ, with $a_{k\ell} > 0$,
3. $a_{km} > 0$ and columns k through $m - 1$ of A are zero.

<u>Proof</u>: Suppose that (i) does not hold so that $a_{km} > 0$. If $i > k$ and $j < m$ then

$$A\begin{pmatrix} k & i \\ j & m \end{pmatrix} \geq 0 ,$$

so that if $a_{kj} = 0$ then $a_{ij} = 0$ for $i > k$. The lemma follows.

<u>Lemma 4.4</u>

Let A be a TP matrix of the form (4.2). Then there exists an upper triangular TP matrix U and a TP matrix $\tilde{A}$ such that:

(i) $A = \tilde{A}U$,

(ii) $\tilde{A}$ is of the form (4.2) and $\tilde{a}_{km} = 0$,

(iii) If A is upper triangular then so is $\tilde{A}$.

<u>Proof</u>: According to Lemma 4.3 three cases can arise. We consider each case separately.

<u>Case 1</u>:

$a_{km} = 0$. Set $U = I$ and $\tilde{A} = A$. All the assertions of the lemma are trivially true.

Case 2:

A is of the form (4.1) with $a_{k\ell} > 0$. Define U and $\tilde{A}$ as in Lemma 4.1. Then the assertions of the lemma follow from Lemma 4.1 .

Case 3:

Columns k through m - 1 of A are zero. It is necessary to proceed differently according to whether or not A is upper triangular.

If A is not upper triangular we define $\tilde{A}$ to be the matrix obtained from A by interchanging columns m and m - 1 . Since column m - 1 of A is zero, $A = \tilde{A}U$ where U is the TP block diagonal matrix

$$U = \mathrm{diag}(I_{m-2}, G_2, I_{n-m}) .$$

It is obvious that $\tilde{A}$ is TP and that assertions (i) and (ii) are satisfied.

If A is upper triangular then the matrix obtained by interchanging rows m and m - 1 may not be upper triangular. In this case we define A and U as in Lemma 4.2 .

We can now consider

Theorem 1.1

A is a TP matrix iff A has an LU-factorization such that L and U are TP .

Proof: It follows from Lemma 4.4 that a TP matrix A can reduced to a lower triangular TP matrix L by postmultiplication by a sequence $U^{(r)}$, r = 1,...,N , of TP upper triangular matrices. Thus A = LU where

$$U = \sum_{r=1}^{N} U^{(r)} .$$

On the other hand, if L and U are given TP matrices, then $A = LU$ is also a TP matrix.

Remark 4.1

Theorem 1.1 was conjectured by Cryer [2, p. 91].

Remark 4.2

The matrices F_r and G_r can be expressed as the product of TP upper tridiagonal matrices. Thus it follows from the proof of Theorem 1.1 that a matrix A is TP iff

$$A = \prod_{r=1}^{N} L^{(r)} \prod_{s=1}^{M} U^{(s)}$$

where each $L^{(r)}$ is a TP lower tridiagonal matrix and each $U^{(s)}$ is a TP upper tridiagonal matrix.

Remark 4.3

We recall (Cryer [2, p. 84]) that a triangular matrix is said to be a ΔSTP matrix if it is a TP matrix and if all "non-trivial" minors are strictly positive.

Let L be a triangular TP matrix which is lower triangular, say. As in Remark 4.2,

$$L = \prod_{r=1}^{N} L^{(r)}$$

where each $L^{(r)}$ is a TP lower tridiagonal matrix. For any $s > 0$ the matrix $L^{(r)} + (1/s)I$ is a nonsingular TP lower tridiagonal matrix, so that $L^{(r)}$ can be approximated arbitrarily closely by nonsingular lower triangular TP matrices. As shown by Cryer [2, p. 87] every nonsingular TP lower triangular matrix can be approximated arbitrarily closely by lower triangular ΔSTP matrices. Thus there exist lower triangular ΔSTP matrices $L_s^{(r)}$ such that $L_s^{(r)} \to L^{(r)}$ as $s \to \infty$. Then

$$L_s = \prod_{n=1}^{N} L_s^{(r)}$$

is a ΔSTP matrix and $L_s \to L$ as $s \to \infty$. We have thus shown (as conjectured by Cryer [2, p. 90]) that a triangular TP matrix can be approximated arbitrarily closely by ΔSTP matrices.

