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# COMPLEXITY CLASSES OF PARTIAL RECURSIVE FUNCTIONS

by

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ABSTRACT

This paper studies possible extensions of the concept of complexity class of recursive functions to partial recursive functions. Many of the well-known results for total complexity classes are shown to have corresponding, though not exactly identical, statements for partial classes. In particular, with two important exceptions, all results on the presentation and decision problems of membership for the two most reasonable definitions of partial classes are the same as for total classes. The exceptions concern presentations of the complements and maximum difficulty for decision problems of the more restricted form of partial classes.

The last section of this paper shows that it is not possible to have an "Intersection Theorem", corresponding to the Union Theorem of McCreight and Meyer, either for complexity classes or complexity index sets.

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## 1. PRELIMINARY DEFINITIONS

The following definitions and notations, many of which are in common usage, are established for this paper.

| | |
|---|---|
| N | the natural numbers $\{0,1,2,...\}$ |
| $\psi(x)\downarrow$ | the computation of the partial function $\psi$ on input $x$ halts or is defined, read "$\psi(x)$ converges" |
| $\psi(x)\uparrow$ | the computation of $\psi(x)$ is not defined, "$\uparrow$" is read "diverges" |
| $\varphi_i$ | the $i^{th}$ partial recursive function in a Gödel indexing $\{\varphi_i\}$, $\varphi_i: N \rightarrow N$ |
| $W_i$ | the domain of $\varphi_i = \{x \mid \varphi_i(x)\downarrow\}$ |
| $\mathcal{P}$ | the partial recursive (pt.r.) functions $= \{\varphi_i \mid i \in N\}$ |
| $\mathcal{R}$ | the total recursive (rec) functions $= \{\varphi_i \mid W_i = N\}$ |
| $\mathcal{P}^{\infty}$ | partial recursive functions with infinite domain |
| $A^C$ | the complement of $A$ (with respect to $N$ or $\mathcal{P}$ as appropriate) |
| $\Omega\mathcal{C}$ | for $\mathcal{C} \subseteq \mathcal{P}$, $\Omega\mathcal{C} = \{i \mid \varphi_i \in \mathcal{C}\}$ |

The "quantifier" $\overset{\infty}{\exists}$ is as an abbreviation such that

$$(\overset{\infty}{\exists} x)[P(x)] \equiv (\forall y)(\ \ x)[x \geq y \ \&\ P(x)],$$

where $P$ is a predicate with one free variable. The usage of "$\overset{\infty}{\exists}$" is similar to that of "$\exists!$" which occurs commonly in mathematical writing. In writing, where the variable quantified over is unspecified or understood, we use "<u>i.o.</u>"

(infinitely often) instead of "$\overset{\infty}{\exists}$". That is, "P i.o." will be taken as synonymous with "$(\overset{\infty}{\exists} x)[P(x)]$". Similarly, "$\overset{\infty}{\forall}$" or "a.e." (almost everywhere) is an abbreviation such that

$$(\overset{\infty}{\forall} x)[P(x)] \equiv (\exists y)(\forall x)[x > y \Longrightarrow P(x)].$$

If $A$ is a predicate over functions, we use "for sufficiently large $f \in \mathscr{R}$, $A(f)$" for "$(\exists g \in \mathscr{R})(\forall f \in \mathscr{R})[f \geq g \text{ a.e.} \Longrightarrow A(f)]$". Similarly, "for arbitrarily large $f \in \mathscr{R}$, $A(f)$" means "$(\forall g \in \mathscr{R})(\exists f \in \mathscr{R})[f \geq g \text{ a.e.} \ \& \ A(f)]$".

We will reserve the word "class" for subsets of $\mathscr{P}$, using "set" to refer to subsets of N, in an attempt to clarify whether functions or a specific algorithm for functions are being considered.

We assume familiarily with the concepts of Turing reducability and 1-1 reducability [13], which will be denoted "$\leq_T$" and "$\leq_1$" respectively (e.g. $A \leq_T B$ is A is Turing reducible to B). Also $\Sigma_n$ and $\Pi_n$ denote the levels of the Kleene hierarchy [13]. Certain standard sets are used as reference points within the Kleene hierarchy. These sets, along with their known positions in the hierarchy, are

| | |
|---|---|
| <u>K</u> = $\{i : i \in W_i\}$ | $\Sigma_1$-complete |
| <u>Total</u> = $\{i \mid W_i = N\}$ | $\Pi_2$-complete |
| <u>Finite</u> = $\{i \mid W_i \text{ finite}\}$ | $\Sigma_2$-complete |
| <u>Equal</u> = $\{\langle i,j\rangle \mid \varphi_i = \varphi_j\}$ | $\Pi_2$-complete |
| <u>Bounded</u> = $\{i \mid W_i = N \ \& \ (\exists z)(\forall x)[\varphi_i(x) \leq z]\}$ | $\Pi_3 \cap \Sigma_3$ |
| <u>Cofinite</u> = $\{i \mid W_i^C \text{ is finite}\}$ | $\Sigma_3$-complete |

where $B$ is $\Sigma_n$ (or $\Pi_n$)-complete if $B \in \Sigma_n$ (resp. $\Pi_n$) and, for all $C \in \Sigma_n$ (resp. $\Pi_n$), $C \leq_1 B$.

Definition:

$\langle\varphi,\Phi\rangle$ will denote an abstract measure of computational complexity [2], where $\varphi = \{\varphi_i\}$ is a Gödel enumeration of $\mathscr{P}$ and $\Phi = \{\Phi_i\}$ satisfies

1) $\varphi_i(x)\downarrow$ iff $\Phi_i(x)\downarrow$

2) the predicate "$\Phi_i(x) = y$" is recursive in $i$, $x$, and $y$.

Unless otherwise stated, we assume a fixed enumeration for $\varphi$ and write $\Phi$ instead of $\langle\varphi,\Phi\rangle$.

Definition

The recursive relation between measures $\Phi$ and $\Phi^*$ is the function

$$r(z) = \max_{i,x\leq z} \{\Phi_i^*(x) \mid \Phi_i(x)\leq z\} + \max_{i,x\leq z} \{\Phi_i(x) \mid \Phi_i^*(x) \leq z\}$$

This function has the important properties [2] that

$$\Phi_i^*(x) \leq r(\max\{i,x,\Phi_i(x)\}) \text{ and}$$

$$\Phi_i(x) \leq r(\max\{i,x,\Phi^*(x)\})$$

for all $i$ and $x$.

In order to simplify notation, we make the following assumption, which will hold for the rest of this paper. Any results in this paper will hold without this assumption making conceptually simple but notationally messy modifications.

Input Representation Assumption:

for any $i, y$ there is an $x_0$ such that $x \geq x_0$ implies $\Phi_i(x) \geq y$.

