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REMARKS ON SINGULAR PERTURBATION OF CERTAIN NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

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1. INTRODUCTION

Consider the nonlinear two-point boundary value problem

(1.1) $$\varepsilon u'' + f(x, u(x), u'(x))u' = 0, \qquad 0 \leq x \leq 1$$

(1.2) $$u'(0) - a\,u(0) = A \geq 0, \qquad (a > 0),$$

(1.3) $$u'(1) + b\,u(1) = B > 0, \qquad (b > 0).$$

Let $\varepsilon > 0$ and assume

H-1: $f(x,u,u')$ is continuous in the region

$$R \equiv \{(x,u,u') \mid 0 \leq x \leq 1, \quad 0 \leq u \leq B/b, \quad 0 \leq u' \leq a + \frac{aB}{b}\}$$

H-2: $f(x,u,u') \geq \beta > 0$ for all $(x,u,u') \in R$.

Recently D. S. Cohen [2] used the "shooting method" to study this problem under somewhat more restrictive hypothesis.

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Our approach is based on a-priori estimates and the Schauder fixed point theorem. The physical motivations for this problem as well as other interesting background facts are discussed in [2].

2. RESULTS

For $\varepsilon > 0$ let H-1' and H-2' be the hypotheses H-1 and H-2 with R replaced by R', where

$$(2.1) \qquad R' \equiv \{(x,u,u') \mid 0 \le x \le 1, \quad -A/a \le u \le B/b, \; 0 \le u' \le A + \frac{aB}{b}\}.$$

Let W be the set of all functions $v(x) \in C^1[0,1]$ which satisfy

$$(2.2) \qquad v'(0) - a\,v(0) = A, \qquad v'(1) + b\,v(1) = B,$$

$$(2.3) \qquad (x, v(x), v'(x)) \in R' \quad \forall x \in [0,1].$$

Let $\varepsilon > 0$ be fixed, let $v(x) \in W$ and let $u(x) \in C^2[0,1]$ be the unique solution of the linear boundary value problem

$$(2.4) \qquad \varepsilon u'' + f(x, v(x), v'(x))u' = 0, \qquad 0 \le x \le 1$$

$$(2.5) \qquad u'(0) - a\,u(0) = A, \qquad u'(1) + b\,u(1) = B.$$

Lemma 1 Assume that H-1' and H-2' hold. Then $u(x) \in W$.

Proof: Since $u(x) \not\equiv$ constant the maximum principle [3] tells us that $|u'(x)| > 0$ for $0 \le x \le 1$. Suppose

$$u'(x) < 0, \qquad 0 \le x \le 1.$$

Then

(2.6) $$u(0) > u(1).$$

On the other hand

$$u(1) = \frac{B - u'(1)}{b} > 0,$$

$$u(0) = \frac{u'(0) - A}{a} < 0$$

which contradicts (2.6). Thus

(2.7) $$u'(x) > 0, \qquad 0 \le x \le 1.$$

Hence

$$-A/a \le -\frac{A}{a} + \frac{u'(0)}{a} = u(0) < u(1) = \frac{B}{b} - \frac{u'(1)}{b} \le B/b$$

and

(2.8) $$-A/a \le u(x) \le B/b.$$

Finally, since

$$u'' = -\frac{1}{\varepsilon} f(x, v(x), v'(x))\, u'(x) \le 0$$

$u'(x)$ assumes its maximum at $x = 0$. Thus

$$0 \le u'(x) \le u'(0) = A + a\, u(0) \le A + \frac{aB}{b} ,$$

which completes the proof.

Let T denote the mapping described above, i.e.

$$T\colon W \to W \tag{2.9}$$

and

$$T(v) = u . \tag{2.10}$$

Lemma 2 T is continuous in the $C^1[0,1]$ topology.

