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ABSTRACT

The paper studies the boundary-value problem arising from the behaviour of a fluid occupying the region $-1 \leq x \leq 1$ between two rotating disks, rotating about a common axis perpendicular to their planes when the disks are rotating with the same speed Ω_0 but in the opposite sense. The equations which describe the axially symmetric similarity solutions of this problem are

$$\varepsilon H^{iv} + HH''' + GG' = 0$$

$$\varepsilon G'' + HG' - H'G = 0$$

with the boundary conditions

$$H(\pm 1) = H'(\pm 1) = 0$$

$$G(-1) = -1, \; G(1) = 1$$

where $\varepsilon = \nu / 2\Omega_0$ and ν is the kinematic viscosity.

The existence of an odd solution $\langle H(x,\varepsilon), G(x,\varepsilon) \rangle$ is established. This particular solution satisfies many special conditions, e.g., $G'(x,\varepsilon) > 0$. Moreover, precise estimates are obtained on the size and behaviour of the solution as $\varepsilon \downarrow 0$.

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I. INTRODUCTION

Following the approach of T. von Kármán [6], G. K. Batchelor [1] and K. Stewartson [15] considered the fluid motion between two rotating disks, rotating about a common axis perpendicular to their planes. The particular case when the two disks are rotating with the same speed but in the opposite sense has generated a great deal of interest. Batchelor conjectured that, in the limit of large Reynolds number (small kinematic viscosity), the main body of the fluid is separated into two parts, rotating with opposite angular velocities with a narrow central transition layer through which the fluid adjusts from one rate of rotation to the other. On the other hand, Stewartson conjectured that the main body of the fluid is only slightly disturbed at large Reynolds number.

Numerical computations have been carried out by Lance and Rogers [8], C. E. Pearson [13] and D. Greenspan [4], but the evidence given by these is conflicting. K. Kuen Tam [16] has applied the method of matched asymptotic expansions to suggest the non-uniqueness of the

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solution. J. Serrin [14] has commented on the computational results and the mathematical difficulty of the problem.

Let the disks be placed at $x = -1$ and $x = 1$ and rotating about the x-axis with angular velocities Ω_0 and $-\Omega_0$ respectively. Let q_r, q_θ, and q_x denote the velocity components in cylindrical polar coordinates (r, θ, x). Following Batchelor [1] we write

$$q_\theta = \frac{1}{2} r Q(x) , \quad q_x = h(x) , \quad q_r = -\frac{r}{2} h'(x)$$

where the prime denotes differentiation with respect to x. The continuity equation is satisfied exactly and the equations of motion become

$$\text{1.1)} \qquad \begin{cases} \nu h^{iv} - hh''' - QQ' = 0 \\ \nu Q'' - hQ' + h'Q = 0 \end{cases}$$

where ν is the kinematic viscosity. The associated boundary conditions are

$$\text{1.2)} \qquad \begin{cases} h(-1) = h'(-1) = h(1) = h'(1) = 0 \\ Q(-1) = 2\Omega_0 , \quad Q(1) = -2\Omega_0 . \end{cases}$$

Let

$$\text{1.3)} \qquad \begin{cases} \varepsilon = \nu/2\Omega_0 \\ H(x,\varepsilon) = -h(x)/2\Omega_0 \\ G(x,\varepsilon) = -Q(x)/2\Omega_0 . \end{cases}$$

Then the equations (1.1), (1.2) become

$$(1.4)\qquad \begin{cases} \varepsilon H^{iv} + HH''' + GG' = 0 , & -1 \le x \le 1 , \\ \varepsilon G'' + HG' - H'G = 0 , & -1 \le x \le 1 , \end{cases}$$

and

$$(1.5)\qquad \begin{cases} H(-1) = H'(-1) = H(1) = H'(1) = 0 \\ G(-1) = -1 , \quad G(1) = 1 . \end{cases}$$

In this report we prove the existence of an odd solution $\langle H(x,\varepsilon), G(x,\varepsilon)\rangle$ of (1.4), (1.5) for all $\varepsilon > 0$. This particular solution, in addition to being odd, satisfies the following additional conditions:

$$H(x,\varepsilon) \le 0 \quad \text{for} \quad 0 \le x \le 1 ;$$

$H'(x,\varepsilon)$ has precisely one zero in $0 < x < 1$ with $H'(0,\varepsilon) < 0$;

$H''(x,\varepsilon)$ has precisely one zero in $0 < x < 1$ with $H''(0,\varepsilon) = 0$ and $H''(1,\varepsilon) < 0$;

$H'''(x,\varepsilon)$ has precisely one zero in $0 < x < 1$ with $H'''(0,\varepsilon) > 0$ and $H'''(1,\varepsilon) < 0$;

$$(1.6)\qquad G'(x,\varepsilon) > 0 \quad \text{for} \quad -1 \le x \le 1 ;$$

$$G''(x,\varepsilon) \ge 0 \quad \text{for} \quad 0 \le x \le 1 .$$

Moreover, for <u>any</u> odd solution which satisfies (1.6), and so for our particular solution, we can obtain precise estimates on the size and behaviour of the solution as $\varepsilon \downarrow 0$. These results are given in detail in the statement of Theorem II in §3 below, but we remark here that the behaviour so found is consistent with Stewartson's predictions and not with Batchelor's. At the same time, the absence of any uniqueness proof amongst our results means that a solution of Batchelor's type is not completely ruled out, although our investigations of the equations enable us to say that certain behaviours are just not consistent with the equations, and that in particular the solution obtained numerically by Greenspan [4] is impossible. We return to this point at the end of Lemma 2.2.

Although much work has been done on existence theory for swirling flow above one rotating disc (see, for example, [3], [5], [7], [10]-[12], [17]), this paper seems to be the first contribution towards existence for the two-disk problem. The proof depends very delicately on the precise boundary conditions that arise in the case of an odd solution, and we do not believe that it will be a triviality to extend it to the case where the rotations are no longer exactly equal and opposite.

The existence theorem is established in §2, and the discussion of behaviour as $\varepsilon \downarrow 0$ is carried through in §3.

We are indebted to Carl de Boor for many patient hours of fruitful discussion on this problem.

2. EXISTENCE THEORY

We restrict our attention to the subinterval $[0,1]$ and seek a pair of functions $\langle f, g \rangle$ which satisfy

$$(2.1) \quad \begin{cases} f^{iv} + ff''' + gg' = 0, & 0 \le x \le 1, \\ g'' + fg' - f'g = 0, & 0 \le x \le 1, \end{cases}$$

$$(2.2) \quad \begin{cases} f(0) = f''(0) = f'(1) = f(1) = 0 \\ g(0) = 0, \quad g(1) = R > 0. \end{cases}$$

If one has such a pair of functions, then, with $\varepsilon = 1/R$, the functions

$$H(x, \varepsilon) = \begin{cases} -\varepsilon\, f(-x), & x < 0, \\ \varepsilon\, f(x), & x > 0, \end{cases}$$

$$G(x, \varepsilon) = \begin{cases} -\varepsilon\, g(-x), & x < 0, \\ \varepsilon\, g(x), & x > 0, \end{cases}$$

satisfy equations (1.4), (1.5).

We shall make frequent use of the function

$$(2.3) \quad m(x) = f''(x)$$

which satisfies the equations

$$(2.4) \quad \begin{cases} \varepsilon\, m'' + fm' = -gg', & 0 \le x \le 1, \\ m(0) = 0. \end{cases}$$

The main result of this section is the following fundamental existence theorem.

<u>Theorem I</u>: <u>There is a pair of functions</u> $\langle f, g \rangle$ <u>which satisfy equations (2.1), (2.2). Moreover</u>

A) $$-\frac{1}{2}R^2 \le f \le 0 ,$$

B) $$|f'| \le \frac{1}{2}R^2 .$$

C) There are three distinguished points x_1, x_2, x_3, with

$$0 < x_1 < x_2 < 1 , \qquad 0 < x_3 < x_2 < 1$$

such that

C.1) $f'(x) < 0 ,$ $\qquad 0 \le x < x_1 ,$

C.2) $f'(x) > 0 ,$ $\qquad x_1 < x < 1 ,$

C.3) $0 < f''(x) \le \frac{1}{2}R^2 ,$ $\qquad 0 < x < x_2 ,$

C.4) $-\frac{1}{2}R^2 e^{\frac{1}{2}R^2} \le f''(x) < 0 ,$ $\qquad x_2 < x \le 1 ,$

C.5) $0 < f'''(x) \le \frac{1}{2}R^2 ,$ $\qquad 0 \le x < x_3 ,$

C.6) $-\frac{1}{2}R^2 e^{\frac{1}{2}R^2} \le f'''(x) < 0 ,$ $\qquad x_3 < x \le 1 .$

Moreover, we also have

D) $0 < g(x) < R ,$ $\qquad 0 < x < 1 ,$

E) $0 < g'(x) \le e^{\frac{1}{2}R^2}(R + \frac{1}{2}R^3) ,$ $\qquad 0 \le x \le 1 ,$

F) $0 \le g''(x) \le \{e^{\frac{1}{2}R^2}(R + \frac{1}{2}R^3) + R\}\frac{R^2}{2} ,$ $\qquad 0 \le x \le 1 .$

The proof follows from a series of lemmas and the Schauder fixed-point theorem.

Lemma 2.1: Let $\bar{g}(x) \in C'[0,1]$ and satisfy

2.5) $$\bar{g}(0) = 0\,, \quad \bar{g}(1) = R\,, \quad \bar{g}'(x) \geq 0\,.$$

Let $\bar{f}(x) \in C'[0,1]$. Then there is a unique $\tilde{f} \in C^4[0,1]$ such that

2.6) $$\begin{cases} \tilde{f}^{iv} + \bar{f}\tilde{f}''' = -\bar{g}\bar{g}'\,, & 0 \leq x \leq 1\,, \\ \tilde{f}(0) = \tilde{f}''(0) = \tilde{f}(1) = \tilde{f}'(1) = 0\,. & \end{cases}$$

Moreover

a) $$\tilde{f}(x) \leq 0\,.$$

c) There exist two distinguished points $\tilde{x}_1$, $\tilde{x}_2$ with

$$0 < \tilde{x}_1 < \tilde{x}_2 < 1\,,$$

such that

c.1) $\tilde{f}'(x) < 0\,,\qquad 0 \leq x < \tilde{x}_1\,,$

c.2) $\tilde{f}'(x) > 0\,,\qquad \tilde{x}_1 < x < 1\,,$

c.3) $0 < \tilde{f}''(x)\,,\qquad 0 < x < \tilde{x}_2\,,$

c.4) $\tilde{f}''(x) < 0\,,\qquad \tilde{x}_2 < x \leq 1\,.$

Further, if $\bar{g}'(x) > 0$, then there exists a third distinguished point $\tilde{x}_3$ with $0 < \tilde{x}_3 < \tilde{x}_2 < 1$ and

c.5) $0 < f'''(x)\,,\qquad 0 \leq x < \tilde{x}_3\,,$

c.6) $\tilde{f}'''(x) < 0\,,\qquad \tilde{x}_3 < x \leq 1\,.$

(The lettering of the various properties corresponds to that in Theorem I. There is no property (b).)

