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TOPICS IN THE COMPUTER SIMULATION OF DISCRETE PHYSICS

by

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I. The Harmonic Oscillator

1. Introduction.

No particle in nature exhibits motion which is exactly periodic or completely free from damping. Nevertheless, in plasma dynamics [3], [4], for example, the harmonic oscillator equation

$$m\ddot{x} + \omega^2 x = 0 \quad , \tag{1.1}$$

any nonzero solution of which is both periodic and undamped, does approximate sufficiently well various aspects of the dynamical behavior under study.

In this self-contained paper, we will show that a prototype discrete approach to mechanics [1], whose aim is to simplify the amount and quality of mathematics necessary to solve nonlinear physical problems by means of computer simulation, is, indeed, energy conserving and stable when applied to harmonic motion.

2. Discrete Mechanics.

In discrete mechanics the dynamical equations are difference equations and the solutions of these equations are discrete functions. In one dimension, a basic form [1] of discrete mechanics can be summarized as follows.

For $\Delta t > 0$ and $t_k = k\Delta t$, $k = 0,1,2,\ldots$, let particle P of mass m be located at x_k at time t_k. If $v_k = v(t_k)$ is the velocity of P at t_k, while $a_k = a(t_k)$ is the acceleration of P at t_k, then we will assume the relationships

$$(2.1)\qquad \frac{v_{k+1} + v_k}{2} = \frac{x_{k+1} - x_k}{\Delta t}, \quad k = 0,1,2\ldots$$

$$(2.2)\qquad a_k = \frac{v_{k+1} - v_k}{\Delta t}, \quad k = 0,1,2\ldots \quad .$$

To relate force and acceleration at each t_k, we assume a discrete Newton's equation of the general form

$$(2.3)\qquad ma_k = F(t_k), \quad k = 0,1,2\ldots \quad .$$

To determine the motion of P from given initial conditions x_0 and v_0, one proceeds recursively, where, for each k, a_k is determined from (2.3), then v_{k+1} is determined from (2.2), and finally x_{k+1} is determined from (2.1).

Existence and uniqueness of the solution of an initial value problem follows directly from the recursive structure described above. The critical computational problem, invariably, is that of instability.

Let us show first that the above form of discrete mechanics is energy conserving when applied to harmonic motion.

3. Energy Conservation.

In order to study energy conservation, it is useful to define the concept of work as follows. From an initial time t_0 to a terminal time t_n, the work W done by a force F is defined by

$$W = \sum_{i=0}^{n-1} (x_{i+1} - x_i) F_i \; . \tag{3.1}$$

From (2.1) - (2.3), then,

$$\begin{aligned} W &= m \sum_0^{n-1} (x_{i+1} - x_i)\, a_i \\ &= m \sum_0^{n-1} (x_{i+1} - x_i) \left(\frac{v_{i+1} - v_i}{\Delta t}\right) \\ &= m \sum_0^{n-1} \left(\frac{v_{i+1} + v_i}{2}\right) (v_{i+1} - v_i) \\ &= \frac{m}{2} \sum_0^{n-1} (v_{i+1}^2 - v_i^2) \\ &= \frac{m}{2} v_n^2 - \frac{m}{2} v_0^2 \; . \end{aligned}$$

If kinetic energy K_i at time t_i is defined as usual by

$$K_i = \frac{1}{2} m v_i^2 \; ,$$

then

$$W = K_n - K_0 \; , \tag{3.2}$$

which is the classical result that the work W is the difference of the terminal and initial kinetic energies.

Note that (3.2) is valid independently of the exact form of F in (2.3). But since the concept of potential energy does depend on the precise structure of F, in order to proceed we must select a discrete analogue of (1.1). The analogue we choose is

$$ma_k + \omega^2 \frac{x_{k+1} + x_k}{2} = 0, \quad k = 0,1,2, \ldots, \tag{3.3}$$

so that

$$F_k = -\omega^2 \frac{x_{k+1} + x_k}{2} . \tag{3.4}$$

Now, consider again (3.1). Then, from (3.4),

$$W = \sum_{0}^{n-1} (x_{i+1} - x_i)\left(-\omega^2 \frac{x_{i+1} + x_i}{2}\right)$$

$$= -\frac{\omega^2}{2} x_n^2 + \frac{\omega^2}{2} x_0^2 .$$

If the potential energy V_k at t_k is defined by

$$V_k = \frac{\omega^2}{2} x_k^2 , \tag{3.5}$$

then

$$W = -V_n + V_0 . \tag{3.6}$$

Hence, from (3.2) and (3.6)

$$K_n + V_n = K_0 + V_0 ,$$

which, since n is arbitrary, implies that $K + V$ is invariant of time, which is indeed the law of conservation of energy.

Note that the discretization

$$ma_k + \omega^2 x_k = 0$$

of (1.1) would not have led to the above derivation.

