Computer Sciences Department
The University of Wisconsin
1210 West Dayton Street
Madison, Wisconsin 53706

Remarks on the Existence Theory for Multiple Solutions of a Singular Perturbation Problem

by

Seymour V. Parter*

Technical Report #125

May 1971

*Sponsored by the Office of Naval Research, under Contract No.: N00014-67-A-0128-0004.

Remarks on the Existence Theory for Multiple Solutions of a Singular Perturbation Problem

by

Seymour V. Parter

I. INTRODUCTION

Consider the singular perturbation problem

1.1) $L_\varepsilon[y] \equiv \varepsilon y'' + y' = g(x,y), \quad 0 < x < 1$

1.2) $y'(0) - a\, y(0) = A, \qquad a > 0,$

1.3) $y'(1) + b\, y(1) = B, \qquad b > 0.$

In [2] D. S. Cohen showed that this problem can have several distinct "asymptotic solutions" for all sufficiently small $\varepsilon > 0$. An asymptotic solution is a function $y(x)$ which satisfies (1.1), (1.2) but only satisfies (1.3) to within $0(\varepsilon)$.

In [8] H. B. Keller extended Cohen's results by weakening the conditions on $g(x,y)$ and considering more general boundary conditions

(1.3a) $f(u'(1),u(1)) = 0$

in place of (1.3). Moreover, under some additional conditions Keller showed the existence of exact solutions of (1.1), (1.2), (1.3a) near the asymptotic solutions discussed by Cohen.

We shall extend these results in several ways. First of all we will further weaken the conditions on $g(x,y)$. We then discuss the existence of exact solutions near the asymptotic solutions. We are able to discuss some cases not treated by Keller. In particular, we discuss certain cases in which there are asymptotic solutions and there is no exact solution nearby.

While we will restrict ourselves to the original boundary conditions (1.3) the extension to boundary conditions of the more general form (1.3a) is relatively easy.

The results of Cohen [2] and Keller [8] are based on the "shooting method". Our analysis is based some results on initial value problems and on the theory of the modified boundary value problem in which the boundary condition (1.3) is replaced by

1.3') $$y(1) = \alpha$$

where α is a specified real number.

However, let us first reformulate our hypothesis.

H.1) $$g(x,y) \in C' \{[0,1] \times \mathcal{R}^1\}$$

H.2) $$|g(x,y)| \le M, \ (x,y) \in \{[0,1] \times \mathcal{R}^1\}$$

Let

H.3) $$H(\alpha) = g(1,\alpha) + b\,\alpha - B$$

have exactly J roots $\alpha_1, \alpha_2, \ldots, \alpha_J$.

2. PRELIMINARY RESULTS

In this section we collect some basic facts about quasi-linear boundary value problems (1.1), (1.2), (1.3') and (1.1), (1.2), (1.3). Many of these results are well known, if not readily accessable in the literature. Most of these results are based on the maximum principle (see [1], [4], [5], [9], [10]). Our first result is a basic a-priori estimate.

Lemma 2.1 Let $\varepsilon > 0$ be fixed. Let $\varphi(x) \in C^2[0,1]$ satisfy

2.1) $$|L_\varepsilon[\varphi]| \le M ,$$

2.2) $$\varphi'(0) - a\varphi(0) = A,$$

2.3) $$\varphi(1) = \alpha .$$

Then

2.4) $$|\varphi(x)| \le \frac{|A|}{a} + M + |\alpha| = K_1(\alpha),$$

2.5) $$|\varphi'(x)| \le M + |A| + a\left(\frac{|A|}{a} + M + |\alpha|\right) = K_2(\alpha).$$

Proof: Let

$$y(x) = Mx - \left(\frac{|A|}{a} + M + |\alpha|\right)$$

Then

2.6a) $$L_\varepsilon(\varphi + y) = L_\varepsilon[\varphi] + M \ge 0$$

2.6b) $$\varphi'(0) + y'(0) - a[\varphi(0) + y(0)] = A + |A| + a(M+|\alpha|) > 0$$