Proof: Let

2.7) $$\tilde{m} = \tilde{f}''(x) .$$

Then

2.8) $$\tilde{m}'' + \bar{f}\tilde{m}' = -\bar{g}\bar{g}' \leq 0 , \qquad 0 \leq x \leq 1 .$$

Integrating (2.7) and using the boundary conditions $\tilde{f}(0) = \tilde{f}'(1) = \tilde{f}(1) = 0$ we obtain

2.9) $$\int_0^1 t\tilde{m}(t)\, dt = 0 .$$

If we set

$$\bar{u}(x) = \int_0^x \bar{f}(t)\, dt ,$$

then we can write (2.8) in the form

$$(e^{\bar{u}} m')' = -e^{\bar{u}} \bar{g}\bar{g}' \leq 0 ,$$

so that $e^{\bar{u}} m'$ is non-increasing, and if $\tilde{m}'(0) \leq 0$, we must have $\tilde{m}(t) \leq 0$ for all $t \in [0,1]$. Hence from (2.9) either $\tilde{f}''(t) \equiv 0$ (which does not satisfy (2.6)), or $\tilde{f}'''(0) > 0$ and $\tilde{f}''(1) < 0$, and then, if $\bar{g}' > 0$, we can assert that there are two distinguished points $\tilde{x}_2$, $\tilde{x}_3$ with $0 < \tilde{x}_3 < \tilde{x}_2 < 1$ and such that (c.3), (c.4), (c.5), (c.6) hold. Indeed, the existence of a well-defined $\tilde{x}_2$ holds when we have only $\bar{g}' \geq 0$.

The above arguments also prove the uniqueness of $\tilde{f}$. For if there are two solutions, then the difference $(\tilde{f}_1 - \tilde{f}_2)''$ satisfies the equation (2.8) with $\bar{g}\bar{g}'$ replaced by zero. Thus $(\tilde{f}_1 - \tilde{f}_2)'' \equiv 0$ (and by integration $\tilde{f}_1 - \tilde{f}_2 \equiv 0$, as we want),

or

$$(\tilde{f}_1 - \tilde{f}_2)'''(0) > 0 ,$$

and of course

$$(\tilde{f}_2 - \tilde{f}_1)'''(0) > 0 ,$$

which is impossible.

It remains to prove (a), (c.1) and (c.2). Integrating (2.7) backwards from $x = 1$ we obtain

$$\tilde{f}'(x) > 0 , \qquad \tilde{x}_2 \leq x < 1 , \tag{2.10}$$

and after one more integration

$$\tilde{f}(x) < 0 , \qquad \tilde{x}_2 \leq x < 1 . \tag{2.11}$$

Consider (2.7) on the interval $[0, \tilde{x}_2]$. The maximum principle implies

$$\tilde{f}(x) < 0 , \qquad 0 < x \leq \tilde{x}_2 ,$$

and there is a unique point $\tilde{x}_1 \in (0, \tilde{x}_2)$ such that (c.1), (c.2) hold. Thus the lemma is proven.

<u>Lemma 2.2</u>: <u>Let</u> $\tilde{f}(x) \in C^2[0,1]$ <u>satisfy</u> $\tilde{f}(0) = 0$ <u>and the properties (a), (c), (c.1), (c.2), (c.3), (c.4) of Lemma 2.1. Then there exists a unique function</u> $\tilde{g}(x) \in C^2[0,1]$ <u>which satisfies</u>

$$\begin{cases} \tilde{g}'' + \tilde{f}\tilde{g}' - \tilde{f}'\tilde{g} = 0 , & 0 \leq x \leq 1 , \\ \tilde{g}(0) = 0 , \ \tilde{g}(1) = R . \end{cases} \tag{2.12}$$

<u>Moreover</u>

d) $0 \le \tilde{g}(x) \le R$, $0 \le x \le 1$,

e) $0 < \tilde{g}'(x)$, $0 \le x \le 1$,

f) $0 \le \tilde{g}''(x)$, $0 \le x \le 1$.

Proof: For every real α let $S(x, \alpha)$ be the unique solution of the initial value problem

2.13)
$$\begin{cases} S'' + \tilde{f} S' - \tilde{f}' S = 0 , & 0 \le x \le 1 , \\ S(0) = 0, \quad S'(0) = \alpha . \end{cases}$$

The existence of $S(x, \alpha)$ follows from standard existence theorems and the linearity of (2.13). Let

$$\tilde{u}(x) = \int_0^x \tilde{f}(t) dt .$$

Then after differentiation of equation (2.13) we have

2.14)
$$\begin{cases} S''' + \tilde{f} S'' = \tilde{f}'' S \\ (e^{\tilde{u}} S'')' = e^{\tilde{u}} \tilde{f}'' S . \end{cases}$$

Since $\tilde{f}(0) = S(0) = 0$, we have

$$S''(0, \alpha) = 0 .$$

Integrating (2.14) we obtain

2.15)
$$e^{\tilde{u}(x)} S''(x) = \int_0^x e^{\tilde{u}(t)} \tilde{f}''(t) S(t) dt .$$

Suppose $\alpha > 0$; then there is an interval $[0, \bar{x}]$ of greatest length in which $S(t) \ge 0$. If

$$x \le \min(\bar{x}, \tilde{x}_2) ,$$

then the representation (2.15) and the fact that $\tilde{f}''(x) \geq 0$ for $0 \leq x \leq \tilde{x}_2$ shows that

$$e^{\tilde{u}(x)} S''(x) \geq 0 , \quad S''(x) \geq 0 .$$

Thus

2.16) $$S'(x) \geq \alpha > 0 , \quad 0 \leq x \leq \min(\bar{x}, \tilde{x}_2) ,$$

and

2.17) $$S(x) \geq \alpha x , \quad 0 \leq x \leq \min(\bar{x}, \tilde{x}_2) .$$

Since either $\bar{x} = 1$ or $S(\bar{x}) = 0$, we may conclude that

2.18) $$\bar{x} > \tilde{x}_2 .$$

Now we rewrite (2.13) as

2.19) $$S'' = -\tilde{f} S' + \tilde{f}' S , \quad \tilde{x}_1 \leq x \leq 1 ,$$

and recall that $\tilde{x}_1 < \tilde{x}_2$. Thus on the interval $[\tilde{x}_1, \tilde{x}_2]$ the right-hand-side of (2.19) is positive, and by continuity there is a largest number y with $\tilde{x}_2 < y \leq 1$ such that

2.20a) $$S(x, \alpha) > 0 , \quad 0 < x < y ,$$

2.20b) $$S'(x, \alpha) > 0 , \quad 0 \leq x < y ,$$

2.20c) $$S''(x, \alpha) > 0 , \quad 0 < x < y .$$

If $y < 1$ then (2.20c) implies that inequalities (2.20a) and (2.20b) apply at $x = y$. Hence returning to (2.19) we see that (2.20c) holds at $x = y$ also.

We have thus shown that $\alpha > 0$ implies that S, S', S'' are non-negative on $[0,1]$. By virtue of the linearity we see that $\alpha < 0$ implies that S, S', S'' are non-positive on $[0,1]$. Moreover, if $\alpha = 0$, then $S(x,0) \equiv 0$. Thus, since $R > 0$, if $S(x,\alpha)$ is to satisfy the conditions on $\tilde{g}(x)$, we must require $\alpha > 0$. At the same time $S(1,\alpha)$ is a continuous (indeed, linear) function of α with

$$S(1,0) = 0 , \qquad S(1,R) > R .$$

Thus there exists exactly one $\alpha \in (0,R)$ such that $S(1,\alpha) = R$.

Remark: This lemma, or, more correctly, the method of proof of this lemma applied to the computations of Greenspan [4] on the interval $[1/2, 1]$ shows that those computations are inconsistent with the problem.

Lemma 2.3: Let $\bar{g}(x) \in C'[0,1]$ and satisfy (2.5). Let $\bar{f}(x) \in C'[0,1]$ and $\bar{f}(x) \leq 0$. Let $\tilde{f}(x)$ be the unique solution of the linear boundary-value problem (2.6). Then

(a') $$-\frac{1}{2}R^2 \leq \tilde{f} \leq 0 ,$$

(b') $$|\tilde{f}'| \leq \frac{1}{2}R^2 ,$$

(c'.3) $$0 < \tilde{f}''(x) \leq \frac{1}{2}R^2 , \qquad 0 < x < \tilde{x}_2 ,$$

(c'.5) $$0 < \tilde{f}'''(x) \leq \frac{1}{2}R^2 , \qquad 0 \leq x < \tilde{x}_3 .$$

(If we have only $\bar{g}' \geq 0$, the point $\tilde{x}_3$ may not be uniquely defined. In this case, $\tilde{x}_3$ is to be interpreted for the purposes of this lemma as any point for which $\tilde{f}'''(x) = 0$, and the inequality in (c'.5) is replaced by equality.

