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THE LU-FACTORIZATION OF TOTALLY POSITIVE MATRICES

by

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1. INTRODUCTION

Let $A = (a_{ij})$ be an $n \times n$ real matrix. The minor of A formed from rows $\alpha_1 < \alpha_2 < \cdots < \alpha_p$ and columns $\beta_1 < \beta_2 < \cdots < \beta_p$ will be denoted by

$$A \begin{pmatrix} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_p \end{pmatrix} . \tag{1.1}$$

Following Karlin [6, p. 46 and p. 87] we say: (i) that A is <u>TP</u> (<u>totally positive</u>) if all the minors of A are non-negative; (ii) that A is <u>NTP</u> if A is non-singular and TP; (iii) that A is <u>STP</u> (<u>strictly totally positive</u>) if all the minors of A are strictly positive; (iv) that A is <u>oscillatory</u> if A is TP and A^m is STP for some positive integer m.

Following Gantmacher and Krein [4, p. 86] we say: (i) that A is <u>ZR</u> (<u>zeichenregulär</u>) if all the signed minors of A, namely

$$(-1)^{\sum_{k=1}^{p} (\alpha_k + \beta_k)} A \begin{pmatrix} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_p \end{pmatrix} ,$$

are non-negative; (ii) that A is <u>NZR</u> (<u>nichtsingulär und zeichenregulär</u>) if A is non-singular and ZR; (iii) that A is <u>SZR</u> (<u>streng zeichenregulär</u>) if all the signed minors of A are strictly positive. It should be noted that the English equivalent of "zeichenregulär", namely "signregular", is used in a slightly different sense by Karlin [6, p. 47].

If A is a lower triangular (upper triangular) matrix, the minors (1.1) for which $\beta_k \le \alpha_k$ $(\beta_k \ge \alpha_k)$ for $1 \le k \le p$ will be called the <u>non-trivial minors</u> of A. The remaining minors of A, the <u>trivial minors</u>, are obviously equal to zero, We say that A is <u>ΔTP</u>, <u>ΔNTP</u>, <u>ΔSTP</u>, <u>Δ-oscillatory</u>, <u>ΔZR</u>, <u>ΔNZR</u>, or <u>ΔSZR</u>, if A is a triangular matrix and the appropriate inequalities are satisfied by the non-trivial minors of A.

We will say that A has an <u>LU-factorization</u> (<u>UL-factorization</u>) if $A = LU$ $(A = UL)$ where L is a lower triangular matrix and U is an upper triangular matrix.

The motivation behind the present work came from the study of finite difference methods for boundary value problems for ordinary differential equations. For example (Henrici [5, p. 347]) the boundary value problem

$$\left.\begin{aligned} \ddot{x}(t) &= f(t), \qquad 0 \le t \le 1, \\ x(0) &= x(1) = 0, \end{aligned}\right\} \tag{1.2}$$

leads to finite difference equations of the form

$$JX = B \tag{1.3}$$

where X and B are n-vectors and J is an $n \times n$ tri-diagonal NZR matrix. The numerical solution of (1.3) is usually carried out by computing the LU-factorization of J (Henrici [5, p. 352]). Theorem 1.1 is a generalization of results obtained by the author while studying (1.3) (Cryer [1]).

The main result of the present paper is the following theorem which is proved in section 5:

<u>Theorem 1.1</u>

Let P denote one of the following properties: NTP, STP, oscillatory, NZR, SZR. Then A has property P iff A has an LU-factorization such that L and U have property ΔP. Also, A has property P iff A has a UL-factorization such that L and U have property ΔP.

