The University of Wisconsin
Computer Sciences Department
1210 West Dayton Street
Madison, Wisconsin 53706

Complete Minimum Discrepancy Distributions for One Through Six Points in the Unit Square*

by

Brian E. White

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ABSTRACT

Given the unit square (0,0) (0,1), (1,0), (1,1) the problem is to find coordinates for a set of N points such that a quantity called the discrepancy is minimized. The discrepancy is defined as the maximum value of $|\nu/N - xy|$ over the square, where ν/N is the fraction of the point set that lies to the left and below (x, y). Precise descriptions of all possible point sets resulting in the minimum discrepancy are presented for $1 \le N \le 6$. The minimum discrepancies are $(\sqrt{5}-1)/2$, $(\sqrt{3}-1)/2$, $(3\sqrt{5}-5)/6$ for N = 1,2,3, respectively, and $1/N$ for $4 \le N \le 6$. A total of 112 optimal distributions (of which 65 are distinct up to reflection about the diagonal y = x) are generated for $4 \le N \le 6$ in about 25 seconds using a FORTRAN program and a UNIVAC 1108 computer. It is shown experimentally that the minimum discrepancy exceeds $1/N$ for N = 7 and 8.

I. INTRODUCTORY REMARKS

The notion of equidistributed sequences of numbers, viewed as points that are in some sense well placed in the k-dimensional unit hypercube, is of particular interest in the Monte Carlo Method for constructing so-called quasirandom sequences. Such sequences can be efficiently used as the nodes of an approximate quadrature formula for multidimensional integration. For detailed background information the reader is referred to the partial list of references, especially [9], pp. 36-47 which contains a recent summary of much of the work in this general area.

The subject treated here is one of very limited scope.* Rather than considering larger questions involving an arbitrary number of dimensions, many points, and any of several meaningful measures of imperfection of equidistribution, only two dimensions (the smallest non-trivial number), a few points, and the "extreme" discrepancy measure are examined. In contrast to the primarily asymptotic investigations in the literature concerning bounds on discrepancies and the construction of good sequences, here the questions are restricted to finding the best possible discrepancies and all associated point distributions.

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*This work began as a term project for a course in Monte Carlo Methods under Professor J. H. Halton at the University of Wisconsin.

II. BASIC DEFINITIONS AND PRELIMINARY RESULTS

Let P_N be a set of N points, labeled with the index set $\{i\}$ and with coordinates $p_i = (x_i, y_i)$, $0 < x_i, y_i < 1$, $1 \le i \le N$, inside the unit square S, where $(x,y) \in S$ iff $0 \le x, y \le 1$. The number of points of P_N in the rectangle $0 \le \xi < x$, $0 \le \eta < y$ defined by the point $p = (x,y) \in S$ is denoted by $\nu\ (P_N, p)$.* Let $p^+ = (x,y)^+ = (x + \delta,\ y + \varepsilon)$, where δ and ε are arbitrarily small but positive, such that

$$\Delta\, \nu(P_N, p) = \nu(P_N, p^+) - \nu(P_N, p) \tag{1}$$

equals the number of points of P_N with $x_i = x$ and/or $y_i = y$.

<u>Definition 1</u>. A ν- plateau is the maximal set of points $\{p\}$ such that every associated rectangle includes the same subset of ν points of P_N, where $0 \le \nu \le N$.

The discrepancy of a given set P_N is defined as

$$D(P_N) = \sup_{p \in S} \left| \frac{\nu\,(P_N, p)}{N} - xy \right| . \tag{2}$$

The objectives are the specification of sets $\{P_N\}$ such that $D(P_N)$ is a minimum and the determination of

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*Because of this definition and for consistency and convenience, the points of P_N are arbitrarily not allowed to lie on the boundary of S.

(3) $$D_N = \inf_{P_N \varepsilon S} D(P_N)$$

for small values of N.

Proposition 1. $D(P_N) \geq 1/N$ if there are two points of P_N such that $x_i \leq x_j$ and $y_i \geq y_j$, $1 \leq i,j \leq N$.

Proof: For such a pair i,j in P_N, $\Delta\, \nu\, (P_N, (x_j, y_i)) \geq 2$, so $D(P_N)$ is no less than that with $p = (x_j, y_i)$ and

(4) $$x_j y_i = \frac{\nu(P_N, (x_j, y_i)) + \nu(P_N, (x_j, y_i)^+)}{2N},$$

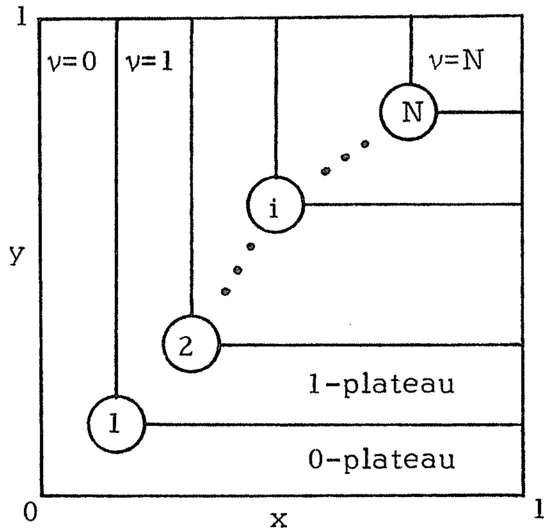
i.e., with the choice of (4), using (1) and (2) it follows that

$$D(P_N) \geq \left| \frac{\nu\,(P_N, p)}{N} - x_j y_i \right| = \frac{\Delta\nu\,(P_N, (x_j, y_i))}{2N} \geq \frac{1}{N}. \qquad \blacksquare$$

Note that for any other choice of $x_j y_i$, the lower bound on $D(P_N)$ is increased.

The question of the existence of sets $\{P_N\}$ with $D_N < \frac{1}{N}$ naturally arises. From Prop. 1 and (3), for $D_N < 1/N$ and $N > 1$, it is necessary that $x_i > x_j$ and $y_i > y_j$ for some labeling of every pair of points in P_N. Obviously, $D_1 < 1$. Thus, for $D_N < 1/N$ the relative positions of the points of P_N must be as depicted in Figure. 1.