Proof: Let

2.21) $$\bar{u}(x) = \int_0^x \bar{f}(t)dt \le 0 ,$$

and

2.22) $$\bar{u}'(x) = \bar{f}(x) \le 0 .$$

We rewrite equation (2.8) as

$$(\tilde{m}' e^{\bar{u}})' = -\frac{1}{2} e^{\bar{u}}(\bar{g}^2)' ,$$

and then

2.23) $$\tilde{m}'(x) = \frac{1}{2} \int_x^{\tilde{x}_3} e^{[\bar{u}(t)-\bar{u}(x)]}[\bar{g}^2(t)]'dt .$$

If $0 \le x \le \tilde{x}_3$, then in the integrand of (2.23) we have

$$\bar{u}(t) - \bar{u}(x) \le 0 .$$

Thus, for $0 \le x \le \tilde{x}_3$,

2.24) $$\tilde{m}'(t) \le \frac{1}{2}[\bar{g}^2(\tilde{x}_3) - \bar{g}^2(x)] \le \frac{1}{2} R^2 ,$$

which proves (c'.5). Since

$$\tilde{m}(\tilde{x}_3) = \max\{\tilde{m}(x) ; \quad 0 \le x \le \tilde{x}_2\}$$

we also have (c'.3). For any $x \in [0, \tilde{x}_2]$, we have

$$|\tilde{f}'(x)| = |\int_{\tilde{x}_1}^{x} \tilde{m}(t)dt| \le \frac{1}{2} R^2 .$$

However,

$$\tilde{f}'(\tilde{x}_2) = \max\{\tilde{f}'(x) ; \quad \tilde{x}_2 \le x \le 1\} ,$$

and so (b') is established. Integration then gives (a').

Proof of Theorem I: Let

$$F = \{\bar{f}(x) \in C^1[0,1];\ -\frac{1}{2}R^2 \leq \bar{f} \leq 0,\ \bar{f}(0) = \bar{f}(1) = 0\},$$

$$G = \{\bar{g}(x) \in C^1[0,1];\ \bar{g}(0) = 0,\ \bar{g}(1) = R,\ \bar{g}'(x) \geq 0\}.$$

Let $\bar{f} \in F$, $\bar{g} \in G$.

Let $\tilde{f}(x)$ be the unique solution of the linear boundary-value problem (2.6). Using (2.23) we see that, in addition to (a'), (b'), (c'.3), (c'.5) we have

$$\text{(c'.6)} \qquad -\frac{1}{2}R^2 e^{\frac{1}{2}R^2} \leq \tilde{f}'''(x) < 0, \qquad \tilde{x}_3 < x \leq 1,$$

and hence

$$-\frac{1}{2}R^2 e^{\frac{1}{2}R^2} \leq \tilde{f}''(x) < 0, \qquad \tilde{x}_2 < x \leq 1.$$

(If we have only $\bar{g}'(x) \geq 0$, we interpret $\tilde{x}_3$ as in Lemma 2.3, and replace the inequality in (c'.6) by equality.) Let $\tilde{g}(x)$ be the unique solution of the linear boundary-value problem (2.12), and let $\tilde{u}(x)$ be defined as in Lemma 2.2. Then we have

$$(e^{\tilde{u}}\tilde{g}')' = e^{\tilde{u}}\tilde{f}'\tilde{g},$$

$$\tilde{g}'(x) = e^{-\tilde{u}(x)}\tilde{g}'(0) + e^{-\tilde{u}(x)}\int_0^x e^{\tilde{u}(t)}\tilde{f}'(t)\tilde{g}(t)dt.$$

Since $\tilde{g}'(0) \leq R$ we have

$$0 < \tilde{g}'(x) \leq e^{\frac{1}{2}R^2} \cdot (R + \frac{1}{2}R^3)$$

and

$$\tilde{g}''(x) = -\tilde{f}(x)\tilde{g}'(x) + \tilde{f}'(x)\tilde{g}(x) \leq \frac{1}{2}R^2\{e^{\frac{1}{2}R^2}(R + \frac{1}{2}R^3) + R\} .$$

Thus the mapping ϕ defined on $F \otimes G$ by

$$\phi\{\bar{f}, \bar{g}\} = \{\tilde{f}, \tilde{g}\}$$

maps the convex set $F \otimes G$ into the compact subset of $F \otimes G$ satisfying all the conditions A, B, C, C.1, C.2, C.3, C.4, C.5, C.6, D, E, F of Theorem I, except that x_3 may not be well-defined. Theorem I follows from the Schauder fixed-point theorem [2], and since for the fixed point we do have $g' > 0$, we also have x_3 well-defined.

Remark: Applying Lemmas 2.1, 2.2 and 2.3 we see that any solution $\langle f, g \rangle$ of equations (2.1), (2.2) which also satisfies

$$g'(x) \geq 0 , \qquad 0 \leq x \leq 1 ,$$

satisfies all the estimates of Theorem I.

3. ASYMPTOTIC BEHAVIOR AS $\varepsilon \downarrow 0$

In this section we return to the functions of physical interest $\langle H(x,\varepsilon), G(x,\varepsilon)\rangle$ which satisfy equations (1.4), (1.5). Whenever we refer to a solution $\langle H, G\rangle$ we shall mean an odd solution, so that we may restrict ourselves to the interval $[0,1]$ with the boundary conditions

$$
(3.1)\qquad \begin{cases} H(0) = H''(0) = H'(1) = H(1) = 0 \\ G(0) = 0 , \qquad G(1) = 1 . \end{cases}
$$

We shall further restrict ourselves to solutions possessing the property that $G' \geq 0$. By the remark at the close of the preceding section, it follows that any such solution, suitably normalized, satisfies all the estimates of Theorem 1, and in particular it ensures the existence of the three distinguished points x_1, x_2, x_3 which are the zeros of H', H'', H'''.

As has already been indicated in the introduction, there may well be solutions of (1.4) and (1.5), even odd solutions, which do not satisfy $G' \geq 0$; there may even be (although we do not consider it likely) more than one odd solution which does satisfy $G' \geq 0$; but what we can assert is that any odd solution satisfying $G' \geq 0$ has a behaviour as $\varepsilon \downarrow 0$ which can be very precisely identified and which is in accordance with Stewartson's prediction [15] rather than Batchelor's [1] in that it displays a boundary layer at $x = 1$ and nowhere else.

We shall use the usual O- and o- notation. In addition, if

X, Y are two functions of ε, we shall use

$$X \succ Y$$

to mean that $Y = o(X)$ as $\varepsilon \downarrow 0$; and we shall use

$$X \asymp Y$$

to mean that both $Y = O(X)$ and $X = O(Y)$ as $\varepsilon \downarrow 0$. The letter K will be used to denote various positive constants not necessarily the same at each appearance, but always independent of ε or of any other variables under discussion; if we wish to indicate that K depends on some parameter, say ε, then we will write $K(\varepsilon)$.

With this notation, we can state the theorem on the behaviour of $\langle H, G\rangle$ as $\varepsilon \downarrow 0$ as follows.

<u>Theorem II</u>: <u>Any odd solution of (1.4) and (1.5) which has</u> $G' \geq 0$ <u>has the following behaviour as</u> $\varepsilon \downarrow 0$.

(i) $x_1, x_2, x_3 \to 1$ <u>with</u>

$$\varepsilon^{\frac{1}{2}} \prec 1 - x_1 = O(\varepsilon^{\frac{1}{2}} \log \varepsilon),$$

$$1 - x_2 \asymp \varepsilon^{\frac{1}{2}}, \qquad 1 - x_3 \asymp \varepsilon^{\frac{1}{2}}.$$ (Lemmas 3.24, 3.26, 3.31)

(ii) $$\sup_{0 \leq x \leq 1} |H(x,\varepsilon)| \asymp \varepsilon^{\frac{1}{2}}.$$ (Lemma 3.29)

(iii) $-H'(0) \asymp \varepsilon^{\frac{1}{2}}$, $H'(x_2) \asymp 1$, $H'(x_2) \leq \frac{1}{2} + O(\varepsilon)$. (Lemmas 3.23, 3.29, 3.30)

(iv) <u>Uniformly in</u> x,

$$H''(x,\varepsilon) = O(\varepsilon^{-\frac{1}{2}} \exp\{-K\varepsilon^{-\frac{1}{2}}(1-x)\}),$$

$$H'''(x,\varepsilon) = O(\varepsilon^{-1} \exp\{-K\varepsilon^{-\frac{1}{2}}(1-x)\}),$$

while $\quad -H''(1) \asymp \varepsilon^{-\frac{1}{2}}, \quad -H'''(1) \asymp \varepsilon^{-1}$. (Lemmas 3.20, 3.21, 3.28)

(v) Uniformly in x,

$$G(x,\varepsilon) = O(\exp\{-K\varepsilon^{-\frac{1}{2}}(1-x)\}),$$

$$G'(x,\varepsilon) = O(\varepsilon^{-\frac{1}{2}} \exp\{-K\varepsilon^{-\frac{1}{2}}(1-x)\}),$$

$$G''(x,\varepsilon) = O(\varepsilon^{-1} \exp\{-K\varepsilon^{-\frac{1}{2}}(1-x)\}),$$

while $\quad G'(1) \asymp \varepsilon^{-\frac{1}{2}}$. (Lemmas 3.22, 3.27)

(vi) The first equation of (1.4) can be integrated to give

3.2) $$\varepsilon H''' + HH'' + \frac{1}{2}(G^2 - H'^2) = \mu,$$

for some constant μ, and as $\varepsilon \downarrow 0$, $-\mu \asymp \varepsilon$. (Lemma 3.29)

(vii) In any fixed interval $0 \le 1-x \le K\varepsilon^{\frac{1}{2}}$, if we set $1-x = \varepsilon^{\frac{1}{2}}\xi$,

then the quantities

$$-\varepsilon^{-\frac{1}{2}}H(x,\varepsilon) - \phi_o(\xi), \quad H'(x,\varepsilon) - \phi_o'(\xi), \quad -\varepsilon^{\frac{1}{2}}H''(x,\varepsilon) - \phi_o''(\xi),$$

$$G(x,\varepsilon) - \psi_o(\xi), \quad -\varepsilon^{\frac{1}{2}}G'(x,\varepsilon) - \psi_o'(\xi)$$

all tend to zero uniformly as $\varepsilon \downarrow 0$, where (ϕ_o, ψ_o) is a solution of the von Kármán single disk problem

3.3) $$\begin{cases} \phi''' + \phi\phi'' + \frac{1}{2}(\psi^2 - \phi'^2) = 0 \\ \psi'' + \phi\psi' - \phi'\psi = 0 \end{cases}$$

with the boundary conditions

3.4) $$\begin{cases} \phi(0) = \phi'(0) = \phi'(\infty) = 0 \\ \psi(0) = 1, \quad \psi(\infty) = 0. \end{cases}$$ (Lemma 3.25)

The last property (vii) is just the precise statement of the fact that in the boundary layer the solution behaves like a suitably scaled version of "the" solution of the von Kármán single disk problem, and at a heuristic level this has been recognized for some time. (We need merely observe that, if we make the change of variables

$$1 - x = \varepsilon^{\frac{1}{2}}\xi \ , \quad -\varepsilon^{\frac{1}{2}}H(x) = \phi(\xi) \ , \quad G(x) = \psi(\xi) \ ,$$

and if μ is in some sense negligible then the equation (3.2) takes the form of the first equation of (3.3).) Further, the solution (ϕ_o, ψ_o) to which $(-\varepsilon^{\frac{1}{2}}H, G)$ tends has, as the discussion in Lemma 3.25 shows, the properties that are associated with "the" solution of the von Kármán problem,

i.e. $$\phi_o \geq 0 \ , \quad \phi_o' \geq 0 \ , \quad \psi_o > 0 \ , \quad \psi_o' < 0 \ ,$$

ϕ_o'' has precisely one zero, being first positive and ultimately negative. (See for example, [10].) At the same time, since there is no uniqueness result for solutions of the von Kármán problem, the use of the phrase "the solution" is not permissible in any rigorous sense. Indeed, if there were more than one solution of the von Kármán problem, it is even possible that by letting $\varepsilon \downarrow 0$ through different sequences, we might have different limits (ϕ_o, ψ_o) in (vii) above.