2. PRELIMINARIES

To simplify the notation we will use multi-subscripts (Marcus and Minc [8, p. 9]). If $1 \leq p \leq n$ then $Q^{(p,n)}$ will denote the set of strictly increasing sequences $\alpha = \{\alpha_1, \ldots, \alpha_p\}$ of p integers chosen from $1, \ldots, n$. If $\alpha \in Q^{(p,n)}$ we set $\ell(\alpha) = p$, $|\alpha| = \sum_{k=1}^{p} \alpha_k$, and $d(\alpha) = \sum_{k=1}^{p-1} (\alpha_{k+1} - \alpha_k - 1) = \alpha_p - \alpha_1 - (p-1)$. The elements of $Q^{(p,n)}$ are partially ordered as follows: if $\alpha, \beta \in Q^{(p,n)}$ then $\alpha \leq \beta$ if $\alpha_k \leq \beta_k$ for $1 \leq k \leq p$. The infinum of the lattice $Q^{(p,n)}$ is denoted by $\sigma^{(p)} = \{1, 2, \ldots, p\}$.

Using multi-subscripts the minor (1.1) will be written as $A(\alpha;\beta)$. Thus, if A is a lower triangular (upper triangular) matrix, the non-trivial minors $A(\alpha;\beta)$ are those for which $\alpha \geq \beta$ $(\alpha \leq \beta)$. In particular, if A has an LU-factorization the Binet-Cauchy expansion (Marcus and Minc [8, p. 14]) takes the form

$$A(\alpha;\beta) = \sum_{\substack{\gamma \in Q^{(p,n)} \\ \gamma \leq \alpha, \beta}} L(\alpha;\gamma)\, U(\gamma;\beta). \tag{2.1}$$

The following important theorems (or variants thereof) are proved by Gantmacher and Krein [4; p. 299, p. 115, and p. 308] and Karlin [6; p. 85, p. 93, and p. 88].

Theorem 2.1

A is STP iff $A(\alpha;\beta) > 0$ for all $\alpha, \beta \in Q^{(p,n)}$, $1 \leq p \leq n$, such that $d(\alpha) = d(\beta) = 0$.

Theorem 2.2

Let A be TP. Then A is oscillatory iff (i) A is non-singular (ii) $a_{i,i+1} > 0$ and $a_{i+1,i} > 0$ for $1 \leq i \leq n-1$.

Theorem 2.3

Let A be a TP matrix. Then A can be approximated arbitrarily closely by STP matrices.

3. ΔSTP MATRICES

In the present section we discuss the equivalent of Theorem 2.1 for triangular matrices.

Restating the results of Karlin [6, p. 85] we immediately obtain the equivalent of Theorem 2.1:

Theorem 3.1

Let A be a lower triangular (upper triangular) matrix. Then A is ΔSTP iff $A(\alpha;\sigma^{(p)}) > 0$ $(A(\sigma^{(p)};\alpha) > 0)$ for all $\alpha \in Q^{(p,n)}$, $1 \leq p \leq n$, such that $d(\alpha) = 0$.

Theorem 2.1 cannot be generalized to TP matrices as is shown by the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

which is non-singular and all of whose minors are non-negative except for the minors

$$A\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = A\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} = -1.$$

..arly, Theorem 3.1 cannot be generalized to ΔTP matrices as is shown by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

which is non-singular and for which the minors mentioned in Theorem 3.1 are non-negative, but for which

$$A\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = A\begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix} = -1.$$

However, one might try to strengthen the hypotheses so as to exclude the above counterexample. In this way we arrive at

<u>Conjecture 3.1</u>

Let A be a non-singular lower triangular (upper triangular) matrix. Then A is ΔTP iff $A(\alpha;\sigma^{(p)}) \geq 0$ $(A(\sigma^{(p)};\alpha) \geq 0)$ for all $\alpha \in Q^{(p,n)}$, $1 \leq p \leq n$.

We have tried hard, without success, to prove this conjecture, and incline to the view that the conjecture is false. However, the conjecture appears to be true if $n \leq 5$, so that the construction of a counterexample is non-trivial.

4. Δ-OSCILLATORY MATRICES

The equivalent of Theorem 2.2 for triangular matrices is

Theorem 4.1

Let $A = (a_{ij})$ be a ΔTP lower triangular (upper triangular) matrix. Then A is Δ-oscillatory iff (i) A is non-singular and (ii) $a_{i+1,i} > 0$ $(a_{i,i+1} > 0)$ for $1 \leq i \leq n-1$.