2.6c) $$\varphi(1) + y(1) = \alpha - \frac{|A|}{a} - |\alpha| \le 0.$$

Applying the maximum principle, we see that $\varphi(x) + y(x)$ assumes its maximum on $\{0,1\}$. Suppose this maximum is assumed at $x = 0$. Then

$$\varphi'(0) + y'(0) \le 0$$

and, using (2.6b) we see that

$$\max\,[\varphi(x) + y(x)] = \varphi(0) + y(0) \le 0.$$

If $\varphi(x) + y(x)$ assumes its maximum at $x = 1$ we apply (2.6c) to find

$$\max[\varphi(x) + y(x)] = \varphi(1) + y(1) \le 0.$$

Thus, in any case

$$\varphi(x) \le -y(x) \le \frac{|A|}{a} + M + |\alpha|$$

A similar argument applied to $\varphi(x) - y(x)$ completes the proof of (2.4).

We rewrite the basic differential equation as

$$(e^{\frac{x}{\varepsilon}} \varphi')' = \frac{1}{\varepsilon} e^{\frac{x}{\varepsilon}} L_\varepsilon[\varphi].$$

After one integration we have

$$|\varphi'(x)| \le |\varphi'(0)| e^{-\frac{x}{\varepsilon}} + \frac{M}{\varepsilon} \int_0^x e^{\frac{s-x}{\varepsilon}} ds$$

Since

$$|\varphi'(0)| \le |A| + a(\frac{|A|}{a} + M + |\alpha|)$$

we obtain (2.5).

<u>Theorem 2.1</u>: Let $\varepsilon > 0$ be fixed. For every $\alpha \in \mathscr{R}^1$ there exists a solution $Z(x,\varepsilon,\alpha)$ of (1.1), (1.2) and (1.3'). Indeed there is a maximal solution $M(x,\varepsilon,\alpha)$ and a minimal solution $m(x,\varepsilon,\alpha)$ in the sense that; if $Z(x,\varepsilon,\alpha)$ is any solution then

2.7) $$m(x,\varepsilon,\alpha) \le Z(x,\varepsilon,\alpha) \le M(x,\varepsilon,\alpha).$$

Moreover, $M(x,\varepsilon,\alpha)$ is monotone non decreasing $m(x,\varepsilon,\alpha)$ is monotone non increasing in α and continuous from the right while $m'(x,a,\alpha)$ is continuous from the left.

Proof: The existence of $m(x,\varepsilon,\alpha)$ and $M(x,\varepsilon,\alpha)$ (which may or may not be equal) follows exactly as in [1], [4], [9],[10]. We now sketch this proof. Using the basic a-priori estimate (2.4) of lemma 2.1 we may modify $g(x,y)$ for large y (see [7]) so that $g(x,y)$ may be assumed to satisfy a uniform Lipschitz condition with constant σ. Let $U_0(x,\varepsilon,\alpha)$ and $v_0(x,\varepsilon,\alpha)$ satisfy

$$\begin{cases} L_\varepsilon U_0 = -M, & 0 \le x \le 1 \\ L_\varepsilon v_0 = M, & 0 \le x \le 1 \\ U_0'(0) - aU_0(0) = A = v_0'(0) - a\, v_0(0) & \\ U_0(1) = \alpha = v_0(1) & \end{cases}$$

Let $U_n(x,\varepsilon,\alpha)$, $v_n(x,\varepsilon,\alpha)$ satisfy

$$\begin{cases} L_\varepsilon U_{n+1} - \sigma U_{n+1} = g(x,U_n) - \sigma U_n, & 0 \le x \le 1, \\ L_\varepsilon v_{n+1} - \sigma v_{n+1} = g(x,v_n) - \sigma v_n, & 0 \le x \le 1, \\ U_{n+1}'(0) - a\; U_{n+1}(0) = A = v_{n+1}'(0) - a\; v_{n+1}(0), & \\ U_{n+1}(1) = \alpha = v_{n+1}(1). & \end{cases}$$

A straight forward argument shows that

$$U_n(x,\varepsilon,\alpha) \searrow M(x,\varepsilon,\alpha);$$
$$v_n(x,\varepsilon,\alpha) \nearrow m(x,\varepsilon,\alpha).$$

The monotonicity of $M(x,\varepsilon,\alpha)$ and $m(x,\varepsilon,\alpha)$ (in α) follows from the same argument as the proof of the similiar theorem 2.1 of [9]. As in [9], the one sided continuity of $M(x,\varepsilon,\alpha)$ and $m(x,\varepsilon,\alpha)$ follows from the definition of maximal and minimal solutions.