The proof of Theorem II is contained in a long series of lemmas, most of which are in themselves relatively simple to prove. For each part of Theorem II, we have indicated the precise lemma or lemmas in which that part is finally proved.

Lemma 3.1: $H''^2 + G'^2$ is a non-decreasing function.

Proof: If we set

$$\Phi = H''^2 + G'^2 ,$$

then

$$\Phi' = 2H''H''' + 2G'G'' , \quad \Phi'' = 2H''H^{iv} + 2G'G''' + 2(H'''^2 + G''^2) ,$$

and by substituting for H^{iv} and G''' from the first equation of (1.4) and the second equation differentiated, we obtain

$$\varepsilon \Phi'' + H\Phi' = 2\varepsilon(H'''^2 + G''^2) ,$$

which implies at once that

$$\Phi'(x) \exp\{\int_0^x \varepsilon^{-1} H(t)dt\}$$

is a non-decreasing function. But the boundary conditions at $x = 0$ say that $\Phi'(0) = 0$, and so we always have $\Phi' \geq 0$, from which the lemma follows at once.

Lemma 3.2: For any x with $0 \leq x < 1$, we have

$$0 < G'(x, \varepsilon) \leq \frac{1}{1 - x} .$$

Proof: For since G' is a non-decreasing function by condition (F) of Theorem I, we have

$$1 \geq G(1) - G(x) = \int_x^1 G'(t)dt \geq G'(x)(1 - x) .$$

Lemma 3.3: If we define μ as in part (vi) of the statement of Theorem II, and $\Psi = (H'^2 + G^2 + 2\mu)/G$, then Ψ' has one and only one zero, at

x_0, *say, and* $\Psi' < 0$ *for* $x < x_0$, $\Psi' > 0$ *for* $x > x_0$. *Further,* $x_0 \leq x_3$.

Proof: It is a routine calculation, using (3.2) and the second equation of (1.4), to prove that

$$\Psi' = -\frac{2\varepsilon}{G^2}(H'''G' - H''G'') . \tag{3.5}$$

Differentiating again, and using the first equation of (1.4) and the second equation differentiated, we see that

$$\Psi'' = -\frac{2G'}{G}\Psi' - \frac{H}{\varepsilon}\Psi' + \frac{2}{G}(H''^2 + G'^2) , \tag{3.6}$$

from which it follows that

$$\Psi'(x)\exp\int_0^x \left\{\frac{H(t)}{\varepsilon} + \frac{2G'(t)}{G(t)}\right\}dt$$

is a non-decreasing function. It follows therefore that Ψ' has at most one zero. (If it had two, it would have to be identically zero between the two, and so everywhere, from the analyticity of the equations; and this is impossible.) Further, the values of the various derivatives at $x = 0$ and $x = 1$ (given either by the boundary conditions or by Theorem I) ensure that $\Psi'(0) < 0$, $\Psi'(1) > 0$, so that Ψ' has precisely one zero, at x_0, say. Finally, at x_3,

$$H''' = 0 , \quad G' > 0 , \quad H'' > 0 , \quad G'' \geq 0 ,$$

so that $\Psi'(x_3) \geq 0$ and $x_0 \leq x_3$.

Lemma 3.4: For $x \geq x_0$, and so in particular for $x \geq x_3$, we have

3.7) $$H'^2 \leq (G - 2\mu)(1 - G) .$$

Proof: By Lemma 3.3, we have, for $x \geq x_0$,

$$\Psi(x) \leq \Psi(1) ,$$

so that

$$H'^2 + G^2 + 2\mu \leq G(1 + 2\mu) ,$$

which rearranges to give (3.7).

Lemma 3.5:

$$0 \leq H'''(x, \varepsilon) \leq \frac{1}{2\varepsilon} \{G^2(x_3, \varepsilon) - G^2(x, \varepsilon)\} , \qquad 0 \leq x \leq x_3 ,$$

$$0 \geq \frac{1}{2\varepsilon} \{G^2(x_3, \varepsilon) - G^2(x, \varepsilon)\} \geq H'''(x, \varepsilon) , \qquad x_3 \leq x \leq 1 .$$

Proof: If $u(x, \varepsilon) = \int_0^x H(t, \varepsilon)dt$, then the first equation of (1.4) can be written in the form

3.8) $$\{H''' \exp(u/\varepsilon)\}' = -\frac{1}{\varepsilon} GG' \exp(u/\varepsilon) .$$

If we now integrate between x and x_3, remember that $H'''(x_3) = 0$ and use the fact that

$$\{u(x) - u(t)\}(x - t) \leq 0 ,$$

we obtain the required result.

Lemma 3.6: $$x_3 \geq \frac{1}{2} x_2 .$$

Proof: Let $$\bar{x} = \max\{0, 2x_3 - 1\} ,$$

$$\bar{y} = \max\{0, 2x_3 - x_2\} .$$

By Lemma 3.5, remembering that $G'' \geq 0$ and that G is therefore convex, we see that

(3.9) $$H'''(t,\varepsilon) \leq - H'''(2x_3 - t, \varepsilon) , \qquad \bar{x} \leq t \leq x_3 .$$

Suppose for contradiction that the lemma is not true. Then $\bar{x} = 0$, and integrating (3.9) over $[0, x_3]$, we obtain

$$H''(x_3, \varepsilon) \leq H''(x_3, \varepsilon) - H''(2x_3, \varepsilon) .$$

But we now have $2x_3 < x_2$, and so $H''(2x_3, \varepsilon) > 0$, which is a contradiction.

<u>Lemma 3.7</u>: $\qquad x_2 \to 1$ <u>as</u> $\varepsilon \downarrow 0$.

<u>Proof</u>: Suppose for contradiction that the lemma is not true, and allow ε to tend to zero through a sequence of values $\{\varepsilon_n\}$ such that $x_2(\varepsilon_n) \to \bar{x}_2 < 1$. Let a be a fixed number such that $\bar{x}_2 < a < 1$, and set

$$G(a)\, G'(a) = \xi(a, \varepsilon) ,$$

say. For $x \geq x_3$, we have

$$|\varepsilon H^{iv}| \geq GG' ,$$

and so for any $x \geq a$, since GG' is increasing, we obtain

$$|H'''(x) - H'''(a)| \geq \varepsilon^{-1}(x-a)\xi(a,\varepsilon) .$$

Since $H'''(x)$, $H'''(a)$ are of the same sign, we in fact have

$$|H'''(x)| \geq \varepsilon^{-1}(x-a)\xi(a,\varepsilon) ,$$

and so on integration, still for $x \geq a$,

$$|H''(x) - H''(a)| \geq \frac{1}{2}\varepsilon^{-1}(x-a)^2\,\xi(a,\varepsilon)\,;$$

and since $H''(x)$, $H''(a)$ are of the same sign,

$$|H''(x)| \geq \frac{1}{2}\varepsilon^{-1}(x-a)^2\,\xi(a,\varepsilon)\,.$$

By a final integration over $[\frac{1}{2}(1+a),1]$, we see that

3.10) $$H'(\tfrac{1}{2}(1+a)) \geq K\varepsilon^{-1}\xi\,,$$

for some suitable positive constant K .

But also $G'(x_2) = O(\xi^{\frac{1}{2}})$ as $\varepsilon \downarrow 0$. For if not, then $G'(x_2)/\xi^{\frac{1}{2}} \to \infty$, and so, since G' is increasing, $G'(a)/\xi^{\frac{1}{2}} \to \infty$; and also, by integration over $[x_2, a]$, $G(a)/\xi^{\frac{1}{2}} \to \infty$, which contradicts the definition of ξ . Thus

3.11) $$(H''^2 + G'^2)(x_2) = O(\xi)\,,$$

and since $H''^2 + G'^2$ is non-decreasing,

$$H''(x) = O(\xi^{\frac{1}{2}})\,,$$

uniformly in x for $x \leq x_2$. By integration between x_1 and x , we obtain

3.12) $$H'(x) = O(\xi^{\frac{1}{2}})\,,$$

uniformly in x for $x \leq x_2$, and so for all x , since x_2 is the positive maximum of H' . Comparing (3.10) and (3.12), we have

$$\varepsilon^{-1}\xi = O(\xi^{\frac{1}{2}})\,,$$

from which

$$\xi = O(\varepsilon^2)\,, \quad H'(x) = O(\varepsilon) \text{ for all } x\,,$$

and by integration

$$H(x) = O(\varepsilon) \qquad \text{for all} \quad x .$$

Also, from (3.11),

$$G'(0) = O(\varepsilon) .$$

Now the equation

$$G'' + \frac{H}{\varepsilon} G' = \frac{H'}{\varepsilon} G$$

is, as an equation in G, a linear equation with coefficients bounded in both x and ε. Let the solution of this equation with initial data

$$G(0) = 0 , \qquad G'(0) = 1$$

have $G(1) = K(\varepsilon)$, say, where, in view of the boundedness of the coefficients, $K(\varepsilon)$ is bounded as $\varepsilon \downarrow 0$. But we are concerned with initial data

$$G(0) = 0 , \qquad G'(0) = O(\varepsilon) ,$$

and since the solution is linear in the value of $G'(0)$, we now have

$$G(1) = O(\varepsilon) ,$$

which is the required contradiction if ε is sufficiently small.

