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QUASILINEAR PARABOLIC BOUNDARY VALUE  
PROBLEMS. APPROXIMATE SOLUTIONS AND  
ERROR BOUNDS BY LINEAR PROGRAMMING

by

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Technical Report #128

June 1971



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ABSTRACT

Constrained minimization problems are formulated from a quasilinear parabolic boundary value problem (probably with non-linear boundary conditions), making use of the latter's (conditional) inverse-positive property. Approximate solutions and three error bounds can be obtained by solving these minimization problems by linear programming and discretization techniques. Numerical results are obtained using splines as basis functions.

\*This research was supported in part by NSF Research Grant GJ-0362.



## 1. Introduction

Let  $S$  (Figure 1) be a simply-connected open domain in the  $xt$ -plane bounded by a closed interval  $B_0$  on  $t = 0$ , an open interval  $B_T$  on  $t = T$ , and two curves  $T_i$  defined by the continuous functions  $x = x_i(t)$ ,  $0 < t \leq T$ ,  $i = 1, 2$ .

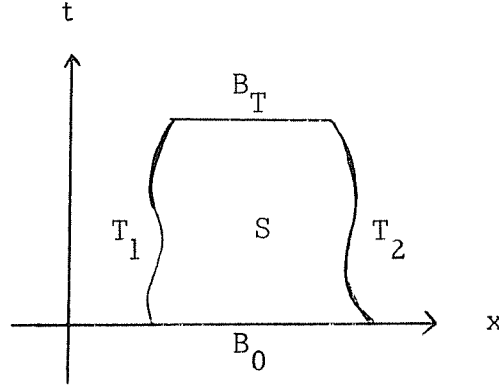


Figure 1

Let  $R_2 = S \cup B_T$  and  $\bar{R} = R_2 \cup T_1 \cup T_2 \cup B_0$ .

Consider two linear operators

$$L_1 \equiv \frac{\partial}{\partial \nu} \quad \text{and} \quad L_2 \equiv \frac{\partial}{\partial t} - a(x, t) \frac{\partial^2}{\partial x^2} + b(x, t) \frac{\partial}{\partial x},$$

where  $a$  and  $b$  are continuous on  $R_2$ ,  $\frac{\partial}{\partial \nu}$  is an outward non-tangential directional derivative and  $L_2$  is a uniform parabolic operator.

In this paper, we apply the linear programming method to determine approximate solutions and error bounds for the first and third boundary value problems (BVP) of the quasilinear parabolic equations

$$Q(g_j) \begin{cases} L_2[u] + g_2(x, t, u) = r_2(x, t) & \text{in } R_2 & (1.1) \\ L_1[u] + g_1(x, t, u) = r_1(x, t) & \text{in } R_1 & (1.2) \\ u = r_0(x, t) & \text{in } R_0 & (1.3) \end{cases}$$

where  $g_j$ ,  $j = 1, 2$ , may be nonlinear in  $x, t$  and  $u$ ;  $r_j$  is bounded in  $R_j$ ,  $j = 0, 1, 2$ ; and  $R_0$  and  $R_1$  are defined as follows:

- i) For first BVP :  $R_0 = T_1 \cup B_0 \cup T_2$ ,  $R_1$  is empty.
- ii) For third BVP :  $R_0 = B_0$ ,  $R_1 = T_1 \cup T_2$  (other combinations are also possible).

In Section 2, we present some definitions and preliminary results. In Section 3, three error bounds are derived for any given approximate solution to the problem  $Q(g_j)$ , making use of the conditionally inverse-positive property of  $L_j$  and the local Lipschitz condition of  $g_j$ ,  $j = 1, 2$ . The first error bound depends on the maximum of the absolute defects of the operator equations. The second error bound is a positively weighted sum of these defects. The

third error bound is a conditional improvement of the second one. In Section 4, by means of these error bound formulas, theoretical constrained minimization problems (CMP) are formulated, by means of which we can determine approximate solutions and error bounds for  $Q(g_j)$ . Section 5 contains some discussions on our approaches. In Section 6, computational schemes making use of linear programming (LP) are suggested to solve these CMP. Section 7 presents some numerical results.

Rosen [1966], at a different approach, seems to be the first one who applied LP to solve the first BVP of nonlinear parabolic equations. By similar approach as this paper, Rosen [1970a] and Cheung [1971] solved BVP of nonlinear elliptic equations.

## 2. Preliminary

Def. 1 Let  $f$  be a function defined on a set  $X \subset E^2$ . Set

$$\|f\|_X \equiv \inf_{(x,t) \in X} |f(x,t)|$$

If there is no ambiguity, the subscript  $X$  may be omitted.

Def. 2  $V(\bar{R}) \equiv C^0(\bar{R}) \cap C^1(\bar{R}_1) \cap C^{2,1}(R_2)$ , where  $\bar{R}_1$  is the closure of  $R_1$  and  $C^{2,1}(R_2)$  denotes the class of functions  $u(x,t)$  whose second order derivative with respect to  $x$  and first order derivative with respect to  $t$  are continuous in  $R_2$ .

Def. 3  $g_j(u) \equiv g_j(x,t,u)$ ;  $g'_j(u) \equiv \frac{\partial g_j(x,t,u)}{\partial u}$ .

Def. 4 For  $j = 1, 2$ , suppose  $g'_j$  exists and is bounded. For a fixed  $v \in V(\bar{R})$  and a constant  $\xi > 0$ , define

$$p_j \equiv p_j(\xi, v, x, t) = \min_{\eta} \{g'_j(\eta) \mid |\eta - v| \leq \xi\} \text{ for each } (x, t) \in R_j,$$

and

$$\hat{g}_j(u) \equiv \hat{g}_j(x, t, u) = \begin{cases} g_j(u) & v + \xi \geq u \geq v - \xi \text{ in } R_j \\ g_j(v + \xi) + (u - v - \xi)g'_j(v + \xi) & u > v + \xi \text{ in } R_j \\ g_j(v - \xi) - (v - \xi - u)g'_j(v - \xi) & v - \xi > u \text{ in } R_j \end{cases} \quad (2.1)$$



The following lemma is obvious:

Lemma 1

Let  $p_j$  and  $\hat{g}_j$  be defined in Def. 4. Then  $\hat{g}_j$  is differentiable with respect to  $u$ , and, for a fixed  $(x, t) \in R_j$  and arbitrary  $u \in V(\bar{R})$ , we have

$$\hat{g}'_j(u) \equiv \frac{\partial \hat{g}_j(x, t, u)}{\partial u} \geq p_j(\xi, v, x, t) \quad (x, t) \in R_j, \quad j = 1, 2 \quad (2.2)$$

or equivalently

$$\hat{g}_j(x, t, v_1) - \hat{g}_j(x, t, v_2) \geq p_j(\xi, v, x, t)(v_1 - v_2) \quad (x, t) \in R_j, \quad j = 1, 2 \quad (2.3)$$

where  $v_1 \geq v_2$ ,  $v_1, v_2 \in V(\bar{R})$ .

In particular,

$$g_j(x, t, v_1) - g_j(x, t, v_2) \geq p_j(\xi, v, x, t)(v_1 - v_2) \quad (x, t) \in R_j, \quad j = 1, 2 \quad (2.4)$$

where  $v_1 \geq v_2$ ,  $|v_i - v| \leq \xi$ ,  $v_i \in V(\bar{R})$ ,  $i = 1, 2$ .

Def. 5 (2.4) is called a one-sided local Lipschitz condition.

Def. 6 The problem  $Q(g_j)$  is said to be inverse-positive if, for

every  $v_1, v_2 \in V(\bar{R})$ , we have

$$\left. \begin{array}{l} L_j[v_1] + g_j(x, t, v_1) \geq L_j[v_2] + g_j(x, t, v_2) \text{ in } R_j, \quad j = 1, 2 \\ v_1 \geq v_2 \text{ in } R_0 \end{array} \right\} \Rightarrow v_1 \geq v_2 \text{ on } \bar{R}.$$

Notation We say:  $[(L_j + g_j, R_j), j = 1, 2; (I, R_0)]$  is inverse-positive, where  $I$  is an identity operator.

An important feature of an inverse-positive problem is that it can have at most one solution. However, the nonlinear problem  $Q(g_j)$  is not always inverse-positive. Let us first consider the linear case with  $g_j(x, t, u) \equiv uk_j(x, t)$ . The following lemma is a direct consequence of Theorem 4 of Protter and Weinberger [1967, p. 172].

Lemma 2

The problem  $Q(uk_j)$  is inverse-positive if  $k_1 > 0$  in  $R_1$  and  $k_2 \geq 0$  in  $R_2$ .

From this lemma we shall derive some other results. First we relax the restriction on  $k_2$ .