Proof: The proof of Theorem 4.1, which is lengthy, is a straightforward modification of the proof of Theorem 2.2 given by Gantmacher and Krein [4, p. 114].

We need the following two lemmas (Gantmacher and Krein [4, p. 108 and p. 114], Karlin [6, p. 89])

Lemma 4.1

Let A be an NTP matrix. Let $\alpha = \{s, s+1, \ldots, t\} \in Q^{(p,n)}$. Then $A(\alpha;\alpha) > 0$.

Lemma 4.2

Let A be a TP $m \times n$ rectangular matrix (that is, all the minors of A are non-negative). Let $\alpha \in Q^{(p,m)}$, $\beta \in Q^{(p,n)}$ be such that $A(\alpha;\beta) = 0$ but

$$A\begin{pmatrix}\alpha_1 \cdots \alpha_{p-1} \\ \beta_1 \cdots \beta_{p-1}\end{pmatrix} \gneq 0 \quad \text{and} \quad A\begin{pmatrix}\alpha_2 \cdots \alpha_p \\ \beta_2 \cdots \beta_p\end{pmatrix} \gneq 0.$$

Then A has rank $p-1$.

Next we establish

Lemma 4.3

Let $A = (a_{ij})$ be Δ-oscillatory. Let $1 \leq s \leq t \leq n$ and $B = (a_{ij})$, $s \leq i,\ j \leq t$. Then B is Δ-oscillatory.

Proof: Only the case when A is a lower triangular matrix will be considered. Furthermore, it suffices to consider the cases $s = 2$, $t = n$ and $s = 1$, $t = n - 1$, since the more general result can be obtained by induction.

We denote by $\hat{Q}^{(p,n)}$ the set of strictly increasing sequences $\hat{\alpha} = \{\hat{\alpha}_1, \ldots, \hat{\alpha}_{p-1}\}$ of $p - 1$ numbers from the set $s, s+1, \ldots, t$. Given $\hat{\alpha} \in \hat{Q}^{(p,n)}$ we construct $\alpha \in Q^{(p,n)}$ by setting $\alpha = \{1, \hat{\alpha}_1, \ldots, \hat{\alpha}_{p-1}\}$ if $s = 2$, $t = n$ and setting $\alpha = \{\hat{\alpha}_1, \ldots, \hat{\alpha}_{p-1}, n\}$ if $s = 1$, $t = n - 1$; we say that α and $\hat{\alpha}$ correspond.

Since A is Δ-oscillatory there is an integer m such that A^m is ΔSTP. It suffices to show that B^m is ΔSTP.

Let $\hat{\alpha}, \hat{\beta} \in \hat{Q}^{(p,n)}$ with $\hat{\alpha} \geq \hat{\beta}$. Let $\alpha, \beta \in Q^{(p,n)}$ correspond to $\hat{\alpha}$ and $\hat{\beta}$. Then $\alpha \geq \beta$ so that, using the Binet-Cauchy expansion, $A^m(\alpha, \beta) = \sum \prod_{s=0}^{m-1} A(\alpha^{(s)}; \alpha^{(s+1)}) > 0$, the sum being taken over all $\alpha^{(s)} \in Q^{(p,n)}$ such that

$$\alpha = \alpha^{(0)} \geq \alpha^{(1)} \geq \cdots \geq \alpha^{(m)} = \beta \qquad (4.1)$$

Since each term in the sum is non-negative, at least one term is positive. Thus, for some choice of $\alpha^{(s)}$ satisfying (4.1) we have

$$A(\alpha^{(s)};\alpha^{(s+1)}) > 0, \qquad 0 \le s \le m-1. \tag{4.2}$$

It follows from (4.1) that to each $\alpha^{(s)}$ there corresponds an $\hat{\alpha}^{(s)}$. Using (4.2) and Lemma 4.1 we see that

$$A(\hat{\alpha}^{(s)};\hat{\alpha}^{(s+1)}) > 0, \qquad 0 \le s \le m-1.$$

Hence

$$B^m(\hat{\alpha};\hat{\beta}) \ge \prod_{s=0}^{m-1} A(\hat{\alpha}^{(s)};\hat{\alpha}^{(s+1)}) > 0 ,$$

and the lemma is proved.