4. Stability.

Next, let us write (2.1) - (2.3), (3.3) in the form

$$m \frac{v_{k+1} - v_k}{\Delta t} + \omega^2 \frac{x_{k+1} + x_k}{2} = 0 \tag{4.1}$$

$$\frac{v_{k+1} + v_k}{2} - \frac{x_{k+1} - x_k}{\Delta t} = 0. \tag{4.2}$$

Consider any nonzero solution of this system of the form

$$v_k = \bar{v} \lambda^k , \quad x_k = \bar{x} \lambda^k . \tag{4.3}$$

Substitution of (4.3) into (4.1) - (4.2) and simplification implies

$$\frac{m}{\Delta t} (\lambda - 1) \bar{v} + \frac{\omega^2}{2} (\lambda + 1) \bar{x} = 0$$

$$\frac{1}{2} (\lambda + 1) \bar{v} - \frac{1}{\Delta t} (\lambda - 1) \bar{x} = 0 .$$

Since the determinant of the above homogeneous system must be zero, it follows that

$$\frac{m}{(\Delta t)^2}(\lambda - 1)^2 + \frac{\omega^2}{4}(\lambda + 1)^2 = 0 ,$$

the roots of which are

$$(4.4) \qquad \lambda_1 = \frac{2\sqrt{m} + i\,\omega\Delta t}{2\sqrt{m} - i\omega\Delta t} , \quad \lambda_2 = \frac{1}{\lambda_1} .$$

But from (4.4), λ_1 and λ_2 are distinct, while $|\lambda_1| = |\lambda_2| = 1$, from which stability follows.

5. Remark.

Finally, let us make a remark about computation in the field of plasmas [4]. Here the well known leap-frog formulas

$$(5.1) \qquad v_{k+\frac{1}{2}} = \frac{x_{k+1} - x_k}{\Delta t} , \quad a_k = \frac{v_{k+\frac{1}{2}} - v_{k-\frac{1}{2}}}{\Delta t}$$

are popular (see, e.g., [4]). Unfortunately, because x and v are never known simultaneously, conservation cannot be established, and (5.1) can be applied to (2.3) only when F is independent of v. Formulas (4.1) - (4.2) or (4.1) and (see [2]) the higher order approximation

$$\frac{3}{2} a_k - \frac{1}{2} a_{k-1} = \frac{v_{k+1} - v_k}{\Delta t}$$

can be applied without such restrictions.

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II. A Discrete Theory of Newtonian Gravitation

1. Introduction.

Discrete mechanics is an approach to the mathematical study of mechanics in which the dynamical equations are difference equations and in which the solutions of these equations are discrete functions [2]. Such an approach is not only compatible with contemporary experimental and theoretical physics [1], but it is in complete harmony with modern digital computer capability. Several forms of discrete mechanics have been developed and applied to nonlinear problems of physical interest [3], [4], and in each case energy conservation was valid when the force considered was only gravity.

In this paper we will explore, in a self-contained fashion, the discrete approach to physical problems in which the acting force is gravitation. Because our interest is in planetary motion, the discussion will be given in two-dimensions.

2. Basic Dynamical Concepts.

For $\Delta t > 0$ and $t_k = k\Delta t$, $k = 0,1,2,\ldots$, let particle P of mass m be located at (x_k, y_k) at time t_k. If $\vec{v}_k = (v_{k,x}, v_{k,y})$ is the velocity of P at t_k, while $\vec{a}_k = (a_{k,x}, a_{k,y})$ is the acceleration of P at t_k, then we will assume first the simple relationships

$$(2.1) \qquad \frac{v_{k+1,x} + v_{k,x}}{2} = \frac{x_{k+1} - x_k}{\Delta t}, \quad \frac{v_{k+1,y} + v_{k,y}}{2} = \frac{y_{k+1} - y_k}{\Delta t};$$

$$k = 0, 1, 2, \ldots$$

$$(2.2) \qquad a_{k,x} = \frac{v_{k+1,x} - v_{k,x}}{\Delta t}, \quad a_{k,y} = \frac{v_{k+1,y} - v_{k,y}}{\Delta t},$$

$$k = 0, 1, 2, \ldots \quad .$$

To relate force and acceleration at each time t_k, we assume a discrete Newtonian equation

$$(2.3) \qquad \vec{F}_k = m\vec{a}_k,$$

where

$$(2.4) \qquad \vec{F}_k = (F_{k,x}, F_{k,y}).$$

The work W done by force $\vec{F}$ on particle P from an initial time t_0 to a terminal time t_n is defined by

$$(2.5) \qquad W = \sum_{k=0}^{n-1} [(x_{k+1} - x_k) F_{k,x} + (y_{k+1} - x_k) F_{k,y}].$$

From (2.1) - (2.4), then,

$$\sum_{k=0}^{n-1} (x_{k+1} - x_k) F_{k,x} = \sum_{0}^{n-1} (x_{k+1} - x_k)\, m\, a_{k,x}$$

$$= m \sum_{0}^{n-1} \left(\frac{x_{k+1} - x_k}{\Delta t}\right) (v_{k+1,x} - v_{k,x})$$

$$= \frac{m}{2} \sum_{0}^{n-1} (v_{k+1,x} + v_{k,x})(v_{k+1,x} - v_{k,x})$$

$$= \frac{m}{2} \sum_{0}^{n-1} (v_{k+1,x}^2 - v_{k,x}^2)$$

$$= \frac{m}{2} v_{n,x}^2 - \frac{m}{2} v_{0,x}^2 \quad .$$

Similarly,

$$\sum_{k=0}^{n-1} (y_{k+1} - y_k) F_{k,y} = \frac{m}{2} v_{n,y}^2 - \frac{m}{2} v_{0,y}^2 \, .$$