Corollary 3

Suppose that  $\frac{\partial}{\partial v} \equiv +\frac{\partial}{\partial x}$  (or  $-\frac{\partial}{\partial x}$ ). The problem  $Q(uk_j)$  is inverse-positive if  $k_1 > 0$  in  $R_1$  and  $k_2$  is bounded below in  $R_2$ .

Proof. Suppose that  $u \in V(\bar{R})$  satisfies

$$u \geq 0 \text{ in } R_0, \quad L_j[u] + k_j u \geq 0 \text{ in } R_j, \quad j = 1, 2 \quad (2.5)$$

Let  $v = ue^{\lambda t}$ , where  $\lambda$  is a constant. Then

$$\left\{ \begin{array}{ll} L_2[v] + (k_2 + \lambda)v = e^{\lambda t} (L_2[u] + k_2 u) \geq 0 & \text{in } R_2 \\ L_1[v] + k_1 v = e^{\lambda t} (L_1[u] + k_1 u) \geq 0 & \text{in } R_1 \\ v = e^{\lambda t} u \geq 0 & \text{in } R_0 \end{array} \right.$$

Since  $k_2$  is bounded below,  $\lambda$  may be chosen so large that  $k_2 + \lambda \geq 0$  in  $R_2$ . By lemma 2, we have  $v \geq 0$  on  $\bar{R}$ . Hence  $u \geq 0$  on  $\bar{R}$ . ■

In particular, the parabolic first BVP ( $R_1$  is empty) is inverse-positive regardless of the sign of  $k_2$ . This is an important difference from the elliptic first BVP which requires  $k_2$  to be nonnegative.

Next, we omit both the requirements  $k_1 > 0$  in  $R_1$  and  $k_2 \geq 0$  in  $R_2$  and consider the following conditionally inverse-positive property.

#### Lemma 4

Suppose there exists a positive  $\mu \in V(\bar{R})$  such that

$$\left\{ \begin{array}{ll} (L_2 + k_2)[\mu] \geq 0 & \text{in } R_2 \\ (L_1 + k_1)[\mu] > 0 & \text{in } R_1 \end{array} \right. \quad (2.6)$$

Then,  $Q(u k_j)$  is inverse-positive.

Proof. Let  $u$  satisfy (2.5). By the transformation  $v = u/\mu$ , we get

$$\left\{ \begin{array}{ll} \frac{\partial v}{\partial t} - a \frac{\partial^2 v}{\partial x^2} + b^* \frac{\partial v}{\partial x} + k_2^* v = \frac{1}{\mu} (L_2 + k_2)[u] \geq 0 & \text{in } R_2 \\ \frac{\partial v}{\partial \nu} + k_1^* v = \frac{1}{\mu} (L_1 + k_1)[u] \geq 0 & \text{in } R_1 \\ v = \frac{u}{\mu} \geq 0 & \text{in } R_0 \end{array} \right.$$

where  $b^* = b - \frac{2a}{\mu} \frac{\partial \mu}{\partial x}$ ,  $k_1^* = \frac{1}{\mu} (\frac{\partial \mu}{\partial \nu} + k_1 \mu) > 0$  in  $R_1$  and  $k_2^* = \frac{1}{\mu} (L_2 + k_2)[u] \geq 0$  in  $R_2$ . By Lemma 2, we have  $v \geq 0$  on  $\bar{R}$ . Hence  $u \geq 0$  on  $\bar{R}$ . ■

For  $k_1 > 0$  and  $k_2 \geq 0$ , the function  $\mu \equiv 1$  satisfies (2.6).

This reduces to Lemma 2 again.

### 3. Derivation of Error Bounds

In this section, three error bounds  $\rho_1$ ,  $\rho_2$  and  $\rho_3$  are derived for any given approximate solution making use of the local Lipschitz condition (2.4) and the conditionally inverse-positive property of Lemma 4.

$\rho_1$  depends on the maximum of the absolute defects of the operation equations.

#### Theorem 5 (Error bound $\rho_1$ )

For the problem  $Q(g_j)$ , the following assumptions are made:

- 1) For any  $u \in V(\bar{R})$ ,  $g'_j(u)$  exists and is bounded on  $\bar{R}_j$  (the closure of  $R_j$ ),  $j = 1, 2$ .
- 2) Let  $\hat{g}_j$  be defined in Def. 4 and  $Q(\hat{g}_j)$  have a solution in  $V(\bar{R})$ .
- 3)  $v \in V(\bar{R})$  is an approximate solution to  $Q(g_j)$ .  $\lambda_0$  is a scalar such that

$$\begin{cases} \lambda_0 \geq |L_j[v] + g_j(v) - r_j| & \text{in } R_j, j = 1, 2 \\ \lambda_0 \geq |v - r_0| & \text{in } R_0 \end{cases} \quad (3.1)$$

- 4) For  $p_j(\xi, v, x, t)$  as defined in Def. 4, there exists a solution  $\mu \in V(\bar{R})$  to

$$\left\{ \begin{array}{l} (L_j + p_j(\xi, v, x, t))[\mu] \geq 1 \quad \text{in } R_j, \quad j = 1, 2 \\ \mu \geq 0 \quad \text{in } R_1 \cup R_2 \end{array} \right\} \quad (3.2)$$

$$\left\{ \begin{array}{l} \mu \geq 1 \quad \text{on } R_0 \end{array} \right\} \quad (3.3)$$

$$\left\{ \begin{array}{l} \xi \geq \lambda_0 \mu \quad \text{on } \bar{R} \end{array} \right\} \quad (3.4)$$

Then, there exists exactly one solution  $u \in V(\bar{R})$  of  $Q(g_j)$  such that

$$|u(x, t) - v(x, t)| \leq \rho_1(x, t) \equiv \lambda_0 \mu(x, t) \quad \text{on } \bar{R} \quad (3.5)$$

Proof. We first prove that

$$[(L_j + \hat{g}'_j, R_j), j = 1, 2; (I, R_0)] \text{ is inverse-positive.} \quad (3.6)$$

In fact, by (2.2) and (3.2), there exists a positive  $u \in V(\bar{R})$  such that

$$(L_j + \hat{g}'_j)[u] \geq (L_j + p_j)[u] \geq 1 \quad \text{in } R_j, \quad j = 1, 2. \quad (3.7)$$

Hence (3.6) follows from Lemma 4 with  $k_j$  replaced by  $\hat{g}'_j$ .

Now, let  $u$  be a solution of  $Q(\hat{g}_j)$ . For  $j = 1, 2$ , let

$$\hat{g}_j(v) - \hat{g}_j(u) = \hat{g}'_j(\bar{v}_j)(v - u) \quad \text{where } \bar{v}_j = \theta_j v + (1 - \theta_j)u, \quad 0 < \theta_j < 1.$$

Since  $\hat{g}_j(v) = g_j(v)$ , it follows from (3.1) and (3.7) that

$$\begin{aligned} (L_j + \hat{g}'_j(\bar{v}_j))[v + \rho_1 - u] &= (L_j + \hat{g}_j)[v] - (L_j + \hat{g}_j)[u] + (L_j + \hat{g}'_j(\bar{v}_j))[\rho_1] \\ &= (L_j + g_j)[v] - r_j + \lambda_0 (L_j + \hat{g}'_j(\bar{v}_j))[u] \geq -\lambda_0 + \lambda_0 = 0 \quad \text{in } R_j \end{aligned} \quad (3.8)$$

On the boundary segment  $R_0$ , (3.1) and (3.3) imply

$$v + \rho_1 - u = v + \lambda_0 \mu - u \geq v + \lambda_0 r_0 \geq 0 \quad \text{in } R_0 \quad (3.9)$$

Hence, by (3.6), (3.8) and (3.9) we get

$$v + \rho_1 \geq u \quad \text{on } \bar{R}.$$

Similarly, we have

$$v - \rho_1 \leq u \quad \text{on } \bar{R}.$$

Next, by (3.4)

$$|v - u| \leq \rho_1 \leq \xi \quad (3.10)$$

But then (2.1) implies that  $u$  is also a solution of  $Q(g_j)$ .

Lastly, suppose that  $Q(g_j)$  has two solutions  $u_1$  and  $u_2 \in V(\bar{R})$ , both satisfying (3.10). For  $j = 1, 2$ , set  $g'_j(\bar{v}_j)(u_1 - u_2) = g_j(u_1) - g_j(u_2)$ , where  $\bar{v}_j = \theta_j u_1 + (1 - \theta_j) u_2$ ,  $0 < \theta_j < 1$ . Then,  $|\bar{v}_j - v| \leq \xi$ ,

$$\hat{g}'_j(\bar{v}_j) = g'_j(\bar{v}_j) \quad \text{in } R_j, j = 1, 2,$$

and

$$\begin{cases} (L_j + \hat{g}'_j)[u_1 - u_2] = (L_j + g_j)[u_1] - (L_j + g_j)[u_2] = 0 & \text{in } R_j, j = 1, 2 \\ u_1 - u_2 = 0 & \text{in } R_0 \end{cases} \quad (3.11)$$

Since  $[(L_j + \hat{g}'_j, R_j), j = 1, 2; (I, R_0)]$  is inverse-positive, (3.11)

has the only solution  $u_1 - u_2 \equiv 0$  on  $\bar{R}$ . ■

The following theorem, which gives the second error bound  $\rho_2$ , can be proved in a similar manner.