<u>Lemma 3.8</u>: <u>If we have a sequence</u> $\{\varepsilon_n\}$, <u>with</u> $\varepsilon_n \downarrow 0$ <u>as</u> $n \to \infty$, <u>and if</u> $|H'(0, \varepsilon_n)| \not\to 0$ <u>as</u> $n \to \infty$, <u>then</u>

$$\limsup_{n \to \infty} x_1(\varepsilon_n) < 1 .$$

<u>Proof</u>: Suppose for contradiction that $x_1(\varepsilon_n) \to 1$ as $\varepsilon_n \downarrow 0$. (It will cause no confusion to drop the subscript n from now on.) By Lemmas 3.6 and 3.7, $x_3(\varepsilon) \geq \frac{1}{4}$, say, for ε sufficiently small.

If now

$$-H'(0,\varepsilon) = c(\varepsilon) ,$$

where $c(\varepsilon) \geq K$ as $\varepsilon \downarrow 0$, we see from the convexity of $H'(x)$ for $0 \leq x \leq x_3$, and so for $0 \leq x \leq \frac{1}{4}$, that

$$-H'(x,\varepsilon) \geq \frac{1}{2} c(\varepsilon) \quad \text{for} \quad 0 \leq x \leq \frac{1}{8} ,$$

and so by integration

$$-H(x,\varepsilon) \geq \frac{1}{16} c(\varepsilon) \quad \text{for} \quad x = \frac{1}{8} , \tag{3.13}$$

and so for $\frac{1}{8} \leq x \leq x_1$. The fact that $H'''(0) > 0$ implies that $\mu + \frac{1}{2} c^2 > 0$, so that

$$-\mu < \frac{1}{2} c^2 .$$

Appealing now to Lemma 3.4, we see that

$$H'^2(x_2) \leq (G(x_2) + c^2)(1 - G(x_2)) \leq 1 + c^2 .$$

Thus

$$|H(x_1)| = \int_{x_1}^{1} H'(t)dt \leq H'(x_2)(1 - x_1)$$

$$\leq \sqrt{1 + c^2}\,(1 - x_1) .$$

Combining this with (3.13) we have

$$\frac{1}{16} \leq \frac{\sqrt{1 + c^2}}{c} (1 - x_1) ,$$

which contradicts the assumption that

$$c \geq K, \quad 1 - x_1 \to 0, \quad \text{as} \quad \varepsilon \downarrow 0 .$$

Lemma 3.9: If $x_1(\varepsilon) \leq \alpha < \beta < 1$, then

$$G'(\alpha) \exp\{\frac{1}{\varepsilon} \int_\alpha^\beta |H(t, \varepsilon)| dt\} \leq G'(\beta) \leq \frac{1}{1 - \beta} . \tag{3.14}$$

Proof: If $x_1(\varepsilon) \leq x$, then $H'(x, \varepsilon) \geq 0$ and

$$G'' = -\frac{1}{\varepsilon} HG' + \frac{1}{\varepsilon} H'G \geq \frac{1}{\varepsilon} |H| G' ,$$

so that (3.14) follows from an integration and Lemma 3.2.

Lemma 3.10: Let $\langle H(x, \varepsilon), G(x, \varepsilon) \rangle$ be a solution of (1.4), (1.5), with

$$m(\varepsilon) = \sup_{0 \leq x \leq x_1(\varepsilon)} H''(x, \varepsilon) . \tag{3.15}$$

Then

$$\sup_{0 \leq x \leq 1} |H(x, \varepsilon)| \leq m(\varepsilon) . \tag{3.16}$$

Proof: This is an immediate computational result.

Lemma 3.11: Suppose we have a sequence $\{\varepsilon_n\}$, with $\varepsilon_n \downarrow 0$ as $n \to \infty$. Let

$$\bar{x}_1 = \limsup_{n \to \infty} x_1(\varepsilon_n) < 1 .$$

Then for any fixed a with $\bar{x}_1 < a < 1$, we have

$$\lim_{n \to \infty} H(a, \varepsilon_n) = 0 .$$

Proof: We assume the contrary. Let $\alpha \in (\bar{x}_1, a)$. Then for ε_n sufficiently small, $x_1(\varepsilon_n) < \alpha < a$. We may suppose (from our assumption for contradiction) that

$$0 < \delta = \liminf |H(a, \varepsilon_n)|(a-\alpha) ,$$

and for all $x \in (\alpha, a)$ we have $|H(x, \varepsilon_n)| \geq |H(a, \varepsilon_n)|$. Thus applying Lemma 3.9 we obtain

$$0 \leq \frac{1}{\varepsilon_n} G'(\alpha, \varepsilon_n) \leq \frac{1}{\varepsilon_n(1-a)} e^{-\delta/\varepsilon_n} \to 0 \quad \text{as} \quad n \to \infty . \tag{3.17}$$

We may also assume that

$$x_3(\varepsilon_n) \to \bar{x}_3 .$$

<u>Case 1</u>: $\bar{x}_1 < a < \bar{x}_3$. Because $H'' \geq 0$ for $x \in [0, x_2(\varepsilon)]$ we have

$$H(x, \varepsilon_n) \leq \frac{x}{a} H(a, \varepsilon_n) \leq x\, H(a, \varepsilon_n) \leq 0 , \qquad 0 \leq x \leq a .$$

Thus in applying Lemma 3.5 we find, for $x \leq \min(t, \alpha)$,

$$u(t) - u(x) = \int_x^t H(s, \varepsilon_n) ds \leq \begin{cases} \dfrac{t^2 - x^2}{2} H(a, \varepsilon_n) , & 0 \leq t \leq \alpha , \\[2ex] \dfrac{\alpha^2 - x^2}{2} H(a, \varepsilon_n) , & \alpha \leq t \leq \bar{x}_3 . \end{cases}$$

Let ε_n be so small that

$$\alpha^2 - x_1(\varepsilon_n)^2 \geq \sigma > 0$$

for some fixed constant σ, and let Δ be a positive constant such that

$$0 < \Delta \leq \frac{\sigma}{2} |H(a, \varepsilon_n)| .$$

Applying (3.8) for $x \leq x_1(\varepsilon_n)$ we have

$$0 \le H'''(x, \varepsilon_n) \le \frac{1}{\varepsilon_n} G'(\alpha, \varepsilon_n) + \frac{1}{2\varepsilon_n} \int_{\alpha}^{x_3(\varepsilon_n)} e^{-\Delta/\varepsilon_n} (G^2)' dt ,$$

$$0 \le H'''(x, \varepsilon_n) \le \frac{e^{-\delta/\varepsilon_n}}{\varepsilon_n(1-a)} + \frac{1}{2\varepsilon_n} e^{-\Delta/\varepsilon_n}, \qquad 0 \le x \le x_1 .$$

After one integration we see that $m(\varepsilon_n) \to 0$. Thus, applying Lemma 3.10, we see that $H(a, \varepsilon_n) \to 0$ also.

<u>Case 2</u>: $\bar{x}_1 < \bar{x}_3 < a < 1$. Let b be any point satisfying $\bar{x}_1 < b < \bar{x}_3 < a < 1$. Then, since $H'(x, \varepsilon_n) > 0$ for $x_1(\varepsilon_n) < x < 1$, we have

$$|H(a, \varepsilon_n)| \le |H(b, \varepsilon_n)| ,$$

and $H(b, \varepsilon_n) \to 0$ by the discussion in Case 1.

<u>Case 3</u>: $\bar{x}_3 \le \bar{x}_1$. For ε_n sufficiently small integration of (3.8) gives, for $x \le x_3(\varepsilon_n)$,

$$0 \le H'''(x, \varepsilon_n) \le \frac{1}{\varepsilon_n} G'(x_3(\varepsilon_n), \varepsilon_n) \le \frac{1}{\varepsilon_n} G'(\alpha, \varepsilon_n) .$$

But in this case

$$m(\varepsilon_n) = \max\{H''(x, \varepsilon_n) ; \qquad 0 \le x \le x_3(\varepsilon_n)\} .$$

Thus Lemma 3.10 again gives desired conclusion.

<u>Lemma 3.12</u>: <u>Let</u> $\langle H(x, \varepsilon), G(x, \varepsilon)\rangle$ <u>be the family of all solutions of (1.4), (1.5) which are odd and have</u> $G'(x, \varepsilon) \ge 0$, <u>i.e. are of the form described in Theorem I. Then</u>

3.18a) $$\lim_{\varepsilon \downarrow 0} \sup_{0 \le x \le 1} |H(x, \varepsilon)| = 0 ,$$

3.18b) $$\lim_{\varepsilon \downarrow 0} |H'(0, \varepsilon)| = 0 .$$

Proof: By the argument of Lemma 3.8 we have

$$\sup_{0 \leq x \leq 1} |H(x, \varepsilon)| \geq \frac{1}{16} |H'(0, \varepsilon)| .$$

On the other hand, the convexity of $H(x, \varepsilon)$ on the interval $0 \leq x \leq x_2$ gives

$$\sup_{0 \leq x \leq 1} |H(x, \varepsilon)| \leq |H'(0, \varepsilon)| .$$

Thus the statements (3.18a) and (3.18b) are equivalent.

Suppose for contradiction that (3.18b) is false. Then there is a sequence $\varepsilon_n \downarrow 0$ and a constant $\eta > 0$ such that

$$\eta \leq \frac{1}{16} |H'(0, \varepsilon_n)| \leq \sup_{0 \leq x \leq 1} |H(x, \varepsilon_n)| .$$

Moreover, we may assume from Lemma 3.8 that

$$x_1(\varepsilon_n) \rightarrow \bar{x}_1 < 1 .$$

Let

$$\alpha = \frac{1}{2}(1 + \bar{x}_1) , \qquad \beta = \frac{1}{2}(\alpha + \bar{x}_1) .$$

When ε_n is small enough, we have

$$\alpha < x_2(\varepsilon_n) ,$$

and, using convexity,

$$|H(\beta, \varepsilon_n)| \geq \frac{\eta(\alpha - \beta)}{\alpha - x_1(\varepsilon_n)} \rightarrow \frac{1}{2}\eta > 0 ,$$

which contradicts Lemma 3.11.