A slightly strong condition, requiring the existence of a non-decreasing and unbounded recursive $f$ such that $\Phi_i(x) \geq f(x)$ for all $i$ and $x$, is the natural condition that some resource is required simply to represent or to read the input. If we are considering as a measure the amount of tape used by a Turing machine, and those machines represents their input as "tallies", then $f = \lambda x[x]$. If the representation is binary, $f = \lambda x\,[\log_2(x)]$. One immediate example of the simplification provided by this assumption involves the recursive relation or between $\Phi$ and $\Phi^*$. The above result may be simply stated, that for all i,

$$\Phi_i^* \leq r \circ \Phi_i \quad \text{a.e.}$$

## 2. COMPLEXITY CALSSES OF TOTAL FUNCTIONS

Almost all of the investigation of abstract complexity measures to date has been concerned only with total functions, and even with certain subclasses of these functions. Important concepts in the development of these investigations have been the $\Phi$-complexity index set of $t$

$$I_t^\Phi = \{i \mid \varphi_i \in \mathcal{R} \ \& \ \Phi_i \leq t \quad \text{a.e.}\}$$

and the <u>$\Phi$-complexity class of $t$</u>,

$$R_t^\Phi = \{\varphi_i \mid i \in I_t^\Phi\},$$

defined for any measure $\Phi$ and total function $t$. In order to have a notation in the integers for a class of functions $\mathcal{C} \subseteq \mathcal{R}$, we say $B \subseteq N$ is a <u>presentation</u> of $\mathcal{C}$ if

$$\mathcal{C} = \{\varphi_i \mid i \in B\}.$$

The value of presentations as notations for complexity classes is indicated by the following results.

<u>Theorem 2.1 [3]</u>. For any complexity measure $\Phi$ there exists $b^\Phi \in \mathcal{R}$ such that, if $t \in \mathcal{R}$ satisfies $t \geq b^\Phi$ a.e., then there is an r.e. set $W_i$ such that $W_i$ is a presentation of $R_t^\Phi$ ($R_t^\Phi$ is then said to be <u>recursively presentable</u>).

<u>Theorem 2.2 [7]</u>. For any complexity measure $\Phi$ and any $t \in \mathcal{R}$, there exists $i \in N$ such that $W_i^C$ is a presentation of $R_t^\Phi$.

<u>Theorem 2.3 [7]</u>. For any $\Phi$ and $t \in \mathcal{R}$, $\mathcal{P} - R_t^\Phi$ is recursively presentable.

## 3. EXTENSIONS OF COMPLEXITY CLASSES TO PARTIAL FUNCTIONS

There has to date been very little study of classes of partial functions. The original motivation for the construction of various hierarchies of computable functions (the "sub-recursive hierarchies") was a problem specifically oriented to total functions.

In Rice [10,11] and Dekker and Myhill [6], the first thorough investigations of questions about algorithms and functions, classification was done for all functions, not only total ones. Thus there is precedent for considering complexity classes and sets of all partial recursive functions.

The first difficulty is, simply: what is a partial complexity class? There are many ways in which partial classes can reflect the properties of total classes, or the properties of partial functions. For this reason two alternative definitions of partial classes are introduced and considered.

Definition: For any measure $\Phi$ and function $\tau$, the set of

$\Phi,\tau$-computable algorithms is

$$I_{\tau}^{\Phi} = \{i \mid \mathrm{Dom}(\tau) \subseteq W_i \ \& \ \Phi_i \leq \tau \text{ a.e.}\}.$$

This is the obvious analogue for some partial function $\tau$ of the set of $\Phi$,t-computable algorithms for a total t. Observe that the notation is consistent, as $I_t^{\Phi}$, t total, is the same class according to either definition. The predicate,

$$(\exists u)(\forall x)[\tau(x)\downarrow \implies \Phi_i(x) \leq \max(u,\tau(x))]$$

expresses "$i \in I_{\tau}^{\Phi}$".

Recall that the Input Representation Assumption (p. 5) is in effect, allowing the simple predicate above.

Definition: For any measure $\Phi$ and any function $\tau$, the partial $\Phi$-complexity class of $\tau$ is

$$P_\tau^\Phi = \{\varphi_i \mid i \in I_\tau^\Phi\}$$

An alternative defintion which will be considered as

$$\hat{P}_\tau^\Phi = \{\psi \mid \psi \in \mathscr{P} \ \& \ (\exists i)[i \in I_\tau^\Phi \ \& \ (\forall x)[\tau(x)\downarrow \Rightarrow \psi(x) = \varphi_i(x)]]\}.$$

Once again, for total $t$, it is true that $P_t^\Phi = R_t^\Phi$ and even $\hat{P}_t^\Phi = R_t^\Phi$. $P_t^\Phi$ was defined as a straight translation of "$R_t^\Phi = \{\varphi_i \mid i \in I_t^\Phi\}$"--by far the most natural way to correspond classes of fuctions to sets of algorithms.

The definition of $\hat{P}_\tau^\Phi$ is motivated by considering $\tau$ to specify types of problems, and conditions on the solution of problems. The domain of interest in the solution of these problems is just the domain of $\tau$; and all those values for which $\tau$ diverges are "don't care" conditions.

It is very easy to see that there are measures with the anomalous conditions that $I_\tau^\Phi = I_\rho^\Phi$, for some $\tau$ and $\rho$, but $\hat{P}_\tau^\Phi \neq \hat{P}_\rho^\Phi$. Say, for example, that no algorithm except $k$ has $\Phi$-complexity equal to zero at any point, but $\Phi_k = \lambda x[0]$. Let $\tau \equiv 0$ and $\rho(2\cdot x) = 0$, $\rho(2x+1)\uparrow$. Then $I_\tau^\Phi = I_\rho^\Phi = \{k\}$, but obviously $\hat{P}_\rho^\Phi$ contains infinitely more functions than $\hat{P}_\tau^\Phi$.

One further complication which arises with partial functions is the cardinality of the domain. In particular if $\text{Dom}(\tau)$ is finite, then

$$I_\tau = \Omega P = \Omega \hat{P}_\tau = \{i \mid Dom(\tau) \subseteq W_i\}.$$

In this case the decision problem is known to be equivalent to $\underline{K}$. Thus all further consideration will be only of $\tau \in \mathcal{P}^\infty$, - partial functions with infinite domain.

Before continuing, we mention several obvious containment results which will be helpful. For any measures $\Phi$, $\Phi^*$; any $\psi$, $\xi \in P$:

*1: $\hat{P}^\Phi_\psi \geq P^\Phi_\psi$

*2: $\psi \geq \xi$ a.e. $\Longrightarrow P^\Phi_\psi \geq P^\Phi_\xi$ & $\hat{P}^\Phi_\psi \geq \hat{P}^\Phi_\xi$

If $r$ is the recursive relation between $\Phi$ and $\Phi^*$, recall the input representation assumption has the consequence that

$\Phi_i \leq r \circ \Phi^*_i$ a.e. and $\Phi^*_i \leq r \circ \Phi_i$ a.e.

Thus, if any $\Phi_i \leq \psi$ a.e., then $\Phi^*_i \leq r \circ \Phi_i \leq r \circ \psi$ a.e. Hence

*3: $P^\Phi_\psi \subseteq P^{\Phi^*}_{r \circ \psi}$ and $\hat{P}^\Phi_\psi \subseteq \hat{P}^{\Phi^*}_{r \circ \psi}$.

The assumption may be avoided if one wishes to replace "$r \circ \Phi_i$" by "$\lambda x(r(max(h(x), x))]$" or some similar horror. Then *3 holds if $\psi(x) \geq x$. For all $x$.

Let L (= $\langle \varphi, L \rangle$) be the "standard"† tape measure, which does satisfy the above assumption.

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†Specifically, the number of squares read or written or by a Turing machine (Davis' model [5]) in the course of its compution.

The following results show that these two alternative versions of partial complexity classes agree for certain natural bounding functions.