Before stating the next result we introduce some notation. If $\alpha \in Q^{(p,n)}$ we set

$$N(\alpha) = \{\beta \in Q^{(p,n)}; \quad \alpha_k - 1 \le \beta_k \le \alpha_k \quad \text{for } 1 \le k \le p$$
$$\text{and} \quad \alpha_k < \beta_{k+1} \quad \text{for } 1 \le k \le p-1\}.$$

Lemma 4.4

If A is a lower triangular NTP matrix and $a_{i+1,i} > 0$ for $1 \le i \le n-1$, then $A(\alpha;\beta) > 0$ if $\beta \in N(\alpha)$.

Proof: The proof proceeds by induction upon the length, $\ell(\alpha)$, of α.

If $\ell(\alpha) = 1$ and $\beta \in N(\alpha)$ then $A(\alpha;\beta) = a_{ii}$ or $A(\alpha;\beta) = a_{i+1,i}$, so that the lemma is true.

Now assume that the lemma is true if $1 \leq \ell(\alpha) < p \leq n$. To prove the lemma if $\ell(\alpha) = p$, assume the contrary. Then there exists $\alpha = \{\alpha_1, \ldots, \alpha_p\} \in Q^{(p,n)}$ and $\beta = \{\beta_1, \ldots, \beta_p\} \subset N(\alpha)$ such that $A(\alpha;\beta) = 0$ but

$$A\begin{pmatrix}\alpha_1 \cdots \alpha_{p-1} \\ \beta_1 \cdots \beta_{p-1}\end{pmatrix} \neq 0 \quad \text{and} \quad A\begin{pmatrix}\alpha_2, \cdots, \alpha_p \\ \beta_2, \cdots, \beta_p\end{pmatrix} \neq 0 .$$

Let $B = (a_{ij})$, $\alpha_1 \leq i \leq \alpha_p$; $\beta_1 \leq j \leq \beta_p$. Applying Lemma 4.2 we conclude that B has rank $p - 1$. Now let $C = (a_{ij})$, $s \leq i, j \leq t$, where $s = \alpha_1$ and $t = \alpha_1 + p - 1$. Since $\beta \in N(\alpha)$, $\beta_p \geq \alpha_{p-1} + 1 \geq \alpha_1 + p - 1 = t$ and $\beta_1 \leq \alpha_1 = s$ so that C is a minor of order p of B. But B has rank $p - 1$ so $\det(C) = 0$. On the other hand, $\det(C) = A(\gamma;\gamma)$ where $\gamma = \{s, s+1, \ldots, t\} \in Q^{(p,n)}$, so that, by Lemma 4.1, $\det(C) > 0$. We have thus arrived at a contradiction, from which we conclude that the lemma is indeed true if $\ell(\alpha) = p$. The proof of the lemma is therefore complete.

We now turn to the proof of Theorem 4.1.

First, assume that A is Δ-oscillatory. To establish condition (i) it suffices to observe that A^m is ΔSTP. To establish condition (ii) we note from Lemma 4.3 that the submatrix

$$\begin{pmatrix} a_{ii} & 0 \\ a_{i+1,i} & a_{i+1,i+1} \end{pmatrix}$$

is Δ-oscillatory; this implies that $a_{i+1,i} > 0$.

It remains to prove that conditions (i) and (ii) of Theorem 4.1 imply that A is Δ-oscillatory. We do so by proving that $B = A^{n-1}$ is ΔSTP.

Let $\alpha, \beta \in Q^{(p,n)}$, $\alpha \geq \beta$. Define $\alpha^{(s)} = \{\alpha_1^{(s)}, \ldots, \alpha_p^{(s)}\}$ for $0 \leq s \leq n-1$ as follows:

$$\alpha_k^{(s)} = \max\{\beta_k, \alpha_k - \max[0, s+1-k]\}.$$

It is easily verified that $\alpha^{(s)} \in Q^{(p,n)}$ and that $\alpha^{(s+1)} \in N(\alpha^{(s)})$.