Lemma 3.13: Let $\langle H(x, \varepsilon), G(x, \varepsilon) \rangle$ be the family of all solutions of

(1.4), (1.5) which are odd and have $G'(x, \varepsilon) \geq 0$. _Let_ c _be any fixed point,_ $0 < c < 1$. _Then_

$$\lim_{\varepsilon \downarrow 0} \sup_{0 \leq x \leq c} |H''(x, \varepsilon)| = 0 .$$

Proof: Suppose the lemma is false. Then there is a sequence $\varepsilon_n \downarrow 0$ for which

$$\sup_{0 \leq x \leq c} |H''(x, \varepsilon_n)| \geq \eta > 0 , \tag{3.19}$$

and we may also without loss of generality suppose that

$$x_3(\varepsilon_n) \to \bar{x}_3 .$$

Case 1: $\bar{x}_3 = 1$. For ε_n sufficiently small, we can find a fixed $\sigma > 0$ such that $c + \sigma < x_3(\varepsilon_n)$. Since $H''' > 0$ on $[0, x_3)$, we deduce from (3.19) that

$$H''(c, \varepsilon_n) \geq \eta > 0 ,$$

and also

$$H''(x, \varepsilon_n) \geq H''(c, \varepsilon_n) \geq \eta \quad \text{for} \quad c \leq x \leq c + \sigma ,$$

which on integration certainly contradicts (3.18a).

Case 2: $\bar{x}_3 \leq c$. Using the concavity of H'' on $[x_3, 1]$, we have

$$\frac{H''(x_3, \varepsilon_n)}{x_2 - x_3} \leq \frac{H''(x, \varepsilon_n)}{x_2 - x} , \qquad x_3 \leq x < x_2 .$$

Thus

$$\frac{x_2 - x}{x_2 - x_3} H''(x_3, \varepsilon_n) \leq H''(x, \varepsilon_n) ,$$

and integration again leads to a contradiction to Lemma 3.12.

Lemma 3.14: $\mu = O(1)$.

Proof: The fact that $H'''(0) > 0$ implies from (3.2) that

$$\mu + \frac{1}{2} H'^2(0) > 0 ,$$

which implies μ bounded below from Lemma 3.12. And the fact that $H'''(1) < 0$ implies similarly that μ is bounded above by $\frac{1}{2}$.

Lemma 3.15: $H'''(x, \varepsilon) = O(\varepsilon^{-1})$, uniformly for x in $[0,1]$.

Proof:
$$\varepsilon H'''(1) = \mu - \frac{1}{2} ,$$

so that

$$|H'''(1)| \leq |\mu - \frac{1}{2}| \varepsilon^{-1} .$$

But from the first equation of (1.4), H''' is negative and monotonic decreasing for $x \geq x_3$, and so we have

$$|H'''(x, \varepsilon)| \leq |\mu - \frac{1}{2}| \varepsilon^{-1} \quad \text{for} \quad x \geq x_3 .$$

For $x \leq x_3$, the result follows from Lemma 3.5.

Lemma 3.16: $H'(x, \varepsilon) = O(1)$, uniformly for $x \in [0,1]$. Moreover, for any fixed c , $0 < c < 1$, $H'(x, \varepsilon) = o(1)$, uniformly for $0 \leq x \leq c$.

Proof: By Lemma 3.12 we have that $H'(0, \varepsilon) = o(1)$, and so by monotonicity $H'(x, \varepsilon) = o(1)$ for $0 \leq x \leq x_1$. For $x > x_1$ Lemma 3.4 and the fact that $\mu = O(1)$ give

$$0 \leq H'(x, \varepsilon) \leq H'(x_2, \varepsilon) = O(1) .$$

Finally,

$$\lim_{\varepsilon \downarrow 0} \sup_{0 \le x \le c} |H'(x,\varepsilon)| = 0$$

because of Lemmas 3.12 and 3.13 and the well-known Landau inequality [9]

$$\sup_{0 \le x \le c} |H'| \le 2\{ \sup_{0 \le x \le c} |H| + \sup_{0 \le x \le c} |H''|\} .$$

Lemma 3.17: $\mu = o(1)$.

Proof: If $\mu < 0$, then the proof follows immediately from the facts $H'''(0,\varepsilon) > 0$, $H'(0,\varepsilon) = o(1)$. Thus we may suppose $\mu > 0$.

Since $G'' \ge 0$, we have $G(x) \le x$, and so, for any fixed c with $0 < c < 1$, and for $0 \le x \le c$, we obtain from (3.2) that

$$\varepsilon H'''(x) \ge \mu - \frac{1}{2} x^2 + o(1) .$$

Now suppose for contradiction that there is a sequence $\varepsilon_n \downarrow 0$ and a constant $\mu_0 > 0$ such that

$$0 < \mu_0 \le \mu(\varepsilon_n) .$$

Choose $c = \sqrt{\mu_0}$. Then

$$\varepsilon H'''(x) \ge \frac{1}{2} \mu_0 + o(1) , \qquad 0 \le x \le c ,$$

and one integration contradicts Lemma 3.13.

Lemma 3.18: $G'(1,\varepsilon) = O(\varepsilon^{-\frac{1}{2}})$.

Proof: If $1 - \varepsilon^{\frac{1}{2}} \le x \le 1$, then

$$H(x,\varepsilon) = - \int_x^1 H'(t,\varepsilon)dt = O(\varepsilon^{\frac{1}{2}}) ,$$

using Lemma 3.16. Hence

$$\varepsilon G'' = H'G - HG' = O(1) + O(\varepsilon^{\frac{1}{2}} G') ,$$

whence by integration over $[x,1]$

$$G'(1) - G'(x) = O(\varepsilon^{-\frac{1}{2}}) + O(\varepsilon^{-\frac{1}{2}}\{G(1) - G(x)\}) = O(\varepsilon^{-\frac{1}{2}}) ,$$

and by integration over $[1 - \varepsilon^{\frac{1}{2}}, 1]$

$$\varepsilon^{\frac{1}{2}} G'(1) - \{G(1) - G(x)\} = O(1) ,$$

giving the required result.

<u>Lemma 3.19</u>: $$H''(1, \varepsilon) = O(\varepsilon^{-\frac{1}{2}}) .$$

<u>Proof</u>: If $1 - \varepsilon^{\frac{1}{2}} \leq x \leq 1$, then

3.20) $$H''(1) - H''(x) = \int_x^1 H'''(t)dt = O(\varepsilon^{-\frac{1}{2}}) ,$$

by use of Lemma 3.15. Integrating over $[1 - \varepsilon^{\frac{1}{2}}, 1]$, we have

$$\varepsilon^{\frac{1}{2}} H''(1) - \{H'(1) - H'(1 - \varepsilon^{\frac{1}{2}})\} = O(1) ,$$

and since $H' = O(1)$, the result follows.

<u>Lemma 3.20</u>: <u>There exists a positive number</u> η, <u>independent of</u> ε, <u>such that</u>

$$-H'''(x, \varepsilon) \asymp \varepsilon^{-1} \quad \underline{\text{for}} \quad 1 - \eta\varepsilon^{\frac{1}{2}} \leq x \leq 1 .$$

<u>Proof</u>: $$G(1) - G(x) = \int_x^1 G'(t)dt \leq K\varepsilon^{-\frac{1}{2}}(1-x) ,$$

by Lemma 3.18 and the monotonic character of G'. Hence, by suitable choice of η, we can arrange that

$$G^2(x) - 2\mu \geq \frac{1}{2}(1-2\mu) \quad \text{for} \quad 1 - \eta\varepsilon^{\frac{1}{2}} \leq x \leq 1 .$$

(We recall from Lemma 3.17 that $\mu = o(1)$.) Similarly, always for $1 - \eta\varepsilon^{\frac{1}{2}} \leq x \leq 1$, we have

$$|H''(1) - H''(x)| = |\int_x^1 H'''(t)dt| \leq K\eta\varepsilon^{-\frac{1}{2}},$$

so that

$$|H''(x)| \leq K\varepsilon^{-\frac{1}{2}},$$

K not necessarily being the same at each appearance. By integration

$$|H'(x)| \leq K\eta, \qquad |H(x)| \leq K\eta^2\varepsilon^{\frac{1}{2}},$$

and from (3.2),

$$-\varepsilon H''' = \frac{1}{2}G^2 - \mu - \frac{1}{2}H'^2 + HH''$$

$$\geq \frac{1}{8}(1-2\mu) \quad \text{if} \quad \eta \quad \text{is sufficiently small.}$$

The lemma is thus proved.

Lemma 3.21: $$-H''(1,\varepsilon) \asymp \varepsilon^{-\frac{1}{2}}.$$

Proof: Repeat the proof of Lemma 3.19, but now restrict x to $[1 - \eta\varepsilon^{\frac{1}{2}}, 1]$ and use

$$-H''' \asymp \varepsilon^{-1} \quad \text{instead of} \quad H''' = O(\varepsilon^{-1}).$$

We obtain

$$-\eta\varepsilon^{\frac{1}{2}} H''(1) - H'(1-\eta\varepsilon^{\frac{1}{2}}) \asymp 1.$$

If we now suppose for contradiction that $H''(1) = o(\varepsilon^{-\frac{1}{2}})$, then

$$-H'(1-\eta\varepsilon^{\frac{1}{2}}) \asymp 1,$$

so that

$$x_2 > x_1 > 1 - \eta\varepsilon^{\frac{1}{2}}$$

and

$$-H'(0) \geq - H'(1-\eta\varepsilon^{\frac{1}{2}}) \asymp 1 .$$

This contradicts Lemma 3.12.