Proposition 3.1 For any function $\Psi$ which can be computed using $\Psi$ tape,

$$\hat{P}^L_\Psi \subseteq P^L_\Psi .$$

Proof: The proof depends heavily on the properties of the tape measure, particularly that many computations can be performed "in parallel" without using any extra tape.

Assume $\xi$, i such that $\xi \in \hat{P}^L_\Psi$, that is there exists $j \in I^L_\Psi$ such that

$$(\forall x)\ [\Psi(x)\downarrow \Longrightarrow \xi(x) = \varphi_j(x)]$$

The following sketches the computation of $\varphi_k$, which can be seen to satisfy $k \in I^L_\Psi$ and $\varphi_k = \xi$. These conditions imply $\xi \in P^L_\Psi$, as required.

To compute $\varphi_k(x)$:

Compute in parallel $\Psi(x)$, $\varphi_j(x)$, and $\xi(x)$, choosing appropriate algorithms and keeping track of the amount of tape used by each: In particular pick a computation of $\Psi$ using exactly $\Psi$ tape,

1) If $\xi(x)$ converges using the least amount of tape, output $\xi(x)$.

2) If $\varphi_j(x)$ converges using the least tape, continue computing until either (a) $\Psi(x)\downarrow$ or (b) $\xi(x)\downarrow$. If (a) occurs first, output $\varphi_i(x)$; otherwise output $\xi(x)$.

3) The case that $\Psi(x)$ converges in (strictly) least amount of tape can only happen finitely often. In this case compute and output $\xi(x)$. ∎

The reader may easily implement on a Turing machine the computation described above in such a way that the desired properties are apparent.

Corollary 3.2 For any i,

$$\hat{P}^{L}_{L_i} = P^{L}_{L_i}$$

Proof: Obvious, since $L_i$ may easily be computed using $L_i$ tape, also *1. ∎

Theorem 3.3: For any measure $\Phi$ there is an $s \in R$ satisfying, for any i,

$$\hat{P}^{\Phi}_{\varphi_i} \subseteq P^{\Phi}_{s \circ \Phi_i} \text{ and } \hat{P}^{\Phi}_{\Phi_i} \subseteq P^{\Phi}_{s \circ \Phi_i}$$

Proof:

Let r be the recursive relation between $\Phi$ and L, and let R be the tape-complexity of some algorithm for r. The property of Davis' model [5] that $L_i(x) \geq \varphi_i(x)$ for all i and x is used to simplify the following argument.

Then the following containments hold for any i,

$$\hat{P}^{\Phi}_{\varphi_i} \subseteq \hat{P}^{L}_{r \circ \varphi_i} \subseteq \hat{P}^{L}_{R \circ L_i} \subseteq P^{L}_{R \circ L_i} \subseteq P^{L}_{R \circ r \circ \Phi_i} \subseteq P^{\Phi}_{r \circ R \circ r \circ \Phi_i}.$$

The first and last of these containments hold using *3 above, the second using *2 and the properties of the model mentioned in 3.2, the fourth using *2 alone, and the third follows from 3.1, since $R \circ L_i$ may be computed using only that much tape if R is choosen to be increasing. Similarly,

$$\hat{P}^{\Phi}_{\Phi_i} \subseteq P^{\Phi}_{r \circ R \circ r \circ \Phi_i}.$$

Hence, the required function is

$$s = r \circ R \circ r. \qquad \blacksquare$$

In light of the previous results, one might hope that a measure could be constructed with sufficiently strong properties so that the two definitions of partial class coincide. The following results show that this is not possible, indicating limits to which conditions can be imposed on measures.

Definition for any function $\psi$, let

$$\alpha_\psi(x) = \begin{cases} x & \text{if } \psi(x)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Theorem 3.4 There exist arbitrarily large functions $\psi$ satisfying

$$\hat{P}^L_{\alpha_\psi} \not\subset P^L_\psi \quad .$$

Proof: Let $f$ be any arbitrarily large function, $f(x) \geq x$. By the usual diagonalization method ([2], Thm. 1) construct a 0-1 valued function $g \in \mathcal{R}$ such that

$$g \notin R^L_f .$$

Define

$$\xi(x) = \begin{cases} 1 & \text{if } g(x) = 1 \\ \uparrow & \text{if } g(x) = 0 \end{cases}$$

and

$$\psi(x) = f(x) \cdot \xi(x) .$$

Claim $\psi$ is as required, since $\xi \in \hat{P}^L_{\alpha_\psi}$ but $\xi \notin P^L_\psi$. Since $\psi(x) \geq x$, $\lambda x[1] \in P^L_\psi$; and $\xi \equiv 1$ on $\mathrm{Dom}(\psi)$. Hence $\xi \in \hat{P}^L_{\alpha_\psi}$.

Assume $\xi \in P^L_\psi$, then there is an $i \in I^L_\psi$, $\varphi_i = \xi$. Define

$$\varphi_k(x) = \begin{cases} 0 & \text{if } L_j(x) < L_i(x) \\ 1 & \text{if } L_i(x) \leq L_j(x). \end{cases}$$

Then it is easy to see that $\varphi_k = g^{\dagger}$, but, using the "parallel computability" property of L[3,7], k may be taken so that $L_k \leq f$ a.e. Hence

$$g = \varphi_k \in R^L_f,$$

a contradiction. ■

Corollary 3.5. For any $\Phi$, any recursive $g$, there exists arbitrarily large partial recursive $\zeta$ such that

$$\hat{P}^\Phi_\zeta \not\subset P^\Phi_{g \circ \zeta}$$

Proof: Let $r$ be the recursive relation between $\Phi$ and L. Pick any recursive $f$, $f(x) \geq x$, to construct $\zeta \geq f$ a.e.

By the previous theorem, construct $\psi \geq r \circ g \circ r \circ f$ a.e. such that

$$\hat{P}^L_{\alpha_\psi} \not\subset P^L_\psi .$$

† Using the above definition, this holds only a.e., but a finite control can be attached to $\varphi_k$ to make the equality hold without affecting the amount of tape used.

and let $\zeta = r \circ f \circ \alpha_\psi$. Now assume the desired property does not hold, that is

$$\hat{P}^\Phi_\zeta \subseteq P^\Phi_{g \circ \zeta} .$$

Then, using $*3$, it follows that

$$\hat{P}^L_{\alpha_\psi} \subseteq \hat{P}^L_{f \circ \alpha_\psi} \subseteq \hat{P}^\Phi_\zeta \subseteq P^\Phi_{g \circ \zeta} \subseteq P^L_{r \circ g \circ \zeta} \subseteq P^L_\psi ,$$

contradicting Theorem 3.4. ∎

The use of presentations as notations is as applicable to either definition of partial class as it is to total classes. With one important exception, the results for presentations carry over exactly from total classes to either definition of partial class. The results for classes $P^\Phi_\tau$ will be presented next, first extending a result of Borodin's [3] by making use of explicit knowledge about when functions are undefined. The results 3.6 - 3.8 were presented in [7] and are included here only for completeness.

Proposition 3.6. For any measure $\langle\varphi,\Phi\rangle$ there is a $b^\Phi \in \mathcal{P}$ such that, for all $\psi \in \mathcal{P}$

$$[\psi = 0 \text{ a.e.}] \Rightarrow (\exists j)[\varphi_j = \psi \ \& \ \Phi_j \leq b^\Phi \text{ a.e.}]$$

Theorem 3.7. For any measure $\Phi$ and all sufficiently large $\varphi_j (\varphi_j \geq b^\Phi \text{ a.e.})$, $P^\Phi_{\varphi_j}$ is recursively presentable.