Theorem 6 (Error bound  $\rho_2$ )

For the problem  $Q(g_j)$ , the following assumptions are made:

- 1) For any  $u \in V(\bar{R})$ ,  $g'_j(u)$  exists and is bounded on  $\bar{R}_j$  (the closure of  $R_j$ ),  $j = 1, 2$ .
- 2) Let  $\hat{g}_j$  be defined in Def. 4 and  $Q(\hat{g}_j)$  have a solution in  $V(\bar{R})$ .
- 3)  $v \in V(\bar{R})$  is an approximate solution to  $Q(g_j)$ .  $\lambda_j$ ,  $j = 0, 1, 2$ , are scalars such that

$$\begin{cases} \lambda_j \geq |L_j[v] + g_j(v) - r_j| & \text{in } R_j, j = 1, 2 \\ \lambda_0 \geq |v - r_0| & \text{in } R_0 \end{cases}$$

- 4) For  $p_j(\xi, v, x, t)$  as defined in Def. 4, let  $\bar{p}_j \equiv \bar{p}_j(\xi, v)$  satisfy  $\bar{p}_j \leq p_j(\xi, v, x, t)$   $(x, t) \in R_j$  and  $\lambda_j - \lambda_0 \bar{p}_j \geq 0$ .
- 5) Let there exist a solution  $u \in V(\bar{R})$  to

$$\left\{ \begin{array}{l} (L_j + p_j(\xi, v, x, t))[u] \geq 1 \quad \text{in } R_j, j = 1, 2 \\ u \geq 0 \quad \text{on } \bar{R} \\ \xi \geq \lambda_0 + u \sum_{j=1}^2 (\lambda_j - \lambda_0 \bar{p}_j) \quad \text{on } \bar{R} \end{array} \right\} \quad (3.12)$$



then, there exists exactly one solution  $u \in V(\bar{R})$  of  $Q(g_j)$  such that

$$\begin{cases} |u(x,t) - v(x,t)| \leq \rho_2(x,t) \equiv \lambda_0 + u(x,t) \sum_{j=1}^2 (\lambda_j - \lambda_0 \bar{p}_j) \\ \text{on } \bar{R}. \end{cases} \quad (3.13)$$

Remark (Error bound  $\rho_3$ )

In addition to  $\rho_2$ , it can be shown by similar proof as Theorem 6 that

$$\rho_3 \equiv \lambda_0 + (\hat{\lambda} - \lambda_0 \hat{p}) u(x,t) \quad (3.14)$$

is also an error bound, where  $\hat{\lambda} = \max(\lambda_1, \lambda_2)$ ,  $\hat{p} = \min(\bar{p}_1, \bar{p}_2)$ . If  $\bar{p}_j \leq 0$ ,  $j = 1, 2$ , then  $(\hat{\lambda} - \lambda_0 \hat{p}) \leq \sum(\lambda_j - \lambda_0 \bar{p}_j)$  and hence  $\rho_3 \leq \rho_2$ . However, this is not necessarily true if some  $\bar{p}_j$  are positive.

The basic idea of Theorem 5 and Theorem 6 is as follows: In case  $[(L_j + g'_j, R_j), j = 1, 2; (I, R_0)]$  is not inverse-positive in the whole domain of  $u$ , we want to find a set  $Z = \{u | v - \rho \leq u \leq v + \rho, (x, y) \in \bar{R}\}$  in which it has this property (i.e. locally inverse-positive) and hence  $Q(g_j)$  has at most one solution in  $Z$ . This is done by first taking an approximate solution  $v$  as the 'center' of  $Z$ . If  $\rho$  is determined in Theorem 5 or Theorem 6, it can be a possible 'radius' of  $Z$ . The constrained minimization problem formulated in the next section is devised for finding the 'smallest possible'  $\rho$  such that  $Z$  may 'trap' a solution of  $Q(g_j)$ .

#### 4. Constrained Minimization Problems

In this section, we formulate from the error bound formulas (3.5) and (3.13) of the last section two constrained minimization problems. In solving them by some numerical techniques and linear programming method, we can obtain approximate solutions and the corresponding error bounds for the problem  $Q(g_j)$ .

##### Notations

For a given function  $v^k$ , define

$$\begin{aligned} g_j(v^k) &\equiv g_j(x, t, v^k); \\ q_j^k &\equiv g'_j(v^k) \equiv \frac{\partial g_j(x, t, v^k)}{\partial u}; \text{ and} \\ G_j^k &\equiv v^k q_j^k - g_j(v^k) + r_j. \end{aligned}$$

##### Constrained minimization problem one (CMP1)

Suppose  $Q(g_j)$  satisfy the assumptions 1) and 2) of Theorem 5.

Step 1 Starting with suitable  $v^0 \in V(\bar{R})$ , let  $v^k \in V(\bar{R})$  and  $\delta_0 \in E^1$

$$\begin{aligned} &\text{solve} \\ \min_{v, \delta_0} &\left\{ \delta_0 \left| \begin{array}{l} \delta_0 \geq (L_j + q_j^{k-1})[v] - G_j^{k-1} \geq -\delta_0 \text{ in } R_j, j=1,2 \\ \delta_0 \geq v - r_0 \geq -\delta_0 \text{ in } R_0 \end{array} \right. \right\} \quad (4.1) \end{aligned}$$

$$k = 1, 2, \dots$$

Step 2 For any approximate solution  $v$  obtained in Step 1, let

$$\lambda_0 = \max \{ \|v - r_0\|_{R_0}, \|L_j[v] + g_j(v) - r_j\|_{R_j}, j=1,2 \} \quad (4.2)$$

Starting with a suitable  $\xi^0$ , compute the sequences  $\{\xi^k\}_{k=1}^\infty$

and  $\{u^k\}_{k=1}^\infty$  by the following iterative process:

Let  $u^k \in V(\bar{R})$  and  $\hat{u}^k \in E^1$  solve

$$\min_{u, \hat{u}} \left\{ \hat{u} \left| \begin{array}{l} (L_j + p_j(\xi^{k-1}, v, x, t))[u] \geq 1 \text{ in } R_j, j=1,2 \\ u \geq 1 \text{ in } R_0 \\ \hat{u} \geq u \geq 0 \text{ on } \bar{R} \end{array} \right. \right\} \quad (4.3)$$

$$\text{Set } \xi^k = \lambda_0 \hat{u}^k \quad (4.4)$$

### Theorem 7

In the above iterative process, if at the  $k^{\text{th}}$  cycle we have

$\xi^k \leq \xi^{k-1}$ , then the error bound (3.5) holds with

$$\rho_1(x, t) \equiv \lambda_0 u^k(x, t) \quad (4.5)$$

Proof: Clearly, we have only to show that (3.2) and (3.4) hold for

$\xi = \xi^k$  and  $u = u^k(x, t)$ .

Since  $\hat{u}^k \geq u^k \geq 0$  and, for fixed  $v$ ,  $p_j$  is a monotonic non-decreasing function of  $\xi$ , it follows from (4.3) and (4.4) that

$$(L_j + p_j(\xi^k, v, x, t))[u^k] \geq (L_j + p_j(\xi^{k-1}, v, x, t))[u^k] \geq 1 \text{ in } R_j, j=1,2$$

$$\xi^k = \lambda_0 \hat{u}^k \geq \lambda_0 u^k(x, t) \text{ on } \bar{R} \quad \blacksquare$$

Constrained minimization problem two (CMP2)

Suppose  $Q(g_j)$  satisfy the Assumptions 1) and 2) of Theorem 6.