Clearly, $\alpha^{(0)} = \alpha$. Since $\beta_k \geq k$ and $\alpha_k \leq n$, $\alpha_k^{(n-1)} = \max\{\beta_k, \alpha_k - n + k\} = \beta_k$, so that $\alpha^{(n-1)} = \beta$. Now,

$$B(\alpha;\beta) \geq \prod_{s=0}^{n-2} A(\alpha^{(s)}; \alpha^{(s+1)}).$$

But, by Lemma 4.4, each of the terms in the above product is strictly positive. Therefore, $B(\alpha;\beta) > 0$ and the proof of the theorem is complete.

5. THE LU-FACTORIZATION OF NON-SINGULAR MATRICES

First we recall the basic result on LU-factorization (Gantmacher [3, p. 35]):

Lemma 5.1

Let $A(\sigma^{(p)};\sigma^{(p)}) \neq 0$ for $1 \leq p \leq n$. Then A has a unique LU-factorization such that L has a unit diagonal. The matrices $L = (\ell_{ij})$ and $U = (u_{ij})$ are defined as follows:

$$A(\sigma^{(p)};\sigma^{(p)})\ell_{ip} = A\begin{pmatrix} 1 & 2 \cdots & p-1 & i \\ 1 & 2 \cdots & p-1 & p \end{pmatrix},$$

for $1 \leq p \leq n$ and $p \leq i \leq n$,

$$A(\sigma^{(p-1)};\sigma^{(p-1)})u_{pj} = A\begin{pmatrix} 1 & 2 \cdots & p-1 & p \\ 1 & 2 \cdots & p-1 & j \end{pmatrix},$$

for $1 \leq p \leq n$ and $p \leq j \leq n$,

where, by convention, $A(\sigma^{(0)};\sigma^{(0)}) = 1$.

Next we prove a special case of Theorem 1.1:

Theorem 5.1

Let A be STP. Then A has an LU-factorization such that L and U are ΔSTP.

Proof: Since A is STP, A is non-singular. Consequently $A(\sigma^{(p)};\sigma^{(p)}) > 0$ for $1 \leq p \leq n$ (Karlin [6, p. 89]) so that, by Lemma 5.1, A has an LU-factorization such that L has a unit diagonal.

Since L has unit diagonal, $L(\sigma^{(p)};\sigma^{(p)}) = 1$ for $1 \leq p \leq n$. Applying the Binet-Cauchy expansion (2.1) we find that if $\alpha \in Q^{(p,n)}$ then

$$\begin{aligned} A(\sigma^{(p)};\alpha) &= \sum_{\gamma \leq \alpha, \sigma^{(p)}} L(\sigma^{(p)};\gamma)\, U(\gamma;\alpha), \\ &= L(\sigma^{(p)};\sigma^{(p)})\, U(\sigma^{(p)};\alpha), \end{aligned}$$

from which it follows that $U(\sigma^{(p)};\alpha) > 0$. Similar arguments show that $L(\alpha;\sigma^{(p)}) > 0$. Using Theorem 3.1 we see that L and U are ΔSTP, and the proof of the theorem is complete.

Next we obtain the equivalent of Theorem 2.3 for ΔNTP matrices.

Theorem 5.2

Let A be a ΔNTP matrix. Then A can be approximated arbitrarily closely by ΔSTP matrices.

Proof: We only consider the case when A is lower-triangular. Let $F_s = (\exp[-s(i-j)^2])$, $1 \leq i, j \leq n$, where $s > 0$. It is known (Karlin [6, p. 88]) that F_s is STP and that $F_s \to I$ (the identity matrix) as $s \to 0$. From Theorem 5.1 it follows that $F_s = L_s U_s$ where L_s and U_s are ΔSTP. The elements of L_s and U_s are of form $M_s/F_s(\sigma^{(p)};\sigma^{(p)})$ where M_s is a minor of F_s (see Lemma 5.1). Since $F_s(\sigma^{(p)};\sigma^{(p)}) \to 1$ as $s \to 0$, it follows that $L_s \to I$ and $U_s \to I$ as $s \to 0$.