Proof: First some notation

$$A(i,j,u,x) \equiv (\forall y < x)[\Phi_j(y) \le x \Rightarrow \Phi_i(y) \le \max(u,\varphi_j(y))]$$

$$B(i,j,u,x) \equiv \Phi_i(x) \le \max(u, \Phi_j(x), \varphi_j(x)).$$

Observe that $A$ is a decidable predicate, since the consequent of the implication need be checked only if $\Phi_j(y) \le x$ holds, in which case it is decidable.

Define $f$ such that

$$\varphi_{f(i,j,j)}(x) = \begin{cases} \varphi_i(x) & \text{if } A(i,j,u,x) \ \& \ B(i,j,u,x) \\ 0 & \text{otherwise} \end{cases}$$

Specifically, $\varphi_{f(i,j,u)}(x)$ is computed first by checking $A$, which always halts, then by checking B. If $\varphi_j(x)$ is defined, the truth or falsity of $B$ will eventually be ascertained. If $\varphi_j(x)$ is undefined, then $\varphi_i(x)\downarrow$ eventually results in an answer to B. The only problem arises if $\varphi_j(x)\uparrow$ and $\varphi_i(x)\uparrow$; but then unless A is false, $\varphi_{f(i,j,j)}(x)$ is undefined. But $\varphi_{f(i,j,j)}(x)\uparrow$ does not disqualify $\varphi_{f(i,j,u)}$ from $P_{\varphi_j}$, since we have assumed $\varphi_j(x)\uparrow$.

We claim $\{f(i,j,u) \mid i,u \in N\}$ is a presentation of $P^{\Phi}_{\varphi_j}$. ∎

That there exist exceptional cases of $\Phi$ and $t$ such that $R^{\Phi}_t$ is not recursively presentable [7,8] obviously carry over to $P^{\Phi}_t$ (and $\hat{P}^{\Phi}_t$).

Theorem 3.8 For any measure $\Phi$ and any $\varphi_k$ there exists a presentation $V$ of $P^{\Phi}_{\varphi_k}$ such that $V^C$ is recursively enumerable

Proof: For $i \in N$, let $B_i$ be a recursive set of indices for $\varphi_i$ such that $i \neq j$ implies $B_i \cap B_j = \emptyset$ and $\cup_i B_i$ is recursive.

We will subsequently show how to effectively enumerate a set $E$ such that $B_i - E \neq \emptyset$ iff $\Phi_i \leq \varphi_k$ a.e. and $W_i \supseteq W_k$. Clearly this implies $\cup_i B_i - E$ is a presentation of $P^{\Phi}_{\varphi_k}$. Since $\cup_i B_i$ is recursive and $E$ is r.e., $(\cup_i B_i - E)^C = (\cup_i B_i)^C \cup E$ is r.e.

Say $B_i = \{b_i^0, b_i^1, \ldots\}$ and define

$$E = \{b_i^n \mid (\exists x, y) [\Phi_k(x) \leq y \ \& \ \Phi_i(x) \leq \varphi_k(x) + (n \dot{-} x)]\}$$

$E$ is clearly r.e. Assume $b_i^j \in B_i - E$, then either for all $x$ either $\varphi_k(x)\uparrow$ or $\Phi_i(x) \leq \varphi_k(x) + (j \dot{-} x)$. Thus $W_i \geq W_k$ and $\Phi_i \leq \varphi_k$ a.e. On the other hand, assume $W_i \supseteq W_k$ and $\Phi_i \leq \varphi_k$ a.e. The second of these conditions implies that there is a $u$ satisfying:

$$x > u \ \& \ \varphi_k(x)\downarrow \implies \Phi_i(x) \leq \varphi_k(x).$$

For this $u$, the first condition then implies $v = \max\{\Phi_i(x) \mid x \in W_k \ \& \ x < u\}$ exists and, for $j = u + v$,

$$(\forall x) [\varphi_k(x)\uparrow \ \vee \ \Phi_i(x) \leq \varphi_k(x) + (j \dot{-} x)]$$

Hence $b_i^j \in B_i - E$. ∎

Theorem 3.9 For any measure $\Phi$ and sufficiently large $\varphi_k$ ($\varphi_k \geq b^\Phi$ a.e.), $\hat{P}^\Phi_{\varphi_k}$ is recursively presentable.

The proof is conceptually similar to that of 3.7, making use of Theorem 3.3.

In particular let $\{a_0, a_1, \ldots\}$ and $\{e_0, e_1 \ldots\}$ be recursive presentations of $P^\Phi_{\varphi_k}$ and $P^\Phi_{s \circ \Phi_k}$ respectively, such that (by a well-known technique of Blum [2]),

$$\Phi_{e_i} > \Phi_k .$$

Define a predicate similar to A above

$$A'(i,j,k,x) = (\forall y \leq x)[\Phi_k(y) \leq x \Rightarrow \varphi_i(y) = \varphi_j(y)]$$

Observe that $A'$ is total if $W_i \supseteq W_k$ and $W_j \supseteq W_k$. Now define

$$\varphi_{f(i,j)}(x) = \begin{cases} \varphi_i(x) & \text{if } A'(i,j,k,x) \ \& \ \Phi_j(x) > \Phi_k(x) \\ \varphi_j(x) & \text{if } A'(i,j,k,x) \ \& \ \Phi_j(x) \leq \Phi_k(x) \\ 0 & \text{if } \neg A'(i,j,k,x) \end{cases}$$

Consider the case that $\varphi_i \in P^\Phi_{\varphi_k}$ and $\varphi_j \in \hat{P}^\Phi_{\varphi_k}$ such that $\Phi_j > \Phi_k$ and $x \in W_k \Rightarrow \varphi_i(x) = \varphi_j(x)$. Then $\varphi_{f(i,j)}$ will equal $\varphi_i$ exactly on $W_k$ (the first line of the definition) and will equal $\varphi_j$ elsewhere (the second line). Thus the reader may easily show

$$\{f(a_i, e_j)\, i, j \in N\}$$

is a presentation of $\hat{P}^{\Phi}_{\varphi_k}$. ∎

Theorem 3.10 For any measure $\Phi$ and $\varphi_k$, there is a presentation $Y$ of $\hat{P}^{\Phi}_{\varphi_k}$ such that $Y^C$ is recursively enumerable.

The proof is similar to that of 3.8, defining

$$E = \{b_i^n \mid (\forall j \leq n)(\exists x, y)\ [\Phi_k(x) \leq y\ \&$$

$$(\Phi_j(x) \geq \varphi_k(x) + (n \dot{-} x)\ \vee\ (\varphi_i(x)\downarrow\ \&\ \varphi_i(x) \neq \varphi_j(x))]\}$$

Theorem 3.11: For any measure $\Phi$ and any $\tau \in \mathcal{P}$; $\mathcal{P} - \hat{P}^{\Phi}_{\tau}$ is recursively presentable

Proof: The following is a sketch of a stage in the operation of a device which enumerates a presentation of $\mathcal{P} - \hat{P}^{\Phi}_{\tau}$. Say $\tau = \varphi_j$. Assume again $\tau \in \mathcal{P}^{\infty}$.

Stage n.

1) If $(\forall x \leq n)[\Phi_j(x) > n]$, go to stage $n + 1$.