Thus, from (2.5),

$$W = \frac{m}{2} (v_{n,x}^2 + v_{n,y}^2) - \frac{m}{2} (v_{0,x}^2 + v_{0,y}^2) \quad . \tag{2.6}$$

If one defines K_i, the kinetic energy at t_i, by

$$K_i = \frac{m}{2} (v_{i,x}^2 + v_{i,y}^2) \, , \tag{2.7}$$

then (2.6) yields, finally, the classical result

$$W = K_n - K_0 \, . \tag{2.8}$$

It should be observed that if $(x_0,y_0) = (x_n,y_n)$, it is not necessary in discrete mechanics that $W = 0$. Though this observation holds also for other forms [3], [4] of discrete mechanics, still, it will have no adverse effects on our development.

3. Planetary Motion. The prototype orbit problem to be considered can be formulated as follows. Let the sun, whose mass is m_1, be positioned at the origin of the XY-coordinate system and let a planet P of mass m be in orbit about the sun. It is assumed that the sun's motion is neglegible and that the gravitational attractive force on the planet acts in a line joining the centers of the two bodies and is given by the discrete law of gravitation

$$\vec{F}_k = (F_{k,x}, F_{k,y}) \,, \tag{3.1}$$

where G is the Newtonian gravitational constant and

$$F_{k,x} = -\frac{Gm_1m_2}{r_kr_{k+1}} \cdot \frac{\frac{x_{k+1}+x_k}{2}}{\frac{2}{\frac{r_k+r_{k+1}}{2}}} \equiv -\frac{Gm_1m_2(x_{k+1}+x_k)}{r_k\,r_{k+1}\,(r_k+r_{k+1})} \tag{3.2}$$

$$F_{k,y} = -\frac{Gm_1m_2(y_{k+1}+y_k)}{r_k\,r_{k+1}\,(r_k+r_{k+1})} \tag{3.3}$$

$$r_k^2 = x_k^2 + y_k^2 \,. \tag{3.4}$$

4. Conservation of Energy.

Let us show now that for the model of planetary motion defined in Section 3, the law of conservation of energy is valid. To do this will require a suitable definition of potential energy, which is arrived at as follows.

Consider again formula (2.5) for work. Then (2.5) and (3.2) - (3.4) imply

$$W = \sum_{k=0}^{n-1} \left[(x_{k+1} - x_k) \left(- \frac{Gm_1m_2(x_{k+1}+x_k)}{r_k r_{k+1} (r_k + r_{k+1})} \right) + (y_{k+1} - y_k) \left(- \frac{Gm_1m_2(y_{k+1}+y_k)}{r_k r_{k+1} (r_k + r_{k+1})} \right) \right]$$

$$= - Gm_1m_2 \sum_{0}^{n-1} \left[\frac{x_{k+1}^2 - x_k^2 + y_{k+1}^2 - y_k^2}{r_k r_{k+1} (r_k + r_{k+1})} \right]$$

$$= - Gm_1m_2 \sum_{0}^{n-1} \left[\frac{r_{k+1}^2 - r_k^2}{r_k r_{k+1} (r_k + r_{k+1})} \right]$$

$$= -Gm_1m_2 \sum_{0}^{n-1} \left[\frac{r_{k+1} - r_k}{r_k r_{k+1}} \right]$$

$$= - Gm_1m_2 \sum_{0}^{n-1} \left[\frac{1}{r_k} - \frac{1}{r_{k+1}} \right]$$

$$= \frac{-Gm_1m_2}{r_0} + \frac{Gm_1m_2}{r_n} .$$

If one defines the potential energy V_k at t_k by

(4.1) $$V_k = -\frac{Gm_1m_2}{r_k} \quad ,$$

then

(4.2) $$W = -V_n + V_0 .$$

Hence, (2.8) and (4.2) imply

(4.3) $$K_n + V_n = K_0 + V_0 .$$

Finally, since n is arbitrary, (4.3) implies that $K + V$ is invariant with respect to time, which is the law of conservation of energy.

5. Examples.

To illustrate the ease with which the discrete approach can be handled on a digital computer, consider the normalized orbit problem [1] in which

(5.1) $$G m_1 = 1$$

and

(5.2) $$x_0 = 0.50,\ y_0 = 0.00,\ v_{0,x} = 0.00,\ v_{0,y} = 1.63 .$$

In the classical continuous formulation, the planet's trajectory is an ellipse with semi-major axis $a = 0.746$ and with period $\tau = 4.04$ seconds.