Given suitable initial approximation  $v^0$  and constants  $\xi^0, \hat{u}^0$  and  $\bar{p}_j^0, j = 1, 2$ , the  $k^{\text{th}}$  cycle of the following iterative process starts with known  $v^{k-1}, \xi^{k-1}, \hat{u}^{k-1}$  and  $\bar{p}_j^{k-1}$ :

Step 1 Let  $v^k(x, t) \in V(\bar{R})$  and  $\delta_j^k \in E^1, j = 0, 1, 2$ , solve

$$\min_{v, \delta_j} \left\{ \left( 1 - \hat{u}^{k-1} \sum_{j=1}^2 \bar{p}_j^{k-1} \right) \delta_0 + \hat{u}^{k-1} \sum_{j=1}^2 \delta_j \right\}$$

$$\left. \begin{aligned} \delta_j &\geq (L_j + q_j^{k-1})[v] - G_j^{k-1} \geq -\delta_j, \text{ in } R_j, j = 1, 2 \\ \delta_0 &\geq v - r_0 \geq -\delta_0 \text{ in } R_0 \end{aligned} \right\} \quad (4.6)$$

Step 2 For  $j = 1, 2$ , evaluate

$$\lambda_0^k = \|v^k - r_0\|_{R_0}, \quad \lambda_j^k = \|L_j[v^k] + g_j(v^k) - r_j\|_{R_j} \quad (4.7)$$

$$\bar{p}_j^k(x, t) \equiv p_j(\xi^{k-1}, v^k, x, t) = \min_{\eta} \{g'_j(\eta) \mid |\eta - v^k| \leq \xi^{k-1}\}$$

for fixed  $(x, t)$  (4.8)

and

$$\bar{p}_j^k = \begin{cases} \min \{ \inf_{(x, t) \in R_j} p_j^k(x, t), \lambda_j^k / \lambda_0^k \} & \text{if } \lambda_0^k \neq 0 \\ \inf_{(x, t) \in R_j} p_j^k(x, t) & \text{if } \lambda_0^k = 0 \end{cases} \quad (4.9)$$

Step 3 Let the scalar  $\hat{u}^k$  and the function  $u^k \in V(\bar{R})$  solve

$$\min_{u, \hat{u}} \left\{ \hat{u} \left| \begin{array}{l} (L_j + p_j^k(x, t))[u] \geq 1 \quad \text{in } R_j, j = 1, 2 \\ \hat{u} \geq u \geq 0 \quad \text{on } \bar{R} \end{array} \right. \right\} \quad (4.10)$$

$$\text{Set } \xi^k = \lambda_0^k + \hat{u}^k \sum_{j=1}^2 (\lambda_j^k - \lambda_0^k \bar{p}_j^k) .$$

Theorem 8

In the above iterative process, if at the  $k^{\text{th}}$  cycle we have  $\xi^k \leq \xi^{k-1}$ , then the error bound (3.13) holds with

$$\rho(x, t) \equiv \lambda_0^k + u^k(x, t) \sum (\lambda_j^k - \lambda_0^k \bar{p}_j(\xi^k, v^k)) \quad (4.11)$$

where  $\bar{p}_j(\xi, v) = \inf_{(x, t) \in R_j} p_j(\xi, v, x, t)$ .

Proof: Clearly we have only to show that (3.12) holds with  $\xi = \xi^k$ ,  $v = v^k$  and  $u = u^k$ .

Since  $\hat{u}^k \geq u^k \geq 0$ , and, for fixed  $v^k$  and  $(x, t)$ ,  $p_j$  and  $\bar{p}_j$  are both monotone non-increasing functions of  $\xi$ , it follows from (4.10) that

$$\begin{aligned} (L_j + p_j(\xi^k, v^k, x, t))[u^k] &\geq (L_j + p_j(\xi^{k-1}, v^k, x, t))[u^k] \geq 1 \quad \text{in } R_j, j = 1, 2. \\ \xi^k = \lambda_0^k + \hat{u}^k \sum (\lambda_j^k - \lambda_0^k \bar{p}_j(\xi^{k-1}, v^k)) &\geq \lambda_0^k + u^k(x, t) \sum (\lambda_j^k - \lambda_0^k \bar{p}_j(\xi^k, v^k)) \quad \text{on } \bar{R} \quad \blacksquare \end{aligned}$$

## 5. Discussion

- (1) Numerical results show that CMP2 gives more accurate approximate solutions than CMP1.
- (2) The error bound  $\rho_1$  (4.5) and  $\rho_2$  (4.11) are monotonic decreasing functions of each of the quantities  $\lambda_j$ ,  $j = 0, 1, 2$ , and  $u(x, t)$ . Hence we may say that each approximate solution is obtained by minimizing its error bound in certain sense. Again, it can be shown both theoretically (see Cheung [1970]) and numerically (see section 7) that  $\rho_2$  is a better error bound than  $\rho_1$ .
- (3) Suppose, instead of the local Lipschitz condition (2.4),  $g_j$  satisfies a global Lipschitz condition, i.e. there exists a bounded function  $k_j$ , independent of  $u$ , such that

$$\begin{aligned} g_j(v_1) - g_j(v_2) &\geq (v_1 - v_2)k_j \quad \text{in } R_j, \text{ for all } v_1 \geq v_2, \\ v_i &\in V(\bar{R}) \end{aligned} \quad (5.1)$$

Theorem 5, Theorem 6 and the constrained minimization problems still hold, with  $p_j = k_j$ . However, a local Lipschitz condition has at least two advantages (for detail, see Cheung [1970]):

- i) A global Lipschitz condition may not exist;
- whereas a local Lipschitz condition can always

be constructed, provided  $g_j'$  exists and is bounded.

- ii) A local Lipschitz condition gives a 'better' error bound than a global one.

## 6. Computational Schemes

This section presents some computational schemes of solving the constrained minimization problems of Section 4. Let  $\{\phi_i(x, t)\}_{i=1}^m \subset V(\bar{R})$  by a set of suitably chosen functions. Assume the function  $u$  and the approximate solution  $v$  to be of the form

$$u(\beta; x, t) = \sum_{i=1}^m \beta_i \phi_i(x, t) \quad \text{and} \quad v(\alpha; x, t) = \sum_{i=1}^m \alpha_i \phi_i(x, t).$$

Let  $D_j$  and  $D_j^*$  be two discretizations of the region  $R_j$ , where  $D_j^*$  has finer grid size than  $D_j$  (Figure 2)

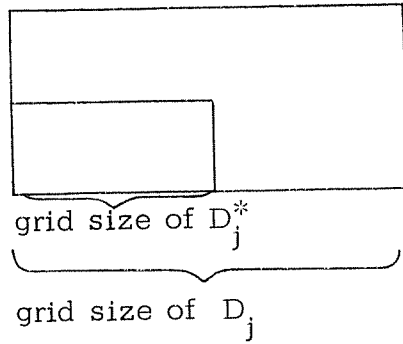


Figure 2

### Computational Schemes One (CS1)

Step 1 Starting with suitable  $v^0 \in V(\bar{R})$ , let  $\alpha_i^k, i = 1, 2, \dots, m$

and  $\delta_0^k$  solve

$$\min_{\alpha_i, \delta_0} \left\{ \delta_0 \mid \begin{cases} \delta_0 \geq \sum_{i=1}^m \alpha_i (L_j + \alpha_j^{k-1}) [\phi_i] - G_j^{k-1} \geq -\delta_0 & \text{in } D_j, j = 1, 2 \\ \delta_0 \geq \sum_{i=1}^m \alpha_i \phi_i - r_0 \geq -\delta_0 & \text{in } D_0 \end{cases} \right. \quad (6.1)$$

$k = 1, 2, 3, \dots$



Step 2 For any approximate solution  $v \equiv \sum \alpha_i^k \phi_i$  obtained in Step 1, let

$$\lambda_0 = \max \{ \|v - r_0\|_{D_0^*}, \|L_j[v] + g_j(v) - r_j\|_{D_j^*}, j = 1, 2 \}.$$

(The superscript  $k$  is omitted from  $v$  and  $\lambda_0$ ).

Step 3 Let  $\varepsilon_j, j = 0, 1, 2$ , be suitably chosen small positive quantities. With a suitable initial  $\xi^0$ , let  $\{\xi^k\}$  and  $\{\beta_j^k\}_{j=0}^m$  be determined by the following iterative process:  
Let  $\beta_i^k, i = 0, 1, \dots, m$ , solve

$$\min_{\beta_i} \left\{ \beta_0 \left| \begin{array}{l} \sum_{i=1}^m \beta_i (L_j + p_j(\xi^{k-1}, v, x, t)) [\phi_i] \geq 1 + \varepsilon_j, \text{ in } D_j, j = 1, 2 \\ \sum \beta_i \phi_i \geq \varepsilon_0 \text{ in } D_1 \cup D_2 \\ \sum \beta_i \phi_i \geq 1 + \varepsilon_0 \text{ in } D_0 \\ \beta_0 \geq \sum \beta_i \phi_i \text{ in } \bar{D} \end{array} \right. \right\} \quad (6.2)$$

where  $\bar{D} = D_0 \cup D_1 \cup D_2$ .