Remark 4.4

Lemma 4.4 provides a very efficient method for determining whether a matrix A with given numerical coefficients is TP . Given A one attempts to construct the sequence of upper triangular matrices $U^{(r)}$, $r = 1,\ldots,N$, which reduce A to a triangular matrix L . One then applies Lemma 4.4 to L^T and attempts to construct the sequence of upper triangular matrices $L^{(s)^T}$, $s = 1,\ldots,M$, which reduce L^T to diagonal form D . If any of the matrices $U^{(r)}$, $(L^{(s)})^T$, D is not TP then A is not TP ; otherwise A is TP .

The number of arithmetic operations required to determine whether A is TP can be computed in the same way that the number of arithmetic operations required to solve n equations by Gaussian elimination is computed (Fox [3]) . To reduce A to the lower triangular matrix L requires

$$\sum_{i=1}^{n-1} i^2 = \frac{1}{6}n(n-1)(2n-1)$$

multiplications and subtractions. To reduce L^T to the diagonal matrix D requires

$$\sum_{i=1}^{n-1} \frac{1}{2}i(i+1) = \frac{1}{6}(n-1)(n)(n+1)$$

multiplications and subtractions. In addition, n^2 sign tests must be made.

When implementing this procedure on a computer one would first scale A so that its coefficients were integers. When eliminating a coefficient a_{km} using column ℓ one would not subtract $a_{km}/a_{k\ell}$ times column ℓ from column m, but multiply column m by $a_{k\ell}$ and then subtract column ℓ. It should be borne in mind that the number of digits in the coefficients may grow as the computation proceeds so that the operation counts given in the preceding paragraph may not give a realistic indication of the amount of computation needed.

<u>Remark 4.5</u>

If A is a TP matrix and $\tilde{A}$ is obtained from A by subtracting a multiple of column ℓ from column m so as to make $\tilde{a}_{1m} = 0$, then $\tilde{A}$ is not necessarily TP. This is illustrated by the example $\ell = 1$, $m = 2$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$$

for which

$$\tilde{A}\begin{bmatrix} 1,2 \\ 2,3 \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} .$$

Remark 4.6

If A is TP with $a_{11} > 0$ and if $\tilde{A}$ is obtained from A by subtracting multiples of column 1 from columns $2,\ldots,n$, then, as shown by Cryer [2, p. 91], $\tilde{A} = AU$ where $\tilde{A}$ is TP but U is not necessarily TP.

5. TRIDIAGONALIZATION OF TP MATRICES

In this section we prove

Theorem 1.2

If A is TP then there exists a TP matrix S and a tridiagonal TP matrix T such that: (i) $TS = SA$; and (ii) the matrices A and T have the same eigenvalues. If A is nonsingular then S is nonsingular.

Theorem 1.2 follows immediately by repeated application of the following lemma:

Lemma 5.1

Let M be an $n \times n$ TP matrix of the form

$$\begin{pmatrix}
m_{11} & m_{12} & 0 & 0 \; \cdots & & & 0 & 0 & 0 & \cdots & 0 \\
m_{21} & m_{22} & m_{23} & 0 & & & 0 & 0 & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdot \; \cdots & & & \cdot & \cdot & \cdot & \cdots & \cdot \\
m_{k-1,1} & m_{k-1,2} & m_{k-1,3} & \cdot \; \cdots & & & 0 & 0 & 0 & \cdots & 0 \\
m_{k1} & m_{k2} & m_{k3} & \cdot \; \cdots & m_{kk} & \cdots & m_{kp} & m_{k,p+1} & 0 & \cdots & 0 \\
m_{k+1,1} & m_{k+1,2} & m_{k+1,3} & \cdot \; \cdots & m_{k+1,k} & \cdots & m_{k+1,p} & m_{k+1,p+1} & m_{k+1,p+2} & \cdots & m_{k+1,n} \\
\cdot & \cdot & \cdot & \cdot \; \cdots & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\
m_{n1} & m_{n2} & m_{n3} & \cdot \; \cdots & m_{nk} & \cdots & m_{np} & \cdot & \cdot & \cdots & m_{nn}
\end{pmatrix} \quad (5.1)$$