From (2.3), (3.2) - (3.4), the discrete equations of motion are

$$a_{k,x} = -\frac{x_{k+1} + x_k}{r_k r_{k+1}(r_k + r_{k+1})} \quad , \quad a_{k,y} = -\frac{y_{k+1} + y_k}{r_k r_{k+1}(r_k + r_{k+1})} \quad ,$$

or, equivalently,

(5.3) $$x_{k+1} = x_k + \frac{\Delta t}{2}(v_{k+1,x} + v_{k,x})$$

(5.4) $$y_{k+1} = y_k + \frac{\Delta t}{2}(v_{k+1,y} + v_{k,y})$$

(5.5) $$v_{k+1,x} = v_{k,x} - \frac{(x_{k+1} + x_k)\Delta t}{(x_k^2+y_k^2)^{\frac{1}{2}}(x_{k+1}^2+y_{k+1}^2)^{\frac{1}{2}}[(x_k^2+y_k^2)^{\frac{1}{2}} + (x_{k+1}^2+y_{k+1}^2)^{\frac{1}{2}}]}$$

(5.6) $$v_{k+1,y} = v_{k,y} - \frac{(y_{k+1} + y_k)\Delta t}{(x_k^2+y_k^2)^{\frac{1}{2}}(x_{k+1}^2+y_{k+1}^2)^{\frac{1}{2}}[(x_k^2+y_k^2)^{\frac{1}{2}} + (x_{k+1}^2+y_{k+1}^2)^{\frac{1}{2}}]} \, .$$

The solution of (5.3) - (5.4) for each value of $k = 0, 1, 2, \ldots$, beginning from initial data (5.2) is found by Newton's method with initial guess $x_{k+1}^{(0)} = x_k$, $y_{k+1}^{(0)} = y_k$, $v_{k+1,x}^{(0)} = v_{k,x}$, $v_{k+1,y}^{(0)} = v_{k,y}$. As a typical example of the calculations done, the orbit was generated for $\Delta t = 0.001$ up to $t_{350,000} = 350$. The total computing time was under 5 minutes on the UNIVAC 1108. There were 86+ orbits, the 86^{th} of which is shown in Figure 5.1. For this particular orbit, the period is $\tau = 4.05$

and the average of the x intercepts is $a = 0.746$.

It is most interesting to note that for the relatively large time step $\Delta t = 0.1$, the formulas (5.3) - (5.6) yielded 85^+ orbits up to $t_{3500} = 350$, a result which improves upon calculations done with higher order approximations for velocity and acceleration [3], [4]. This improvement is attributed to the validity of conservation for the discrete formulation of gravitation given by (3.2) - (3.3).

