Computer Sciences Department
The University of Wisconsin
1210 West Dayton Street
Madison, Wisconsin 53706

NEW FORMS OF DISCRETE MECHANICS

by

Donald Greenspan

Technical Report #122

May 1971

NEW FORMS OF DISCRETE MECHANICS

by

Donald Greenspan

ABSTRACT

Stability limitations, inherent in a previously developed discrete approach to mechanics, are relaxed by introducing two alternative approaches, one explicit and the other implicit. Computer results for motion in a nonlinear force field, planetary motion, nondegenerate three body interaction, and nonlinear string vibration are described and compared.

NEW FORMS OF DISCRETE MECHANICS

1. Introduction

Discrete mechanics is an approach to the mathematical study of mechanics in which the dynamical equations are difference equations and in which the solutions of these equations are discrete functions [3]. Such an approach is not only compatible with contemporary experimental and theoretical physics, but it is in complete harmony with modern digital computer methodology.

In [3] an ultrasimplistic, physically motivated discrete mechanics was developed in which the fundamental dynamical equation was explicit and in which the classical conservation laws followed. In one dimension, the particular explicit formulas used were equivalent to

$$(1.1) \qquad F(x_k, v_k, t_k) = m\, a(t_k) \qquad , \; k = 0,1,2,\dots$$

$$(1.2) \qquad a(t_{k-1}) = [v(t_k) - v(t_{k-1})]/\Delta t \qquad , \; k = 1,2,3,\dots$$

$$(1.3) \qquad \frac{v_k + v_{k-1}}{2} = \frac{x_k - x_{k-1}}{\Delta t} \qquad , \; k = 1,2,3,\dots$$

Feasibility of the model was established by developing applications to the nonlinear pendulum [1], [3], nonlinear string vibrations [4] nondegenerate three-body problems [5], shock waves [6], van der Pol's equation [7], and free surface fluid flow [8]. Stability problems were ever present, but existence and uniqueness were immediate consequences of the recursive structure of the dynamical equations, thus requiring no topological or algebraic considerations.

In this paper we will develop more sophisticated discrete model approaches. The aim will be to improve on the stability conditions inherent in [3] by introducing implicit and other explicit formulas. The resulting practical and theoretical consequences will be illustrated and discussed.

2. Implicit Formulation

One of the usual approaches to improve a difference equation formulation of a given model is to introduce implicit formulas, and this will be done in this section. Consider first, for simplicity, motion in a fixed, say X, direction. For $\Delta t > 0$, let particle P be at x_k at time $t_k = k\Delta t$, $k = 0,1,2,\ldots$. Let the particle's velocity v_k and acceleration a_k at time t_k be defined by the implicit, smoothing formulas

$$(2.1) \qquad \frac{v_k + v_{k-1}}{2} = \frac{x_k - x_{k-1}}{\Delta t}, \quad k = 1,2,\ldots,$$

$$(2.2) \qquad \frac{a_k + a_{k-1}}{2} = \frac{v_k - v_{k-1}}{\Delta t}, \quad k = 1,2,\ldots,$$

and let the motion of P be governed by the discrete Newton's equation

$$(2.3) \qquad ma_k = f(x_k, v_k, t_k), \quad k = 0,1,2,\ldots .$$

From (2.3), then,

$$ma_k + ma_{k-1} = f(x_k, v_k, t_k) + f(x_{k-1}, v_{k-1}, t_{k-1}), \quad k = 1,2,\ldots ,$$

so that

$$(2.4) \qquad m\,\frac{a_k + a_{k-1}}{2} = \frac{f(x_k, v_k, t_k) + f(x_{k-1}, v_{k-1}, t_{k-1})}{2}, \quad k = 1,2,\ldots .$$

By means of (2.2), then, we can rewrite (2.4) in the form

$$(2.5) \qquad m\,\frac{v_k - v_{k-1}}{\Delta t} = \frac{f(x_k, v_k, t_k) + f(x_{k-1}, v_{k-1}, t_{k-1})}{2}, \quad k = 1,2,\ldots$$

which will serve as our fundamental equation of motion.

For later purposes, it will be of value to note also that since (2.1) is a first order difference equation in v_k, it can be solved explicitly [3] to yield

$$(2.6) \qquad \begin{cases} v_1 = \dfrac{2}{\Delta t}(x_1 - x_0) - v_0 \\ v_k = \dfrac{2}{\Delta t}\left\{x_k + (-1)^k x_0 + 2\sum_{j=1}^{k-1}[(-1)^j x_{k-j}]\right\} + (-1)^k v_0; \; k \geq 2. \end{cases}$$

That the usual conservation laws are valid can be established as follows. At time t_k, $k = 0,1,2,\ldots,n$, let particle P be located at point (x_k, y_k), which is on the straight line segment directed from

$A(x_0, y_0)$ to $B(x_n, y_n)$, one possible arrangement of which is shown in Figure 2.1. Let S_k be the directed distance from (x_0, y_0) to (x_k, y_k) and let f_k, $k = 0, 1, \ldots, n$, be the component in direction $\overrightarrow{AB}$ of force $\overrightarrow{F}$ applied to P. Then the work W, done by moving P from A to B, is defined by

$$(2.7) \qquad W = \sum_{k=1}^{n} \left[\left(\frac{f_k + f_{k-1}}{2}\right) \quad (S_k - S_{k-1})\right] \quad .$$

Hence, from (2.5),

$$W = \sum_{k=1}^{n} \left[\left(m \cdot \frac{v_k - v_{k-1}}{\Delta t}\right) \quad (S_k - S_{k-1})\right]$$

$$= m \sum_{k=1}^{n} \left[(v_k - v_{k-1}) \left(\frac{S_k - S_{k-1}}{\Delta t}\right)\right]$$

$$= \frac{m}{2} \sum_{k=1}^{n} \left[(v_k - v_{k-1})(v_k + v_{k-1})\right]$$

$$= \frac{m}{2} \sum_{k=1}^{n} \left[v_k^2 - v_{k-1}^2\right]$$

$$= \frac{m}{2} v_n^2 - \frac{m}{2} v_0^2 \; .$$

If one defines the kinetic energy K_k at time t_k by

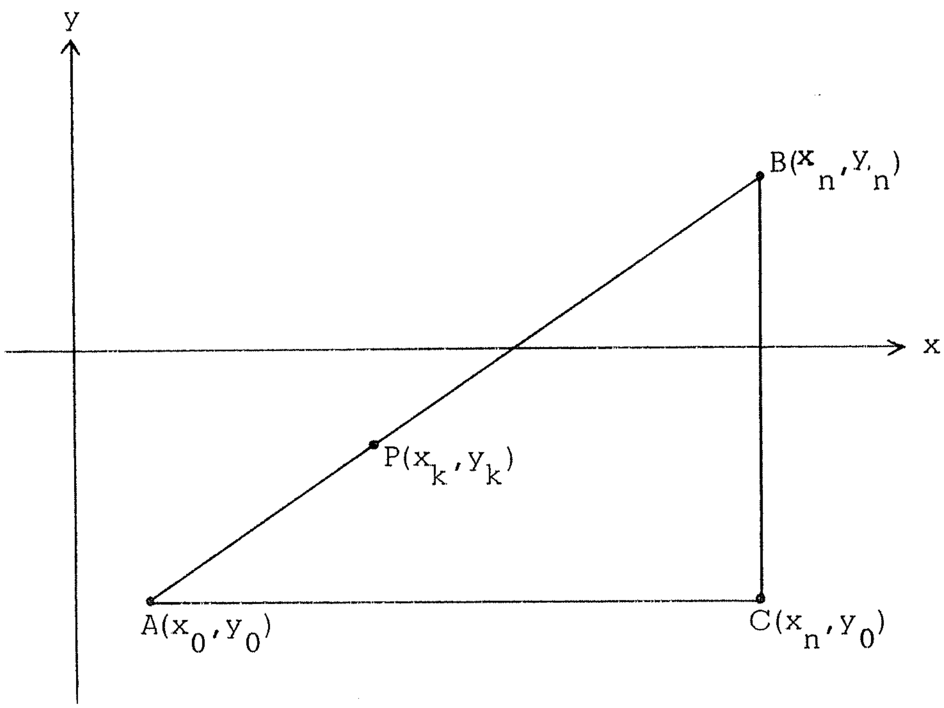

Figure 2.1

(2.8) $$K_k = \frac{1}{2} m v_k^2 ,$$

then the formula for work can be written as

(2.9) $$W = K_n^2 - K_0^2 .$$

The availability of (2.8) and (2.9) imply, then, in the usual way [3], the conservation of energy and momentum. The extension of these results to n-dimensions follows readily by the introduction of vectors.