Let  $\hat{\mu}^k = \max_{(x,t) \in D^*} \sum \beta_i^k \phi_i(x,t)$  and  $\xi^k = \lambda_0 \hat{\mu}^k$ , where

$$D^* = D_0^* \cup D_1^* \cup D_2^*.$$

#### Computational Scheme Two (CS2)

Given suitable initial approximation  $v^0$  and constants  $\xi^0, \hat{\mu}^0$  and  $\bar{p}_j^0$ , the  $k^{\text{th}}$  cycle of the following iterative process starts with

known  $\xi^{k-1}$ ,  $\hat{u}^{k-1}$ ,  $\bar{p}_j^{k-1}$  and  $v^{k-1}(x,t) \equiv \sum \alpha_i^{k-1} \phi_i(x,t)$ :

Step 1 Let  $\alpha_i^k$ ,  $i = 1, \dots, m$ , and  $\delta_j^k$ ,  $j = 0, 1, 2$ , solve

$$\min_{\alpha_i, \delta_j} \left\{ (1 - \hat{u}^{k-1} \sum_{j=1}^2 \bar{p}_j^{k-1}) \delta_0 + \hat{u}^{k-1} \sum_{j=1}^2 \delta_j \right\}$$

$$\left. \begin{aligned} \delta_j &\geq \sum \alpha_i (L_j + q_j^{k-1}) [\phi_i] - G_j^{k-1} \geq -\delta_j \text{ in } D_j, j = 1, 2 \\ \delta_0 &\geq \sum \alpha_i \phi_i - r_0 \geq -\delta_0 \text{ in } D_0 \end{aligned} \right\} \quad (6.3)$$

Step 2 For  $j = 1, 2$ , evaluate

$$\lambda_0^k = \|v^k - r_0\|_{D_0^*}, \lambda_j^k = \|L_j[v^k] + g_j(v^k) - r_j\|_{D_j^*}$$

$$p_j^k(x,t) \equiv p_j(\xi^{k-1}, v^k, x, t) \approx \min_{\eta} \{g_j'(\eta) \mid |\eta - v^k| \leq \xi^{k-1}\} \text{ for fixed } (x, t)$$

and

$$\bar{p}_j^k = \begin{cases} \min \{ \min_{(x,t) \in D_j^*} p_j^k(x,t), \lambda_j^k / \lambda_0^k \} & \text{if } \lambda_0^k \neq 0 \\ \min_{(x,t) \in D_j^*} p_j^k(x,t) & \text{if } \lambda_0^k = 0 \end{cases}$$

Step 3 Choose suitable small positive constants  $\varepsilon_j$ ,  $j = 0, 1, 2$ . Let

$$\beta_i^k, i = 0, 1, \dots, m, \text{ solve}$$

$$\min_{\beta_i} \left\{ \beta_0 \mid \begin{aligned} \sum_{i=1}^m \beta_i (L_j + p_j^k(x,t)) [\phi_i] &\geq 1 + \varepsilon_j \text{ in } D_j, j = 1, 2 \\ \beta_0 &\geq \sum_{i=1}^m \beta_i \phi_i \geq \varepsilon_0 \text{ on } \bar{D} \equiv D_0 \cup D_1 \cup D_2 \end{aligned} \right\} \quad (6.4)$$

Let  $\hat{u}^k = \max_{D^*} \sum_{i=1}^m \beta_i^k \phi_i^k$  and  $\xi^k = \lambda_0^k + \hat{u}^k \sum_{j=1}^2 (\lambda_j^k - \lambda_0^k \bar{p}_j^k)$ , where  
 $D^* = D_0^* \cup D_1^* \cup D_2^*$ .

Choice of initial values for the parameters  $\xi^0, \hat{u}^0, \bar{p}_j^0$  and approximation  $v^0$

Theorems 7 and 8 imply that the initial  $\xi^0$  should overestimate the error of the initial approximation  $v^0$ . Usually if  $\xi^0$  is large enough we should have  $\xi^0 \geq \xi^1$  and hence an error bound is obtained at a single cycle. However difficulty may arise that if  $\xi^0$  is too large, (6.2) or (6.4) may have no feasible solution.

Without better values, we may set  $\hat{u}^0 = 1$ ,  $\bar{p}_j^0 = 0$ ,  $j = 1, 2$  and  $v^0(x, t) = 0$  (or 1).

Choice of the parameters  $\varepsilon_j$ ,  $j = 0, 1, 2$

In (6.2) and (6.4), the positive quantities  $\varepsilon_j$  are added to the right sides. If the density of discretization is fine enough and the differential operators satisfy some Lipschitz conditions, it can be shown (for detail, see Cheung [1970]) that a solution  $u = \sum \beta_i \phi_i$  of the discretized problem (6.2) (or (6.4)) also satisfies the inequalities (4.3) (or (4.10)) over the whole region. Therefore, the error bound is valid over  $\bar{R}$ .

Criterion for terminating the iterative process

The iterative process may stop whenever  $\xi^k \leq \xi^{k-1}$ . However, in practice, this is usually satisfied at the first cycle. To obtain better accuracy, we may use the following criterion:

Let  $\delta_2^k = \|L_2[v^k] + q_2^{k-1} v^k - G_2^{k-1}\|_{D_2}$  and  $\lambda_2^k = \|L_2[v^k] + g_2(v^k) - r_2\|_{D_2^*}$ .

For a preassigned quantity  $\epsilon_N$  (convergence tolerance of Newton's method), the iterative process is stopped at the  $k^{\text{th}}$  cycle if

$$\frac{|\lambda_2^k - \delta_2^k|}{\lambda_2^k} \leq \epsilon_N. \quad (6.5)$$

Since  $\lambda_2^k$  and  $\delta_2^k$  are quantities obtained during the iterative process, only little additional computation is required.

#### Linear programming formulation

It is easy to show that (6.1), (6.2), (6.3) and (6.4) are linear programs of the form (for detail, see Rosen [1970]):

$$\min_{\pi} \{d'\pi \mid A'\pi \geq -c\} \quad (6.6)$$

where  $d$ ,  $\pi$  and  $c$  are vectors and  $A$  is a matrix.  $'$  denotes the transpose. With  $w = -\pi$ , (6.6) is equivalent to the dual problem of a standard linear program (see Dantzig [1963]):

$$\max_w \{d'w \mid A'w \leq c\} \quad (6.7)$$

Instead of solving (6.7) directly, most available computer linear programming code (e.g. SIMPLX [1969]) are devised so as to solve its primal problem

$$\min_z \{c'z \mid Az = d, z \geq 0\} \quad (6.8)$$

(6.6) and (6.8) have the following relations which are well known in the duality theory of linear programming:

(i) If (6.8) has an optimal solution  $z^*$  with optimal base  $B^*$ , then  $\pi^* = -w^* = -(B^*)^{-1}z^*$  is an optimal solution to (6.6). In some computer linear programming codes,  $\pi^*$  is one of the output data. Hence, we can directly obtain an optimal solution to (6.6) by solving (6.8).

(ii) If (6.8) has an infinite (negative) solution, (6.6) has no feasible solution. This fact can be used to test the inverse-positive property of the given problem  $Q(g_j)$ .

#### Sizes of the linear programs

Let  $m$  = total number of basis functions  $\{\phi_i\}_{i=1}^m$ ;

$n$  = total number of grid points over  $\bar{D} = D_0 \cup D_1 \cup D_2$ ; and

$n_0$  = number of grid points over  $D_0$ .

The following table shows the dimensions of the different linear programs:

Table 1

linear program	dimension of A
(6.1)	$(m + 1) \times 2n$
(6.3)	$(m + 3) \times 2n$
(6.2) or (6.4)	$(m + 1) \times (3n - n_0)$

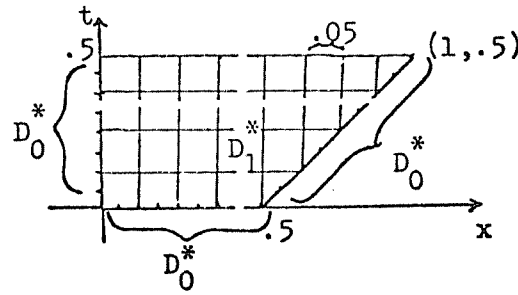
## 7. Numerical Results

### Example 1 (first BVP on a Trapezoid)

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} - (2+t) \frac{\partial^2 u}{\partial x^2} + g(u) = r & 0 < x < t + .5, 0 < t \leq .5 \text{ --- } R_0 \\ u = x^2 + \cos x & 0 \leq x \leq .5, t = 0 \\ u = \cos t^2 & x = 0, 0 \leq t \leq .5 \\ u = x^2 + \cos(x+t^2) & x = t + .5, 0 \leq t \leq .5 \end{array} \right\} R_0$$

where  $g(u) = \exp(.1(u - \cos(x+t^2)))$  and  $r = -2t \sin(x+t^2) - (2+t)(2 - \cos(x+t^2)) + \exp(.1x^2)$ .