Fig. 1. Plateau Configuration Required for $D_N < 1/N$.



Definition 2. For convenience, points of P_N are lexicographically ordered in accordance with their abscissas:

$x_1 \leq \ldots \leq x_i \leq \ldots \leq x_N$ with $x_i = x_j$ and $y_i < y_j$ implying $i < j$. A given ordering for P_N is completely specified by the ordering of the y coordinates of $\{p_i\}$ denoted by the N-tuple $\pi = n_1 \ldots n_k \ldots n_N, 1 \leq n_k \leq N$, i.e., π indicates that $y_{n_1} \leq \ldots \leq y_{n_k} \leq \ldots \leq y_{n_N}$.

Proposition 2. $D_N < 1/N$ only if $N < 4$.

Proof: Referring to Fig. 1 and (2), $D_N < 1/N$ implies that $|\nu(P_N, p)/N - xy| < 1/N$ for every point p on any of the $N+1$ ν-plateaus. Since $\nu(P_N, p)$ is constant on a given plateau, only the extreme values of the product xy on the plateau are of interest. The following definition is convenient.

Definition 3. The southwest (SW) corner of each ν-plateau is any arbitrarily small region where $\nu = \nu(P_N, p^+)$, and $p = (x, y)$ is the (unique) limit point not in the set $\{p^+\}$ on that plateau, i.e., $\nu(P_N, p) < \nu$. A northeast (NE) corner of each ν-plateau is any (not necessarily unique) point $p = (x, y)$, where $\nu = \nu(P_N, p)$ and neither x nor y can be increased without increasing ν or leaving S.

Continuing the proof, for the $(i-1)$ - plateau, $1 \le i \le N$, the largest value of xy can occur only at one of the two NE corners $(x_i, 1)$ or $(1, y_i)$. This imposes the conditions

(5a) $$\left|\frac{i-1}{N} - y_i\right| , \left|\frac{i-1}{N} - x_i\right| < \frac{1}{N}.$$

For the i-plateau inf $xy = x_i y_i$ occurs at the SW corner, so the condition

(5b) $$\left|\frac{i}{N} - x_i y_i\right| < \frac{1}{N}$$

is obtained. Conditions (5ab) and $p_i \in S$ imply that

(6a) $$\frac{i-1}{N} < x_i y_i < \frac{i^2}{N^2}$$

(6b) $$\frac{i-1}{N} < x_i, y_i < \frac{i}{N} .$$

Note that (5) and (6) hold with $\leq$'s if $D_N = 1/N$. From (6a) it is seen that $D_N < 1/N$ only if $i - 1 < i^2/N$, $1 \leq i \leq N$, i.e., only if the minimum value $N(1 - N/4)$ of the parabola $i^2 - Ni + N$ is positive. This occurs only for $N < 4$. ∎

The possibility of $D_N = 1/N$ still exists for $N = 4$.

Definition 4. An optimal distribution is a complete specification of the coordinates of P_N for which $D_N = D(P_N)$ and the ordering π of Def. 2 is fixed.

The optimal distributions and the corresponding values of D_N for N = 1, 2 and 3 are computed in the Appendix. The results are summarized in Table 1 and sketched in Fig. 2. Point 1 is fixed for N = 1 and 2, as is point 2 for N = 3. Point 2 for N = 2 and points 1 and 3 for N = 3 can be positioned anywhere in their respective regions.

Proposition 3. $D_4 = 1/4$ for $\pi = 1234$.

Proof: Using the same technique as that of the Appendix for N = 1, 2 and 3, the "critical index" for $\pi = 1234$ is $i = 2$, whence $x_2 = y_2 = 1/2$ and $D_4 = 1/4$. ∎

The optimal distribution is given in Table 2 and sketched in Fig. 3. Although point 2 is fixed, the other three points can be positioned anywhere in their respective regions.

Table 1. D_N and Optimal Distributions for N = 1, 2 and 3.

N = 1: $x_1 = y_1 = D_1 = \frac{\sqrt{5}-1}{2} \approx 0.618$

N = 2: $x_1 = y_1 = D_2 = \frac{\sqrt{3}-1}{2} \approx 0.366$

$x_2, y_2 \le \frac{\sqrt{3}}{2} \approx 0.866$

$x_2 y_2 \ge \frac{3-\sqrt{3}}{2} \approx 0.634$

N = 3: $x_1, y_1 \le D_3 = \frac{3\sqrt{5}-5}{6} \approx 0.285$

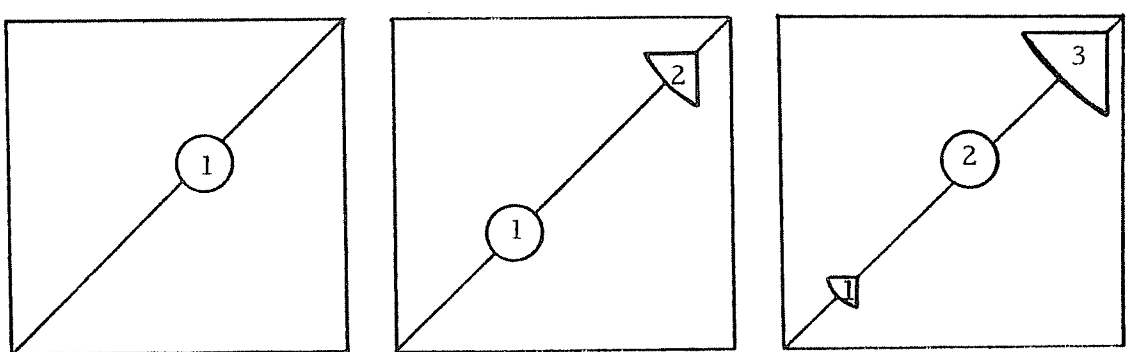
$x_1 y_1 \ge \frac{7-3\sqrt{5}}{6} \approx 0.049$

$x_2 = y_2 = D_1$

$x_3, y_3 \le \frac{3\sqrt{5}-1}{6} \approx 0.951$

$x_3 y_3 \ge \frac{11-3\sqrt{5}}{6} \approx 0.715$

Fig. 2. Optimal-Distribution Configurations for N = 1, 2 and 3.