2) Enumerate functions diverging at some value where $\varphi_j$ converges. This requires listing the domain of $\varphi_j$, which is done in stages corresponding to the stages of the larger device.

3) Enumerate the index of an algorithm which is equal to $\varphi_n$ if indeed $\varphi_n \in \mathcal{P} - \hat{P}^{\Phi}_{\varphi_j}$, and which is almost everywhere undefined otherwise.

In particular, note that, for a fixed $n$ such that $W_n \supseteq W_j$, the following predicate $P(x) \equiv$

$$(\forall k \leq x) \left[ \begin{array}{l} (\exists y \leq x)\,[\Phi_j(y) > x \;\vee\; \Phi_k(y) > x \;\vee\; \varphi_k(y) \neq \varphi_n(n)] \;\vee \\ (\exists w > x)(\exists y)[x < y \leq w \;\&\; \Phi_j(y) \leq w \;\&\; (\Phi_k(y) > \varphi_j(y) \;\vee\; \varphi_k(y) \neq \varphi_n(y)] \end{array} \right]$$

is true for all $x$ only if there is no $k \in I^{\Phi}_{\varphi_j}$ such that $\varphi_k = \varphi_n$ on $W_j$. Thus at stage $n$ an index for

$$\psi(x) = \begin{cases} \varphi_n(x) & \text{if } P(x) \\ \uparrow & \text{otherwise (since } P(x) \text{ is undefined if not true)} \end{cases}$$

is enumerated. ∎

## 4. Decision Problems for Partial Classes

An interesting approach to the study of total complexity classes was that taken by F. Lewis [8], who investigated the structure of an individual complexity class via a classical tool for studying complexity of another sort - the Kleene hierarchy. The following results show how the results for total classes carry over, or fail to carry over, to partial classes.

Theorem 4.1. For any measure $\Phi$ and $\varphi_k$

$$\Omega\hat{P}^{\Phi}_{\varphi_k} \text{ is a } \Pi_3 \cap \Sigma_3 \text{ set.}$$

Proof: The predicate

$$(\forall x)[\varphi_k(x)\downarrow \implies \varphi_i(x)\downarrow] \quad \&$$

$$(\exists j)[j \in I^{\Phi}_{\varphi_k} \quad \& \quad (\forall x)[\varphi_k(x)\downarrow \implies \varphi_i(x) = \varphi_j(x)]]$$

clearly expresses. "$\varphi_i \in \hat{P}^{\Phi}_{\varphi_k}$". Noting that "$\varphi_i(x)\downarrow$" is $\Sigma_1$ and "$\varphi_i(x) = \varphi_j(x)$" may be rewritten as "$\varphi_i(x)\downarrow \;\&\; \varphi_j(x)\downarrow \implies \varphi_i(x) = \varphi_j(x)$"†, the predicate is clearly a conjunction of $\Pi_2$ and $\Sigma_2$, and hence $\Sigma_3 \cap \Pi_3$. ∎

Under certain conditions a similar extension may be made for classes $P^{\Phi}_{\tau}$.

Theorem 4.2: For any measure $\Phi$ and $\tau \in \mathcal{P}^{\infty}$. If

1. $P^{\Phi}_{\tau} \neq \emptyset$ and 2. $\mathcal{P} - P^{\Phi}_{\tau}$ has a $\Pi_3 \cap \Sigma_3$ presentation,

then $\Omega P^{\Phi}_{\tau} \equiv_T$ Equal.

Proof: Let $\tau \in \mathcal{P}^{\infty}$ satisfying 1 and 2. First show $\Omega P^{\Phi}_{\tau} \leq_T$ Equal.

We describe a machine with an "oracle" for Equal which performs two processes in parallel, the first of which will halt if $\varphi_i \notin P^{\Phi}_{\tau}$ (but which by itself does not converge if $\varphi_i \in P^{\Phi}_{\tau}$), the second determines $\varphi_i \in P^{\Phi}_{\tau}$.

To determine if $\varphi_i \notin P^{\Phi}_{\tau}$, enumerate $e_0, e_1, \ldots$ a presentation of $\mathcal{P} - P^{\Phi}_{\tau}$ and ask the oracle if $\langle i,e_0\rangle \in$ Equal, $\langle i,e_1\rangle \in$ Equal, .... By

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†This is sufficient, since if $i$ and $j$ do indeed satisfy the rest of the expression, then $\varphi_i(x)\downarrow$ and $\varphi_j(x)\downarrow$.

assumption such an enumeration is possible, since any $\Pi_3 \cap \Sigma_3$ set can be enumerated with Equal as an oracle.

$\varphi_i \; \varepsilon \; P_\tau^\Phi$ is determined in a similar manner, with the enumeration always possible since $P_\tau^\Phi$ is $\Pi_1$-presentable (3.7).

Now to show Equal $\leq_T \Omega P_\tau^\Phi$. It was assumed that $\Omega P_\tau^\Phi$ is non-empty, so say $\rho \; \varepsilon \; P_\tau^\Phi$. Define $f$ so that

$$\varphi_{f(i,j)}(x) = \begin{cases} \rho(x) & \text{if } (\forall y \leq x)[(\Phi_i(y) \leq x \vee \Phi_j(y) \leq x) \Longrightarrow \\ & (\varphi_i(y)\downarrow \;\&\; \varphi_j(y)\downarrow \;\&\; \varphi_i(y) = \varphi_j(y))] \\ \uparrow & \text{otherwise} \end{cases}$$

It is easy to see that

$$\langle i,j\rangle \; \varepsilon \; \underline{\text{Equal}} \Longleftrightarrow \varphi_{f(i,j)} = \rho \Longleftrightarrow f(i,j) \; \varepsilon \; \Omega P_\tau^\Phi .$$

Thus, if $f$ is made 1-1, Equal $\leq_1 \Omega P_\tau^\Phi$. ∎

Theorem 4.3: There exists a measure $\Phi^*$ such that, for arbitrarily large $\tau \; \varepsilon \; \mathscr{P}^\infty$,

$$\underline{\text{Cofinite}} \leq_1 P_\tau^{\Phi^*} .$$

Proof: Let $\Phi$ be any measure, with $\gamma$ an effective procedure as described in the Honesty Theorem [9] such that, for all i, x, y

1) "$\Phi_{\gamma(i)}(x) \leq y$" is decidable

2) $I_{\varphi_i}^\Phi = I_{\varphi_{\gamma(i)}}^\Phi$ and hence $P_{\varphi_i}^\Phi = P_{\varphi_{\gamma(i)}}^\Phi$.

The following construction has the desired effect if $\varphi_{(i)}$ converges on an infinite set of even $x$. A similar construction must be done for the odds.

For each $i$, let $e^i$ be the index of a function computed, for input $x$, in the following manner [after 2, Thm. 7].

If $x$ is odd, output $0$.

If $x$ is even, compute $\varphi_{\gamma(i)}(x)$ until (and if) it halts. Let $k$ be the least integer such that $\Phi_k(x) \leq \varphi_{\gamma(i)}(x)$, but for no even $x' < x$ is it the case that

$$\Phi_{\gamma(i)}(x') \leq x \quad \& \quad \Phi_k(x') \leq \varphi_{\gamma(i)}(x) \quad \&$$

$$k \text{ is checked off at } x' \text{ by stage } x .$$

This $k$ is now said to be "checked off at $x$ by stage $\varphi_{\gamma(i)}(x)$." If $\varphi_k(x) = 0$, output $1$. If $\varphi_k(x) \neq 0$ or no such $k$ exists, output $0$.