Figure 3



Exact solution  $u = x^2 + \cos(x+t^2)$

Algorithm Since  $g'(u)$  is bounded below, Example 1 is inverse-positive.

We determine approximate solutions and error bounds by CS1 and CS2. (See Section 6).

Both algorithms are started with  $v^0 \equiv 0$ ,  $\xi^0 = 1$ ,  $\hat{u}^0 = 1$  and  $\bar{p}^0 = 0$ .

$\|\cdot\|_{R_j}$  is approximately by  $\|\cdot\|_{D_j^*}$ , where  $D_j^*$  are meshes described in

Figure 3. The termination criterion (6.5) is applied with  $\epsilon_N = .0002$ .

Discretization method The linear programs included in CS1 and CS2 are solved with  $\epsilon_0 = \epsilon_1 = .0001$ .  $D_0^n$  and  $D_1^n$  are the same as  $D_0^*$  and  $D_1^*$  but with their grid sizes doubled.

Function space  $S_2^{(3+4,3+2)}$  (bicubic B-spline with 5 knots in the x-direction and 3 knots in the t-direction. See Appendix.)

Computer and LP code CDC 3600, SIMPLX [1969].

Time 6 minutes 57 seconds.

Numerical results After 2 iterations the termination criterion is satisfied and the following results are obtained (omitting superscripts):

	CS 1	CS 2
$\lambda_0$	.025593	.000088
$\lambda_1$		.020331
$\bar{p}$		.079323
max. value of $\mu$	1.040319	.044913
max. error	5.49 E-3 at (.1,.0)	-1.09 E-4 at (.25,.15)
max. error bound	2.66 E-2 at (.30,.0)	1.00 E-3 at (.35,.0)
min. error bound	2.56 E-2 on $t = 0$	8.98 E-5 on $x = 0$

Table 2

Table 2 shows that CS2 gives a much smaller error function  $\mu$  than CS1. This is the main reason why the error bound  $\rho_2$  obtained by CS2 is much better than the error bound  $\rho_1$  obtained by CS1 (by a factor of  $10^{-1.5}$  on the average). The error in the approximate solution obtained by CS2 is also much better than that obtained by CS1 (by a factor of  $10^{-2}$  on the average (Table 3).)

Figure 4 and Figure 5 show respectively the errors of the approximate solutions  $v_1$  and  $v_2$  along 5 lines parallel to the x-axis, Table 4 contains the coefficients of  $v_1$  and  $v_2$ . Within each lattice, the upper entry is for  $v_1$  while the lower for  $v_2$ . Each entry is the coefficient of the product of functions on the top row and leftmost column.



Table 3

Approximate Solution  $v$ , Errors  $(u-v)$ , and Error Bounds  $\rho$ 

		CS 1			CS 2		
x	t	$v_1$	$u-v_1$	$\rho_1$	$v_2$	$u-v_2$	$\rho_2$
%.05	.00	.99577	5.48 E-3	2.59 E-2	1.00124	1.40 E-5	3.16 E-4
%.10	.00	.99952	5.48 E-3	2.61 E-2	1.00499	1.05 E-5	5.12 E-4
%.35	.00	1.05706	4.81 E-3	2.66 E-2	1.06188	-5.43 E-6	1.00 E-3
.45	.00	1.09878	4.16 E-3	2.65 E-2	1.10295	-4.93 E-7	9.08 E-4
.10	.05	1.00132	3.43 E-3	2.59 E-2	1.00472	2.77 E-5	3.83 E-4
.30	.05	1.04201	2.59 E-3	2.62 E-2	1.04460	-6.15 E-7	6.41 E-4
.45	.05	1.10048	1.38 E-3	2.61 E-2	1.10184	1.96 E-5	5.18 E-4
.25	.15	1.02742	-1.82 E-3	2.61 E-2	1.02571	-1.09 E-4	5.55 E-4
%.00	.20	1.00264	-3.44 E-3	2.56 E-2	.99522	-1.77 E-5	8.89 E-5
.25	.20	1.02432	-3.57 E-3	2.62 E-2	1.02083	-8.95 E-5	6.54 E-4
.55	.20	1.13810	-4.66 E-3	2.62 E-2	1.13350	-5.56 E-5	6.36 E-4
%.70	.20	1.23391	-5.44 E-3	2.57 E-2	1.22854	-7.31 E-5	1.80 E-4
.25	.25	1.01870	-4.63 E-3	2.63 E-2	1.01412	-5.02 E-5	7.54 E-4
.50	.25	1.10058	-4.66 E-3	2.65 E-2	1.09594	-1.44 E-5	8.86 E-4
%.75	.25	1.25481	-4.63 E-3	2.59 E-2	1.25020	-1.77 E-5	3.79 E-4
%.00	.40	.99013	-2.91 E-3	2.56 E-2	.98725	-1.77 E-5	8.98 E-5
.35	.40	.99735	-2.10 E-3	2.65 E-2	.99524	-3.69 E-6	9.26 E-4
%.70	.40	1.14160	8.39 E-4	2.63 E-2	1.14242	1.98 E-5	7.04 E-4
.90	.40	1.29543	3.44 E-3	2.56 E-2	1.29885	1.77 E-5	8.98 E-5
.15	.45	.96105	-3.39 E-5	2.61 E-2	.96100	1.38 E-5	5.66 E-4
%.00	.45	.96348	5.44 E-3	2.56 E-2	.96893	-1.77 E-5	8.98 E-5
%.30	.45	.93932	3.20 E-3	2.65 E-2	.94257	-4.10 E-5	8.60 E-4
.65	.45	1.04036	3.75 E-3	2.65 E-2	1.04416	-4.88 E-5	8.72 E-4
.95	.45	1.30989	5.44 E-3	2.60 E-2	1.31534	-1.77 E-5	8.98 E-4

\*grid points

Table 4

Coefficients of  $v_1$  and  $v_2$  (multiplied by 100)

	$\beta(4x+1)$	$\beta(4x)$	$\beta(4x-1)$	$\beta(4x-2)$	$\beta(4x-3)$	$\beta(4x-4)$	$\beta(4x-5)$
$\beta(4t+1)$	2.8455	2.7370	2.7889	3.0084	3.4225	4.0828	0
	2.8714	2.7585	2.8130	3.048	3.4806	4.1875	0
$\beta(4t)$	2.7971	2.7213	2.8253	3.1138	3.5000	4.3093	4.9201
	2.8173	2.7456	2.8472	3.1257	3.5970	4.2870	5.1306
$\beta(4t-1)$	2.8971	2.7777	2.8340	3.0715	3.5075	4.1654	5.1514
	2.8689	2.7532	2.8107	3.0470	3.4814	4.1400	5.0900
$\beta(4t-2)$	2.9481	2.7087	2.670	2.7493	3.0719	3.6377	4.4269
	2.9484	2.7070	2.6310	2.7508	3.0868	3.6715	4.5129
$\beta(4t-3)$	2.6774	2.2746	2.0692	2.0343	2.2755	2.8223	3.9031
	2.8738	2.4085	2.1405	2.0925	2.3009	2.8035	3.7437

Fig. 4 Error Curves of  $(u-v_1)$   
(Example 1, using CS)