where

$$1 \le k < p < n \; .$$

That is,

$$m_{ij} = 0 \quad \text{if} \begin{cases} 1 \le i < k & \text{and} \quad i+1 < j \le n \; , \\ \text{or } i = k & \text{and} \quad p+1 < j \le n \; . \end{cases}$$

Then there exist TP matrices S' and M' such that

(i) $S'M = M'S'$

(ii) M' is of the form (5.1) and $m'_{k,p+1} = 0$,

(iii) if M is upper Hessenberg so is M' .

(iv) if M is nonsingular then S' is nonsingular.

Proof: If $m_{k,p} \neq 0$ then the lemma is proved by Rainey and Habetler [16, p. 124]. If $m_{k,p+1} = 0$ we set $M = M'$ and $S = I$ and the lemma is trivially true.

Finally, if $m_{k,p} = 0$ and $m_{k,p+1} \neq 0$ we define S' to be the TP block diagonal matrix

$$S' = \text{diag}\ (I_{p-1}, H_{n+1-p})$$

and let M' be the matrix obtained by deleting row p and column p of M and then adjoining a zero column at the right and a zero row at the bottom. It is clear that $S'M = M'S'$, while it is shown by Rainey and Habetler [16, p. 124] that M' is TP and has the same eigenvalues as M . (It should be noted that we denote by M' the matrix denoted by $\hat{M}$ by Rainey and Habetler). Finally, since M is TP , $m_{kp} = 0$, and $m_{k,p+1} > 0$ it follows that column p of M is zero. Hence, the fact that S' is singular does not invalidate the lemma.

Theorem 1.2 partially answers the conjecture of Rainey and Habelter [16, p. 123] that if A is TP then there exists a nonsingular TP matrix S and a tridiagonal matrix T with the same eigenvalues as A such that $SA = TS$. This conjecture is not true even if the condition that T have the same eigenvalues as A is removed, as is shown by the following lemma:

Lemma 5.2

Let A be the TP matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A nonsingular TP matrix S and a TP tridiagonal matrix T such that SA = TS do not exist.

<u>Proof</u>: If S and T exist then

$$SA = \begin{pmatrix} 0 & 0 & s_{11} \\ 0 & 0 & s_{21} \\ 0 & 0 & s_{31} \end{pmatrix} = TS = \begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & t_{23} \\ 0 & t_{32} & t_{33} \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix},$$

$$= \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix}, \text{ say .}$$

Since S and T are both TP matrices, they have nonnegative coefficients. Since S is nonsingular

$$s_{11}s_{22}s_{33} = S\binom{1}{1} \quad S\binom{2}{2} \quad S\binom{3}{3} \geq S\binom{123}{123} > 0 ,$$

so that the diagonal elements of S are strictly positive. But

$$w_{11} = t_{11}s_{11} + t_{12}s_{21} = 0 ,$$
$$w_{12} = t_{11}s_{12} + t_{12}s_{22} = 0 ,$$

so that $t_{11} = t_{12} = 0$.

But then,

$$s_{11} = w_{13} = t_{11}s_{13} + t_{12}s_{23} = 0$$

which contradicts the fact that $s_{11} > 0$.

6. DETERMINATAL CRITERIA FOR TOTAL POSITIVITY

The main purpose of this section is to prove Theorems 1.3 and 1.4.

Theorem 1.3

If A is an $n \times n$ matrix of rank m then A is TP iff every minor of A formed from any columns β satisfying $d(\beta) \leq n - m$ is nonnegative.