Proof: Let $v_1(x)$, $v_2(x) \in W$ and let

$$T(v_1) = u_1, \quad T(v_2) = u_2, \quad w = T(v_1) - T(v_2). \tag{2.11}$$

Then $w(x)$ satisfies the equation

$$\begin{cases} \varepsilon w'' + f(x,v_1,v_1')\, w' = [f(x,v_2,v_2') - f(x,v_1,v_1')]u_2'(x) & 0 \le x \le 1 \\ w'(0) - a\, w(0) = 0, \quad w'(1) + b\, w(1) = 0. \end{cases} \tag{2.12}$$

The lemma now follows from standard estimates. That is, as $v_2 \to v_1$ and $v_2' \to v_1'$ w and $w' \to 0$.

We now remind the reader of the well-known Schauder fixed-point theorem (see [1, p. 97]).

Theorem (Schauder): If T is a continuous mapping of a closed convex set W in a Banach space X into a compact set $W_0 \subset W$, then T has a fixed point in W_0.

Theorem 1 For every $\varepsilon > 0$ there exists (at least one) a solution $u(x,\varepsilon)$ of (1.1), (1.2), (1.3) and that solution $u(x,\varepsilon) \in W$.

Proof: For fixed $\varepsilon > 0$ let

$$K = \frac{1}{\varepsilon}\left(A + \frac{aB}{b}\right) \max \{ |f(x,u,u')| ; (x,u,u') \in R' \}.$$

Let X be the Banach space $C'[0,1]$ and let W be the W defined above. Let W_0 be the set of all $w(x) \in W$ for which

$$|w''| \leq K.$$

Then, using the Ascoli-Arzela lemma (see [1]) we see that W_0 is a compact subset of the closed convex set $W \subset X$. Thus we may apply the Schauder fixed point theorem and the theorem follows.

Lemma 3 There is a sequence $\varepsilon_n \to 0+$ and a constant $\bar{u}$ such that

$$\text{(2.13)} \qquad \max_{0 \leq x \leq 1} |u(x,\varepsilon_n) - \bar{u}| \to 0 \quad \text{as} \quad \varepsilon_n \to 0+.$$

Proof: The solutions $u(x,\varepsilon)$ are uniformly bounded and equicontinuous. Hence there is a sequence $\varepsilon_n \to 0+$ and a function $U(x)$ such that

$$\text{(2.14)} \qquad \max_{0 \leq x \leq 1} |u(x,\varepsilon_n) - U(x)| \to 0 \quad \text{as} \quad \varepsilon_n \to 0+.$$

However, we claim $U(x) \equiv \text{const}$. Consider the function

$$\phi(x,\varepsilon_n) = e^{\frac{\beta x}{\varepsilon_n}} [u(x,\varepsilon_n) - u(1,\varepsilon_n)].$$

Then $\phi(x, \varepsilon_n)$ satisfies the equations

$$(2.15) \quad \begin{cases} \varepsilon \phi'' + [f - 2\beta]\phi - \frac{1}{\varepsilon}[f\beta - \beta^2]\phi = 0 \\ \phi(1, \varepsilon_n) = 0, \qquad |\phi(0, \varepsilon_n)| \leq 2B/b. \end{cases}$$

Applying H.1' we see that

$$|\phi(x, \varepsilon_n)| \leq 2B/b$$

which implies that

$$|u(x, \varepsilon_n) - u(1, \varepsilon_n)| \leq \frac{2B}{b} e^{-\frac{\beta x}{\varepsilon n}} .$$

Thus, for all $x \in (0,1)$

$$(2.16) \qquad u(x, \varepsilon_n) \to \lim u(1, \varepsilon_n) \quad \text{as} \quad \varepsilon_n \to 0^+ .$$

But because of the uniform convergence in (2.14) we see that

$$U(x) \equiv U(1),$$

and the lemma is proven.