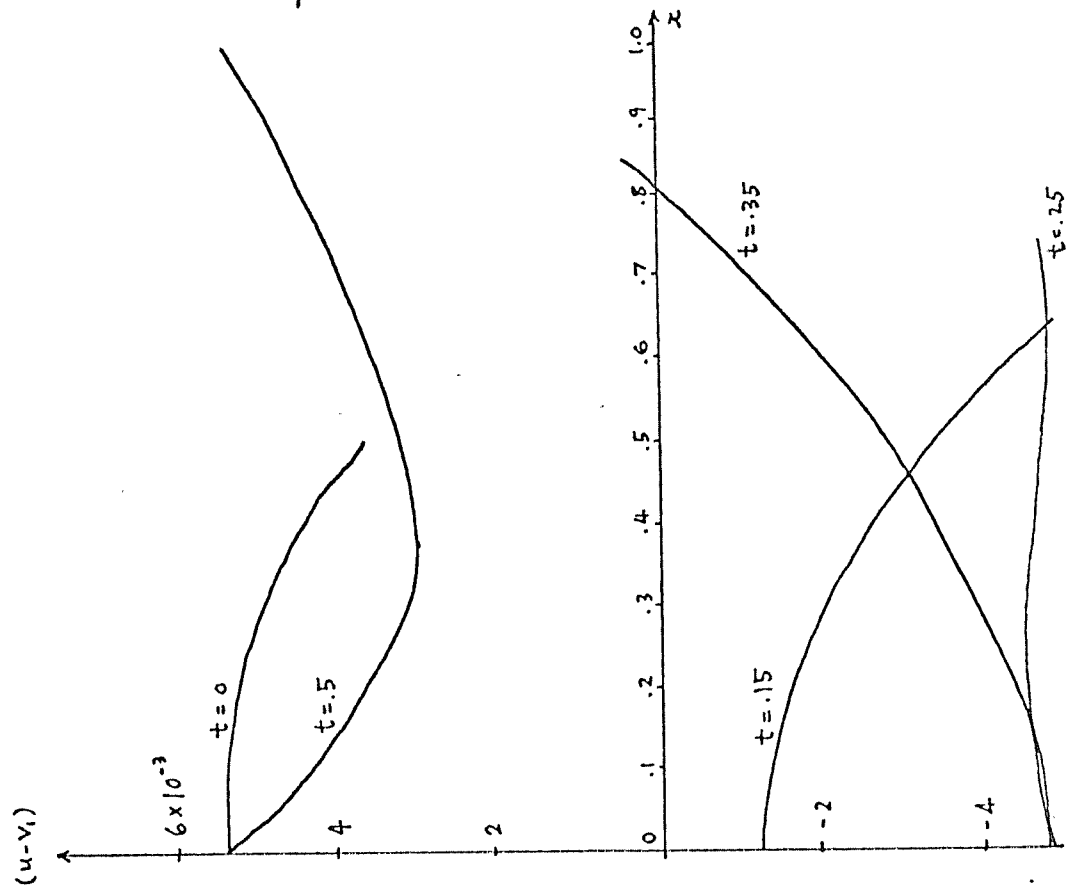
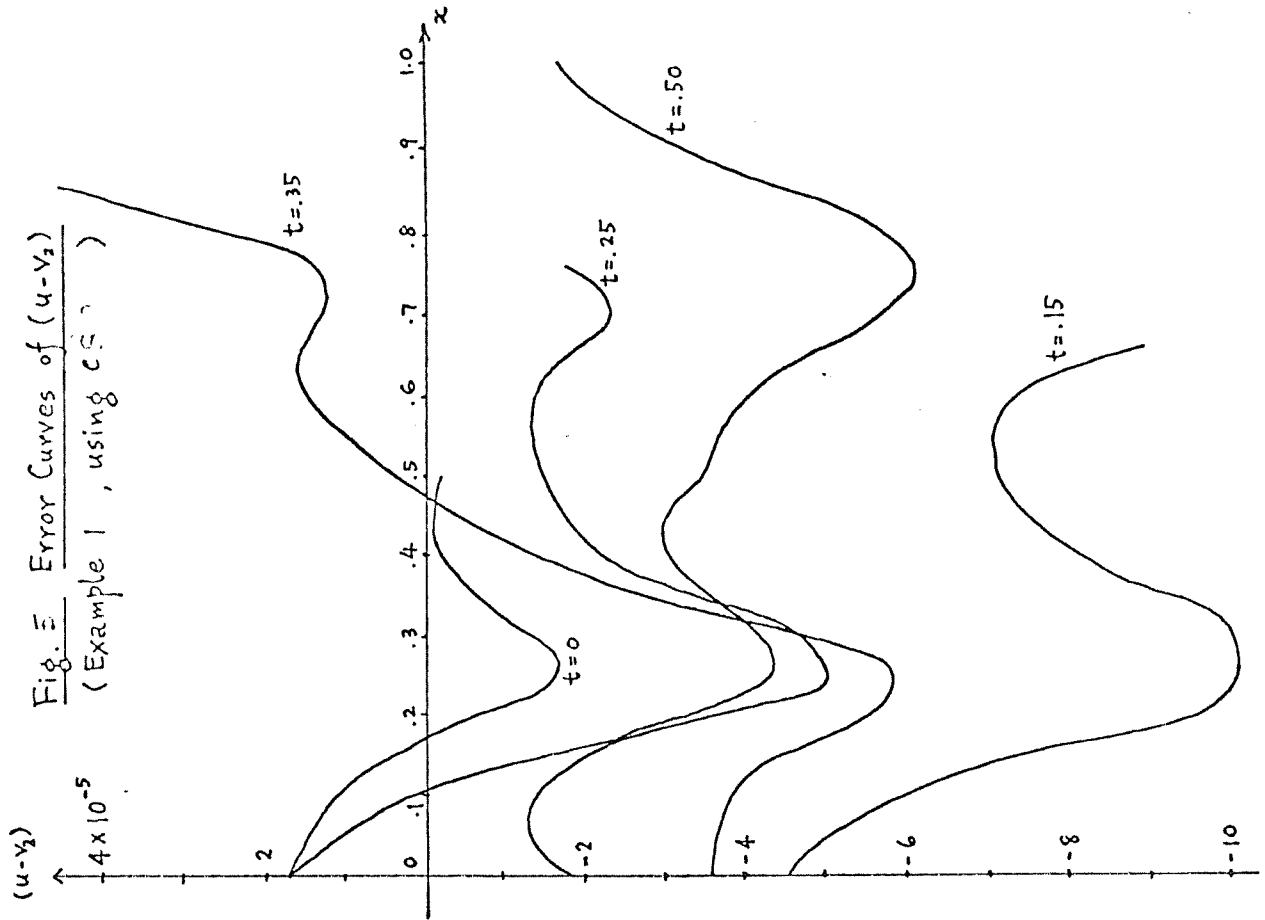


Fig. 5 Error Curves of  $(u-v_2)$   
(Example 1, using CS)



Example 2 (third BVP on a rectangular domain)

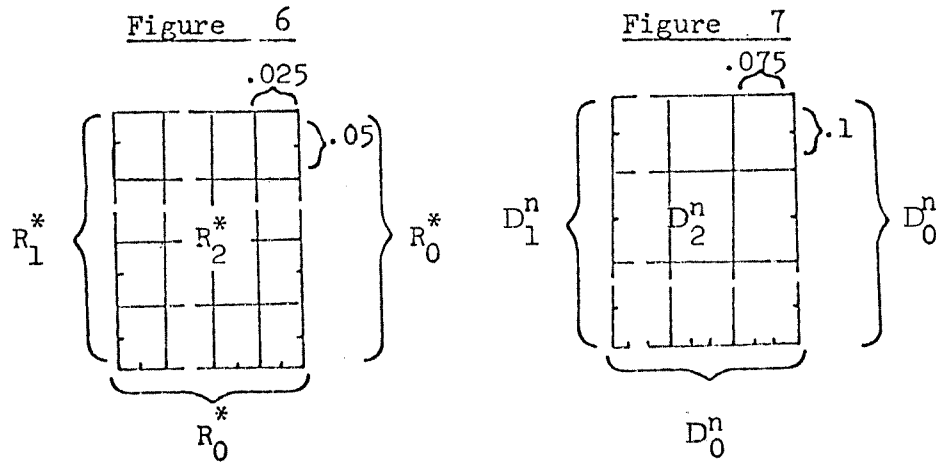
$$\left\{ \begin{array}{ll} (x+1) \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + g_2(u) = r_2 & 0 < x < .45, \quad 0 < t \leq .8 \text{---} R_2 \\ -\frac{\partial u}{\partial x} + g_1(u) = r_1 & x = 0, \quad 0 < t \leq .8 \text{---} R_1 \\ \left. \begin{array}{l} u = 2 \cos x \quad 0 \leq x \leq .45, \quad t = 0 \\ u = 2 \cos (.45+t^2) \quad x = .45, \quad 0 < t \leq .8 \end{array} \right\} & R_0 \end{array} \right.$$

where  $g_1 = -.01tu^3$ ,  $g_2 = -.1t \sin u$ ,  $r_1 = 2 \sin t^2 - .08t^2 \cos^3 t^2$  and  $r_2 = 2 \cos (x+t^2) - 4(x+1)t \sin (x+t^2) - .1t \sin (2 \cos (x+t^2))$ .

Exact solution  $u = 2 \cos (x+t^2)$

Algorithm Since  $g_1'(u) = -.03tu^2$  may vanish somewhere in  $R_1$ , inverse-positivity is not guaranteed. We apply CS2 to obtain an approximate solution and an error bound  $\rho_2$ .

CS2 is applied with  $v^0 \equiv 1$ ,  $\xi^0 = .5$ ,  $\hat{u}^0 = 1$ , and  $\bar{p}_1^0 = \bar{p}_2^0 = 0$ .  $\|\cdot\|_{R_j}$  is approximated by  $\|\cdot\|_{D_j^*}$ , where  $D_j^*$  are meshes described in Figure 6. The termination criterion (6.5) is applied with  $\epsilon_N = .0002$ . Besides the error bound  $\rho_2$ , we also calculate the error bound  $\rho_3$  (3.14).



Discretization method The linear programs included in CS2 are solved with  $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = .0001$ . The meshes  $D_j^n$  are described in Figure 7.