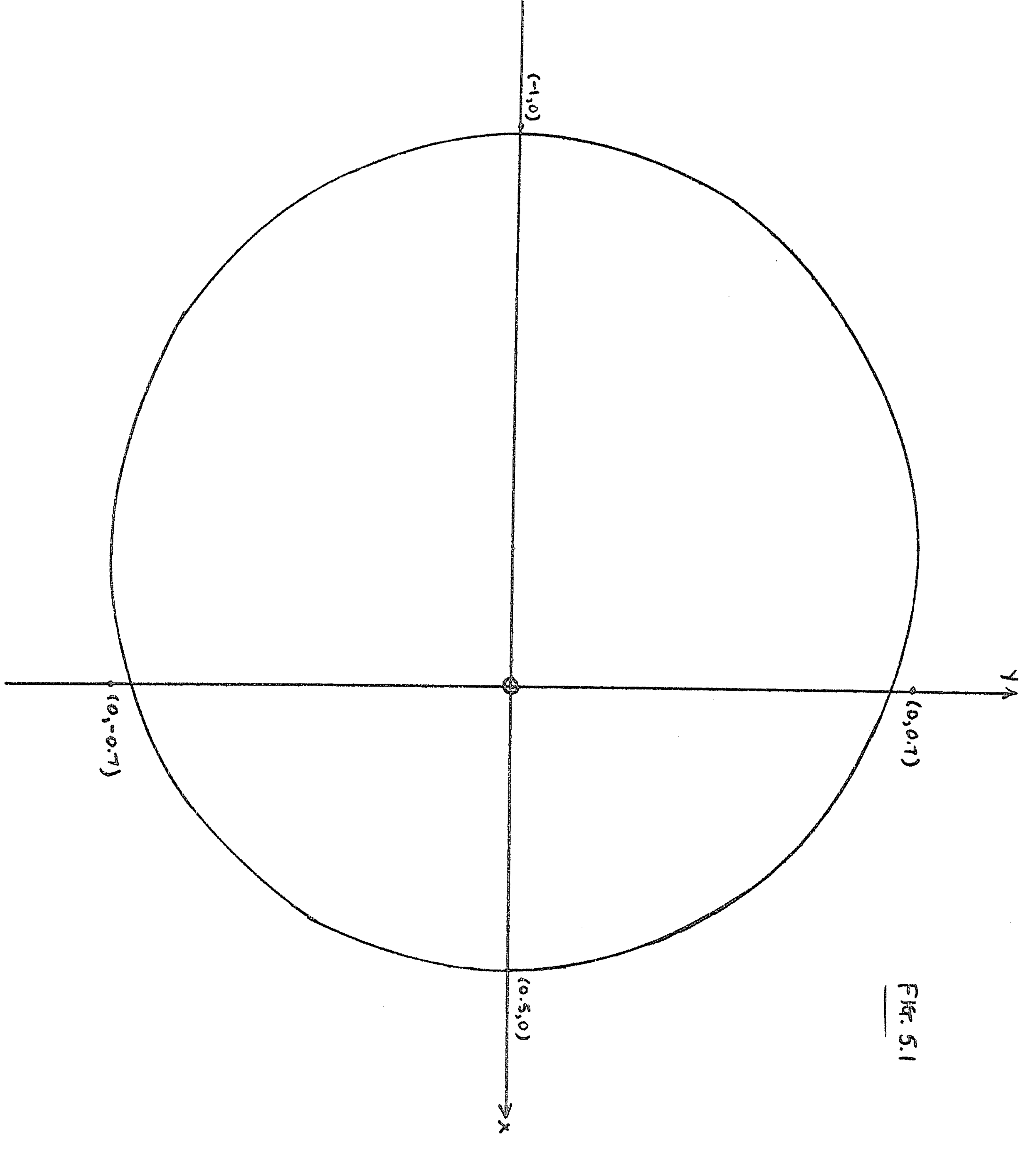

Fig. 5.1

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III. Symmetry in Discrete Mechanics

1. Introduction.

Symmetry, in applied science, means that the physical laws of that science are invariant with respect to the coordinate system used. The objective of this paper is to show, in a self-contained fashion, that various forms of discrete mechanics possess symmetry under translation, under rotation, and with respect to certain coordinate frames in uniform motion.

2. Discrete Mechanics.

Discrete mechanics, which is motivated by modern digital computer capability, is a form of mechanics in which the basic concepts are defined in terms of differences. The fundamental equations are difference equations, and the solutions of initial value problems are discrete functions [2a] - [2h]. In two dimensions, one such simple form can be summarized as follows. For $\Delta t > 0$, let particle P of mass m be at (x_k, y_k) at time $t_k = k\Delta t$, $k = 0,1,2,\ldots$. The velocity $\vec{v}_k$ of P at t_k is defined by

$$\vec{v}_k = (v_{k,x}, v_{k,y}), \quad k = 0,1,2,\ldots \quad , \tag{2.1}$$

where

(2.2) $$\frac{v_{k+1,x} + v_{k,x}}{2} = \frac{x_{k+1} - x_k}{\Delta t} \; ; \quad k = 0,1,2,\ldots$$

(2.3) $$\frac{v_{k+1,y} + v_{k,y}}{2} = \frac{y_{k+1} - y_k}{\Delta t} \; ; \quad k = 0,1,2,\ldots \quad ,$$

while acceleration $\vec{a}_k$ of P at t_k is defined by

(2.4) $$\vec{a}_k = (a_{k,x}, a_{k,y}), \quad k = 0,1,2,\ldots \; ,$$

where

(2.5) $$a_{k,x} = \frac{v_{k+1,x} - v_{k,x}}{\Delta t} \; , \quad k = 0,1,2,\ldots$$

(2.6) $$a_{k,y} = \frac{v_{k+1,y} - v_{k,y}}{\Delta t} \; , \quad k = 0,1,2,\ldots \quad .$$

An equation of the form

(2.7) $$\vec{F}_k = m\vec{a}_k, \quad k = 0,1,2,\ldots \quad ,$$

which is valid componentwise with

(2.8) $$\vec{F}_k = (F_{k,x}, F_{k,y}), \quad k = 0,1,2,\ldots$$

is called a discrete Newton's equation. The determination of the motion of P from (2.7) when (x_0, y_0) and $\vec{v}_0$ are given is called an initial value problem.

It is interesting to note that, with the appropriate definition of the concept of work, the classical laws of conservation of energy and momentum follow readily (see, e.g., 2(c)).

3. Symmetry With Respect To Translation.

Let us show first that the discrete formulation (2.1) - (2.8) is invariant with respect to translation. To do this requires, essentially, proving that $\vec{a}_k$ is, in fact, a vector under the assumption that $\vec{F}_k$ is a vector [1].

Consider then the translation

(3.1) $$x' = x - a, \quad y' = y - b ,$$

where a and b are constants. Assume $\vec{F}_k$ is a vector, so that

(3.2) $$F_{k,x} = F_{k,x'} , F_{k,y} = F_{k,y'} .$$

Thus, from (2.7), (2.8), and (3.2),

(3.3) $$m\, a_{k,x} = F_{k,x'}$$

(3.4) $$m\, a_{k,y} = F_{k,y'} .$$

To complete the proof, one need only show that $a_{k,x} = a_{k,x'}$ and $a_{k,y} = a_{k,y'}$. To do this, we will show only that $a_{k,x} = a_{k,x'}$, since an analogous proof holds for the other component.