For each $\varphi_{\gamma(i)}$, $\varphi_{e^i}$ is a $0-1$ valued function with a domain of $\mathrm{Dom}\,(\varphi_{\gamma(i)}) \cup \{\text{odds}\}$ satisfying

$$(\forall j)[(\forall x)\,[\varphi_j(2x) = \varphi_{e^i}(2x)] \Rightarrow (\overset{\infty}{\forall} x)\,[\Phi_j(2x) > \varphi_{\gamma(i)}(2x) \vee \varphi_{\gamma(i)}(2x)\uparrow]].$$

Now define $e^i_k$ such that

$$\varphi_{e^i_k}(x) = \begin{cases} \varphi_{e^i}(x) & \text{if } x \text{ even} \\ 2^{i+1} & \text{if } x \text{ odd } \& \ x \notin D_k \\ \uparrow & \text{if } x \text{ odd } \& \ x \in D_k, \end{cases}$$

where $D_k$ is the $k^{th}$ set in an effective listing of all finite sets. The use of $2^{i+1}$ assures that modifications associated with one $i$ do not interfere with other modifications. Finally, $\{e_k^i | i,k \in N\}$ should be recursive.

Similarly, define $\{o_k^i\}$ which will handle the cases where $\varphi_{\gamma(i)}$ converges i.o. for odd $x$ by interchanging even and odd, in the above definition. Define a new measure by

$$\Phi_n^*(x) = \begin{cases} \Phi_n(x) & \text{if } n \notin \{e_k^i\} \cup \{o_k^i\} \\ \varphi_{\gamma(i)}(x) & \text{if } (n = e_k^i \ \& \ x \text{ even}) \vee (n = o_k^i \ \& \ x \text{ odd}) \\ 0 & \text{if } x \notin D_k \ \& \ ((n = e_k^i \ \& \ x \text{ odd}) \vee (n = o_k^i \ \& \ x \text{ even})) \\ \uparrow & \text{if } x \in D_k \ \& \ ((n = e_k^i \ \& \ x \text{ odd}) \vee (n = o_k^i \ \& \ x \text{ even})) \end{cases}$$

$\Phi^*$ is a measure since membership in $D_k$ and in $\{e_k^i\} \cup \{o_k^i\}$ is decidable, and since $\varphi_{\gamma(i)}$ is honest.

Now pick some arbitrarily large $\varphi_{\gamma(i)} \in \mathcal{P}^{\infty}$. Assume $\varphi_{\gamma(i)}$ converges i.o. on even input and define

$$\tau(x) = \begin{cases} \varphi_{\gamma(i)}(x) & \text{if } x \text{ even} \\ \uparrow & \text{if } x \text{ odd.} \end{cases}$$

Then define $h$ so that

$$\underline{\text{Cofinite}} \leq_1 \Omega P_{\tau}^{\Phi^*}$$

via $h$. This is accomplished by $h$ satisfying

$$\varphi_{h(j)}(x) = \begin{cases} \varphi_{\gamma(i)}(x) & \text{if } x \text{ even} \\ 2^{i+1} & \text{if } x \text{ odd } \& \ \varphi_j([x/2])\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Observe that, if $\varphi_n$ agrees with $\varphi_{e^i}$ on all even inputs then, by construction, $\varphi_n \in P_\tau^{\Phi^*}$ iff $\varphi_n = \varphi_{e_k^i}$ for some $k$. But $\varphi_{h(j)} = \varphi_{e_k^i}$ iff $W_j = N - D_k$. Hence

$$h(j) \in \Omega P_\tau^{\Phi^*} \iff j \in \underline{\text{Cofinite}} \quad \blacksquare$$

<u>Corollary 4.4</u> For $\Phi$ and $\tau$ as in 4.3, $\mathcal{P} - P_\tau^\Phi$ is not recursively presentable.

<u>Proof</u>: By 4.2, if $\mathcal{P} - P_\tau^\Phi$ were recursively presentable, then, using 4.3 also

$$\underline{\text{Cofinite}} \leq_1 \Omega P_\tau^\Phi \leq_T \underline{\text{Equal}},$$

which is a contradiction to known fact. $\blacksquare$

The above theorem (4.3) states that $\Omega P_\tau^\Phi$ may be $\Sigma_3$-complete for some measures and functions. It is an open question whether such functions exist for all measures. It is easy to see, however, that this is worst case.

<u>Proposition 4.5</u>. For any measure $\Phi$ and $\varphi_k \in \mathcal{P}$,

$$\Omega P_{\varphi_k}^\Phi \text{ is } \Sigma_3.$$

Proof: The reader may easily see that the following predicate is $\Sigma_3$ and expresses "$\varphi_i \in P^{\Phi}_{\varphi_k}$":

$$(\exists j)[j \in I^{\Phi}_{\varphi_k} \ \& \ \varphi_i = \varphi_j]. \ \blacksquare$$

A stronger lower bound on the hierarchy classification of partial class is the following suggested by and including Lewis' [8] result for total classes.

Theorem 4.6: If $\tau \in \mathscr{P}^{\infty}$ and $P^{\Phi}_{\tau}$ is finitely invariant,[†] then

$\underline{\text{Bound}} \leq_1 \Omega P^{\Phi}_{\tau}$.

Proof: Let $W_j$ be an infinite recursive subset of $\text{Dom}(\tau)$, which may be found from $\tau$ by a standard construction. By 3.7 there exists $f \in \mathscr{R}$ which presents $P^{\Phi}_{\tau}$. Now define $g$ such that

$$\varphi_{g(i)}(x) = \begin{cases} \varphi_{f(k)}(x) & \text{if } x \notin W_j \text{ or } \varphi_i(y) \in \{\varphi_i(0), \ldots, \varphi_i(y-1)\} \\ \varphi_{f(k)}(x)+1 & \text{otherwise} \end{cases}$$

where $y = |w_j \cap \{0, \ldots, x-1\}|$ and $k = |\varphi_i(0), \ldots \varphi_i(y-1)\}|$, which is obviously undefined if $\varphi_i$ diverges at some input less than $y$, causing $\varphi_{g(i)}(x)$ to diverge.

Now claim: $\underline{\text{Bound}} \leq_1 \Omega P^{\Phi}_{\tau}$ via $g$, which follows from

$$\varphi_i \in \underline{\text{Bound}} \iff \varphi_i \text{ total } \& \ (\exists z)(\forall y)[y > z \Longrightarrow \varphi_i(y) \in \{\varphi_i(0), \ldots \varphi_i(z)\}]$$

$$\iff (\exists k)[\varphi_{g(i)} = \varphi_{f(k)} \text{ a.e.}] \iff \varphi_{g(i)} \in P^{\Phi}_{\tau} . \ \blacksquare$$

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[†] A class of functions $\mathscr{C} \subseteq \mathscr{P}$ is finitely invariant if $\psi \in \mathscr{C}$ and $\xi = \psi$ a.e. and $\text{Dom}(\xi) = \text{Dom}(\psi)$ together imply $\xi \in \mathscr{C}$ .

Finally, we show a result concerning decision problems not for functions but for algorithms.