Function space  $S_1^{(3+3, 2+4)}$  (product of elementary cubic spline in  $x$  and quadratic spline in  $t$ , with 4 knots in the  $x$ -direction and 5 knots in the  $t$ -direction. See Appendix).

Computer and LP code CDC 3600, RS MSUB (Clasen [1961]).

Time 7 minutes 49 seconds.

Numerical results After 3 cycles, the termination criterion is satisfied and the following results are obtained (omitting superscripts):

$$\lambda_0 = 2.11 \text{ E-}2, \lambda_1 = 4.04 \text{ E-}2, \lambda_2 = 3.16 \text{ E-}2; \xi = 4.32 \text{ E-}2;$$

$$\bar{p}_1 = -.1, \bar{p}_2 = -.08; \text{ max. value of error function } \mu = 5.69 \text{ E-}1.$$

Each entry in Table 5 is the coefficient of a basis function which is the product of the corresponding functions on the top row and the left-most column.

Table 5Coefficients of Approximate Solution

	1	x	x <sup>2</sup>	(x) <sub>+</sub> <sup>3</sup>	(x-.15) <sub>+</sub> <sup>3</sup>	(x-.3) <sub>+</sub> <sup>3</sup>
1	1.99857	0.00074	-1.00116	0.01102	0.08195	-0.01331
t	-0.03019	0.00940	0.15975	-0.33221	0.61872	0.49820
(t) <sub>+</sub> <sup>2</sup>	0.09610	-2.05745	-0.42922	2.08791	-3.42521	-1.00641
(t-.2) <sub>+</sub> <sup>2</sup>	-0.66026	0.12379	0.58758	-3.33028	6.36580	0.17891
(t-.4) <sub>+</sub> <sup>2</sup>	-0.96210	0.23835	0.61809	2.70660	-4.69479	2.31576
(t-.6) <sub>+</sub> <sup>2</sup>	-1.04285	0.77683	1.23289	-2.38896	2.93268	-3.42271

Table 6

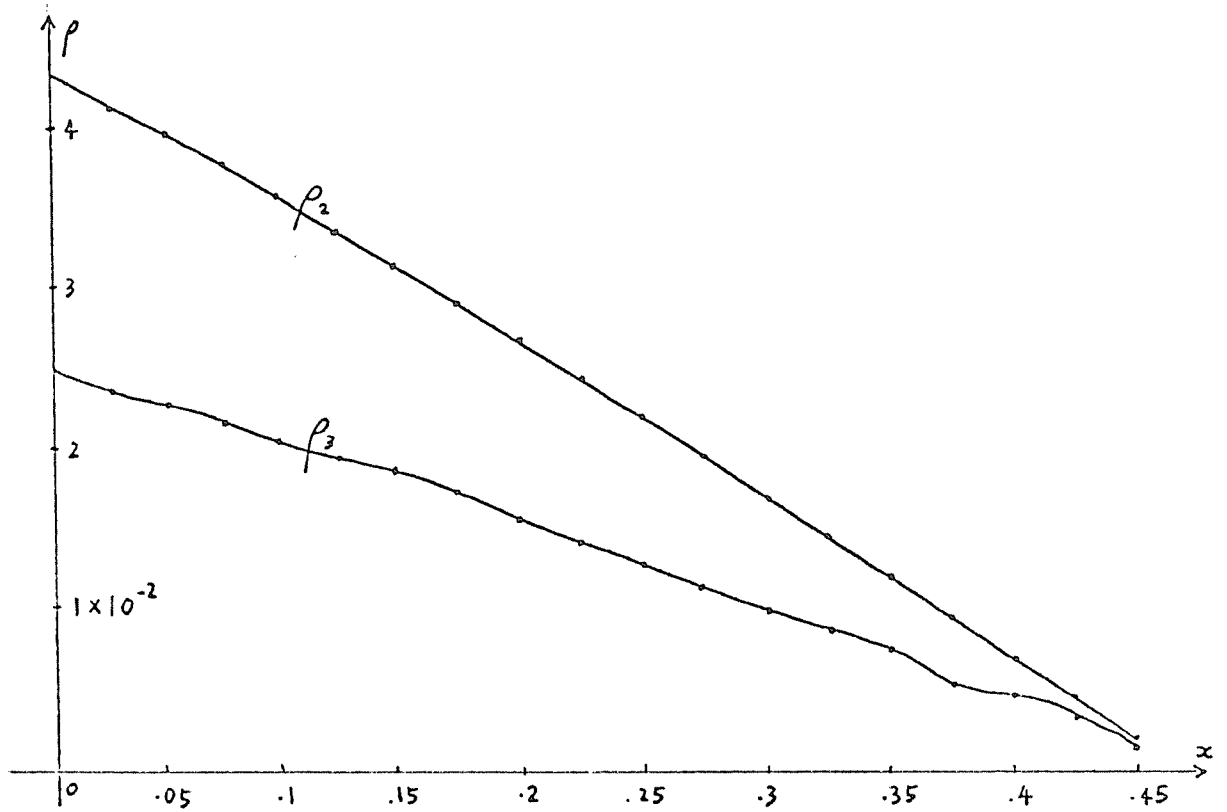
Approximate Solution  $v$ , Error  $(u-v)$  and  
Error Bounds  $\rho_2$  and  $\rho_3$

	x	t	v	u-v	$\rho_2$	$\rho_3$
*	.00	.00	1.998573	-1.43 E-3	2.96 E-2	1.76 E-2
*	.20	.00	1.958772	-1.36 E-3	1.65 E-2	1.02 E-2
*	.30	.00	1.909264	-1.41 E-3	1.07 E-2	6.95 E-3
*	.45	.00	1.799342	-1.55 E-3	2.12 E-2	2.11 E-3
o*	.00	.10	1.996514	-3.39 E-3	4.00 E-2	2.34 E-2
*	.30	.10	1.902116	-2.55 E-3	1.59 E-2	9.83 E-3
	.25	.20	1.915299	-1.19 E-3	2.22 E-2	1.34 E-2
	.40	.20	1.809574	7.03 E-5	7.46 E-3	5.11 E-3
	.00	.35	1.984922	-9.08 E-5	4.29 E-2	2.50 E-2
	.15	.35	1.926220	1.83 E-5	3.10 E-2	1.83 E-2
	.30	.35	1.824817	6.84 E-4	1.74 E-2	1.07 E-2
	.45	.35	1.682757	1.66 E-3	2.19 E-3	2.15 E-3
	.25	.45	1.800130	1.42 E-3	2.21 E-2	1.33 E-2
	.45	.45	1.590241	1.10 E-3	2.13 E-3	2.12 E-3
*	.38	.50	1.621948	2.18 E-5	9.95 E-3	6.51 E-3
*	.45	.50	1.529429	-2.56 E-4	2.12 E-3	2.11 E-3
	.20	.60	1.693365	-1.15 E-3	2.67 E-2	1.59 E-2
	.43	.60	1.413037	-1.74 E-3	4.77 E-3	3.60 E-3
	.08	.75	1.604364	-2.81 E-3	3.74 E-2	2.19 E-2
	.20	.75	1.443795	-2.43 E-3	2.68 E-2	1.59 E-2
	.40	.75	1.141855	-1.09 E-3	7.41 E-3	5.08 E-3
‡*	.00	.80	1.602582	-1.61 E-3	4.32 E-2	2.52 E-2
	.33	.80	1.138715	-1.19 E-4	1.50 E-2	9.35 E-3
*	.45	.80	0.926523	1.55 E-3	2.11 E-3	2.11 E-3

\*grid points, o max. error, ‡ max. error bounds.

Figure 8

Comparison of the Error Bound Curves  $\rho_2$  and  $\rho_3$   
Along the Line  $t = .4$





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APPENDIXMonovariate splines

In the following we present two convenient ways of forming a basis for the space  $B^{m+n}$  of spline functions of degree  $m$  with  $n+1$  knots in terms of the basic splines defined by

$$(x)_+^m = \begin{cases} x^m & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

1) Elementary splines

Let  $x_0, x_1, \dots, x_n$  be a set of knots over  $[x_0, x_n]$ . An arbitrary spline of degree  $m$  is given by

$$\alpha_i \phi_i(x) = \sum_{i=0}^{m-1} \alpha_i x^i + \sum_{i=0}^{n-1} \alpha_{m+i} (x-x_i)_+^m$$

This space is of dimension  $m+n$ .

2) B-splines

Define the following function

$$\beta_m(x) = \sum_{j=-k}^k (-1)^{j+k} \binom{2k}{j+k} (j-x)_+^m \quad m = 2k-1.$$

It looks like Figure A.1 and has the following properties:

$$\beta_m(x) > 0 \quad \text{for } -k < x < k$$

$$\beta_m(x) = 0 \quad \text{for } |x| \geq k$$

Figure A.1  $k = 2$