Finally, the one sided continuity of $M'(x,\varepsilon,\alpha)$ and $m'(x,\varepsilon,\alpha)$ follows from the Green's function representation of L_ε^{-1} and the one sided continuity of $M(x,\varepsilon,\alpha)$ and $m(x,\varepsilon,\alpha)$ respectively.

Remark: It is not too difficult to construct examples and see that $M(x,\varepsilon,\alpha)$ and $m(x,\varepsilon,\alpha)$ need not be continuous.

Lemma 2.2: Let $\varphi(x) \in C^2[0,1]$ satisfy

$$2.8) \qquad \begin{cases} L_\varepsilon[\varphi] = g(x,\varphi), & 0 \le x \le 1 \\ |\varphi(1)| \le M_0 & \\ \varphi'(0) - a\,\varphi(0) = A & \end{cases}$$

Then there is a constant $K_3 = K_3(M, M_0, |A|)$ depending on M, M_0, $|A|$, but not on ε, such that

$$2.9) \qquad |\varphi''(x)| \le \frac{K_3}{x}, \qquad 0 < x \le 1$$

Proof: Let $v(x) = \varphi'(x)$. Then

$$L_\varepsilon v = g_x(x,\varphi) \quad g_y(x,\varphi)\varphi'.$$

Applying lemma 2.1 we see that the right hand side of this equation is bounded by a constant defending only an M, M_0, $|A|$. Similarly, $|v(0)|$, $|v(1)|$ is bounded. Thus, we may apply Theorem 2.7 of [5].

Theorem 2.2 Let α_j be a root of

$$H(\alpha) = 0.$$

Then all solutions of (1.1), (1.2), (1.3') with

$$\alpha = \alpha_j$$

are asymptotic solutions of (1.1), (1.2) and (1.3) in the sense of [2].

Proof: We need merely check (1.3). We have

$$Z'(1,\varepsilon,\alpha_j) + bZ(1,\varepsilon,\alpha_j) = H(\alpha_j) + B - \varepsilon Z''(1,\varepsilon,\alpha_j),$$

hence

$$|Z'(1,\varepsilon,\alpha_j) + bZ(1,\varepsilon,\alpha_j) - B| < K_3 \cdot \varepsilon$$

Theorem 2.3: Let α be fixed. There exists a unique function $W(x,\alpha)$ which satisfies

2.10a) $\quad W' = g(x,W), \quad 0 < x < 1,$

2.10b) $\quad W(1,\alpha) = \alpha,$

2.10c) $\quad |W(x,\alpha)| \leq K_1(\alpha).$

Moreover, let $Z(x,\varepsilon,\alpha)$ be any solution of (1.1), (1.2) and (1.3'). Then

2.11) $\quad \text{Max}\{|Z(x,\varepsilon,\alpha) - W(x,\alpha)|;\ 0 \leq x \leq 1\} \to 0$ as $\varepsilon \to 0+$.