3. Explicit Formulation

Another way to improve on a difference equation formulation of a given model is to utilize special properties in the formulation when they are available, and this will be done in this section. We will explore a particularly simple, yet remarkably accurate approach discussed by Feynman [2], and used extensively by physicists.

For $\Delta t > 0$, let particle P be at x_k at time $t_k = k\Delta t$, $k = 0, 1, 2, \ldots$. Consider the special Newtonian equation

$$(3.1) \qquad F(x_k, t_k) = m\, a_k, \quad k = 0, 1, 2, \ldots .$$

From (3.1), determine first

$$(3.2) \qquad a_0 = \frac{1}{m} F(x_0, t_0),$$

and define $v_{1/2}$ by means of

$$(3.3) \qquad v_{1/2} = v_0 + \frac{\Delta t}{2} a_0 .$$

The sequence x_k, $k = 1, 2, \ldots,$ is defined now recursively by

$$(3.4) \qquad x_{k+1} = x_k + (\Delta t)\, v_{k+(1/2)}, \quad k = 0, 1, 2 \ldots .$$

(3.5) $$v_{k+(1/2)} = v_{k-(1/2)} + (\Delta t)a_k , \qquad k = 1,2,\cdots$$

(3.6) $$a_k = \frac{1}{m} F(x_k, t_k), \qquad k = 1,2,\cdots .$$

Since the "leap-frog" formulas (3.4) - (3.6) yield position at times $k\Delta t$ and velocity at times $(k + \frac{1}{2})\Delta t$, $k = 1,2,\cdots$, the classical kinetic energy formula can only be derived if one modifies the definition of work by the introduction of average distances. Such a definition of work is as reasonable for (3.4) - (3.6) as (2.7) is for (2.1) - (2.3). A major advantage of (3.4) - (3.6) is that existence and uniqueness theorems follow immediately for any explicit formulation [3], but not for any implicit formulation. If one requires that F in (3.1) also depend on v_k, then (3.4) - (3.6) are not, per se, applicable.

4. One Dimensional Motion in a Nonlinear Force Field

To illustrate the application of the methods of Sections 2-3, consider first a prototype problem in nonlinear mechanics, that of motion in a nonlinear force field. As shown in Figure 4.1, let a particle of unit mass be constrained to move with its center Q on the

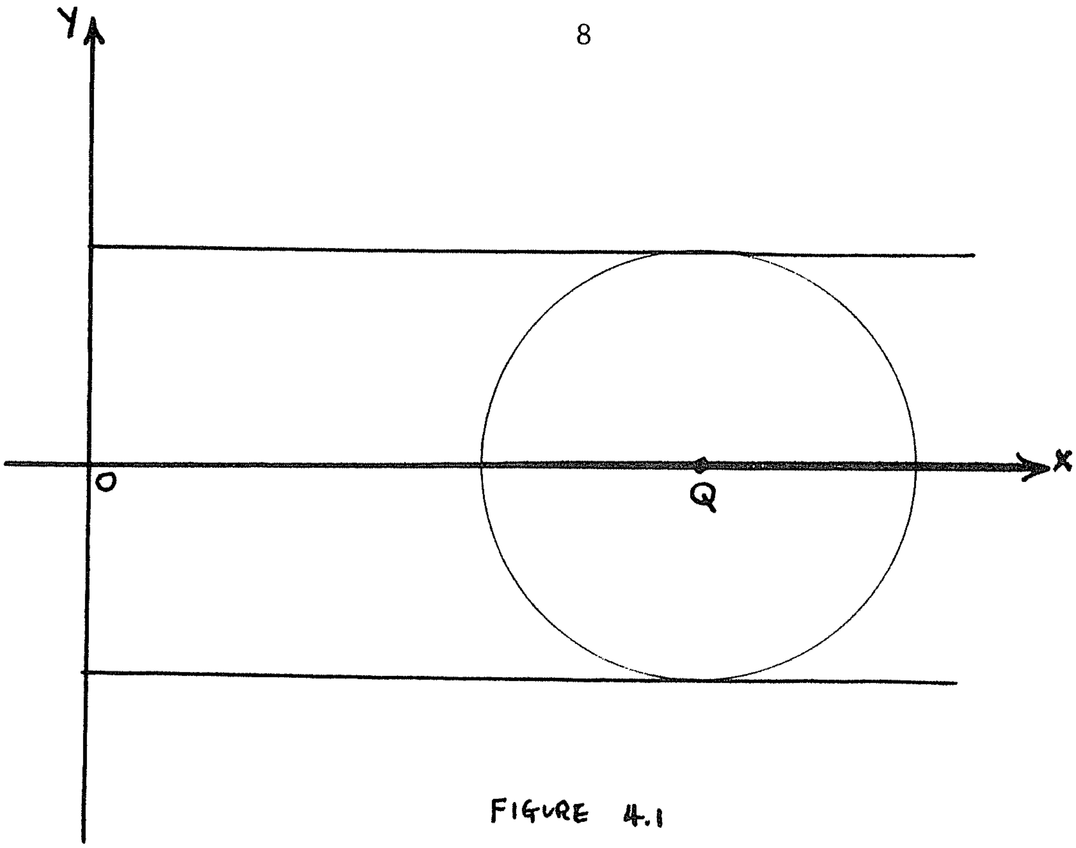

FIGURE 4.1

X-axis. Assume, for example, that a displacement of the particle such that the directed distance OQ is x is opposed by a field force of magnitude $\sin x$ and a viscous damping force of magnitude αv, where $\alpha \geq 0$. Then, the equation of motion, from (2.5), is

$$(4.1) \qquad \frac{v_k - v_{k-1}}{\Delta t} = -\frac{(\alpha v_k + \sin x_k + \alpha v_{k-1} + \sin x_{k-1})}{2}, \quad k = 1, 2, \ldots .$$

Hence,

$$(4.2) \qquad v_k\left(1 + \frac{\alpha\Delta t}{2}\right) + \frac{\Delta t}{2}\sin x_k - v_{k-1}\left(1 - \frac{\alpha\Delta t}{2}\right) + \frac{\Delta t}{2}\sin x_{k-1} = 0,$$

which, from (2.6), implies

$$(4.3) \quad x_1\left(1 + \frac{\alpha\Delta t}{2}\right) + \frac{(\Delta t)^2}{4}\sin x_1 - x_0\left(1 + \frac{\alpha\Delta t}{2}\right) + \frac{(\Delta t)^2}{4}\sin x_0 - v_0\Delta t = 0$$

$$(4.4) \quad x_2\left(1 + \frac{\alpha\Delta t}{2}\right) + \frac{(\Delta t)^2}{4}\sin x_2 - x_1\left(3 + \frac{\alpha\Delta t}{2}\right) + \frac{(\Delta t)^2}{4}\sin x_1 + 2x_0 + v_0\Delta t = 0$$

$$(4.5)\quad x_k\left(1+\frac{\alpha\Delta t}{2}\right)+\frac{(\Delta t)^2}{4}\sin x_k - x_{k-1}\left(3+\frac{\alpha\Delta t}{2}\right)+\frac{(\Delta t)^2}{4}\sin x_{k-1}$$

$$+\,4\sum_{j=2}^{k-1}\left[(-1)^j x_{k-j}\right]+2(-1)^k x_0+(-1)^k v_0\Delta t = 0,\quad k \geq 3.$$