Set $A_s = L_s A$. Then A_s is a lower triangular matrix and $A_s \to A$ as $s \to 0$. Using the Binet-Cauchy formula it follows that if $\alpha, \beta \in Q^{(p,n)}$ and $\alpha \geq \beta$ then

$$
\begin{aligned}
A_s(\alpha;\beta) &= \sum_{\gamma \leq \alpha,\beta} L_s(\alpha;\gamma)\, A(\gamma;\beta) \\
&\geq L_s(\alpha;\beta)\, A(\beta;\beta) \\
&> 0 ,
\end{aligned}
$$

since (Karlin [6, p. 89]), $A(\beta;\beta) > 0$. Hence A_s is ΔSTP and the proof of the theorem is complete.

<u>Theorem 5.3</u>

Let A be an NTP matrix. Then A has a LU-factorization such that L and U are ΔNTP matrices.

<u>Proof:</u> According to Theorem 2.3 there exists a sequence of STP matrices A_s which converge to A as $s \to 0$. According to Theorem 5.1, A_s has an LU-factorization, $A_s = L_s U_s$, where L_s and U_s are Δ STP.

The elements of L_s and U_s are of the form $M_s/A_s(\sigma^{(p)};\sigma^{(p)})$ where M_s is a minor of A_s (see Lemma 5.1). Since

$A_s(\sigma^{(p)};\sigma^{(p)}) \to A(\sigma^{(p)};\sigma^{(p)}) > 0$, it follows that $L_s \to L$ and $U_s \to U$ where L and U are ΔTP matrices. Clearly, $A = LU$ and L and U are nonsingular, so that the proof is complete.

Before proceeding to the proof of Theorem 1.1 we introduce some notation.

We denote by Λ the $n \times n$ matrix

$$\Lambda = \begin{pmatrix} 0 & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & 0 \end{pmatrix},$$

and set $A^{\Lambda} = \Lambda A \Lambda$; that is, A^{Λ} is the matrix obtained from A by inverting the order of the rows and columns of A. Then we have

<u>Lemma 5.2</u>

(i) $\Lambda^2 = I$; (ii) if A is non-singular, $(A^{\Lambda})^{-1} = (A^{-1})^{\Lambda}$; (iii) if A is lower triangular (upper triangular) then A^{Λ} is upper triangular (lower triangular); (iv) $(A^{\Lambda})^{\Lambda} = A$; (v) if A is NTP, STP, oscillatory, NZR, SZR, ΔNTP, ΔSTP, Δ-oscillatory, ΔNZR, or ΔSZR then so is A^{Λ}.

We can now prove Theorem 1.1. We consider each case separately.

If A is STP then, by Theorem 5.1, $A = LU$ where L and U are ΔSTP. On the other hand, suppose that $A = LU$ where L and U are ΔSTP. Then, if $\alpha, \beta \in Q^{(p,n)}$,

$$A(\alpha;\beta) = \sum_{\gamma \leq \alpha,\beta} L(\alpha;\gamma)\ U(\gamma;\beta) > 0$$

so that A is STP.

If A is NTP, then by Theorem 5.3 $A = LU$ where L and U are ΔNTP. On the other hand, suppose that $A = LU$ where L and U are ΔNTP. Then A is non-singular, and, since the product of TP matrices is a TP matrix (Gantmacher and Krein [4, p. 86]) A is NTP.

If A is oscillatory, then A is NTP so that, by the preceding arguments, $A = LU$ where L and U are ΔNTP. Let $L = (\ell_{ij})$. Then, by Lemma 5.1,

$$\ell_{p+1,p} = A(\alpha^{(p)};\sigma^{(p)})/A(\sigma^{(p)};\sigma^{(p)})$$

where $\alpha^{(p)} = \{1,2,\ldots,p-1, p+1\}$. Since A is oscillatory and $A(\alpha^{(p)};\sigma^{(p)})$ is a "quasi-principal minor", $A(\alpha^{(p)};\sigma^{(p)}) > 0$ (Gantmacher and Krein [4, p. 115]), so that $\ell_{p+1,p} > 0$. From Theorem 4.1 we conclude that L is Δ-oscillatory. Similar arguments show that U is Δ-oscillatory. On the other hand, suppose that $A = LU$ where L and U are Δ-oscillatory. Then L and U