From (2.2), it follows [2c] that

(3.5) $$v_{1,x} = \frac{2}{\Delta t} [x_1 - x_0] - v_{0,x}$$

(3.6) $$v_{k,x} = \frac{2}{\Delta t} [x_k + (-1)^k x_0 + 2 \sum_{j=1}^{k-1} (-1)^j x_{k-j}] + (-1)^k v_{0,x}; \; k \geq 2,$$

where $v_{0,x}$ is the given, x-component of $\vec{v}_0$. Since $v_{0,x}$ is invariant under translation,

(3.7) $$v_{0,x} = v_{0,x'} \ .$$

Thus, from (3.1) and (3.5)

(3.8) $$\begin{aligned} v_{1,x} &= \frac{2}{\Delta t}\,[(x'_1 + a) - (x'_0 + a)] - v_{0,x'} \\ &= \frac{2}{\Delta t}\,[x'_1 - x'_0] - v_{0,x'} \\ &= v_{1,x'} \ , \end{aligned}$$

while from (3.1) and (3.6) for $k \geq 2$

(3.9) $$v_{k,x} = \frac{2}{\Delta t}\,[x'_k + a) + (-1)^k (x'_0 + a) + 2 \sum_{j=1}^{k-1} (-1)^j (x'_{k-j} + a)] + (-1)^k v_{0,x'} \ .$$

For k odd, (3.9) yields

(3.10) $$v_{k,x} = \frac{2}{\Delta t}\,[x'_k - x'_0 + 2 \sum_{j=1}^{k-1} (-1)^j x_{k-j}] - v_{0,x'} = v_{k,x'} \ ,$$

while, for k even, (3.9) yields

(3.11) $$v_{k,x} = \frac{2}{\Delta t}[x'_k + x'_0 + 2a + 2 \sum_{j=1}^{k-1} (-1)^j x'_{k-j} + 2 \sum_{j=1}^{k-1} (-1)^j a] + v_{0,x'} = v_{k,x'} \ .$$

Thus, from (3.7), (3.8), (3.10) and (3.11)

(3.12) $$v_{k,x} = v_{k,x'}, \quad k = 0,1,2,\ldots \ .$$

Hence, from (2.4)

$$a_{k,x} = \frac{v_{k+1,x} - v_{k,x}}{\Delta t} = \frac{v_{k+1,x'} - v_{k,x'}}{\Delta t} = a_{k,x'} ,$$

and the proof is complete.

4. Symmetry With Respect To Rotation

Next, let us show that the discrete formulation (2.1) - (2.8) is symmetric with respect to the rotation

(4.1) $$x' = x \cos\theta + y \sin\theta$$

(4.2) $$y' = y \cos\theta - x \sin\theta ,$$

where θ is the smallest positive angle measured in the counterclockwise direction from the X to the X' axes. Assume again that $\vec{F}_k$ is a vector. Then, under rotation,

(4.3) $$F_{k,x'} = F_{k,x} \cos\theta + F_{k,y} \sin\theta$$

(4.4) $$F_{k,y'} = F_{k,y} \cos\theta - F_{k,x} \sin\theta .$$

In addition,

(4.5) $$v_{0,x'} = v_{0,x} \cos\theta + v_{0,y} \sin\theta$$

(4.6) $$v_{0,y'} = v_{0,y} \cos\theta - v_{0,x} \sin\theta$$

Now,

$$(4.7) \qquad v_{1,x'} = \frac{2}{\Delta t}\,[x'_1 - x'_0] - v_{0,x'}$$

$$= \frac{2}{\Delta t}\,[(x_1 \cos\theta + y_1 \sin\theta) - (x_0 \cos\theta + y_0 \sin\theta)]$$

$$- v_{0,x} \cos\theta - v_{0,y} \sin\theta$$

$$= v_{1,x} \cos\theta + v_{1,y} \sin\theta \;,$$

and, similarly,

$$(4.8) \qquad v_{1,y'} = v_{1,y} \cos\theta - v_{1,x} \sin\theta.$$

In the same fashion as above, (3.6) implies

$$(4.9) \qquad v_{k,x'} = v_{k,x} \cos\theta + v_{k,y} \sin\theta \quad , \quad k \geq 2$$

$$(4.10) \qquad v_{k,y'} = v_{k,y} \cos\theta - v_{k,x} \sin\theta \;, \quad k \geq 2 \;.$$

Finally, from (2.5), and (4.3) - (4.10), one has

$$(4.11) \qquad m a_{k,x'} = m\,[\frac{v_{k+1,x'} - v_{k,x'}}{\Delta t}]$$

$$= \frac{m(v_{k+1,x} \cos\theta + v_{k+1,y} \sin\theta) - (v_{k,x} \cos\theta + v_{k,y} \sin\theta)}{\Delta t}$$

$$= m a_{k,x} \cos\theta + m a_{k,y} \sin\theta$$

$$= F_{k,x} \cos\theta + F_{k,y} \sin\theta$$

$$= F_{k,x'} \;.$$

Similarly,

(4.12) $$m\, a_{k,y'} = F_{k,y'} \quad ,$$

and the proof is complete.