For each of the twelve examples with $v_0 = 0$; $x_0 = \pi/4$; $\alpha = 0.003, 0.002, 0.001, 0.000$; $\Delta t = 1.0, 0.1, 0.01$, equations (4.3) - (4.5) were solved by Newton's method. The total number of time steps for each case was 10^4 and the resulting motion was always stable, which is clearly an improvement over the explicit method of [2], for which the stability condition (see [1]):

$$\Delta t < \max\left[2\alpha, \frac{2}{\alpha}\right].$$

For illustrative purposes, the motion for the case $\alpha = 0.003$, $\Delta t = 0.01$, is shown in Figure 4.2. In all cases with $\alpha = 0$, the relative maxima $\bar{X}_i$ and relative mimima $\underline{X}_i$ of sequence x_k satisfied

$$|\bar{X}_i - \frac{\pi}{4}| < 10^{-5}, \quad |\underline{X}_i + \frac{\pi}{4}| < 10^{-5}.$$

To apply the leap-frog formulas, we are limited immediately to undamped motion, i.e., to problems with $\alpha = 0$. Of those problems above, then, solved by the implicit method, we considered only the three cases $v_0 = 0$; $x_0 = \frac{\pi}{4}$; $\alpha = 0$; $\Delta t = 1.0, 0.1, 0.01$. Each run was stable for 10,000 times steps and the running time was one fourth of that required by the implicit method. No relative

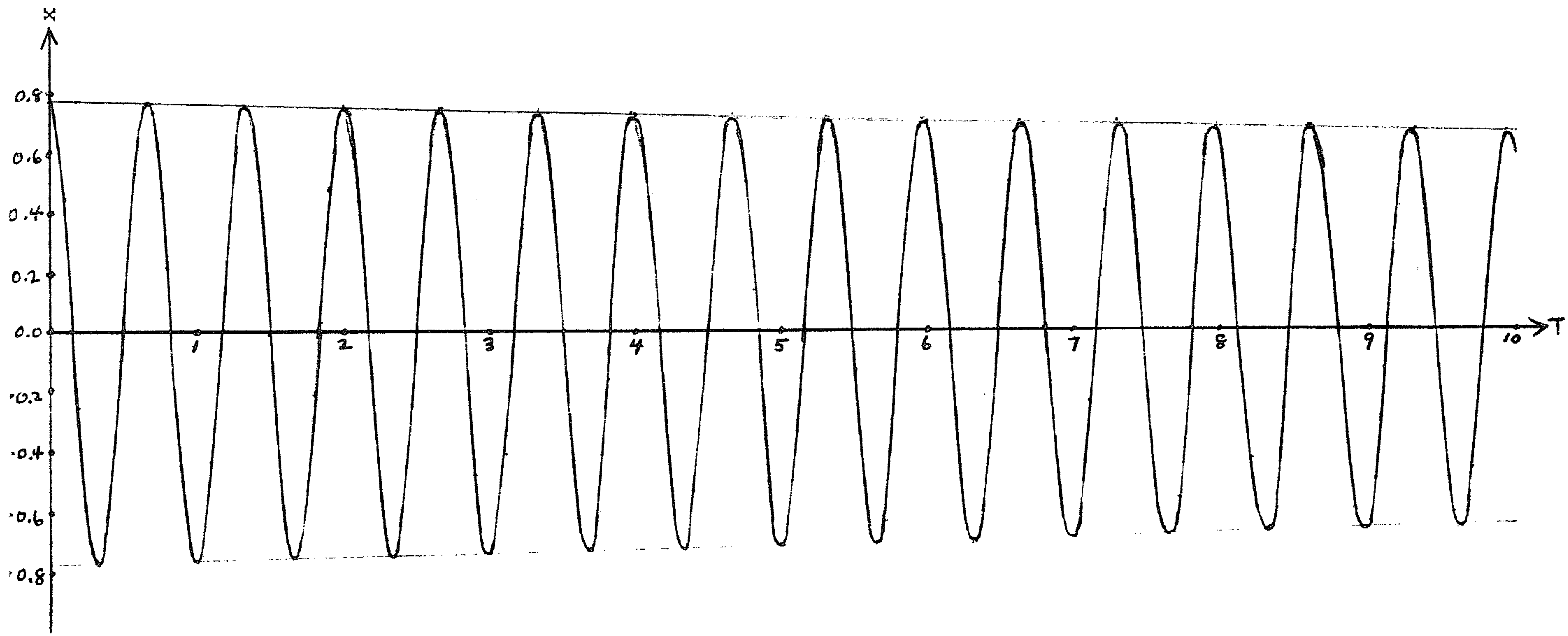

FIGURE 4.2

max $\bar{X}$ or relative min $\underline{X}$ of the resulting sequences x_n ever, in absolute value, exceeded $\pi/4$ and the motion was periodic and undamped, as was to be expected.

5. A Classical Orbit Problem

Consider next a two-body problem which is modeled to approximate planetary motion and whose initial values are identical to ones proposed by Feynman [2]. Let the sun have mass m_1 and let a planet, in orbit about the sun, have mass m_2. It is assumed that the sun's motion may be neglected and that the gravitational attractive force on the planet is given by Newton's Law of Gravitation, so that

$$(5.1) \qquad \vec{F} = (F_x, F_y) ,$$

where

$$(5.2) \qquad F_x = -\frac{G m_1 m_2}{r^2} \cdot \frac{x}{r} = -\frac{G m_1 m_2}{r^3} x ,$$

$$(5.3) \qquad F_y = -\frac{G m_1 m_2}{r^2} \cdot \frac{y}{r} = -\frac{G m_1 m_2}{r^3} y$$

$$(5.4) \qquad r^2 = x^2 + y^2 ,$$

and where the sun is positioned at the origin of the coordinate system. If one sets

(5.5) $$G m_1 = 1 ,$$

then the normalized initial conditions are

(5.6) $$\begin{cases} x = 0.50 , & v_x = 0.00 \\ y = 0.00 , & v_y = 1.63, \end{cases}$$

and the planet's trajectory is known [9] to be an ellipse with the sun at a focus, with semi-major axis $a = 0.746$, and with a period τ = 4.04 seconds. For this problem, then, let us examine the application of the methods of [3], of Section 2 and of Section 3.

From (5.2) and the formulas of [3], the desired trajectory is defined by

(5.7) $$a_{k+1,x} = \frac{-x_k}{(x_k^2 + y_k^2)^{3/2}} , \quad a_{k+1,y} = \frac{-y_k}{(x_k^2 + y_k^2)^{3/2}} , \quad k = 0,1,2\ldots ,$$

where

(5.8) $$a_{1,x} = \frac{2}{(\Delta t)^2} [x_1 - x_0 - v_{0,x} \Delta t]$$

(5.9) $$a_{2,x} = \frac{2}{(\Delta t)^2} [x_2 - 3x_1 + 2x_0 + v_{0,x} \Delta t]$$

(5.10) $$a_{k,x} = \frac{2}{(\Delta t)^2} \left\{ x_k - 3x_{k-1} + 2(-1)^k x_0 + 4 \sum_{j=2}^{k-1} [(-1)^j x_{k-j}] + (-1)^k v_{0,x} \Delta t \right\} \; ; \quad k \geq 3$$

(5.11) $$v_{1,x} = \frac{2}{\Delta t} [x_1 - x_0] - v_{0,x}$$

(5.12) $$v_{k,x} = \frac{2}{\Delta t} [x_k + (-1)^k x_0 + 2 \sum_{j=1}^{k-1} (-1)^j x_{k-j}] + (-1)^k v_{0,x}; \quad k \geq 2,$$

and where (5.8) - (5.12) are also valid with x replaced by y. From (5.7) - (5.12), it then follows that the trajectory (x_k, y_k), $k \geq 0$, of the planet is given explicitly by the formulas

(5.13) $$x_1 = x_0 + v_{0,x} \Delta t - \frac{(\Delta t)^2}{2} \left[\frac{x_0}{(x_0^2 + y_0^2)^{3/2}} \right]$$

(5.14) $$y_1 = y_0 + v_{0,y} \Delta t - \frac{(\Delta t)^2}{2} \left[\frac{y_0}{(x_0^2 + y_0^2)^{3/2}} \right]$$