Lemma 3.22: $$G'(1, \varepsilon) \asymp \varepsilon^{-\frac{1}{2}} .$$

Proof: Using (3.20) with $1 - \eta\varepsilon^{\frac{1}{2}} \leq x \leq 1$ and $-H''' \asymp \varepsilon^{-1}$, we see that $-H''(x, \varepsilon) \asymp \varepsilon^{-\frac{1}{2}}$ if η is chosen sufficiently small. This implies that $x_2 < 1 - \eta\varepsilon^{\frac{1}{2}}$, and so, by Lemma 3.3, $(H'^2 + G^2 + 2\mu)/G$ is non-decreasing in $[1 - \eta\varepsilon^{\frac{1}{2}}, 1]$, so that $H''G'' - H'''G' \geq 0$, or replacing G'' from (1.4),

$$H''H'G - (\varepsilon H''' + HH'')G' \geq 0 ,$$

3.21) $$|\varepsilon H''' + H H''|G' \geq |H''H'G| .$$

We have already seen that $-H'' \asymp \varepsilon^{-\frac{1}{2}}$ in $[1 - \eta\varepsilon^{\frac{1}{2}}, 1]$, and so by integration

$$H'(1-\eta\varepsilon^{\frac{1}{2}}) \asymp 1 , \quad -H(1-\eta\varepsilon^{\frac{1}{2}}) \asymp \varepsilon^{\frac{1}{2}} .$$

Further, by Lemma 3.20 and its proof,

$$-H'''(1-\eta\varepsilon^{\frac{1}{2}}) \asymp \varepsilon^{-1} , \quad G(1-\eta\varepsilon^{\frac{1}{2}}) \asymp 1 ,$$

and substituting in (3.21) we obtain

$$G'(1-\eta\varepsilon^{\frac{1}{2}}) \geq K\varepsilon^{-\frac{1}{2}} .$$

This, combined with the monotonicity of G' and Lemma 3.18, gives the required result.

Lemma 3.23: $$H'(x_2) \asymp 1 .$$

Proof: As in Lemma 3.22, we show $x_2 < 1 - \eta\varepsilon^{\frac{1}{2}}$ and that

$$H'(1-\eta\varepsilon^{\frac{1}{2}}) \asymp 1 .$$

Then

$$H'(x_2) \geq H'(1-\eta\varepsilon^{\frac{1}{2}}) \asymp 1 ,$$

and this with Lemma 3.16 gives the required result.

Lemma 3.24: $$1 - x_1 \succ \varepsilon^{\frac{1}{2}} .$$

Proof: We make the transformation

$$1 - x = \varepsilon^{\frac{1}{2}}\xi , \quad -H(x,\varepsilon) = \varepsilon^{\frac{1}{2}}\phi(\xi,\varepsilon) , \quad G(x,\varepsilon) = \psi(\xi,\varepsilon) .$$

Then the equations (1.4) become

3.22) $$\begin{cases} \phi^{iv} + \phi\phi''' = -\psi\psi' \\ \psi'' + \phi\psi' = \phi'\psi , \end{cases}$$

with the boundary conditions (amongst others)

3.23) $$\phi(0) = \phi'(0) = 0 , \quad \psi(0) = 1 .$$

Further, Lemmas 3.21, 3.22 show that in the limit as $\varepsilon \downarrow 0$ the initial values $\phi''(0)$, $\psi'(0)$ are bounded, and we may therefore suppose that as $\varepsilon \downarrow 0$, if necessary through some sequence,

$$\phi''(0) \to \alpha , \quad \psi'(0) \to -\beta ,$$

where α and β are strictly positive. Further, (3.2) and $\mu = o(1)$ show that

$$\phi'''(0) \to -\frac{1}{2} .$$

Because of the continuous dependence of the solutions of a differential equation on the initial conditions, we can say that, if (ϕ_0, ψ_0) is the solution of (3.22) satisfying (3.23) and

$$\phi''(0) = \alpha \; , \quad \psi'(0) = -\beta \; , \quad \phi'''(0) = -\frac{1}{2} \; ,$$

then in any fixed interval $[0, K]$ the functions ϕ, ψ and their derivatives tend uniformly to ϕ_0, ψ_0 and their derivatives, and we remark in passing that this proves (vii) of Theorem II, except that we still have to show that

$$\phi_0'(\infty) = 0 \; , \quad \psi_0(\infty) = 0 \; .$$

Now suppose for contradiction that $1 - x_1 \asymp \varepsilon^{\frac{1}{2}}$. (We certainly know that $1 - x_1 \geq K\varepsilon^{\frac{1}{2}}$, since we saw in the proof of Lemma 3.22 that $1 - x_2 \geq K\varepsilon^{\frac{1}{2}}$.) Thus we may suppose that $(1-x_1)/\varepsilon^{\frac{1}{2}} \to K_0$, say. Since Lemma 3.16 implies that $H' = o(1)$ for $x \leq x_1$, and since the convergence of ϕ' to ϕ_0' is uniform for $\frac{1}{2} K_0 \leq (1-x)/\varepsilon^{\frac{1}{2}} \leq \frac{3}{2} K_0$, we conclude that $\phi_0' = 0$ in $[K_0, \frac{3}{2} K_0]$, and so everywhere, which is impossible and gives the required contradiction.

<u>Lemma 3.25</u>: <u>Part (vii) of Theorem II holds</u>.

<u>Proof</u>: We have already remarked in the course of Lemma 3.24 that all that is necessary is to prove that

$$\phi_0'(\infty) = 0 \; , \quad \psi_0(\infty) = 0 \; .$$

We first observe that $\phi_0' \geq 0$. For if ϕ_0' changes sign, say at

$\xi = \xi_0$, then, for ε sufficiently small, ϕ must change sign near $\xi = \xi_0$, and this contradicts Lemma 3.24. We thus have

$$\phi_0' \geq 0 , \qquad \phi_0 \geq 0 .$$

Similar arguments show that

$$\psi_0 > 0 , \quad \psi_0' < 0 , \quad \psi_0'' \geq 0 ,$$

while ϕ_0'' and ϕ_0''' change sign at most once.

From the second equation of (3.22) we deduce that $\psi_0'(\xi) \ \exp(\int_0^\xi \phi_0(t)dt)$ is negative and non-decreasing, and so tends to a finite limit as $\xi \to \infty$. Since ϕ_0 is positive and non-decreasing, we deduce that ψ_0' is exponentially small as $\xi \to \infty$.

Now the first equation of (3.22) tells us that

$$\{\phi_0'''(\xi) \ \exp(\int_0^\xi \phi_0(t)dt)\}' \ \text{is bounded,}$$

and so

$$|\phi_0'''(\xi) \ \exp(\int_0^\xi \phi_0(t)dt)| \leq K(\xi^2+1)^{\frac{1}{2}} ,$$

and ϕ_0''' is exponentially small as $\xi \to \infty$. It follows that ϕ_0'' tends exponentially to a limit, ℓ say, and $\ell \geq 0$ since $\ell < 0$ would contradict $\phi_0' \geq 0$. Also, $\ell > 0$ is impossible, for this would imply that ϕ_0' is unbounded as $\xi \to \infty$, and this contradicts the uniform boundedness, for all x and ε, of $H'(x, \varepsilon)$. Thus ϕ_0'' is exponentially small as $\xi \to \infty$, and ϕ_0' tends exponentially to a limit,

m say. The fact, already established, that ψ_0' is exponentially small implies that ψ_0 tends exponentially to a limit, n say, and substitution in (3.22), the first equation having been first integrated, gives

$$m^2 - n^2 = 0 , \quad mn = 0 ,$$

from which it follows, as required, that $m = n = 0$.

We remark in passing that $m = 0$ certainly implies that ϕ_0'' has to change sign, and we have already seen that it does so at most once. A similar remark applies to ϕ_0'''.

<u>Lemma 3.26</u>: $\quad 1 - x_2 \asymp \varepsilon^{\frac{1}{2}} , \quad 1 - x_3 \asymp \varepsilon^{\frac{1}{2}} .$

<u>Proof</u>: This is now an immediate consequence of the previous lemma. For since ϕ_0'', ϕ_0''' change sign at points $\xi = \xi_2$, $\xi = \xi_3$, say, it follows for ε sufficiently small that ϕ'', ϕ''' change sign near ξ_2, ξ_3, and this is what is required.

<u>Lemma 3.27</u>: <u>Uniformly in</u> x,

$$G(x, \varepsilon) = O(\exp\{-K\varepsilon^{-\frac{1}{2}}(1-x)\}) ,$$

$$G'(x, \varepsilon) = O(\varepsilon^{-\frac{1}{2}}\exp\{-K\varepsilon^{-\frac{1}{2}}(1-x)\}) ,$$

$$G''(x, \varepsilon) = O(\varepsilon^{-1}\exp\{-K\varepsilon^{-\frac{1}{2}}(1-x)\}) .$$

<u>Proof</u>: By differentiating the second equation of (1.4), we obtain

$$\{G''(x) \exp(\int_1^x \varepsilon^{-1}H(t)dt)\}' \geq 0 \quad \text{for} \quad x \leq x_2 ,$$

so that

3.24) $$G''(x) \le G''(x_2)\exp(-\int_{x_2}^{x} \varepsilon^{-1} H(t)dt) \quad \text{for} \quad x \le x_2 .$$

But using Lemma 3.25, we see that certainly $\psi''(\xi) = O(1)$ in any bounded interval of ξ, and so

$$G''(x) = O(\varepsilon^{-1}) \quad \text{in any interval} \quad 1 - x \le K\varepsilon^{\frac{1}{2}} .$$

In particular, $G''(x_2) = O(\varepsilon^{-1})$, and so (3.24) yields

$$G''(x) = O\{\varepsilon^{-1}\exp(-\int_{x_2}^{x} \varepsilon^{-1} H(t)dt)\} .$$

Again appealing to Lemma 3.25, we see that

$$-H(x_2) \asymp \varepsilon^{\frac{1}{2}} ,$$

and since H is convex for $x \le x_2$, we have

3.25) $$\frac{H(x)}{H(x_2)} \ge \frac{x}{x_2} , \quad -H(x) \ge Kx\varepsilon^{\frac{1}{2}} ,$$

for $x \le x_2$, and so, still for $x \le x_2$,

$$-\int_{x_2}^{x} \varepsilon^{-1} H(t)dt \le -K\varepsilon^{-\frac{1}{2}}(x_2^2 - x^2) \le -K\varepsilon^{-\frac{1}{2}}(x_2 - x) .$$

We conclude that

$$G''(x) = O(\varepsilon^{-1}\exp\{-K\varepsilon^{-\frac{1}{2}}(x_2 - x)\})$$

3.26) $$= O(\varepsilon^{-1}\exp\{-K\varepsilon^{-\frac{1}{2}}(1-x)\}) , \quad \text{since} \quad 1 - x_2 \asymp \varepsilon^{\frac{1}{2}} ,$$

and though this estimate on G'' is in the first place for $x \le x_2$, it in fact holds for all x since we have already seen that $G''(x) = O(\varepsilon^{-1})$

for $1 - x \leq K\varepsilon^{\frac{1}{2}}$. Finally, by integration we have

$$G'(x) = G'(0) + O(\varepsilon^{-\frac{1}{2}} \exp\{-K\varepsilon^{-\frac{1}{2}}(1-x)\}) , \tag{3.27}$$

$$G(x) = xG'(0) + O(\exp\{-K\varepsilon^{-\frac{1}{2}}(1-x)\}) .$$

We next remark that if $x_1 \to \bar{x}_1$ as $\varepsilon \downarrow 0$ through some sequence, then

$$\bar{x}_1 \geq \frac{1}{2} . \tag{3.28}$$

For since H is convex for $x \leq x_2$, it follows that

$$|H(x_1-t) - H(x_1)| \leq |H(x_1+t) - H(x_1)|$$

$$\text{for } 0 \leq t \leq \min(x_1, x_2-x_1) ,$$

and so

$$H(x_1-t) \leq H(x_1+t) .$$

The case $x_2 - x_1 > x_1$ would lead, with $t = x_1$, to $H(2x_1) \geq 0$, which is impossible, and so $x_2 - x_1 \leq x_1$, which leads to $\bar{x}_1 \geq \frac{1}{2}$.