5. Symmetry Under Uniform Motion.

In this section it will be assumed that one coordinate system moves with a constant velocity relative to the first. For simplicity, we assume that motion is in the x-direction only and is defined by

(5.1) $$x'_k = x_k - ct_k, \quad k = 0,1,2,\ldots \ ,$$

where c is the speed of the X' axis relative to the X axis. If $v_{0,x}$ is the initial velocity of a particle on the X axis, then its initial velocity on the X' axis is

(5.2) $$v_{0,x'} = v_{0,x} - c \ .$$

Hence, from (3.5), (3.6), (5.1) and (5.2),

(5.3) $$\begin{aligned} v_{1,x} &= \frac{2}{\Delta t}\,[(x'_1 + ct_1) - (x'_0 + ct_0)] - v_{0,x} \\ &= \frac{2}{\Delta t}\,[x'_1 - x'_0] - v_{0,x} + 2c \\ &= \frac{2}{\Delta t}\,[x'_1 - x'_0] - (v_{0,x'}) + c \\ &= v_{1,x'} + c \ , \end{aligned}$$

while, for $k \geq 2$,

$$(5.4)\qquad v_{k,x} = \frac{2}{\Delta t}\ \{(x'_k + ct_k) + (-1)^k\ (x'_0 + ct_0) + 2\sum_{j=1}^{k-1}\ [(-1)^j\ (x'_{k-j} + c\,t_{k-j})]\} + (-1)^k\ v_{0,x}$$

$$= \frac{2}{\Delta t}\ \{x'_k + (-1)^k x'_0 + 2\sum_{j=1}^{k-1}\ [(-1)^j\ x'_{k-j}]\} + (-1)^k\ v_{0,x}$$

$$+ \frac{2c}{\Delta t}\ \{t_k + (-1)^k\ t_0 + 2\sum_{j=1}^{k-1}\ [(-1)^j\ t_{k-j}]\}\ .$$

But,

$$(5.5)\qquad t_k + (-1)^k\ t_0 + 2\sum_{j=1}^{k-1}\ [(-1)^j\ t_{k-j}] = \begin{cases} 0\ , & k\ \text{even} \\ \Delta t\ , & k\ \text{odd}\end{cases}\ .$$

Thus, from (5.2) and (5.4), for k even,

$$(5.6)\qquad v_{k,x} = \frac{2}{\Delta t}\ \{x'_k + x'_0 + 2\sum_{j=1}^{k-1}\ [(-1)^j\ x'_{k-j}]\} + (v_{0,x'} + c)$$

$$= v_{k,x'} + c\ ,$$

while from (5.2) and (5.4), for k odd,

$$(5.7)\qquad v_{k,x} = \frac{2}{\Delta t}\ \{x'_k - x'_0 + 2\sum_{j=1}^{k-1}\ [(-1)^j\ x'_{k-j}] - v_{0,x} + 2c$$

$$= v_{k,x'} + c\ .$$

Finally, under the assumption that $F_{k,x} = F_{k,x'}$, one has from (5.3), (5.4), (5.6) and (5.7) that

$$m\,a_{k,x} = m\,\frac{v_{k+1,x} - v_{k,x}}{\Delta t}$$

$$= m\,\frac{(v_{k+1,x'} + c) - (v_{k,x'} + c)}{\Delta t}$$

$$= m\,a_{k,x'}\ ,$$

from which the symmetry follows readily.

6. Remark.

Other forms of discrete mechanics, stated in terms of one dimensional formulas, can be summarized as follows, where

(6.1) $$F_k = m\,a_k$$

is the discrete Newton's equation:

(a) $\dfrac{v_{k+1} + v_k}{2} = \dfrac{x_{k+1} - x_k}{\Delta t}\ ,\quad \dfrac{a_{k+1} + a_k}{2} = \dfrac{v_{k+1} - v_k}{\Delta t}$

(b) $\dfrac{v_{k+1} + v_k}{2} = \dfrac{x_{k+1} - x_k}{\Delta t}\ ,\quad \dfrac{3}{2}\,a_k - \dfrac{1}{2}\,a_{k-1} = \dfrac{v_{k+1} - v_k}{\Delta t}$

(c) $v_k = \dfrac{x_{k+1} - x_k}{\Delta t}\ ,\quad a_k = \dfrac{v_{k+1} - v_k}{\Delta t}$

(d) $v_{k+1} = \dfrac{x_{k+1} - x_k}{\Delta t}\ ,\quad a_{k+1} = \dfrac{v_{k+1} - v_k}{\Delta t}$

(e) $v_{k+\frac{1}{2}} = \dfrac{x_{k+1} - x_k}{\Delta t}\ ,\quad a_k = \dfrac{v_{k+\frac{1}{2}} - v_{k-\frac{1}{2}}}{\Delta t}$, (where F_k is independent of v_k).

Using the methods of Sections 3-5, it follows easily that each of the above forms also possesses the symmetry properties discussed in these sections, where, special starting procedures, when necessary ([2g], [2h]), define initial acceleration directly from (6.1).

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IV. A Finite Difference Proof That $E = mc^2$

In order to reach the reader whose background was minimal, Taylor and Wheeler [2] developed the theory of special relativity using differences, whenever possible, rather than derivatives. In this note we will show that the classical formula $E = mc^2$ can, in fact, be established entirely without the concept of a derivative. Such a result is not only of interest in itself, but it also affirms the intrinsic role of finite differences in the development of physical models, a result already substantiated by the application of high speed computers in solving nonlinear problems of applied science [1].