(5.15) $$x_2 = 3x_1 - 2x_0 - v_{0,x} \Delta t - \frac{(\Delta t)^2}{2} \left[\frac{x_1}{(x_1^2 + y_1^2)^{3/2}} \right]$$

(5.16) $$y_2 = 3y_1 - 2y_0 - v_{0,y} \Delta t - \frac{(\Delta t)^2}{2} \left[\frac{y_1}{(x_1^2 + y_1^2)^{3/2}} \right],$$

(5.17) $$x_k = 3x_{k-1} + 2(-1)^{k+1} x_0 + 4 \sum_{j=2}^{k-1} [(-1)^{j+1} x_{k-j}] + (-1)^{k+1} v_{0,x} \Delta t - \frac{(\Delta t)^2}{2} \left[\frac{x_{k-1}}{(x_{k-1}^2 + y_{k-1}^2)^{3/2}} \right], \quad k > 3$$

$$(5.18)\qquad y_k = 3y_{k-1} + 2(-1)^{k+1} y_0 + 4 \sum_{j=2}^{k-1} [(-1)^{j+1} y_{k-j}] + (-1)^{k+1} v_{0,y} \Delta t - \frac{(\Delta t)^2}{2} \left[\frac{y_{k-1}}{(x_{k-1}^2 + y_{k-1}^2)^{3/2}}\right],\ k \geq 3.$$

To apply the implicit method of Section 2, one begins with

$$(5.19)\qquad a_{k,x} = -\frac{x_k}{r_k^3},\quad a_{k,y} = -\frac{y_k}{r_k^3},\quad k = 0,1,2,\ldots .$$

Hence,

$$\frac{a_{k,x} + a_{k-1,x}}{2} = -\frac{1}{2}\left(\frac{x_k}{r_k^3} + \frac{x_{k-1}}{r_{k-1}^3}\right),\quad k = 1,2,\ldots ,$$

or,

$$\frac{v_{k,x} - v_{k-1,x}}{\Delta t} = -\frac{1}{2}\left(\frac{x_k}{r_k^3} + \frac{x_{k-1}}{r_{k-1}^3}\right),\quad k = 1,2,\ldots .$$

Thus,

$$(5.20)\qquad v_{k,x} - v_{k-1,x} = -\frac{\Delta t}{2}\left(\frac{x_k}{r_k^3} + \frac{x_{k-1}}{r_{k-1}^3}\right),\quad k = 1,2,\ldots ,$$

and similarly

$$(5.21)\qquad v_{k,y} - v_{k-1,y} = -\frac{\Delta t}{2}\left(\frac{y_k}{r_k^3} + \frac{y_{k-1}}{r_{k-1}^3}\right),\quad k = 1,2,\ldots .$$

By means of (2.6), finally, (5.20) and (5.21) can be rewritten as

(5.22) $$x_1\left[1+\frac{(\Delta t)^2}{4r_1^{\,3}}\right]-x_0\left[1-\frac{(\Delta t)^2}{4r_0^{\,3}}\right]-v_{0,x}\,\Delta t=0$$

(5.23) $$y_1\left[1+\frac{(\Delta t)^2}{4r_1^{\,3}}\right]-y_0\left[1-\frac{(\Delta t)^2}{4r_0^{\,3}}\right]-v_{0,y}\,\Delta t=0$$

(5.24) $$x_2\left[1+\frac{(\Delta t)^2}{4r_2^{\,3}}\right]-x_1\left[3-\frac{(\Delta t)^2}{4r_1^{\,3}}\right]+2x_0+v_{0,x}\,\Delta t=0$$

(5.25) $$y_2\left[1+\frac{(\Delta t)^2}{4r_2^{\,3}}\right]-y_1\left[3-\frac{(\Delta t)^2}{4r_1^{\,3}}\right]+2y_0+v_{0,y}\,\Delta t=0$$

and

(5.26) $$x_k\left[1+\frac{(\Delta t)^2}{4r_k^{\,3}}\right]-x_{k-1}\left[3-\frac{(\Delta t)^2}{4r_{k-1}^{\,3}}\right]+2(-1)^k\,x_0$$
$$+4\sum_{j=2}^{k-1}\left[(-1)^j\,x_{k-j}\right]+(-1)^k\,v_{0,x}\,\Delta t=0,\quad k\geq 3$$

(5.27) $$y_k\left[1+\frac{(\Delta t)^2}{4r_k^{\,3}}\right]-y_{k-1}\left[3-\frac{(\Delta t)^2}{4r_{k-1}^{\,3}}\right]+2(-1)^k\,y_0$$
$$+4\sum_{j=2}^{k-1}\left[(-1)^j\,y_{k-j}\right]+(-1)^k\,v_{0,y}\,\Delta t=0,\quad k\geq 3,$$

where, recall, $r_k=\sqrt{x_k^{\,2}+y_k^{\,2}}$, $k=0,1,2,\ldots$. The trajectory path (x_k,y_k), $k=1,2,3,\ldots$, will be generated from (5.22) - (5.27) by solving each nonlinear system by Newton's method with the initial guess $(\bar{x}_k,\bar{y}_k)=(x_{k-1},y_{k-1})$.

To apply the explicit method of Section 3, one extends the leap-frog formulas as follows. The equations of motion are (5.19) and one defines

$$(5.28) \qquad v_{k+\frac{1}{2},x} = \frac{x_{k+1} - x_k}{\Delta t} , \quad v_{k+\frac{1}{2},y} = \frac{y_{k+1} - y_k}{\Delta t} , \quad k = 0,1,2 \ldots$$

$$(5.29) \qquad a_{k,x} = \frac{v_{k+\frac{1}{2},x} - v_{k-\frac{1}{2},x}}{\Delta t} , \quad a_{k,y} = \frac{v_{k+\frac{1}{2},y} - v_{k-\frac{1}{2},y}}{\Delta t} , \quad k = 1,2,\ldots .$$

The iteration is begun by means of the special formulas

$$(5.30) \qquad v_{\frac{1}{2},x} = v_{0,x} + \frac{\Delta t}{2} a_{0,x} , \quad v_{\frac{1}{2},y} = v_{0,y} + \frac{\Delta t}{2} a_{0,y} ,$$

where $a_{0,x}$ and $a_{0,y}$ are calculated from (5.19). Next, we determine x_1 and y_1 from (5.28). One now proceeds iteratively as follows. For each value of $k = 1,2,\ldots$,

(a) determine $a_{k,x}$ and $a_{k,y}$ from (5.19), then

(b) determine $v_{k+\frac{1}{2},x}$ and $v_{k+\frac{1}{2},y}$ from (5.29), and, finally,

(c) determine x_{k+1} and y_{k+1} from (5.28).

All three methods discribed above were run for $\Delta t = 10^{-1}$, 10^{-2}, and 10^{-3} for 3500, 35000, and 350000 time steps, respectively, that is, through $t = 350$. The results of the first orbit for each case are displayed in Table 1, where, by semi-major axis we mean half the distance between consecutive x-intercepts after the planet has left its initial position. The results generated by the explicit formulas (5.13) - (5.18) show a low order of accuracy and the orbits for $\Delta t = 0.1$ and 0.01 actually had an outward spiral, unstable character. The results obtained by both the implicit formulas (5.22) - (5.27) and the leap-frog formulas (5.28) - (5.30) were completely comparable, relatively accurate, and always stable. Again, calculation with (5.28) - (5.30) was about five times faster than with the implicit ones. In Figure 5-1 is shown the 86^{th} orbit determined by the leap-frog formulas with $\Delta t = 0.001$. The 86^{th} orbit determined by the implicit formulas was essentially identical to that given in Figure 5-1.