are ΔNTP and so A is NTP. Let $L = (\ell_{ij})$ and $U = (u_{ij})$. From Theorem 4.1 it follows that $\ell_{i+1,i} > 0$ and $u_{i,i+1} > 0$. Hence $a_{i+1,i} \geq \ell_{i+1,i}\, u_{ii} > 0$ and $a_{i,i+1} \geq \ell_{ii}\, u_{i,i+1} > 0$. From Theorem 2.2 we conclude that A is Δ-oscillatory.

With the aid of the Λ-transformation we use the above results to show that A is NTP, STP, or oscillatory iff $A = UL$ where U and L are ΔNTP, ΔSTP, or Δ-oscillatory, respectively. For example, if A is NTP then, by Lemma 5.2, so is A^{Λ}. Hence $A^{\Lambda} = LU$ where L and U are ΔNTP. Thus $A = (A^{\Lambda})^{\Lambda} = L^{\Lambda} U^{\Lambda}$ where L^{Λ} is an upper triangular ΔNTP matrix and U^{Λ} is a lower triangular ΔNTP matrix.

Finally, the validity of Theorem 1.1 for NZR and SZR matrices follows from the observation (Gantmacher and Krein [4, p. 87]) that A is NZR, SZR, ΔNZR, or ΔSZR iff A^{-1} is NTP, STP, ΛNTP, or ΔSTP, respectively.

We conclude this section with two remarks on LU-factorization.

First, it seems to the author that the representation of a TP matrix as an LU product makes most of the "determinantal" properties of a TP matrix quite obvious. For example, an important property of TP matrices is the inequality

$$A(\sigma^{(n)};\sigma^{(n)}) \leq A(\sigma^{(p)};\sigma^{(p)})\; A(\alpha^{(p)};\alpha^{(p)})$$

where $\alpha^{(p)} = \sigma^{(n)} - \sigma^{(p)}$. Using the LU-factorization of A we obtain the following trivial proof:

$$
\begin{aligned}
&A(\sigma^{(n)};\sigma^{(n)}) \\
&= L(\sigma^{(n)};\sigma^{(n)})\ U(\sigma^{(n)};\sigma^{(n)}), \\
&= L(\sigma^{(p)};\sigma^{(p)})\ L(\alpha^{(p)};\alpha^{(p)})\ U(\sigma^{(p)};\sigma^{(p)})\ U(\alpha^{(p)};\alpha^{(p)}), \\
&= A(\sigma^{(p)};\sigma^{(p)})\ L(\alpha^{(p)};\alpha^{(p)})\ U(\alpha^{(p)};\alpha^{(p)}), \\
&\leq A(\sigma^{(p)};\sigma^{(p)}) \sum_{\gamma \leq \alpha^{(p)}} L(\alpha^{(p)};\gamma)\ U(\gamma;\alpha^{(p)}), \\
&= A(\sigma^{(p)};\sigma^{(p)})\ A(\alpha^{(p)};\alpha^{(p)}).
\end{aligned}
$$

The following result due to Karlin [6, p. 89] can be proved in the same way.

<u>Theorem 5.4</u>

Let A be a NTP matrix. Then all the principal minors of A are strictly positive.

Second, it seems to the author that the LU-factorization will lead to more efficient algorithms for determining whether or not a matrix A with given numerical coefficients is an STP matrix. Using Theorem 2.1 requires that $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$ minors be evaluated and checked for sign. On the other hand, the LU-factorization of A requires only $n^3/3$ multiplications and additions (Fox [2 , p. 175]) and using Theorem 3.1 to check whether L and U are ΔSTP requires that only $2\sum_{i=1}^{n} i = n(n+1)$ minors be evaluated and checked for sign.

6. SINGULAR TP MATRICES

Theorem 6.1

Let A be TP. Then A has an LU-factorization.