Now, since $G'(t) \geq G'(0)$ and $G(t) \geq tG'(0)$, we can deduce from (1.4) that, for $0 \leq x \leq X$,

$$H'''(x) - H'''(X) \exp(\int_x^X \varepsilon^{-1}H(t)dt) \geq G'^2(0) \int_x^X \frac{t}{\varepsilon} \exp(\int_x^t \varepsilon^{-1}H(u)du)dt$$

$$\geq \frac{G'^2(0)}{H'(0)} \int_x^X \frac{H(t)}{\varepsilon} \exp(\int_x^t \varepsilon^{-1}H(u)du)dt$$

for $X \leq x_1$, since in $[0, x_1]$ we have $t \geq H(t)/H'(0)$,

$$= - \frac{G'^2(0)}{H'(0)} \{1 - \exp(\int_x^X \varepsilon^{-1}H(u)du)\} .$$

Letting $X = x_1$, and recalling (3.25) and (3.28), we see that certainly

$$H'''(x) \geq -\frac{3}{4} G'^2(0)/H'(0) \quad \text{for} \quad x \leq \frac{1}{4}$$

if ε is sufficiently small. Hence

$$H''(x) \geq -\frac{3}{4} \frac{G'^2(0)}{H'(0)} x ,$$

$$\varepsilon G'''(x) = H''G - HG'' \geq H''G \geq -\frac{3}{4} \frac{G'^3(0)}{H'(0)} x^2 ,$$

$$\varepsilon G''(x) \geq -\frac{1}{4} \frac{G'^3(0)}{H'(0)} x^3 .$$

Letting $x = \frac{1}{4}$, we have from (3.26) that

$$G'(0) = O(\exp\{-\frac{1}{4} K\varepsilon^{-\frac{1}{2}}\})$$

$$= O(\exp\{-\frac{1}{4} K\varepsilon^{-\frac{1}{2}}(1-x)\}) , \qquad 0 \leq x \leq 1 .$$

Substituting this in (3.27), we obtain the statement of the lemma, with K replaced by $\frac{1}{4} K$.

Lemma 3.28: Uniformly in x,

$$H''(x, \varepsilon) = O(\varepsilon^{-\frac{1}{2}} \exp\{-K\varepsilon^{-\frac{1}{2}}(1-x)\}) ,$$

$$H'''(x, \varepsilon) = O(\varepsilon^{-1} \exp\{-K\varepsilon^{-\frac{1}{2}}(1-x)\}) .$$

Proof: The result for H'' follows by integration from that for H''', so that we need prove only the latter.

For H''', the result is certainly true for $x \geq x_3$, since the fact that $1 - x_3 \asymp \varepsilon^{\frac{1}{2}}$ means that the effect of the lemma for $x \geq x_3$

is just that $H''' = O(\varepsilon^{-1})$, and this we already have in Lemma 3.15.

Also, for $x \leq \frac{1}{2}$, we have by integrating (1.4) that

$$\varepsilon H'''(\tfrac{1}{2}) - \varepsilon H'''(x) \geq -\tfrac{1}{2}\{G^2(\tfrac{1}{2}) - G^2(x)\},$$

$$H'''(x) \leq H'''(\tfrac{1}{2}) + \tfrac{1}{2}\varepsilon^{-1}\{G^2(\tfrac{1}{2}) - G^2(x)\}.$$

If now the result is true for $x = \frac{1}{2}$, then, for $x \leq \frac{1}{2}$,

$$H'''(x) = O(\varepsilon^{-1}\exp\{-K\varepsilon^{-\frac{1}{2}}\})$$

for some suitable K, and this necessarily implies that the lemma holds for $x \leq \frac{1}{2}$ with the same value of K.

It remains to prove the lemma for $\frac{1}{2} \leq x \leq x_3$, and to do this we choose K_0 positive and sufficiently small that

$$|H(x)| \geq 2K_0\varepsilon^{\frac{1}{2}} \quad \text{for} \quad \tfrac{1}{2} \leq x \leq x_3 .$$

(This is possible by virtue of (3.25).) Then

$$\begin{aligned}
&(\varepsilon H''' \exp\{K_0\varepsilon^{-\frac{1}{2}}(1-x)\})' \\
&\quad = (-HH''' - GG' - K_0\varepsilon^{\frac{1}{2}}H''')\exp\{K_0\varepsilon^{-\frac{1}{2}}(1-x)\} \\
&\quad \geq (K_0\varepsilon^{\frac{1}{2}}H''' - GG')\exp\{K_0\varepsilon^{-\frac{1}{2}}(1-x)\} \\
&\quad \geq -GG'\exp\{K_0\varepsilon^{-\frac{1}{2}}(1-x)\} .
\end{aligned}$$

If K_0 is fixed sufficiently small, as we may suppose, then we can integrate, using the results of Lemma 3.27, to conclude that

$$[\varepsilon H''' \exp\{K_0\varepsilon^{-\frac{1}{2}}(1-x)\}]_x^{x_3}$$

is bounded below in ε, uniformly for $\frac{1}{2} \leq x \leq x_3$, and so the lemma is proved.

Lemma 3.29:
$$\sup_{0 \leq x \leq 1} |H| \asymp \varepsilon^{\frac{1}{2}}, \quad -H'(0) \asymp \varepsilon^{\frac{1}{2}}, \quad -\mu \asymp \varepsilon .$$

Proof: By integrating the result of Lemma 3.28, we see that for $x \geq x_1$,

$$H' = O(\exp\{-K\varepsilon^{-\frac{1}{2}}(1-x)\}) ,$$

so that, integrating again over $[x_1, 1]$,

$$\sup_{0 \leq x \leq 1} |H| = |H(x_1)| = O(\varepsilon^{\frac{1}{2}}) .$$

But we have already seen in (3.25) that $\sup|H| \geq K\varepsilon^{\frac{1}{2}}$, and the first part of the lemma is proved.

The second part follows at once from the remarks at the beginning of Lemma 3.12, and the third part from an evaluation of (3.2) at $x = 0$, remembering that $H'''(0)$ is exponentially small.

Lemma 3.30:
$$H'(x_2) \leq \frac{1}{2} + O(\varepsilon) .$$

Proof: This inequality, which is surprisingly precise when compared with the numerical computations, is a consequence of Lemma 3.4. For using (3.7) and the fact that $\mu = O(\varepsilon)$, we have for $x \geq x_3$, and so for $x = x_2$,

$$H'^2 \leq G(1-G) + O(\varepsilon)$$
$$\leq \frac{1}{4} + O(\varepsilon) ,$$

from which the result follows.

Lemma 3.31: $$1 - x_1 = O(\varepsilon^{\frac{1}{2}} \log \varepsilon) .$$

Proof: We first observe that

$$H''(x_1) \geq K\varepsilon^{\frac{1}{2}} .$$

For if contrarily $H''(x_1) = o(\varepsilon^{\frac{1}{2}})$, then from monotonicity $H''(x) = o(\varepsilon^{\frac{1}{2}})$ for $x \leq x_1$, and this on integration contradicts $-H'(0) \asymp \varepsilon^{\frac{1}{2}}$, $H'(x_1) = 0$. Thus, from Lemma 3.28 we have

$$\varepsilon^{\frac{1}{2}} = O(\varepsilon^{-\frac{1}{2}} \exp\{-K\varepsilon^{-\frac{1}{2}}(1-x_1)\}) ,$$

from which

$$1 - x_1 = O(\varepsilon^{\frac{1}{2}} \log \varepsilon) ,$$

as required.

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15. Supplementary Notes

16. Abstracts

The paper studies the boundary-value problem arising from the behaviour of a fluid occupying the region $-1 \leq x \leq 1$ between two rotating disks, rotating about a common axis perpendicular to their planes when the disks are rotating with the same speed Ω_0 but in the opposite sense. The equations which describe the axially symmetric similarity solutions of this problem are

$$\varepsilon H^{iv} + HH''' + GG' = 0$$
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with the boundary conditions

$$H(\pm 1) = H'(\pm 1) = 0$$
$$G(-1) = -1, \; G(1) = 1$$

where $\varepsilon = \nu/2\Omega_0$ and ν is the kinematic viscosity.

The existence of an odd solution $\langle H(x,\varepsilon), G(x,\varepsilon)\rangle$ is established. This particular solution satisfies many special conditions, e.g., $G'(x,\varepsilon) > 0$. Moreover, precise estimates are obtained on the size and behaviour of the solution as $\varepsilon \downarrow 0$.

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13. ABSTRACT

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$$\varepsilon H^{iv} + HH''' + GG' = 0$$
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with the boundary conditions

$$H(\pm 1) = H'(\pm 1) = 0$$
$$G(-1) = -1, \; G(1) = 1$$

where $\varepsilon = \nu / 2\Omega_0$ and ν is the kinematic viscosity.

The existence of an odd solution $\langle H(x,\varepsilon), G(x,\varepsilon)\rangle$ is established. This particular solution satisfies many special conditions, e.g., $G'(x,\varepsilon) > 0$. Moreover, precise estimates are obtained on the size and behaviour of the solution as $\varepsilon \downarrow 0$.

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