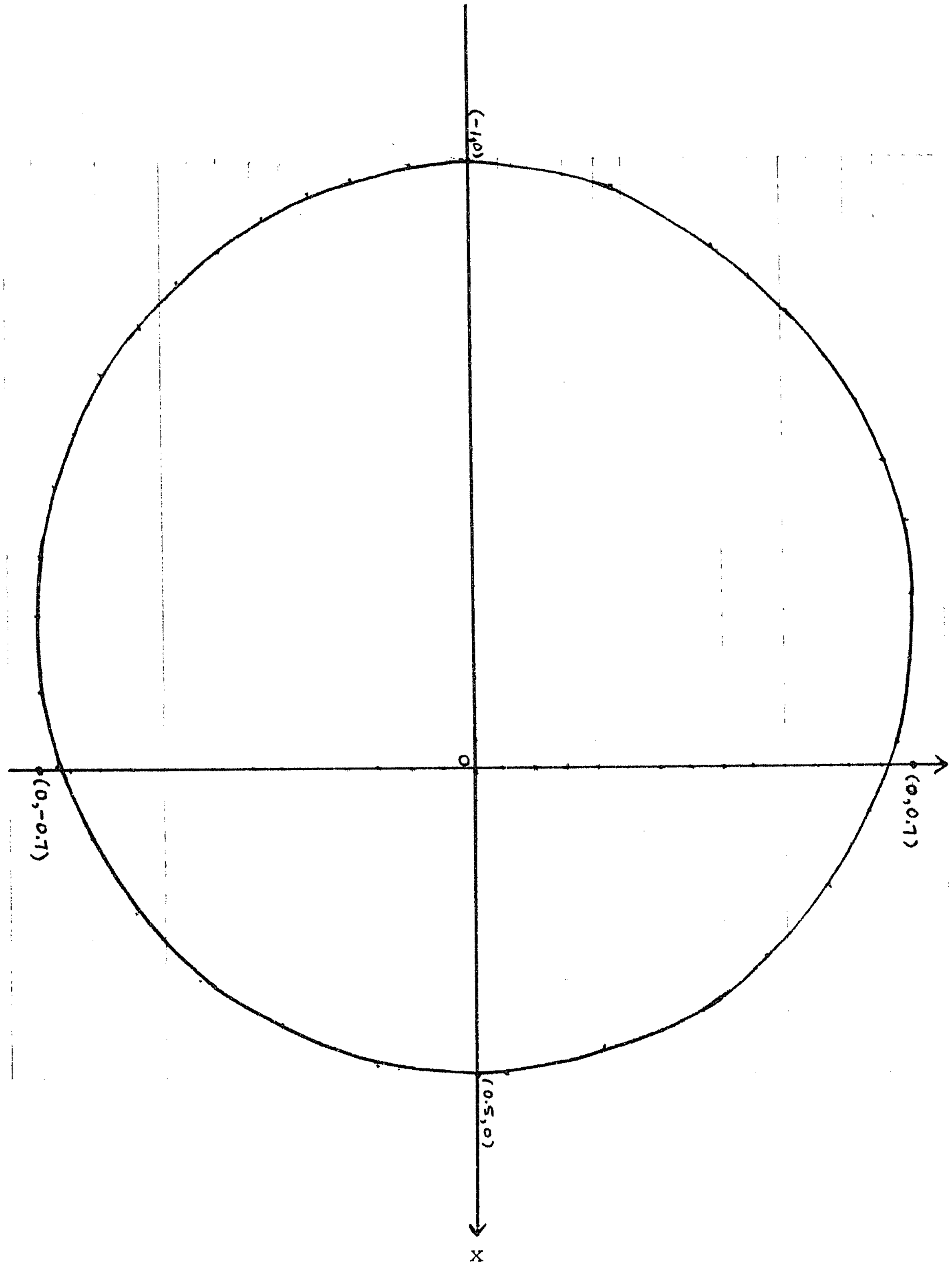

FIGURE 5.1

TABLE 1

	Time for first orbit	Semi-major axis of first orbit	X-intercept at end of first orbit	Number of orbits up to t = 350
$\Delta t = 0.1$				
Explicit	16.4	1.6	0.82	2+
Implicit	4.3	0.772	0.501	81+
Leap-frog	4.1	0.776	0.493	83+
$\Delta t = 0.01$				
Explicit	4.48	0.87	0.55	19+
Implicit	4.03	0.745	0.4998	86+
Leap-frog	4.03	0.745	0.4999	86+
$\Delta t = 0.001$				
Explicit	4.08	0.75	0.505	85+
Implicit	4.035	0.7445	0.4996	86+
Leap-frog	4.036	0.745	0.500	86+

6. A Nondegenerate Three-Body Problem

In this and in the next sections, we will examine two non-trivial problems, the first a three body problem, and the second a nonlinear vibration problem. However, for the sake of economy, only the implicit method of Section 2 will be applied in this section and only the explicit method of Section 3 will be applied in the next one.

From the discrete point of view, all dynamical behavior is the result of molecular interaction, or, in other words, all problems are n-body problems. Fundamental to such problems is, of course, the non-degenerate three-body problem, which we will formulate as follows. Consider three circular particles P_1, P_2, P_3. For each of $i = 1,2,3$, let P_i have mass m_i. At time $t_k = k\Delta t$, $k = 0,1,2,\ldots$, let P_i have center C_i at $(x_{i,k}, y_{i,k})$, have velocity $(v_{i,k,x}, v_{i,k,y})$, and have acceleration $(a_{i,k,x}, a_{i,k,y})$. Finally, let $r_{ij,k}$ be the distance between P_i and P_j at time t_k.

As in Section 2, let

$$(6.1)\quad \begin{cases} \dfrac{v_{i,k,x}+v_{i,k-1,x}}{2} = \dfrac{x_{i,k}-x_{i,k-1}}{\Delta t}, & i = 1,2,3 \\ & k = 1,2,\ldots \\ \dfrac{v_{i,k,y}+v_{i,k-1,y}}{2} = \dfrac{y_{i,k}-y_{i,k-1}}{\Delta t}, & i = 1,2,3 \\ & k = 1,2,3,\ldots \end{cases}$$

$$(6.2)\quad \begin{cases} \dfrac{a_{i,k,x} + a_{i,k-1,x}}{2} = \dfrac{v_{i,k,x} - v_{i,k-1,x}}{\Delta t}, & i = 1,2,3 \\ & k = 1,2,\ldots \\ \dfrac{a_{i,k,y} + a_{i,k-1,y}}{2} = \dfrac{v_{i,k,y} - v_{i,k-1,y}}{\Delta t}, & i = 1,2,3 \\ & k = 1,2,\ldots \end{cases} .$$

If the motions of P_i, $i = 1,2,3$, are governed by the Newtonian equations

$$(6.3)\quad \begin{cases} m_i a_{i,k,x} = f_{1,k}, & i = 1,2,3 \\ m_i a_{i,k,y} = f_{2,k}, & i = 1,2,3 \end{cases}$$

then, for $i = 1,2,3$,

$$(6.4)\qquad m_i \frac{v_{i,k,x} - v_{i,k-1,x}}{\Delta t} = \frac{f_{1,k} + f_{1,k-1}}{2}$$

$$(6.5)\qquad m_i \frac{v_{i,k,y} - v_{i,k-1,y}}{\Delta t} = \frac{f_{2,k} + f_{2,k-1}}{2}$$

and, as in (2.6),

$$(6.6)\quad \begin{cases} v_{i,1,x} = \dfrac{2}{\Delta t}(x_{i,1} - x_{i,0}) - v_{i,0,x} \\ v_{i,1,y} = \dfrac{2}{\Delta t}(y_{i,1} - y_{i,0}) - v_{i,0,y} \end{cases}, \quad i = 1,2,3$$

$$(6.7)\quad \begin{cases} v_{i,k,x} = \dfrac{2}{\Delta t}\{x_{i,k} + (-1)^k x_{i,0} + 2\sum_{j=1}^{k-1}[(-1)^j x_{i,k-j}]\} \\ \qquad + (-1)^k v_{i,0,x}\ ; \quad i = 1,2,3;\ k \geq 2 \\ v_{i,k,y} = \dfrac{2}{\Delta t}\{y_{i,k} + (-1)^k y_{i,0} + 2\sum_{j=1}^{k-1}[(-1)^j y_{i,k-j}]\} \\ \qquad + (-1)^k v_{i,0,y}\ ; \quad i = 1,2,3;\ k \geq 2. \end{cases}$$