$\pi = 1$ $\pi = 12$ $\pi = 123$

Table 2. D_4 and Optimal Distribution for $\pi = 1234$.

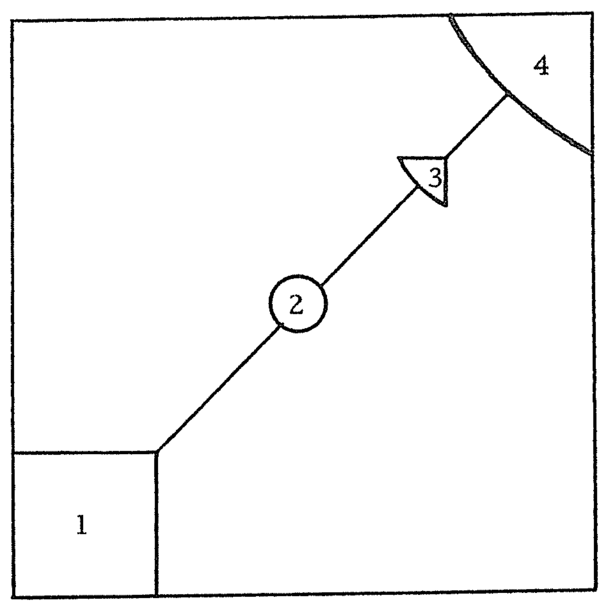
$0 < x_1, y_1 \leq D_4 = 1/4$

$x_2 = y_2 = 1/2$

$x_3, y_3 \leq 3/4; x_3 y_3 \geq 1/2$

$x_4 y_4 \geq 3/4$.

Fig. 3. Optimal-Distribution Configuration for $\pi = 1234$.



Observation 1. In an optimal distribution with $D(P_N) = 1/N$, there may be two, but no more, points of P_N that agree exactly in either their x or y coordinates.*

Proof: For two such points see Fig. 3 with $x_3 = x_4 = 3/4$ or $y_3 = y_4 = 3/4$. More than two such points means there is a pair i,j such that $\Delta\nu(P_N, (x_j, y_i)) > 2$ and $D(P_N) > 1/N$ (see the proof of Prop. 1). ∎

The sequel is concerned with optimal distributions with $D_N = 1/N$ for $N \geq 4$. Typically, there are many optimal distributions for a given N. For example, $N = 4$ yields a total of twenty with $D_4 = 1/4$ (including the optimal distribution of Prop. 3). Some characteristics of optimal distributions are explored before describing an algorithm for generating $D_N = 1/N$ solutions.

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*Evidentally, no two points of P_N should share the same coordinate, in general, since this sharing cannot decrease but may increase D_N.

III. SOME NECESSARY CONDITIONS FOR $D_N = 1/N$

Observation 2. For $N \geq 4$, $D_N = 1/N$ only if for every pair i,j in P_N such that $x_i \leq x_j$ and $y_i \geq y_j$, (a) $x_j y_i = (A+1)/N$, where A is the number of points $\{k\}$ in P_N with $x_k < x_j$ and $y_k < y_i$, and (b) no p_ℓ $(\ell \neq i,j)$ can satisfy ($x_\ell = x_j$ and $y_\ell \leq y_i$) or ($x_\ell \leq x_j$ and $y_k = y_i$).

Proof: This follows from the proof of Prop. 1, where $\Delta \nu (P_N, (x_j, y_i))$ must be only two. With $\nu(P_N, (x_j, y_i)) = A$ and $\nu(P_N, (x_j, y_i)^+) = A + 2$ in (4), one obtains $x_j y_i = (A+1)/N$ as a necessary condition. ∎

Lemma 1. For $N \geq 4$, if P_N contains a subset of four points such that p_i, p_j, p_k, p_ℓ satisfy

$$\left.\begin{matrix} x_i < \\ x_j \leq \end{matrix}\right\} x_k < x_\ell; \quad \left.\begin{matrix} y_k < \\ y_\ell \leq \end{matrix}\right\} y_i < y_j, \text{ then } D_N = 1/N \text{ only if}$$

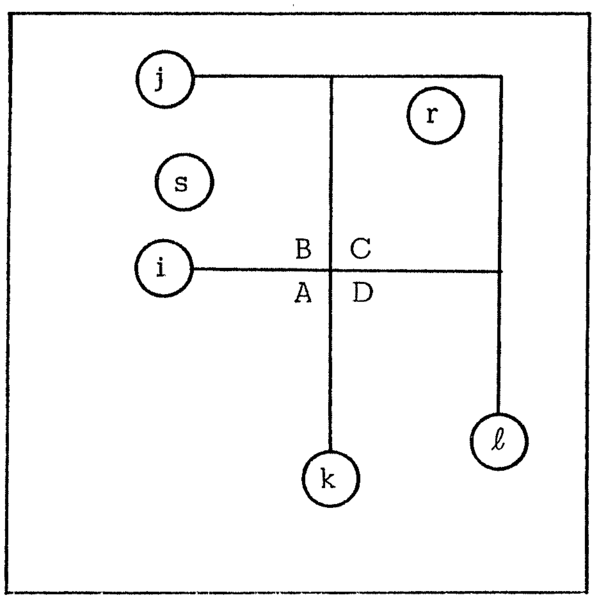
$(A+1)C = (B+1)(D+1)$, where

$A = \nu(P_N, (x_k, y_i))$, $B = \nu(P_N, (x_k, y_j)) - A - 1$,

$D = \nu(P_N, (x_\ell, y_i)) - A - 1$, and $C = \nu(P_N, (x_\ell, y_j)) - A - B - D - 2$

(see Fig. 4).

Fig. 4. Configuration of Points in Lem. 1 and Th. 1.