Lemma 4 Under the hypothesis above,

$$\text{Lim } u(x, \varepsilon_n) = \bar{u} = B/b.$$

Proof: Let $x \in (0,1)$. Then

$$\frac{u(x,\varepsilon_n) - u(1,\varepsilon_n)}{x-1} = u'(1,\varepsilon) + \frac{1}{2} u''(\xi,\varepsilon_n)(x-1).$$

Since $u''(\xi,\varepsilon_n)(x-1) > 0$, we have

$$\frac{u(x,\varepsilon_n) - u(1,\varepsilon_n)}{x-1} \geq u'(1,\varepsilon_n) \geq 0.$$

Then, using Lemma 3, we have

$$0 \geq \limsup u'(1,\varepsilon_n) \geq \liminf u'(1,\varepsilon_n) \geq 0,$$

and

$$u'(1,\varepsilon_n) \to 0.$$

But the

$$u(1,\varepsilon_n) = \frac{B - u'(1,\varepsilon_n)}{b} \to B/b.$$

Theorem 2 Let $\{u(x,\varepsilon)\}$ be solutions of (1.1), (1.2), (1.3) which lie in W. Then

(2.17) $$u(1,\varepsilon) \to B/b \quad \text{as} \quad \varepsilon \to 0+,$$

and

(2.18) $$\max_{0 \leq x \leq 1} |u(x,\varepsilon) - B/b| \to 0 \quad \text{as} \quad \varepsilon \to 0+$$

Proof: Suppose (2.17) is false. Then there is a sequence $\varepsilon_n \to 0^+$ such that

(2.19) $$u(1,\varepsilon_n) \to c_0 \neq B/b.$$

However, we may extract a subsequence $\varepsilon_{n'}$ which converges as in Lemma 3. Then applying Lemma 4

$$u(1,\varepsilon_{n'}) \to B/b$$

which contradicts (2.19). Thus (2.17) is established. Then the argument of Lemma 3 using the comparison function $\phi(x,\varepsilon)$ leads to the conclusion that

$$u(x,\varepsilon) \to B/b \qquad \forall x \in (0,1].$$

But, an equicontinuous and bounded family which converges on a dense set converges uniformly.

Remark: We cannot expect that $u'(x,\varepsilon)$ will converge to 0 uniformly on the entire interval $[0,1]$. Indeed

$$u'(0,\varepsilon) = A + a\,u(0,\varepsilon) \to A + \frac{aB}{b}.$$

However, we easily obtain the following result.

Theorem 3. Let $\delta > 0$. Then

$$\text{Max}\,\{|u'(x,\varepsilon)|, \quad \delta \leq x \leq 1\} \to 0 \text{ as } \varepsilon \to 0.$$

Proof: Observe that

$$u'' < 0, \qquad u' > 0.$$

Hence, if $\delta \leq x \leq 1$, then

$$|u'(x,\varepsilon)| \leq u'(\delta,\varepsilon).$$

Thus it suffices to prove that

$$u'(\delta,\varepsilon) \to 0. \tag{2.20}$$

But we now proceed as in the proof of Lemma 4. Let $y \in (0,\delta)$. Then

$$\frac{u(y,\varepsilon) - u(\delta,\varepsilon)}{y - \delta} \geq u'(\delta,\varepsilon) \geq 0$$

and we see that (2.20) holds.

Finally, let us return to our original problem. Suppose we do not have (H.1') but only H.1. Let

$$\tilde{f}(x,u,u') = \begin{cases} f(x,u,u') & (x,u,u') \in R \\ f(x,0,u') & (x,u,u') \in R' \text{ but } u \leq 0. \end{cases}$$

Let us replace $f(x,u,u')$ by $\tilde{f}(x,u,u')$. Then the solution $\tilde{u}(x,\varepsilon)$ obtained in Theorem 1 are solutions of the original problem if $\tilde{u}(0,\varepsilon) \geq 0$. However since $\tilde{u}(0,\varepsilon) \to B/b > 0$ we have: under the hypothesis H.1 and H.2 there is an $\varepsilon_0 > 0$ such that there exists a solution of (1.1), (1.2), (1.3) for all $\varepsilon \in (0,\varepsilon_0)$.

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