Now, for the classical Newtonian 3-body problem, the force components f_1 and f_2 have factors of the form $-G/r^2$. For more comprehensive purposes, it will be of advantage to replace such terms by more general expressions of the form

$$-\frac{G}{r^2} + \frac{H}{r^m} - \alpha\sqrt{v_x^2 + v_y^2}\ ,$$

where $G \geq 0$, $H \geq 0$, $\alpha \geq 0$, and $m > 2$. Such a formulation allows the inclusion of gravitational attraction, collision in the form of repulsion, and viscous damping [5], when such considerations are important. The formulas (6.3) therefore take the particular forms

(6.8)
$$a_{1,k,x} = \frac{m_2(x_{1,k} - x_{2,k})}{r_{12,k}} \left\{ -\frac{G}{(r_{12,k})^2} + \frac{H}{(r_{12,k})^m} - \alpha[(v_{1,k,x})^2 + (v_{1,k,y})^2]^{1/2} \right\} + \frac{m_3(x_{1,k} - x_{3,k})}{r_{13,k}} \times \left\{ -\frac{G}{(r_{13,k})^2} + \frac{H}{(r_{13,k})^m} - \alpha[(v_{1,k,x})^2 + (v_{1,k,y})^2]^{1/2} \right\},$$

$k = 0,1,2 \ldots$

(6.9)
$$a_{1,k,y} = \frac{m_2(y_{1,k} - y_{2,k})}{r_{12,k}} \left\{ -\frac{G}{(r_{12,k})^2} + \frac{H}{(r_{12,k})^m} - \alpha[(v_{1,k,x})^2 + (v_{1,k,y})^2]^{1/2} \right\} + \frac{m_3(y_{1,k} - y_{3,k})}{r_{13,k}} \times \left\{ -\frac{G}{(r_{13,k})^2} + \frac{H}{(r_{13,k})^m} - \alpha[(v_{1,k,x})^2 + (v_{1,k,y})^2]^{1/2} \right\},$$

$k = 0,1,2 \ldots$

(6.10)
$$a_{2,k,x} = \frac{m_1(x_{2,k} - x_{1,k})}{r_{12,k}} \left\{ -\frac{G}{(r_{12,k})^2} + \frac{H}{(r_{12,k})^m} - \alpha[(v_{2,k,x})^2 + (v_{2,k,y})^2]^{1/2} \right\} + \frac{m_3(x_{2,k} - x_{3,k})}{r_{23,k}} \times \left\{ -\frac{G}{(r_{23,k})^2} + \frac{H}{(r_{23,k})^m} - \alpha[(v_{2,k,x})^2 + (v_{2,k,y})^2]^{1/2} \right\},$$

$k = 0,1,2 \ldots$

$$(6.11)\quad a_{2,k,y} = \frac{m_1(y_{2,k} - y_{1,k})}{r_{12,k}} \left\{ -\frac{G}{(r_{12,k})^2} + \frac{H}{(r_{12,k})^m} - \alpha[(v_{2,k,x})^2 + (v_{2,k,y})^2]^{1/2} \right\} + \frac{m_3(y_{2,k} - y_{3,k})}{r_{23,k}} \times \left\{ -\frac{G}{(r_{23,k})^2} + \frac{H}{(r_{23,k})^m} - \alpha[(v_{2,k,x})^2 + (v_{2,k,y})^2]^{1/2} \right\},$$

$k = 0,1,2 \ldots$

$$(6.12)\quad a_{3,k,x} = \frac{m_1(x_{3,k} - x_{1,k})}{r_{13,k}} \left\{ -\frac{G}{(r_{13,k})^2} + \frac{H}{(r_{13,k})^m} - \alpha[(v_{3,k,x})^2 + (v_{3,k,y})^2]^{1/2} \right\} + \frac{m_2(x_{3,k} - x_{2,k})}{r_{23,k}} \times \left\{ -\frac{G}{(r_{23,k})^2} + \frac{H}{(r_{23,k})^m} - \alpha[(v_{3,k,x})^2 + (v_{3,k,y})^2]^{1/2} \right\},$$

$k = 0,1,2 \ldots$

$$(6.13)\quad a_{3,k,y} = \frac{m_1(y_{3,k} - y_{1,k})}{r_{13,k}} \left\{ -\frac{G}{(r_{13,k})^2} + \frac{H}{(r_{13,k})^m} - \alpha[(v_{3,k,x})^2 + (v_{3,k,y})^2]^{1/2} \right\} + \frac{m_2(y_{3,k} - y_{2,k})}{r_{23,k}} \times \left\{ -\frac{G}{(r_{23,k})^2} + \frac{H}{(r_{23,k})^m} - \alpha[(v_{3,k,x})^2 + (v_{3,k,y})^2]^{1/2} \right\},$$

$k = 0,1,2 \ldots,$

and the algorithm proceeds as follows.

Select parameter values m_1, m_2, m_3, G, H, m, and α, and fix initial data $x_{i,0}$, $y_{i,0}$, $v_{i,0,x}$, $v_{i,0,y}$, $i = 1,2,3$. Eliminate $a_{i,k,x}$, $a_{i,k,y}$, $a_{i,k-1,x}$, $a_{i,k-1,y}$ from (6.2) by means of (6.8) - (6.13). In the resulting equations, eliminate $v_{i,k,x}$, $v_{i,k,y}$, $v_{i,k-1,x}$ and $v_{i,k-1,y}$ by means of (6.6) - (6.7), to yield six equations in the six unknowns $x_{i,k}$, $y_{i,k}$, $i = 1,2,3$. Using the initial data, generate $(x_{i,k}, y_{i,k})$, $i = 1,2,3$, $k = 1,2,\ldots$ by solving the resulting system for each k by Newton's method with initial guess $(\bar{x}_{i,k}, \bar{y}_{i,k}) = (x_{i,k-1}, y_{i,k-1})$.

As an example of the method, consider the three body problem with $m_1 = m_2 = m_3 = 10$, $G = H = 1$, $m = 3$, $\alpha = 0.08$, $x_{1,0} = 0$, $y_{1,0} = 100$, $x_{2,0} = 100$, $y_{2,0} = 0$, $x_{3,0} = -100$, $y_{3,0} = 0$, $v_{1,0,x} = 0$, $v_{1,0,y} = -10$, $v_{2,0,x} = -10$, $v_{2,0,y} = 0$, $v_{3,0,x} = 1$, $v_{3,0,y} = 0$, $\Delta t = 0.01$. The resulting motion was determined for 15000 time steps, the first 1700 of which are shown in steps of 50 in Figure 6.1. Relative particle positions are marked by a particle number followed by a letter, so that, for example, 1A, 2A and 3A mark the positions of P_1, P_2 and P_3, respectively, at the same time. Though the system has a complex motion, it is interesting to note that after repulsion,the three particles always gravitate toward each other, so that the motion is marked by constant speeding up or