Proof: From Fig. 4 and Obs. 2(b), $D_N > 1/N$ if any of $x_i = x_k$, $x_k = x_\ell$, $y_k = y_i$, $y_i = y_j$ hold. The number of points of P_N in the rectangle $0 \le \xi < x_k$, $0 \le \eta < y_i$ is A; B(D) is the number of points {r} in P_N with $x_r < x_k$ and $y_i < y_r < y_j$ ($x_k < x_r < x_\ell$ and $y_r < y_i$); C is the number with $x_k < x_r < x_\ell$ and $y_i < y_r < y_j$. By Obs. 2(a),

(7a) $$x_k y_i = \frac{A + 1}{N}$$

(7b) $$x_k y_j = \frac{A + B + 2}{N}$$

(7c) $$x_\ell y_i = \frac{A + D + 2}{N}$$

(7d) $$x_\ell y_j = \frac{A + B + C + D + 3}{N} .$$

The two ratios (7a)/(7b) and (7c)/(7d), for example, must be equal or the necessary conditions (7) cannot all hold, i.e.,

$$\frac{y_i}{y_j} = \frac{A + 1}{A + B + 2} = \frac{A + D + 2}{A + B + C + D + 3}$$

is necessary. Crossmultiplying and simplifying yields

(8) $$(A + 1)\, C = (B + 1)(D + 1) .$$ ∎

Theorem 1. Under the conditions of Lem. 1, $D_N = 1/N$ is impossible unless $A = B = D = 0$ and $C = 1$.

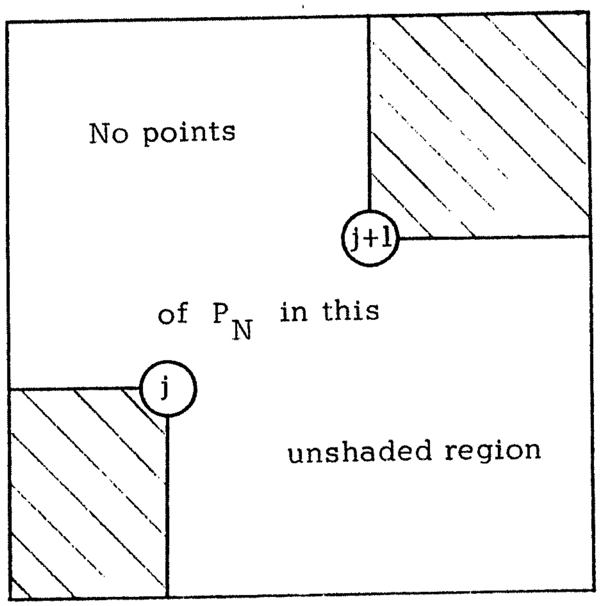
Proof: From Lem. 1, (8) must hold for all subsets of four points of P_N with the relative positions of Fig. 4 for the points i,j,k,ℓ. Clearly, $C = 0$ can be eliminated from consideration since the right-hand side of (8) is at least one. Suppose $C \geq 1$ and $B \geq 1$. Then there exists the six-point configuration of Fig. 4. From Obs. 2(b), it is necessary that $x_s < x_k, y_i < y_s < y_r$ or $y_r < y_s < y_j$, $x_k < x_r < x_\ell$, and $y_i < y_r < y_j$. If $y_i < y_s < y_r$, then a $C = 0$ situation exists for the four points i,s,k,ℓ. (p_s replaces p_j in Lem. 1). If $y_r < y_s < y_j$, then a $C = 0$ situation exists for the four points s,j,r,ℓ (p_s replaces p_i and p_r replaces p_k in Lem. 1).

Consequently, $C \geq 1$ implies that $B = 0$ for $D_N = 1/N$; by symmetry, $D = 0$ is also necessary. Hence, (8) is reduced to $(A + 1)C = 1$ which implies that $A = 0$ and $C = 1$ are required. ∎

Observation 3. For $N > 4$, $D_N = 1/N$ is impossible if for two points $j, j+1$, $1 \leq j < N$, of P_N with $y_j < y_{j+1}$, there exists no point k of P_N satisfying either $x_k \leq x_j$ and $y_j \leq y_k$ or $x_{j+1} \leq x_k$ and $y_k \leq y_{j+1}$ (see Fig. 5).

Proof: This follows directly from the proof of Prop. 2 and Def. 2. If $A = \nu(P_N, p_{j+1})$, the i of (6a) is replaced by $A + 1$ to yield $D_N = 1/N$ only if $A/N \leq x_{j+1}y_{j+1} \leq ((A+1)/N)^2$. This just amounts to a horizontal shift of the parabola in the proof of Prop. 2, so the minimum value is the same, and a non-empty range for $x_{j+1}y_{j+1}$ is impossible for $N > 4$. ∎

Fig. 5. An Impossible Point Configuration for $D_N = 1/N$ if $N > 4$.



Lemma 2. For $N \geq 4$, if P_N contains a subset of three points such that p_i, p_j, p_k satisfy $x_i < x_j < x_k$ and $y_k < y_j < y_i$, where only the first of the two $<$'s in either condition may be replaced by $\leq$, then $D_N = 1/N$ only if $x_j y_j = \frac{(A+B+1)(A+D+1)}{(A+B+C+D+2)N}$, where $A = \nu(P_N, (x_j, y_j))$, $B = \nu(P_N, (x_j, y_i)) - A$, $D = \nu(P_N, (x_k, y_j)) - A$, and $C = \nu(P_N, (x_k, y_i)) - A - B - D - 1$ (See Fig. 6).