Proof: The theorem is trivially true if $n = 1$. It therefore suffices to show that there exist matrices L_1 and U_1 such that,

$$A = L_1 \begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix} U_1 \tag{6.1}$$

where A_1 is a TP $(n-1) \times (n-1)$ matrix.

Set

$$A = \begin{pmatrix} a_{11} & r \\ c & \tilde{A} \end{pmatrix}$$

where $r = (a_{12}, \ldots, a_{1n})$ and $c = (a_{21}, \ldots, a_{n1})^T$ and $\tilde{A}$ is the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the first column and first row of A. Two cases must be considered depending on whether a_{11} is equal to zero or greater than zero.

First let $a_{11} = 0$. Since

$$A\begin{pmatrix} 1 & i \\ 1 & j \end{pmatrix} = -a_{i1}\, a_{1j} \geq 0, \quad \text{for } 2 \leq i, j \leq n,$$

it follows that either $c = 0$ or $r = 0$. If $c = 0$ set $L_1 = I_n$ and

$$U_1 = \begin{pmatrix} 0 & r \\ 0 & I_{n-1} \end{pmatrix} ;$$

if $r = 0$ set $U_1 = I_n$ and

$$L_1 = \begin{pmatrix} 0 & 0 \\ c & I_{n-1} \end{pmatrix} ;$$

and in either case set $A_1 = \widetilde{A}$. Then A_1 is TP and (6.1) holds

Next consider the case when $a_{11} \not\leq 0$. We apply one step of the usual LU-factorization process. That is, we set

$$L_1 = \begin{pmatrix} 1 & 0 \\ c/a_{11} & I_{n-1} \end{pmatrix}, \quad U_1 = \begin{pmatrix} a_{11} & r \\ 0 & I_{n-1} \end{pmatrix},$$

and $A_1 = \widetilde{A} - cr/a_{11}$. To see that A_1 is a TP matrix, we note (Gantmacher [3, p. 26]) that $A_1 = B/a_{11}$ where B is the $(n-1) \times (n-1)$ matrix with elements

$$b_{ij} = A\begin{pmatrix} 1 & i+1 \\ 1 & j+1 \end{pmatrix}, \quad 1 \leq i, \ j \leq n-1.$$

Now let $\alpha, \beta \in Q^{(p,n-1)}$. Let $\widetilde{\alpha} \in Q^{(p+1,n)}$ be defined by

$$\widetilde{\alpha}_k = \begin{cases} 1 , & \text{if } k = 1 \\ \alpha_k + 1, & \text{if } 1 < k \leq p+1 , \end{cases}$$

and let $\widetilde{\beta} \in Q^{(p+1,n)}$ be defined similarly. From Sylvester's identity (Gantmacher and Krein [4, p. 15]), or from first principles, it follows that

$$B(\alpha;\beta) = (a_{11})^{p-1} A(\widetilde{\alpha};\widetilde{\beta}) \geq 0,$$

so that B, and hence A_1, is TP. The proof is thus complete.

We conclude this section with some remarks and conjectures concerning singular TP matrices.

First we draw the reader's attention to the work of Koteljanskii [7] which may be of use in the study of such matrices.

Next we note that Conjecture 3.1 is not true if A is allowed to be singular as is shown by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

Finally, corresponding to Theorems 5.2 and 5.3 we have

<u>Conjecture 6.1</u>:

Let A be a ΔTP matrix. Then A can be approximated arbitrarily closely by ΔSTP matrices.

<u>Conjecture 6.2</u>:

Let A be a TP matrix. Then A has an LU-factorization such that L and U are ΔTP matrices.

Regarding Conjecture 6.2 it should be pointed that the matrices L and U constructed in Theorem 6.1 are not necessarily ΔTP matrices. For example, if

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

then Theorem 6.1 leads to the factorization

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

for which L is not TP. This does not disprove Conjecture 6.2, however, since for singular matrices the LU-factorization is not unique. Indeed, the above matrix A possesses several other LU-factorizations including

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

for both of which L and U are ΔTP matrices.

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