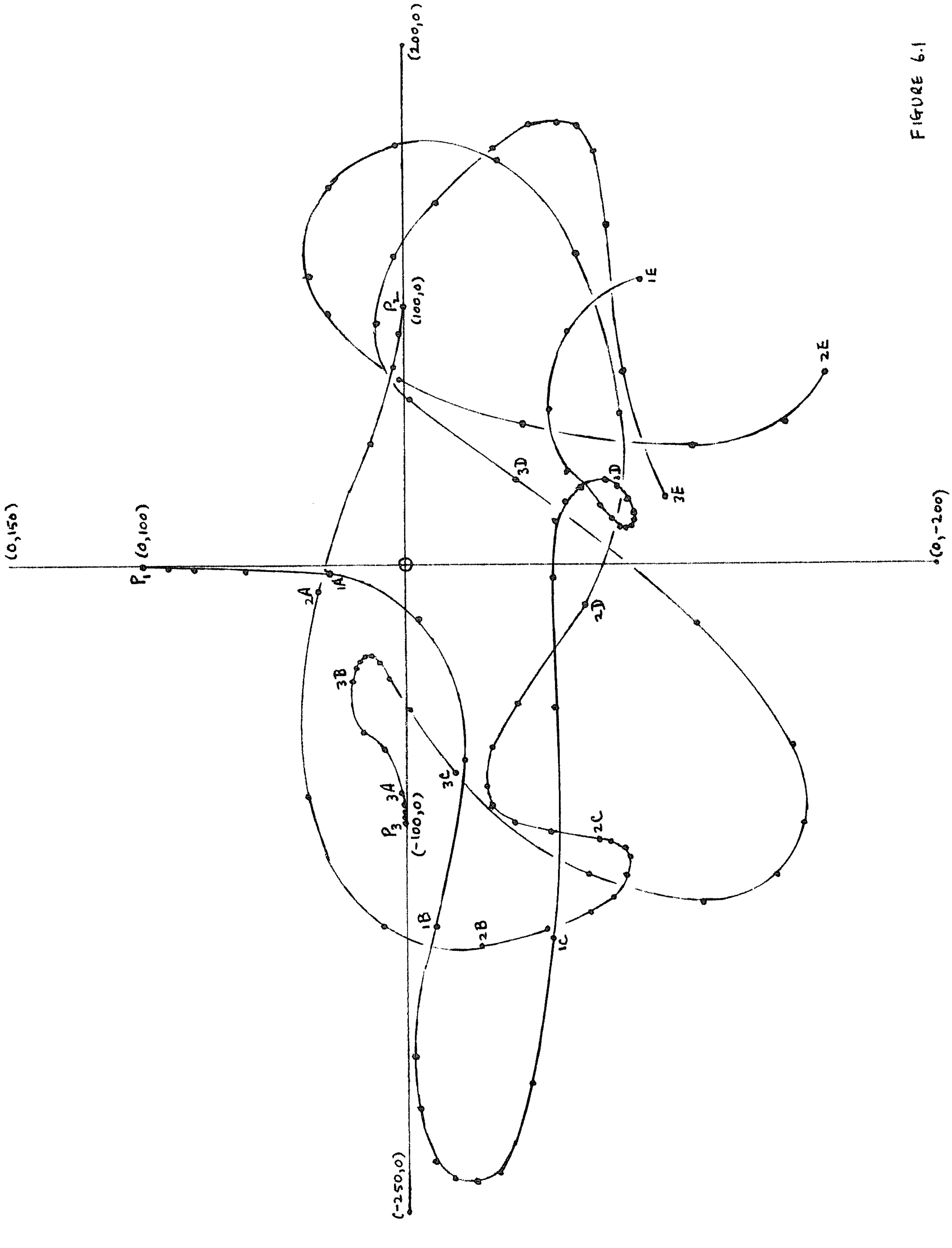
(200,0)
(100,0)
(-100,0)
(-250,0)
(0,150)
(0,100)
(0,-200)
P1
P2
P3
1A
2A
3A
1B
2B
3B
1C
2C
3C
1D
2D
3D
1E
2E
3E

FIGURE 6.1

slowing down by the individual particles.

The running time for the entire 15000 time steps was under 14 minutes.

7. The Vibrating String

Finally, let us illustrate the application of the method of Section 3 to nonlinear string vibration. Except for the exclusion of viscous damping terms, the example to be given will be more sophisticated than any in [4]. We proceed under the popular assumption that each particle of a discrete string can move in the vertical direction only.

Consider, then, a string of $n + 2$ particles $C_0, C_1, C_2, \ldots, C_n, C_{n+1}$, with C_0 fixed at $(0,0)$ and with C_{n+1} fixed at $(x_{n+1}, 0)$. Assume that $x_0 < x_1 < x_2 < \cdots < x_{n+1}$ and that $x_{j+1} - x_j = \Delta x$, $j = 0,1,\ldots,n$. Let each particle have mass m. At time $t_k = k\Delta t$, $k = 0,1,2,\ldots$, let C_j, $j = 1,2,\ldots,n$, have center $(x_j, y_{j,k})$, velocity $(0, v_{j,k,y})$ and acceleration $(0, a_{j,k,y})$. In modeling the motion of each C_j, we will consider only tensile and gravitational forces. For this purpose, let $T_{j-1,j}$ be the tensile force between C_{j-1} and C_j, while $T_{j,j+1}$ is the tensile force between C_j and C_{j+1}. Then, as in [4], the equations of motion are

$$(7.1)\qquad m a_{j,k,y} = |T_{j,j+1}| \frac{y_{j+1,k} - y_{j,k}}{[(\Delta x)^2 + (y_{j+1,k} - y_{j,k})^2]^{1/2}}$$

$$- |T_{j-1,j}| \frac{y_{j,k} - y_{j-1,k}}{[(\Delta x)^2 + (y_{j,k} - y_{j-1,k})^2]^{1/2}} - mg ,$$

For simplicity, define $T_{j,j-1}$ and $T_{j+1,j}$ by

$$(7.2)\qquad T_{j,j-1} = T_0 \left[1 + \left| \frac{y_{j,k} - y_{j-1,k}}{\Delta x} \right| + \varepsilon \left| \frac{y_{j,k} - y_{j-1,k}}{\Delta x} \right|^2 \right]$$

$$(7.3)\qquad T_{j+1,j} = T_0 \left[1 + \left| \frac{y_{j+1,k} - y_{j,k}}{\Delta x} \right| + \varepsilon \left| \frac{y_{j+1,k} - y_{j,k}}{\Delta x} \right|^2 \right] ,$$

where T_0 and ε are non-negative constants.

The initial value problem to be considered is that of fixing $(x_j, y_{j,0})$ and $v_{j,0,y}$, $j = 1,2,\ldots,n$, and of then describing the resulting motion of each particle. This will be done as follows. For each of $j = 1,2,\ldots,n$, determine $a_{j,0,y}$ from (7.1) and the initial conditions. Next, determine $v_{j,1/2,y}$ from

$$(7.4)\qquad v_{j,1/2,y} = v_{j,0,y} + \frac{\Delta t}{2} a_{j,0,y} .$$

Then, generate $y_{j,k}$, $j = 1,2,\ldots,n$, $k = 1,2,\ldots$, recursively from

$$(7.5)\qquad y_{j,k+1} = y_{j,k} + \Delta t \, v_{j,k+(1/2),y} , \quad k = 0,1,\ldots\ldots ,$$

where each $a_{j,k,y}$ is determined for $k = 1, 2, \ldots$ from (7.1), while each $v_{j,k+(1/2),y}$ is determined from

$$(7.6) \qquad v_{j,k+(1/2),y} = v_{j,k-(1/2),y} + \Delta t \cdot a_{j,k,y}, \quad k = 1, 2, \ldots .$$

In particular consider a "heavy", 201-particle string with 199 moving particles and with $x_j = \frac{j}{100}$, $j = 0, 1, \ldots, 200$, $\Delta x = 0.01$, $m = 0.005$, $T_0 = 20$, $\varepsilon = 0.01$, $\Delta t = 0.001$, $g = 32.2$. The string is placed and held in a position of tension, so that each $v_{j,0,y} = 0$ by setting the particles whose centers have x coordinates which satisfy $x_j \leq 1$ on the line $y = x$, while the remaining particles are centered on $y = -x + 1$. This initial configuration is marked t_0 in Figure 7.1. The string is then released from this initial position and its downward motion from t_0 to t_{280} is shown typically in Figure 7.1, while its upward motion from t_{320} to t_{520} is shown typically in Figure 7.2.