Proof: The $\leq$ restriction follows from Obs. 2(b). Referring to Fig. 6, the number of points of P_N in the rectangle $0 \leq \xi < x_j$, $0 \leq \eta < y_j$ is A; $B(D)$ is the number of points $\{r\}$ in P_N with $x_r < x_j$ and $y_j < y_r < y_i$ ($y_r < y_j$ and $x_j < x_r < x_k$); C is the number with $x_j < x_r < x_k$ and $y_j < y_r < y_i$. By Obs. 2 (a),

(9a) $$x_j y_i = \frac{A + B + 1}{N}$$

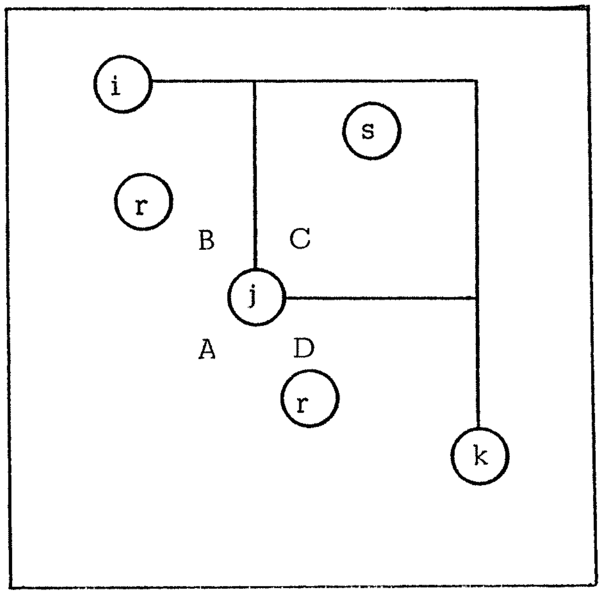
(9b) $$x_k y_j = \frac{A + D + 1}{N}$$

(9c) $$x_k y_i = \frac{A + B + C + D + 2}{N}$$

Multiplying (9a) and (9b) and dividing by (9c) yields

(10) $$x_j y_j = \frac{(A + B + 1)(A + D + 1)}{(A + B + C + D + 2)N} .$$

Fig. 6. Configuration of Points in Lem. 2 and Th. 2.



<u>Theorem 2</u>. Under the conditions of Lem. 2, $D_N = 1/N$ is impossible unless either $x_j y_j = (A + 1)^2/(A + C + 2)N$ when $B = D = 0$, or $x_j y_j = (A^2 + 3A + 2)/(A + 4)N$ when $C = B + D = 1$, with the point r of B(D) satisfying $x_r < x_j$ and $y_j < y_r < y_s (y_r < y_j$ and $x_j < x_r < x_s)$.

<u>Proof:</u> From Lem. 2, (10) must hold for all subsets of three points of P_N with the relative positions of Fig. 6 for the points i,j,k. Suppose C = 1, i.e., s is the only point of P_N satisfying $x_j < x_s < x_k$ and $y_j < y_s < y_i$. Then either B = D = 0, in which

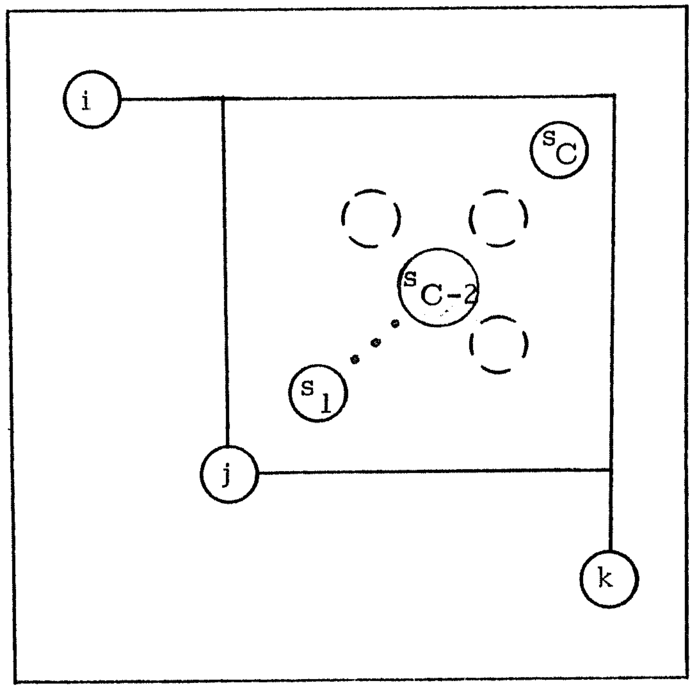
case $x_j y_j = (A + 1)^2/(A + 3)N$, or $B + D = 1$ with p_r satisfying the stated conditions, in which case $x_j y_j = (A^2 + 3A + 2)/(A + 4)N$; every other possibility with $C = 1$ leads to $D_N > 1/N$. For example, with $B \geq 1$, if y_r equals y_j or y_s, then Obs. 2(b) does not hold; if $y_s < y_r < y_i$, then a $C = 0$ situation (forbidden by Th. 1) exists for the four points r,i,s,k; if there are two points r_1, r_2 in P_N satisfying $x_{r_1}, x_{r_2} < x_j$ and $y_j < y_{r_1} < y_{r_2} < y_s$, then a $C = 0$ situation exists for the four points r_1, r_2, j, k; if $y_{r_1} = y_{r_2}$, then Obs. 2(b) does not hold. Similarly, for $D \geq 1$ by symmetry. Now supposing $C \neq 1$, in the same fashion it is readily seen that either Obs. 2(b) does not hold or a $C \neq 1$ situation (forbidden by Th. 1) exists for all possibilities but $B = D = 0$. ∎

Th. 2 is now refined via Lem. 3 and Lem. 4. See Cor. 1 and Cor. 2.

Lemma 3. In Th. 2, any points $s_1, \ldots, s_\ell, \ldots, s_C$ of P_N satisfying $x_j < x_{s_\ell} < x_k$ and $y_j < y_{s_\ell} < y_k$ $(1 \leq \ell \leq C)$ must also satisfy $x_{s_1} < \ldots < x_{s_\ell} < \ldots < x_{s_C}$ and $y_{s_1} < \ldots < y_{s_\ell} < \ldots < y_{s_C}$ except when $C > 2$, in which case either the x or the y coordinates of s_{C-2} and s_{C-1} may be interchanged in the ordering (see Fig. 7).