<u>Theorem 4.7</u>: For any measure $\Phi$ there is a $g \in \mathcal{R}$ such that, for all $\tau \in \mathcal{P}^{\infty}$,

$$\tau \geq g \quad \text{a.e.} \implies \underline{\text{Finite}} \leq_1 \Omega I_{\tau}^{\Phi} .$$

<u>Proof</u>: Unlike $b^{\Phi}$ (of 3.8), it is not possible to specify $g$ by some simple criteria. Instead, $g$ must majorize (in certain cases where convergence is guaranteed) functions which reduce <u>Finite</u> to $I_{\tau}^{\Phi}$ for all possible $\tau$.

First, introduce the notation

$Q(i,k,x) \equiv$

$$(\exists y,w \leq x)[\Phi_i(y) = w \ \& \ (\forall z)[\max(w,y) < z < x \implies \Phi_k(z) > x]].$$

This predicate says "in a limited search, $\varphi_i$ has been found to converge at some point after $\varphi_k$."

Assume $\varphi_k \in \mathcal{P}^{\infty}$ and examine the implications of this for $Q$. In particular, say $\Phi_i(y_o) = w_o$. By assumption $\varphi_k$ converges above $\max(y_o, w_o)$, say $z_o$ is the least such value for which this occurs. Then $Q(i,k,z_o)$ holds. On the other hand, for $x > \max(z_o, \Phi_k(z_o))$, $Q(i,k,x)$ can hold only if $\varphi_i(y)\uparrow$ for some $y > y_o$ and not because $\varphi_i(y_o)\downarrow$. Hence

i. $\varphi_k \in \mathcal{P}^{\infty} \ \& \ \varphi_i \in \mathcal{P}^{\infty} \implies (\overset{\infty}{\exists} x)[\varphi_k(x)\downarrow \ \& \ Q(i,k,x)]$

ii. $\varphi_k \in \mathcal{P}^{\infty} \ \& \ \varphi_i \notin \mathcal{P}^{\infty} \implies (\exists x_o)(\forall x > x_o)[\neg Q(i,k,x)]$

Now define $f \in \mathcal{R}$ so that $\varphi_{f(e)}$ enumerates (increasing) indices of functions such that

$$\varphi_{\varphi_{f(e)}(i,k)}(x) = \begin{cases} \uparrow & \text{if } \ Q(i,k,x) \ \& \ \varphi_{\varphi_e(i,k)}(x) \le \max(\varphi_k(x), \Phi_k(x)) \\ 0 & \text{otherwise} \end{cases}$$

Pick $e_o$ a fixed-point of f. Since $\varphi_{e_o}$ enumerates indices, it is evidently total and, letting $h = \varphi_{e_o}$, the above may be rewritten as

$$\varphi_{h(i,k)}(x) = \begin{cases} \uparrow & \text{if } \ Q(i,k,x) \ \& \ \Phi_{h(i,k)}(x) \le \max(\Phi_k(x), \varphi_k(x)) \\ 0 & \text{otherwise} \end{cases}$$

The function $\lambda i[h(i,k)]$ will ultimately be shown to provide the reduction for appropriate $\varphi_k = \tau$.

Observe that

iii. $\neg Q(i,k,x) \Longrightarrow \varphi_{h(i,k)}(x)\downarrow$

iv. $Q(i,k,x) \ \& \ \varphi_k(x)\downarrow \Longrightarrow \varphi_{h(i,k)}(x)\uparrow \ \& \ \Phi_{h(i,k)}(x) > \varphi_k(x)$

iii is immediate and iv must hold to avoid the contradiction that $\Phi_{h(i,k)}(x)$ is both bounded and undefined. Together i and iii imply

v. $\varphi_k \in \mathscr{P}^\infty \ \& \ \varphi_i \in \mathscr{P}^\infty \Longrightarrow (\overset{\infty}{\exists} x)[\varphi_k(x)\downarrow \ \& \ \Phi_{h(i,k)}(x) > \varphi_k(x)]$

vi. $\neg Q(i,k,x) \ \vee \ \varphi_k(x)\downarrow \Longrightarrow \varphi_{h(i,k)}(x)\downarrow$

From the latter if follows that

vii. $$\varphi_k \in \mathscr{P}^\infty \implies (\forall i)[W_{h(i,k)} \supset W_k]$$

Finally, define

$$g(x) = \max(0, \{\Phi_{h(i,k)}(x) \mid i,k \le x \ \& \ \neg\ Q(i,k,x)\}).$$

By iii, g is total. From ii and the definition of g ,

viii. $$\varphi_k \in \mathscr{P}^\infty \ \& \ \varphi_1 \notin \mathscr{P}^\infty \implies \Phi_{h(i,k)} \le g \quad \text{a.e.}$$

If $\tau = \varphi_k \in \mathscr{P}^\infty$ and $\tau \ge g$ a.e., v and viii imply

$$\varphi_{h(i,k)} \in I_\tau^\Phi \iff W_i \text{ finite},$$

which is exactly what is required for $\lambda i[h(i,k)]$ to reduce

$$\underline{\text{Finite}} \le_1 \Omega I_\tau^\Phi . \quad ∎$$

As was noted above, $I_\tau^\Phi$ is $\Sigma_2$ for any $\tau \in \mathscr{P}^\infty$, and thus under the above conditions it actually holds that

$$\underline{\text{Finite}} \equiv_1 \Omega I_\tau^\Phi ,$$

that is that $I_\tau^\Phi$ is $\Sigma_2$-complete.

Although an original intent of this investigation was to suggest a definition for partial complexity class, there seems to be no absolute justification for choosing one of $P_\tau^\Phi$ or $\hat{P}_\tau^\Phi$ over the other. Further study in this area is obviously desirable. It would be especially interesting to discover whether or not Theorem 4.3 can be generalized to all measures.

## 5. INFINITE INTERSECTIONS OF TOTAL COMPLEXITY CLASSES

In this section we return to complexity classes bounded by total functions to answer negatively two important questions. McCreight and Meyer [9] showed that the family of complexity classes defined by total recursive functions was closed under the infinite union of "upward nested" sequences of complexity classes. It was then natural to ask whether the same result held for "downward nested" sequences under infinite intersections. This question was originally suggested to the author by L. Bass (personal communication).

Definition: A sequence of functions $\{f_i\}$ is an r.e. sequence of total functions if $\lambda i,x[f_i(x)]$ is recursive. Such a sequence is said to be increasing (decreasing) if $f_i(x) \leq f_{i+1}(x)$ $(f_i(x) \geq f_{i+1}(x))$ for all $i, x \in N$.

Theorem 5.1. (Union Theorem) [9]. For any measure $\Phi$ and any r.e. increasing sequence of total functions $\{f_i\}$, there is a $g \in \mathcal{R}$ such that

$$\bigcup_{i \in N} I^{\Phi}_{f_i} = I^{\Phi}_{g}$$

This result extends immediately to classes $R^{\Phi}_{f}$ and to weaker conditions on the sequences $\{f_i\}$. On the other hand,

Theorem 5.2.† For $L$ = the tape measure, there is an r.e. decreasing sequence of functions $\{g_i\}$ such that, for no $h \in \mathcal{R}$ is it true that

$$\bigcap_{i \in N} I^{L}_{g_i} = I^{L}_{h} .$$

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† A similar result has been independently discovered by M. S. Paterson.

Proof: Let $g$ be a function such that $g(x) > x$ and which is computable in $g(x)$ squares. Define a recursive set of indices $\{e_j\}$ such that

$$\varphi_{e_j} \equiv 0$$

and the computation of $\varphi_{e_j}(x)$ operates as follows.