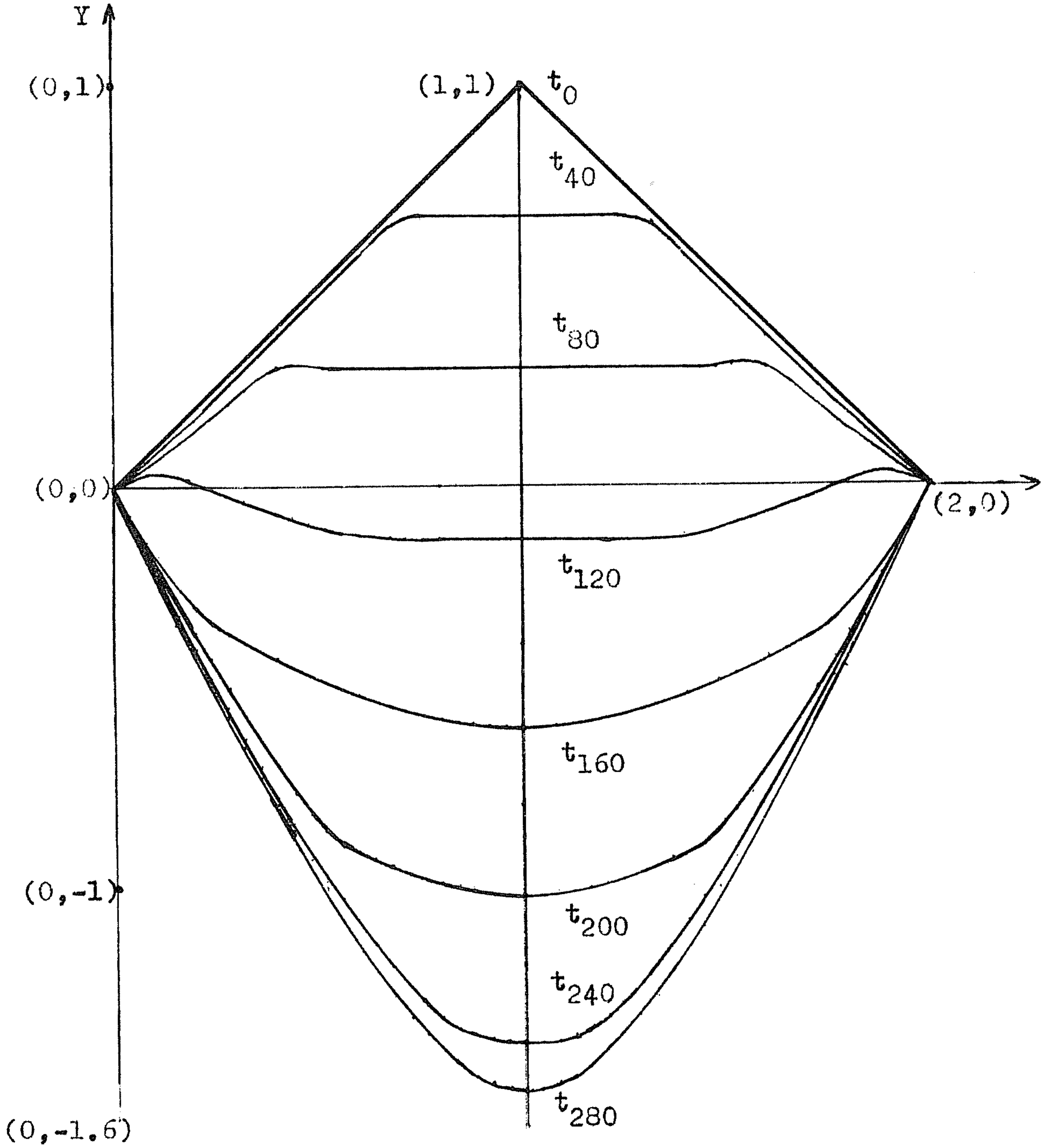

FIGURE 7.1

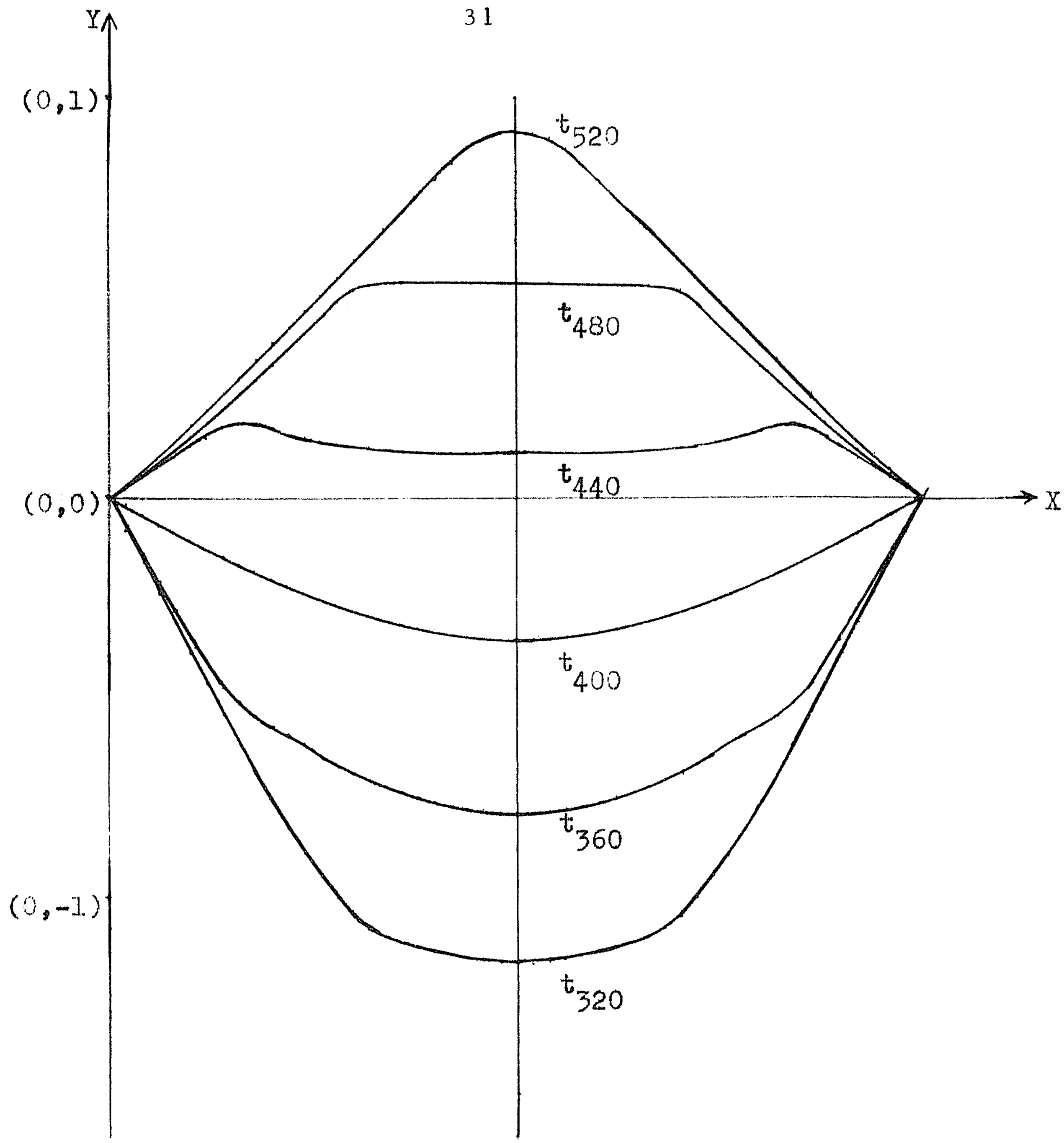

FIGURE 7.2

REFERENCES

1. C. Cryer, "Stability analysis in discrete mechanics," Tech. Report #67, Dept. of Comp. Sci., Univ. of Wis., 1969.

2. R. D. Feynman, R. B. Leighton, and M. Sands, The Feynman Lectures on Physics, Vol. 1, Addison-Wesley, Reading, Mass., Chapter 9.

3. D. Greenspan, "Discrete Mechanics," Tech. Report #49, Dept. of Comp. Sci., Univ. of Wis., 1968.

4. ______________, "Discrete nonlinear string vibrations," The Comp. Jour., 13 (1970), 195-201.

5. ______________, "Numerical studies of the three-body problem," SIAM Journ. Appl. Math., 20 (1971), 67-78.

6. ______________, "Discrete shock waves," Tech. Rpt. #84, Dept. of Comp. Sci., Univ. of Wisconsin, 1970.

7. ______________, "Numerical approximation of periodic solutions of van der Pol's equation," Tech. Rpt. #89, Dept. of Comp. Sci., Univ. of Wis., 1970, to appear in Jour. Math. Anal. Appl.

8. ______________, "Discrete liquid flow," Tech. Rpt. #14, Computing Center, Univ. of Wis., 1970.

9. K. Symon, Mechanics, Addison-Wesley, Reading, Mass., 1960.