Proof: We assume that A has rank m and that $A(\alpha;\beta) \geq 0$ for all $\alpha,\beta \in Q^{(q,n)}$, $1 \leq q \leq n$, such that $d(\beta) \leq n - m$. We prove that $A(\alpha;\beta) \geq 0$ for all $\alpha,\beta \in Q^{(r,n)}$, $1 \leq r \leq n$.

The proof proceeds by induction upon the following hypothesis:

H1(p): $A(\alpha;\beta) \geq 0$ for all $\alpha,\beta \in Q^{(r,n)}$, $1 \leq r \leq p$.

Hypothesis H1(1) simply states that the elements of A are nonnegative, which is certainly true. We assume that H1(q-1) is true for some $q \geq 2$ and prove that H1(q) is true.

The proof of H1(q) proceeds by induction upon the hypothesis.

H2(s): $A(\alpha;\beta) \geq 0$ for all $\alpha,\beta \in Q^{(q,n)}$ satisfying $d(\beta) \leq s$.

Hypothesis H2(n-m) is true by assumption. We assume that H2(t-1) is true for some $t \geq n - m + 1$ and prove H2(t).

If H2(t) is not true then there exist $\alpha,\beta \in Q^{(q,n)}$ such that $d(\beta) = t$ and $A(\alpha;\beta) < 0$. Thus there are t columns of A, $\mu_1,\mu_2,\ldots,\mu_t$, say, which lie between columns β_1 and β_q and do not belong to β.

Let $\nu \in Q^{(q+1,n)}$ be obtained by adjoining μ_ℓ to β so that for some k satisfying $1 < k < q + 1$ we have $\nu_k = \mu_\ell$ and

$$\beta = \{\nu_1, \ldots, \hat{\nu}_k, \ldots, \nu_{q+1}\} .$$

Clearly $d(\nu) = d(\beta) - 1 = t - 1$.

We denote by B the $n \times (q+1)$ array

$$B = A\begin{bmatrix} 1, \ldots, n \\ \nu_1, \ldots, \nu_{q+1} \end{bmatrix}$$

consisting of columns ν of A .

By H1(q-1) all minors of B of order less than q are nonnegative. Since

$$d(\nu_1, \ldots, \nu_q) \le d(\nu_1, \ldots, \nu_{q+1}) \le t - 1$$

it follows from H2(t-1) that

$$B\begin{pmatrix} \alpha_1, \ldots, \alpha_q \\ \nu_1, \ldots, \nu_q \end{pmatrix} \ge 0 .$$

Thus the first q columns of B form a TP matrix. Similarly, the last q columns of B form a TP matrix. Finally,

$$B\begin{pmatrix} \alpha_1, \ldots\ldots\ldots\ldots, \alpha_q \\ \nu_1, \ldots, \hat{\nu}_k, \ldots, \nu_{q+1} \end{pmatrix} = A(\alpha;\beta) < 0 .$$

Applying Lemma 3.3 we conclude that column ν_k of B depends linearly upon columns $\nu_2, \ldots, \hat{\nu}_k, \ldots, \nu_q$ of B . That is, column μ_ℓ of A depends linearly upon columns $\beta_2, \ldots, \beta_q$. Since this is true for $1 \le \ell \le t$ we conclude that A has rank at most

$$n - t \leq n - (n-m+1) = m - 1$$

which contradicts the assumption that A is of rank m. It follows that $H2(t)$ is true and the proof of the theorem is complete.

We now prove

Theorem 1.4

If A is a nonsingular lower triangular matrix then A is TP iff every minor of A formed from consecutive initial columns is nonnegative.

Proof: We assume that A is nonsingular lower triangular matrix satisfying

$$A\begin{pmatrix}\alpha_1,\ldots,\alpha_q \\ 1,\ldots,q\end{pmatrix} \geq 0$$

for all $\alpha \in Q^{(q,n)}$, $1 \leq q \leq n$.

Since A is lower triangular and nonsingular the diagonal elements of A are nonzero. Since

$$A\begin{pmatrix}1,\ldots,q \\ 1,\ldots,q\end{pmatrix} = \prod_{i=1}^{q} a_{ii} \geq 0$$

the diagonal elements of A are strictly positive.