And, for any $\delta \in (0,1)$

2.12) $$\begin{cases} \text{Max}\{|Z'(x,\varepsilon,\alpha) - W'(x,\alpha)|;\ \delta \leq x \leq 1\} \to 0 \text{ as } \varepsilon \to 0+ \\ \text{Max}\{|Z''(x,\varepsilon,\alpha) - W''(x,\alpha)|;\ \delta \leq x \leq 1\} \to 0 \text{ as } \varepsilon \to 0+ \end{cases}$$

Proof: Using lemma 2.1 we may again modify $g(x,y)$ for large $|y|$ so that $g(x,y)$ satisfies a uniform Lipschitz condition. Hence the solutions of (2.10a), (2.10b), (2.10c) are unique. The theorem follows along the lines of the proof of theorem 4.1 of [5] based on lemma 2.1. The functions $\{Z(x,\varepsilon,\alpha)\}$ are equicontinuous and uniformly bounded. Thus a subsequence converges to a weak solution $W(x,\alpha)$ of (2.10a), (2.10b), (2.10c). However, using a theorem of Friedrichs [6] we see that $W(x,\alpha)$ is a genuine solution. The unicity of the limit function allows us to dispense with the subsequence. Finally, (2.12) is proven by using an argument of Coddington and Levinson [3] (see [5] also).

Corollary: Suppose $g(x,y) \in C^2\{[0,1] \times \mathcal{R}\}$. Then, for every $\eta > 0$ and every $M_0 > 0$ there exists an $\varepsilon_0 > 0$ such that $0 < \varepsilon \le \varepsilon_0$ implies

2.13) $$|Z''(1,\varepsilon,\alpha) - [g_x(1,\alpha) + g_y(1,\alpha)g(1,\alpha)]| < \eta$$

for all solutions $Z(x,\varepsilon,\alpha)$ of (1.1), (1.2), (1.3') with

$$|\alpha| \le M_0 \ .$$

Proof: For each fixed α the corollary is true without assuming the additional smoothness of $g(x,y)$. An application of theorem 2.7 of [5] as in lemma 2.2 gives the uniform result under this additional hypothesis.

Remark: It is of interest to observe that the functions $Z(x,\varepsilon,\alpha)$ exhibit no "boundary layer" behaviour near $x = 0$ while the derivatives $Z'(x,\varepsilon,\alpha)$ may do so.

Our next result is a basic existence theorem which is fundamental to the shooting method used in [2], [8].

Lemma 2.3: Let H.1 and H.2 hold. For every value h there exists a unique solution $I(x,\varepsilon,h)$ which satisfies

$$2.14)\qquad \begin{cases} L_\varepsilon I = g(x,I), & 0 \le x \le 1 \\ I(0,\varepsilon,h) = h \\ I'(0,\varepsilon,h) = ah + A. \end{cases}$$

Moreover $I(x,\varepsilon,h)$, $I'(x,\varepsilon,h)$ are continuous functions of h. Finally

$$2.15)\qquad |I(x,\varepsilon,h) - h| \le |ah + A| + M.$$

Proof: The estimate (2.15) follows via the argument of lemma 2.1. Once one has this a-priori estimate we may modify $g(x,y)$ for large $|y|$ and thus assure that $g(x,y)$ satisfies a uniform Lipschitz condition. The lemma now follows from the standard Picard iteration procedure (see [8, theorem 2.3]).

Corollary. As h runs over the real axis so does $I(1,\varepsilon,h)$.

Proof: For every $\alpha \in \mathbb{R}^1$ let

$$h = m(0,\varepsilon,\alpha).$$

We see that

$$I(x,\varepsilon,h) = m(x,\varepsilon,\alpha).$$

3. EXISTENCE OF SOLUTIONS

In this section we are concerned with the existence of solutions of (1.1), (1.2) and (1.3) for small $\varepsilon > 0$. Our first result shows that if $\varepsilon > 0$ is small enough and $u(x,\varepsilon)$ is a solution, then $u(1,\varepsilon)$ must be near a zero of $H(\alpha)$.