1) Simulate the computation of $\varphi_j(0)$, $\varphi_j(1), \ldots$ on $x$ squares to find the least $z$ such that

$$\sum_{y=0}^{z} (L_j(y) + 1) > x.$$

2) Calculate $g(x)$ (by the method using $g(x)$ squares of course), and move so that exactly $g(x) + (x \dot{-} z)$ on squares are used by the entire computation, then halt with output $0$.

Obviously $\varphi_{e_j}$ is total and is the identically zero function. Now consider the relationship between the computations of $\varphi_{e_j}$ and $\varphi_j$. In particular, observe that, if $\varphi_j(z)\uparrow$ and this is the least $z$ for which $\varphi_j$ diverges, then for all $x$,

$\sum_{y=0}^{z} (L_j(y) + 1) > x$, and $z$ is the least value for which this is true for arbitrarily large $x$. Thus $L_{e_j}(x) = g(x) + x \dot{-} z$ for almost all $x$. In general,

$$\varphi_j \uparrow \Rightarrow L_{e_j}(x) \geq g(x) + x \dot{-} z \quad \text{a.e.}$$

On the other hand, if $\varphi_j$ is total, then for every $z$ there is an $x_0$ such that

$$x_0 \geq \sum_{y=0}^{z} (L_j(y) + 1).$$

Thus $$x \geq x_0 \Rightarrow L_{e_j}(x) \leq g(x) + x \dot{-} z.$$

Now define $g_i(x) = g(x) + x \dot{-} i$ and it follows immediately from the above that

$$\varphi_j \text{ total} \iff (\forall i)[L_{e_j} \leq g_i \text{ a.e.}] \iff \varphi_{e_j} \varepsilon \bigcap_{i \varepsilon N} I^L_{g_i} .$$

Now assume the existance of $h \varepsilon \mathcal{R}$ such that

$$I^L_h = \bigcap_{i \varepsilon N} I^L_{q_i}$$

Then $j \varepsilon$ Total $\iff e_j \varepsilon I^L_h$. Since we may easily make $\lambda j[e_j]$ a 1-1 function, this implies

$$\underline{\text{Total}} \leq_1 I^L_h .$$

But this is a contradiction, since Total and $I^L_h$ are respectively $\Pi_2$- and $\Sigma_2$-complete [7,8]. ∎

This extends either by a direct proof or from Theorem 5.2 using the recursive relation between measures[†], to

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[†]The author is indebted to A. Borodin for observing this method of proof was applicable.

Theorem 5.3. For any measure $\Phi$, and any $t \in \mathcal{R}$, there is an r.e. decreasing sequence of functions $\{g_i\}$ such that, for all $i$,

$$g_i \geq t \quad \text{a.e.},$$

and such that there is no $h \in \mathcal{R}$ satisfying

$$I_h^{\Phi} = \bigcap_{i \in N} I_{g_i}^{\Phi} .$$

Proof: Let $\varphi_i = t$ and let $\varphi_j$ be the recursive relation between $L$ and $\Phi$, define

$$T(x) = \max(\varphi_i(x), \Phi_i(x), x) \quad \text{and}$$

$$R(x) = \max(\varphi_j(x), \Phi_j(x)).$$

Using $R^n$ to denote the n-fold composition of $R$, define

$$g_i(x) = R^{(x \dot{-} i)} \circ T(x).$$

These functions $g_i$ may be seen to be as required, arguing largely as before in the measure $L$, but shifting to $\Phi$ for the final steps.

First observe that $\lambda x, z[g_z(x)]$ is computable by a Turing machine using exactly that amount of tape. Hence we may redefine 2) in the computation of $\varphi_{e_i}$

2') use exactly $g_z(x)$ steps and output zero.

Then it follows as before that, if $z$ is the least number such that $\varphi_j(z)\uparrow$,

$$L_{e_j} = \lambda x[g_z(x)] \qquad \text{a.e.}$$

$$\varphi_j \text{ total} \Longrightarrow (\forall i)[L_{e_j} \leq g_i \quad \text{a.e.}].$$

Using notation freely, we may carry the above facts over to the measure $\Phi$ obtaining: if $z$ is the least number such that $\varphi_j(z)\uparrow$ (using $R^{-1}\ g_z$ for $\lambda x[R^{x \dot- z \dot- 1} \circ T]$),

$$e_j \in I^{\Phi}_{R \circ g_z} - I^{\Phi}_{R^{-1} \circ g_z}$$

and

$$\varphi_j \text{ total} \Longleftrightarrow (\forall i)[\Phi_{e_j} \leq g_i \quad \text{a.e.}]$$

The existance of an $h$ yeilds the same contradiction as before. ∎

Unlike the case with the Union Theorem, there can be no general implications concerning infinite intersections from $I^{\Phi}_t$ to $R^{\Phi}_t$ or vise versa. Hence the following theorem must be proved independently of the previous theorem. This important result is due to L. Bass [1].

Theorem 5.4. For any measure $\Phi$ there is an r.e. decreasing sequence of total function $\{q_i\}$ such that, for no $t \in \mathcal{R}$,

$$\bigcap_{i \in N} R^{\Phi}_{q_i} = R^{\Phi}_t .$$

Proof: The following sketch depends upon Blum's proof ([2], 327-330), freely substituting unary functions for Blum's binary ones as permitted by the input representation assumption. A complete proof along similar lines can be found in [1].

Let $\varphi_e$ be as in Lemma 3 (all such references are to [2]) and define

$$q_i = (\lambda x)[\varphi_\ell (x \dot{-} i)].$$

To see that the sequence $\{q_i\}$ is as required, assume

$$\bigcap_{i \in N} R^{\Phi}_{q_i} = R^{\Phi}_t ,$$

for some $t \in \mathcal{R}$.

Since the f of Lemma 1 belongs to each $R^{\Phi}_{q_i}$, $f \in R^{\Phi}_t$ and hence there is a j such that $\varphi_j = f$ and $\Phi_j \leq t$ a.e. Then there exists an index k for f satisfying

$$\Phi_j \geq r \circ \Phi_R , \qquad \text{a.e.} ,$$

for the "speed-up" $r_j$ and, by Lemma 2, an i such that

$$\Phi_k > q_i \qquad \text{a.e.}$$

Hence, by the properties of complexity classes,

$$R^{\Phi}_t \supseteq R^{\Phi}_{\Phi_j} \supseteq R^{\Phi}_{r \circ \Phi_k} \supseteq R^{\Phi}_{\Phi_k} \supseteq R^{\Phi}_{q_i} .$$

Yet by assumption

$$R^{\Phi}_t \subseteq R^{\Phi}_{q_i}$$

and hence

$$R^{\Phi}_{r \circ \Phi_k} = R^{\Phi}_{\Phi_k} .$$

Since Blum's result holds for any $r \in \mathscr{R}$, pick $r$ to be the compression function ([2], Thm 8) for the measured set $\{\Phi_i\}$. Then the compression theorem requires

$$R^{\Phi}_{r \circ \Phi_k} \underset{+}{\supset} R^{\Phi}_{\Phi_k} ,$$

a contradiction. ∎

The previous result may also be shown using an arguement like that in 5.3, in which the existence of such a $t$ contradicts well-known properties of the Kleene heirarchy. Although Theorems 5.2, 5.3, and 5.4 prohibit the existence of recursive functions with certain properties, functions higher in the Kleene heirarchy always exist with these properties [12].

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