Suppose that $\alpha,\beta \in Q^{(q,n)}$ and that $d(\beta) = 0$. We assert that $A(\alpha;\beta) \geq 0$. To prove this assertion we observe that if $\alpha_1 < \beta_1$ then $A(\alpha;\beta) = 0$ because A is lower triangular. If $\alpha_1 \geq \beta_1$ and $\beta_1 = 1$ then

$$A(\alpha;\beta) = A\begin{pmatrix}\alpha_1,\ldots,\alpha_q \\ 1,\ldots,q\end{pmatrix} \geq 0$$

by assumption. Finally, if $\alpha_1 \geq \beta_1$ and $\beta_1 > 1$ set $\tau = \{1,\ldots,\beta_1 - 1\}$. Then

$$A\begin{pmatrix}\tau,\alpha\\ \tau,\beta\end{pmatrix} = A\begin{pmatrix}1,\ldots,\beta_1-1,\alpha_1,\ldots,\alpha_q\\ 1,\;\ldots\;\ldots\;,\beta_q\end{pmatrix}$$

$$= A(\alpha;\beta)\prod_{r=1}^{\beta_1-1} a_{rr}$$

$$\geq 0\ ,$$

since $d(\tau,\beta) = 0$, from which it follows that $A(\alpha;\beta) \geq 0$.

The theorem now follows immediately from Theorem 1.3.

Remark 6.1

It is an immediate consequence of Theorem 1.3 that if A is nonsingular then A is TP iff every minor of A formed from consecutive rows is nonnegative.

Remark 6.2

Theorem 1.3 was suggested by the result (Gantmacher and Krein [5, p. 299]) that A is strictly totally positive iff every minor formed from consecutive rows and consecutive columns is strictly positive. Theorem 1.4 was conjectured by Cryer [2, p. 86] .

Remark 6.3

The matrix

$$A = \begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

is nonsingular and all minors formed from consecutive initial columns are nonnegative, but A is not TP . It is also the case that all minors formed from consecutive rows and consecutive columns are nonnegative.

The matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

is nonsingular and all minors formed from consecutive initial columns or consecutive initial rows are nonnegative, but A is not TP .

The matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

is nonsingular and lower triangular and all minors formed from consecutive initial columns and consecutive rows are nonnegative, but A is not TP .

Remark 6.4

If A is singular then all elements of A may be zero except for two elements $a_{ij} = 1$ and $a_{rs} = 1$ say. In this case A is TP iff

$$A \begin{pmatrix} i,r \\ j,s \end{pmatrix}_N \geq 0 .$$

It is therefore clearly not sufficient to examine minors of A formed from consecutive rows or consecutive columns.

Remark 6.5

It is not clear how to generalize Theorem 1.4 to the case of singular matrices. The matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

is lower triangular and of rank 3. Furthermore, $A(\alpha;\beta) \geq 0$ for all $\alpha,\beta \in Q^{(q,n)}$ such that $\beta_1 = 1$. But A is not TP .

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13. ABSTRACT

Let A be a real $n \times n$ matrix. A is TP (totally positive) if all the minors of A are non-negative. A has an LU-factorization if $A = LU$ where L is a lower triangular matrix and U is an upper triangular matrix.

The following results are proved:

Theorem 1: A is TP iff A has an LU-factorization such that L and U are TP. Theorem 2: If A is TP then there exists a TP matrix S and a tridiagonal TP matrix T such that: (i) $TS = SA$; and (ii) the matrices A and T have the same eigenvalues. If A is nonsingular then S is also nonsingular. Theorem 3: If A is an $n \times n$ matrix of rank m then A is TP iff every minor of A formed from any columns

$$\beta_1, \ldots, \beta_p \quad \text{satisfying} \quad \sum_{i=2}^{p} |\beta_i - \beta_{i-1}| \le n - m$$

is nonnegative. Theorem 4: If A is a nonsingular lower triangular matrix then A is TP iff every minor of A formed from consecutive initial columns is nonnegative.

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