First, let us summarize in a convenient way the basic concepts which are necessary for the discussion. Consistently, we will measure not only length, but also time, in the same unit, meters, as follows. A meter of time, denoted by 1 meter/c, is the time it takes for light to travel one meter. Thus,

$$\text{(1)} \qquad 1 \text{ meter/c} = (3.335640)10^{-9} \text{ sec.}$$

It will be assumed that at every point in Euclidean three-space there is a clock which is synchronized with the clock at the origin. When one observes an event and records not only its position but also the time on the clock at that position, one says that an observation has been made in space-time. The coordinates of an event are of the form (x,y,z,t). With regard to the observation of events in space-time, it will be assumed that the coordinate system is inertial and that all laws of physics are the same in every inertial reference frame.

Though time will always be measured in meters, it is sometimes convenient to measure speed conventionally as v meters per second, or in light-time as β meters per meter. Thus, if t_1 and t_2 are any two time readings such that

$$t_2 - t_1 = 1 \text{ meter}/c ,$$

and if a particle in motion along an X-axis is at x_1 at time t_1 and at x_2 at time t_2, then we define β and v at t_1 by the forward differences

(2) $$\beta = \frac{x_2 - x_1}{t_2 - t_1}$$

(3) $$v = \frac{x_2 - x_1}{(t_2 - t_1)(3.335640)10^{-9}} .$$

The units of β are then meters per meter, while the units of v are meters per second. From (2) and (3) one has

(4) $$\beta = v/c .$$

Of course, the speed of light β^* is given by

$$\beta^* = 1 \text{ meter per meter.}$$

Note also that if a particle has a <u>constant</u> speed β, then (2) does yield this exact value from t_1, t_2, x_1 and x_2.

Next, consider two inertial frames moving relative to each other in such a way that their X-axes are collineal. Call one the laboratory frame and call the second, which moves in a positive direction relative to the first, the rocket frame. A light flashes and is recorded in both systems. The problem is to relate the coordinates (x, y, z, t) in the lab frame to the coordinates (x', y', z', t') in the rocket frame. Under the simplifying assumptions that the flash occurs on the X-axes with $y = z = y' = z' = 0$, and that the origins of the two systems are coincident at $t = 0$, then, if β_r is the constant speed of the rocket frame relative to the lab frame, and if $\beta_r < 1$, the desired relationships are a special case of the well-known Lorentz transformation and are given by

(5) $$x = [x' + \beta_r t'] [1 - \beta_r^2]^{-1/2}$$

(6) $$t = [\beta_r x' + t'] [1 - \beta_r^2]^{-1/2} .$$

With regard to the time of an event, observe that the variable τ, given by

(7) $\tau = [t^2 - x^2]^{1/2}$,

can be rewritten by means of (3) and (4) as

(8) $\tau = [(t')^2 - (x')^2]^{1/2}$.

Since τ is the same in both coordinate frames, it is an invariant which, when $t^2 - x^2 > 0$, is defined to be the proper time of an event. In observing two events, say E_1 with $x = x_1$, $t = t_1$ and E_2 with $x = x_2$, $t = t_2$, then

(9) $\Delta\tau = [(t_2 - t_1)^2 - (x_2 - x_1)^2]^{1/2}$

is called the proper time between the two events and is also an invariant under transformation (5) - (6).

Finally, let us now turn to the concept of energy. Consider a particle P of mass m which, for simplicity, is in motion only on an X-axis of, say, a lab frame. Its position is observed at every $\Delta t = (3.335640)10^{-9}$ seconds. Let t_1 and t_2 be the times of two consecutive observations and let x_1 and x_2 be the respective X-coordinates of P at these times. Then the particle's relativistic energy E^* at time t_1 is defined by the forward difference formula

(10) $E^* = m \dfrac{t_2 - t_1}{\Delta\tau}$,

where the units of E^* are units of mass. To convert relativistic energy

E^* to energy E in conventional units requires ([2], p. 103) multiplication of E^* by c^2, so that

(11) $$E = E^* c^2 .$$

By means of (4), (9), and (10), one can then rewrite (11) as

$$\begin{aligned} E &= m \frac{t_2 - t_1}{\Delta \tau} c^2 \\ &= mc^2 / \left(\frac{\Delta\tau}{t_2 - t_1}\right) \\ &= mc^2 / \left[1 - \left(\frac{x_2 - x_1}{t_2 - t_1}\right)^2\right]^{1/2} \\ &= mc^2/(1 - \beta^2)^{1/2} . \end{aligned}$$

If $\beta < 1$, then

$$\begin{aligned} E &= mc^2\left(1 + \frac{\beta^2}{2} + \frac{3}{8}\beta^4 + \cdots\right) \\ &= mc^2 + \frac{m\beta^2 c^2}{2} + \cdots \\ &= mc^2 + \frac{mv^2}{2} + \cdots . \end{aligned}$$

For β small, then,

(12) $$E \sim mc^2 + \frac{mv^2}{2} ,$$

where $mv^2/2$ is the kinetic energy of the particle and mc^2 is called its rest energy, because, when $v = 0$,

$$(13) \qquad E = mc^2 .$$

Thus, the well known formula (13) has followed directly from difference formulations (2), (3) and (10) of the basic physical concepts of velocity and energy.

It should be noted that other relativistic concepts, like momentum, can be defined, similarly, in terms of forward differences.

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