Lemma 3.1. Suppose $[x_0, x_1]$ is a finite interval such that

3.1) $$|H(\alpha)| \geq H_0 > 0, \quad x_0 \leq \alpha \leq x_1 .$$

Then there is an $\varepsilon_0 > 0$ such that $0 < \varepsilon \leq \varepsilon_0$ implies that there is no solution $u(x,\varepsilon)$ of (1.1), (1.2) and (1.3) such that

$$u(1,\varepsilon) = \alpha \in [x_0, x_1].$$

Proof: Let ε_0 be so small that (using lemma 2.2) any solution $y(x,\varepsilon)$ of (1.1), (1.2) and (1.3') with

$$\alpha \in [x_0, x_1], \quad 0 < \varepsilon \leq \varepsilon_0$$

satisfies

$$|\varepsilon_0 \, y''(1,\varepsilon)| < \frac{1}{2} H_0 .$$

Then

$$y'(1,\varepsilon) + b\,y(1,\varepsilon) = B + H\,(u(1,\varepsilon)) - \varepsilon u''(1,a)$$
$$\neq B.$$

Thus, the lemma is proven.

Theorem 3.1 Let α_j be a zero of $H(\alpha) = 0$ and suppose α_j is a nodal zero. That is, there is a $\delta > 0$ such that, for all δ, $0 < \delta < \delta_0$,

3.2) $$H(\alpha_j + \delta) \cdot H(\alpha_j - \delta) < 0$$

Then there is an $\varepsilon_0 > 0$ such that

$$0 < \varepsilon \leq \varepsilon_0$$

implies that there is at least one solution $u(x,\varepsilon)$ of (1.1), (1.2) and (1.3) which also satisfies

3.3) $$\alpha_j - \delta_0 \leq u(1,\varepsilon) \leq \alpha_j + \delta_0$$

Proof: Let $Z(x,\varepsilon,\alpha_j+\delta_0)$ and $Z(x,\varepsilon,\alpha_j-\delta_0)$ be solutions of (1.1), (1.2) and (1.3') with

$$\alpha = \alpha_j \pm \delta_0.$$

Then, applying lemma 2.2 we see that: if ε is small enough

$$(3.4)\begin{cases} [Z'(1,\varepsilon,\alpha_j+\delta_0) + bZ(1,\varepsilon,\alpha_j+\delta_0) - B] \times [Z'(1,\varepsilon,\alpha_j-\delta_0) + bZ(1,\varepsilon,\alpha_j-\delta_0) - B] \\ = [H(\alpha_j+\delta_0) - \varepsilon Z''(1,\varepsilon,\alpha_j+\delta_0)] \times [H(\alpha_j-\delta_0) - \varepsilon Z''(1,\varepsilon,\alpha_j-\delta_0)] \\ < 0. \end{cases}$$

Let

$$h_1 = m(0,\varepsilon,\alpha_j-\delta_0)$$
$$h_2 = M(0,\varepsilon,\alpha_j+\delta_0).$$

Then $h_1 < h_2$. Moreover, as h runs over the interval $[h_1,h_2]$, $I(1,\varepsilon,h)$ covers (at least) the interval $[\alpha_j-\delta_0,\ \alpha_j+\delta_0]$. The inverse image

$$J \equiv [h;\ I(1\varepsilon,h) \in (\alpha_j-\delta_0,\ \alpha_j+\delta_0)]$$

is an open set, and hence a countable union of intervals. The continuous function

$$I'(1,\varepsilon,h) + bI(1,\varepsilon,h) - B$$

must change sign in one of these intervals.

Remark: This proof is essentially due to Keller [8]. He uses another method to show that the interval $[\alpha_j-\delta_0,\ \alpha_j+\delta_0]$ is covered by $I(1,\varepsilon,h)$. His proof seems to depend on the additional conditions $g(x,y) \ge 0$, $a \ge 0$, $A \ge 0$.

Remark: As $\varepsilon \to 0$ and $\delta_0 \to 0$ theorem 2.3 shows that

$$0 \le h_2 - h_1 \to 0 \text{ as } \delta_0 \to 0.$$

Theorem 3.2 Suppose α_j is a zero of $H(\alpha) = 0$ and there is a $\delta_0 > 0$ such that

3.5) $$H(\alpha_j \pm \delta) > 0, \qquad 0 < \delta \leq \delta_0$$

and

3.6) $$g_x(1, \alpha_j) + g_y(1, \alpha_j)\, g(1, \alpha_j) > 0.$$

Then, there exists an $\varepsilon_0 > 0$ such that, for every ε, $0 < \varepsilon \leq \varepsilon_0$ there is at least one solution $u(x, \varepsilon)$ of (1.1), (1.2) and (1.3) which also satisfies