Proof: By Obs. 2(b) no two points of $\{s_\ell\}$ may agree in either their x or y coordinates, while $x_j < x_{s_1}$ and $y_j < y_{s_1}$ are also

Fig. 7. Possible Point Configurations for $C > 2$ in Lem. 3 Showing the Three Allowable Positions of s_{C-1}.



necessary conditions. If $C = 2$, then s_2 must be located above and to the right of s_1; any other arrangement would lead to a $C = 0$ situation (forbidden by Th. 1) involving i, s_1, s_2, k. If $C = 3$, then s_2 can assume exactly three locations relative to s_1 and s_3 not forbidden by Th. 1:

$$x_{s_1} < x_{s_2} < x_{s_3} \text{ and } y_{s_1} < y_{s_2} < y_{s_3}$$

$$x_{s_2} < x_{s_1} < x_{s_3} \text{ and } y_{s_1} < y_{s_2} < y_{s_3}$$

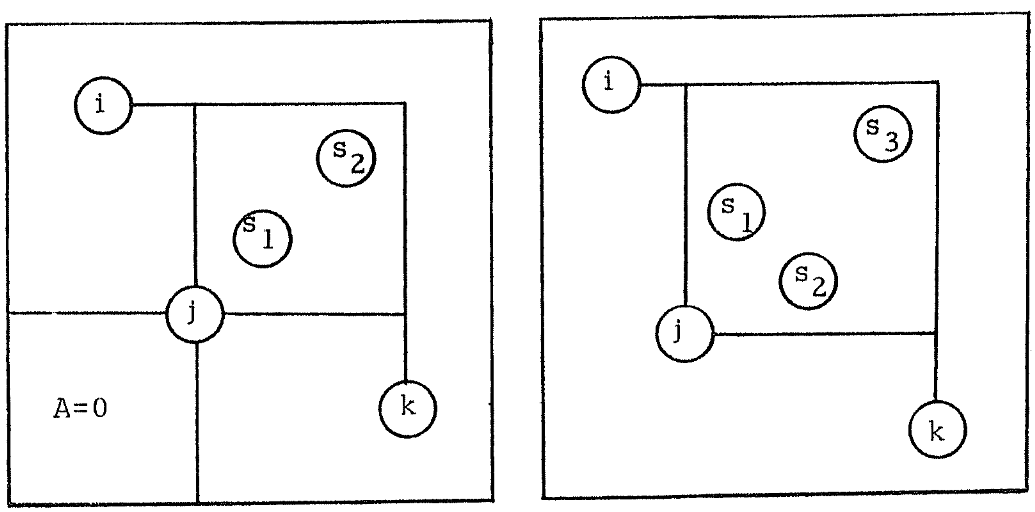
$$x_{s_1} < x_{s_2} < x_{s_3} \text{ and } y_{s_2} < y_{s_1} < y_{s_3}.$$

Similarly, for $C > 3$, it is easy to see that the x or y coordinates of s_{C-2} and s_{C-1} may be interchanged but no other exceptions to the stated ordering are permissible for $D_N = 1/N$. Thus, s_{C-1} has three distinct relative locations for $C > 2$ (see Fig. 7). ∎

Lemma 4. $C \leq 3$ in Th. 2 and Lem. 3; if $C = 2$, then A must be zero and s_1, s_2 must satisfy $x_{s_1} < x_{s_2}$ and $y_{s_1} < y_{s_2}$; if $C = 3$, then s_1, s_2, s_3 must satisfy $x_{s_1} < x_{s_2} < x_{s_3}$ and $y_{s_2} < y_{s_1} < y_{s_3}$ (see Fig. 8).

Proof: See Appendix. ∎

Fig. 8. Only Possible Point Configurations for $C \geq 2$ in Th. 2, Lem. 3 and Lem. 4.



C = 2 C = 3

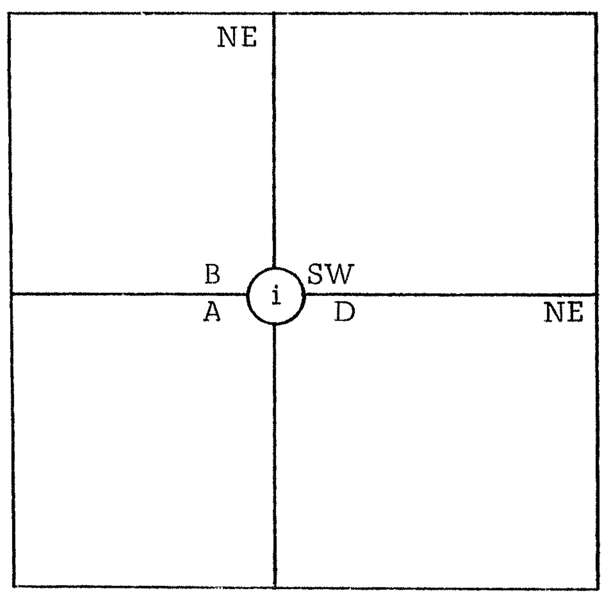
Corollary 1. Under the conditions of Lem. 2, $D_N = 1/N$ is impossible unless

$$x_j y_j = \begin{cases} \dfrac{(A+1)^2}{(A+C+2)N}, & \text{if } C = 0,1,3;\ B = D = 0; \\[2ex] \dfrac{A^2+3A+2}{(A+4)N}, & \text{if } C = B + D = 1; \\[2ex] \dfrac{1}{4N}, & \text{if } C = 2;\ A = B = D = 0. \end{cases}$$

Proof: This follows directly from Th. 2 and Lem. 4. ∎

It is emphasized that all these cases may not necessarily occur for $D_N = 1/N$.

Fig. 9. Illustrating the Two NE Corners at the Boundary and SW Corner at p_i for Any Point i of P_N.