3.7a) $$\alpha_j \leq u(1, \varepsilon) \leq \alpha_j + \delta_0$$

and at least one solution $u(x, \varepsilon)$ of (1.1), (1.2), (1.3) which also satisfies

3.7b) $$\alpha_j - \delta_0 \leq u(1, \varepsilon) \leq \alpha_j$$

Proof: The proof is based on the same argument as in theorem 3.1. We merely observe that if $\varepsilon > 0$ is small enough

$$\begin{cases} Z'(1, \varepsilon, \alpha_j - \delta_0) + b\, Z(1, \varepsilon, \alpha_j - \delta_0) - B > 0, \\ Z'(1, \varepsilon, \alpha_j) + b\, Z'(1, \varepsilon, \alpha_j) - B = -\varepsilon Z''(1, \varepsilon, \alpha_j) < 0, \\ Z'(1, \varepsilon, \alpha_j + \delta_0) + b\, Z'(1, \varepsilon, \alpha_j + \delta_0) - B > 0. \end{cases}$$

In a completely analogous way we obtain the next result.

Theorem 3.3 Suppose α_j is a zero of $H(\alpha) = 0$ and there is a $\delta_0 > 0$ such that

3.8) $$H(\alpha_j \pm \delta) < 0, \qquad 0 < \delta \leq \delta_0$$

and

3.9) $$g_x(1, \alpha_j) + g_y(1, \alpha_j) g(1, \alpha_j) < 0.$$

Then there is an $\varepsilon_0 > 0$ such that, for every ε, $0 < \varepsilon \le \varepsilon_0$ there is at least one solution $u(x,\varepsilon)$ of (1.1), (1.2), (1.3) which also satisfies

3.10a) $$\alpha_j \le u(1,\varepsilon) \le \alpha_j + \delta_0$$

and at least one solution $u(x,\varepsilon)$ of (1.1), (1.2), (1.3) which also satisfies

3.10b) $$\alpha_j - \delta_0 \le u(1,\varepsilon) \le \alpha_j .$$

Of equal interest are nonexistence theorems.

Theorem 3.4 Suppose α_j is a zero of $H(\alpha) = 0$ and there is a $\delta_0 > 0$ such that

3.11) $$H(\alpha_j \pm \delta) \ge 0, \qquad 0 < \delta \le \delta_0$$

Suppose $g(x,y) \in C^2 \{[0,1] \times \mathcal{R}^1\}$ and

3.12) $$g_x(1,\alpha_j) + g_y(1,\alpha_j) \cdot g(1,\alpha_j) < 0$$

Then, there exists an ε_0 and a δ_1 such that, for all $Z(x,\varepsilon,\alpha)$ with $0 < \varepsilon \le \varepsilon_0$ and $\alpha \in [\alpha_j - \delta_1, \alpha_j + \delta_1]$

3.13) $$Z'(1,\varepsilon,\alpha) + bZ(1,\varepsilon,\alpha) > B$$

Proof: Applying the corollary to theorem 2.3 there exists an ε_0, δ_1 such that

$$\varepsilon Z''(1,\varepsilon,\alpha) < 0$$

for all ε, $0 < \varepsilon \le \varepsilon_0$ and all $\alpha \in [\alpha_j - \delta_1, \alpha_j + \delta_1]$. Thus

3.14) $$Z'(1,\varepsilon,\alpha) + bZ(1,\varepsilon,\alpha) = B + H(\alpha) - \varepsilon Z''(1,\varepsilon,\alpha) > B.$$

and the theorem is proven.

In the same way we obtain a non existence theorem when the inequalities (3.11) and (3.12) are reversed.

4. ITERATIVE METHODS

The shooting methods of [2], [8] can be used to obtain iterative methods in which the successive iterates are solutions of certain initial value problems. In this section we discuss iterative methods in which the successive iterates are solutions of certain boundary value problems.