Theorem 3. For $N \geq 4$ and no two points of P_N sharing the same coordinate, $D_N = 1/N$ is impossible unless for each point $1 \leq i \leq N$ of P_N

(a) $$\frac{A+B}{N} = \frac{i-1}{N} \leq x_i \leq \frac{i}{N} = \frac{A+B+1}{N};$$

(b) $$\frac{A+D}{N} = \frac{k-1}{N} \leq y_i \leq \frac{k}{N} = \frac{A+D+1}{N},^{*} \quad 1 \leq k \leq N;$$

(c) $$\frac{A}{N} \leq x_i y_i \leq \frac{A+1}{N},$$

where $A = \nu(P_N, p_i)$, $B = \nu(P_N, (x_i, 1)) - A$, and $D = \nu(P_N, (1, y_i)) - A$ (see Fig. 9). If points i,j of P_N share the same x(y) coordinate (see Obs. 1), then $x_i = x_j = (A + B + 1)/N$ ($y_i = y_j = (A + D + 1)/N$), and if $j < i$, then $x_i y_i = (A + 1)/N$.

Proof: Suppose no two points of P_N share the same x or y coordinate. Referring to Fig. 9, $\nu(P_N, p_i) = A$, while in the SW corner (see Def. 3), $\nu(P_N, p_i^+) = A + 1$. For $D_N = 1/N$, (2) implies that $(A - 1)/N \leq x_i y_i \leq (A + 1)/N$ and $A/N \leq x_i y_i \leq (A + 2)/N$, at p_i and p_i^+, respectively; (c) follows from these constraints. The NE corner $(x_i, 1)$ yields $\nu(P_N, (x_i, 1)) = A + B$, while $\nu(P_N, (x_i + \delta, 1)) = A + B + 1$, where δ is arbitrarily small but positive. Again, for $D_N = 1/N$, (2) implies that

---

* k is the number of points in P_N with y coordinates $\leq y_i$; see Def. 2.

$(A + B - 1)/N \le x_i \le (A + B + 1)/N$ and $(A + B)/N \le x_i \le (A + B + 2)/N$ at $(x_i, 1)$ and $(x_i + \delta, 1)$, respectively; (a) follows. Similarly, (b) follows from the constraints imposed at $(1, y_i + \varepsilon)$ and the NE corner $(1, y_i)$. Now suppose P_N contains two points i, j such that $x_j = x_i$ or $y_j = y_i$. If $x_j = x_i$, then the constraint at $(x_i + \delta, 1)$ becomes $(A + B + 1)/N \le x_i \le (A + B + 3)/N$, so $x_i = x_j = (A + B + 1)/N$ is required with the constraint at the NE corner $(x_i, 1)$; similarly for $y_j = y_i$ at $(1, y_i)$. In either instance, if $j < i$, with Def. 2, the constraint at p_i^+ in the SW corner becomes $(A + 1)/N \le x_i y_i \le (A + 3)/N$ which, along with the previous p_i constraint, yields $x_i y_i = (A + 1)/N$. ∎

Corollary 2. Under the conditions of Lem. 2, $D_N = 1/N$ is impossible unless

$$x_j y_j = \begin{cases} \dfrac{(A+1)^2}{(A + C + 2)N}, & \text{if } C = 0, 1 (A = 0, 1), 3 (A = 0); B = D = 0; \\[2ex] \dfrac{A^2 + 3A + 2}{(A+4)N}, & \text{if } C = B + D = 1; A = 0, 1, 2; \\[2ex] \dfrac{1}{4N}, & \text{if } C = 2; A = B = D = 0. \end{cases}$$

Proof: Cor. 1 suggests that under certain conditions $x_j y_j = (A + 1)^2/(A + C + 2)N$, while Th. 3 (c) implies that $A/N \le x_j y_j \le (A + 1)/N$ i.e., the question of compatibility of these constraints arises:

(11) $$\frac{A}{N} \overset{?}{\leq} \frac{(A+1)^2}{(A+C+2)N} \overset{?}{\leq} \frac{A+1}{N} .$$

The right-hand $\leq$ yields $A + 1 \overset{?}{\leq} A + C + 2$, which always holds. The left-hand $\leq$ yields $A^2 + AC + 2A \overset{?}{\leq} A^2 + 2A + 1$; this holds iff $AC \leq 1$. Similarly, the other formula involving A in Cor. 1 leads to $A \leq 2$. Cor. 1 is modified accordingly. ∎

Hence, the remark following Cor. 1 is justified.

IV. $D_N > 1/N$ IF $N \geq 7$?

Lemma 5. For $N \geq 7$, $D_N > 1/N$, if P_N contains a subset of four points specified in Lem. 1.

Proof: Assume that $D_N = 1/N$ is possible. If such a subset occurs, then it must appear in the lower left-hand region of S with $A = B = D = 0$ and $C = 1$, as shown in Fig. 10, by Th. 1. Using the same point labels as those of Fig. 4., $i = 1(2)$, $j = 2(1)$, $k = 3$ and $r = 4$ results from Def. 2. There can be no point t of P_N with $x_t \leq x_4$ and $y_{2(1)} < y_t$ or a situation (forbidden by Th. 1) exists.* The label $\ell \geq 5$, however, since $x_4 < x_t < x_\ell$ and $y_{2(1)} < y_t$ can occur. From Cor. 2

(12a) $$x_4 y_4 = \frac{(2+1)^2}{(2+2)N} = \frac{9}{4N};$$

Th. 3 (ab) implies that

(12b) $$x_4, y_4 \leq \frac{2+1+1}{N} = \frac{4}{N};$$

Thus,

(12c) $$\frac{9}{4} \overset{?}{\leq} \frac{16}{N}$$

must hold for $D_N = 1/N$; but (12c) holds only for $N \leq 7$, so it remains to investigate only $N = 7$. By Obs. 2 (a),

(13a) $$x_\ell y_4 = x_4 y_{2(1)} = \frac{2+1}{7} = \frac{3}{7};$$

---

*By symmetry, there can be no point t of P_N with $x_\ell < x_t$ and $y_t \leq y_4$; these forbidden regions are indicated by X's in Fig. 10.