Lemma 4.1 Let $\sigma > 0$ be a constant. Suppose $\varphi(x) \in C^2[0,1]$ and

4.1) $$\begin{cases} L_\varepsilon[\varphi] - \sigma\varphi \geq 0, & 0 \leq x \leq 1 \\ \varphi'(0) - a\,\varphi(0) \geq 0, & \\ \varphi'(1) + b\,\varphi(1) \leq 0. & \end{cases}$$

Then

4.2) $$\varphi(x) \leq 0.$$

Proof: Applying the maximum principle we see that $\varphi(x)$ cannot possess an interior positive maximum. Suppose $\varphi(x)$ assumes its maximum at $x = 0$. Then

$$\varphi'(0) \leq 0 \Longrightarrow \varphi(0) \leq 0.$$

On the other hand, if $\varphi(x)$ assumes its maximum at $x = 1$, we have

$$\varphi'(1) \geq 1 \Longrightarrow \varphi(1) \leq 0.$$

In either case

$$\max \varphi(x) \leq 0.$$

<u>Lemma 4.2</u>. Suppose $g(x,y)$ satisfies a uniform Lipschitz condition with constant σ. Let $V(x)$ satisfy

4.3)
$$\begin{cases} L_\varepsilon[V] \geq g(x,V), & 0 \leq x \leq 1 \\ V'(0) - a\,V(0) \geq A, \\ V'(1) + b\,V(1) \leq B, \end{cases}$$

Let $U(x)$ satisfy

4.4a)
$$V(x) \leq U(x), \qquad 0 \leq x \leq 1$$

and

4.4b)
$$\begin{cases} L_\varepsilon[U] \leq g(x,U), & 0 \leq x \leq 1 \\ U'(0) - a\,U(0) \leq A, \\ U'(1) + b\,U(1) \geq B. \end{cases}$$

Let V_1 be the unique solution of the linear boundary value problem

4.5)
$$\begin{cases} (L_\varepsilon - \sigma)\,V_1 = g(t,V) - \sigma V, & 0 \leq x \leq 1 \\ V_1'(0) - a\,V_1(0) = A, \\ V_1'(1) + b\,V_1(1) = B. \end{cases}$$

Then

4.6)
$$L_\varepsilon V_1 \geq g(x,V_1), \qquad 0 \leq x \leq 1$$

and

4.7)
$$V(x) \leq V_1(x) \leq U(x), \qquad 0 \leq x \leq 1$$

<u>Proof</u>: Let

$$\varphi(x) = V(x) - V_1(x).$$

Then we may apply lemma 4.1 to obtain

4.8) $$V(x) \leq V_1(x), \quad 0 \leq x \leq 1$$

And

$$L_\varepsilon V_1 = g(t,V_1) + [g(t,V) - g(t,V_1)] - \sigma(V - V_1).$$

Since σ is a Lipschitz constant for $g(x,y)$ and (4.8) holds we obtain (4.6). Finally, if

$$\varphi(x) = V_1(x) - U(x)$$

we have

$$L_\varepsilon[\varphi] - \sigma\varphi \geq g(t,V) - \sigma V - g(x,U) + \sigma U$$

and using (4.4a) and the definition of σ we see that we may apply lemma 4.1 together with (4.8) and obtain (4.7).

Theorem 4.1: Suppose $\hat{\alpha}_1 < \hat{\alpha}_2$ are two values such that

4.9) $$H(\hat{\alpha}_1) < H(\hat{\alpha}_2)$$

and $0 < \varepsilon \leq \varepsilon_0$ implies that

4.10) $$\begin{cases} Z'(1,\varepsilon,\hat{\alpha}_1) + bZ(1,\varepsilon,\hat{\alpha}_1) < B \\ Z'(1,\varepsilon,\hat{\alpha}_2) + bZ(1,\varepsilon,\hat{\alpha}_2) > B \end{cases}$$

Let σ be a uniform Lipschitz constant for $g(x,y)$. Let $Z(x,\varepsilon,\hat{\alpha}_1)$ be any solution of (1.1), (1.2) and (1.3') with $\alpha = \hat{\alpha}_1$. For example, let

4.11) $$v_0(x) = M(x,\varepsilon,\hat{\alpha}_1), \quad \text{or} \quad v_0(x) = m(x,\varepsilon,\hat{\alpha}_1).$$

Let $V_n(x)$ be defined by the linear boundary value problems

$$\text{4.12)}\quad \begin{cases} L_\varepsilon V_{n+1} - \sigma V_{n+1} = g(x, V_n) - \sigma V_n \\ V_{n+1}'(0) - a\,V_{n+1}(0) = A \\ V_{n+1}'(1) + b\,V_{n+1}(1) = B \end{cases}$$

Then the functions $\{V_n(x)\}_{n=1}^{\infty}$ increase to a function $Z(x, \varepsilon, \alpha)$ which satisfies (1.1), (1.2), (1.3) and

$$\text{4.13)}\qquad \hat{\alpha}_1 \le \alpha \le \hat{\alpha}_2 .$$

Proof: Let

$$U(x) = M(x, \varepsilon, \hat{\alpha}_2).$$

Applying lemma 4.2 we see that

$$\text{4.14)}\qquad V_n(x) \le V_{n+1}(x) \le U(x), \qquad 0 \le x \le 1$$

Remark: Glancing back at theorems 3.2, 3.3 we see that "every other" solution of (1.1), (1.2), (1.3) can be obtained via these iterative methods

Remark: If $g(x,y)$ does not satisfy a uniform Lipschitz condition we may modify $g(x,y)$ for y out of the region of interest, i.e. we modify $g(x,y)$ for

$$y \le m(x, \varepsilon, \hat{y}_1)$$

and

$$M(x, \varepsilon, \hat{\alpha}_2) \le y.$$

Remark: Clearly one may find an approximant to $Z(x, \varepsilon, \hat{\alpha}_1)$ by the use of theorem 2.1. This approximant will be a perfectly good 1st guess in the iteration described by (4.12).

Remark: Using another first iterate (see theorem 2.1) we can construct a decreasing sequence which would also provide a solution $Z(x, \varepsilon, \alpha_1)$ of (1.1), (1.2), (4.13) Moreover, $Z(x, \varepsilon, \alpha)$ and $Z(x, \varepsilon, \alpha_1)$ would be minimal and maximal solution of (1.1), (1.2), (1.3) which also satisfy (4.13).

REFERENCES

[1] D. S. Cohen: Multiple stable solutions of nonlinear boundary-value problems arising in chemical reactor theory. SIAM J. App. Math. 20 (1971)

[2] ____________: Multiple Solutions of Singular Perturbation Problems. To appear.

[3] E. A. Coddington and N. Levinson, A boundary value problem for a non-linear differential equation with a small parameter. Proc. A.M.S. 3, 73-81 (A52)

[4] R. Courant: "Partial Differential Equations" (Courant-Hilbert Vol. II) Interscience, New York (1962)

[5] F. W. Dorr, S. V. Parter and L. F. Shampine: Applications of the maximum principle to singular perturbation problems. Technical Report #101, C Computer Sciences Dept., Univ. of Wis. (1970).

[6] K. O. Friedrichs: The identity of weak and strong extensions of differential operators. Trans. A.M.S. 55 132-141 (1944)

[7] D. Greenspan and S. V. Parter: Mildly nonlinear elliptic partial differential equations and their numerical solution II, Numerische Math. 7 129-146 (1965)

[8] H. B. Keller: Existence theory for multiple solutions of a singular perturbation problem. To appear.

[9] S. V. Parter: Maximal solutions of mildly non-linear elliptic equation in "Numerical Solutions of Nonlinear Differential Equations" (1966) Editor Donald Greenspan, 213-238, John Wiley and Sons, New York.

[10] ____________: Nonlinear eigenvalue problems for some fourth order equations I: Maximal Solutions, SIAM J. MATH. Anal 1, 437-457 (1970).