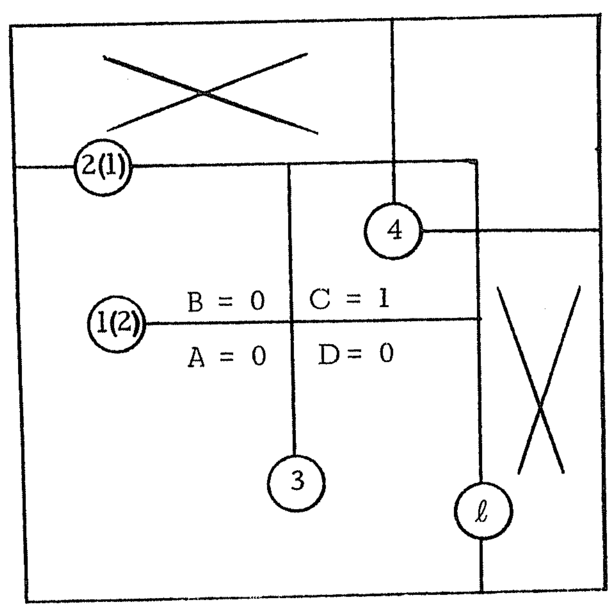
Th. 3(ab) implies that

(13b) $x_\ell \le \frac{\ell}{7}; \quad y_4 \le \frac{4}{7};$

(13c) $x_4 \le \frac{4}{7}, \quad y_{2(1)} \le \frac{5}{7},$

where the condition on $y_{2(1)}$ follows by observing that $\ell = 6$ or 7 is necessary to satisfy (13ab), while if $\ell = 6$, point 7 of P_N cannot satisfy $y_4 < y_7 \le y_{2(1)}$ or a $C = 0$ situation (forbidden by Th. 1) occurs. Unfortunately, (13c) is insufficient to satisfy (13a), since $20/7 < 3$. Consequently, $D_N = 1/N$ for $N \ge 7$ only if the subset of Lem. 1 does not appear in P_N. Otherwise, $D_N > 1/N$ by Prop. 2. ∎

Fig. 10. Only Possible Point Configuration with the Hypotheses of Lem. 5.



Lemma 6. For $N \geq 7$, $D_N > 1/N$, if P_N contains a subset of three points specified in Lem. 2, unless $B = C = D = 0$ and $N \leq (A + 3)(A + 2)/(A + 1) = A + 4 + 2/(A + 1)$.

Proof: See Appendix. ∎

Note that $N \geq 7$ implies that $A \geq 3$, and that the minimum value of A increases by one with an increase in N of one.

Conjecture. $D_N > 1/N$, for $N \geq 7$.

This is based in part on some experimental results that are discussed in the sequel. Lems. 5 and 6 are nearly enough to prove the conjecture. If it is shown for the exceptional case of Lem. 6, then the conjecture follows with a modicum of additional effort to eliminate the remaining cases not covered by Lems. 5 and 6.

V. AN ALGORITHM FOR GENERATING $D_N = 1/N$ SOLUTIONS

Prop. 4 and Obs. 4 are useful for the systematic generation of ordered N-point sets and the elimination of one of every symmetric pair of orderings in the N! possible permutations.

Proposition 4. The P^{th} permutation $\pi_P = n_1 \ldots n_N$ of the y coordinates of P_N (see Def. 2) in a list of N! possible orderings for N points can be uniquely specified by

$$P = P(N) + \sum_{i=2}^{N-1} \frac{N!}{i!} [P(i) - 1], \quad 3 \le N, \tag{14}$$

where $P(j)$, $2 \le j \le N$, is the place (numbering from left to right) of element j in π_P with all elements $> j$ removed.

Proof: This follows from a standard inductive definition of permutation generation:

Definition 5. Given a permutation $\pi = n_1 \ldots n_{N-1}$ of $1, \ldots, N-1$, the next N permutations in the list are:

$$\begin{array}{llll} N & n_1 \ldots & & n_{N-1} \\ n_1 & N \ldots & & n_{N-1} \\ & \vdots & & \\ n_1 & \ldots & N & n_{N-1} \\ n_1 & \ldots & n_{N-1} & N \end{array}.$$

Continuing the proof, the i^{th} permutation of the N indicated above is that with N in the i^{th} place. This accounts for $P(N)$ in (14). Now select any one, say π_p, of these permutations and delete N to get π again. This permutation has a weight of N in (14), since it generates N permutations of length N. The other factor of (14) for $i = N - 1$ is $P(N - 1) - 1$ because $N - 1$ in the $P(N - 1)^{th}$ place of π implies that $N[P(N - 1) - 1]$ permutations of length N appear earlier in the list than those generated by π. Then delete $N-1$ from π to obtain a permutation π' of length $N - 2$; π' receives a weight of $N(N - 1)$ since it generates $N(N - 1)$ permutations of length N, etc. $\blacksquare$

As clarification the permutations for $1 \leq N \leq 4$ are listed in Table 3; (14) is easily checked for low-order permutations.

Table 3. List of Permutations for $1 \leq N \leq 4$.

$1 \leq N \leq 3$

P	π_P
1	1
1	2 1
2	1 2
1	3 2 1
2	2 3 1
3	2 1 3
4	3 1 2
5	1 3 2
6	1 2 3

$N = 4$

P	π_P	P	π_P	P	π_P
1	4 3 2 1	9	4 2 1 3	17	4 1 3 2
2	3 4 2 1	10	2 4 1 3	18	1 4 3 2
3	3 2 4 1	11	2 1 4 3	19	1 3 4 2
4	3 2 1 4	12	2 1 3 4	20	1 3 2 4
5	4 2 3 1	13	4 3 1 2	21	4 1 2 3
6	2 4 3 1	14	3 4 1 2	22	1 4 2 3
7	2 3 4 1	15	3 1 4 2	23	1 2 4 3
8	2 3 1 4	16	3 1 2 4	24	1 2 3 4

<u>Observation 4</u>. Given an ordering $\pi_P = n_1 \ldots n_N$ for P_N (see Def. 2), there is a unique inverse permutation

(15) $$\pi_P^{-1} = \pi_Q = P(1) \cdots P(i) \cdots P(N),$$

where $P(i)$ is the place of i in π_P; π_Q corresponds to a reflection of P_N about the diagonal $y = x$ in S; if $Q = P$, then π_P is its own inverse, and P_N is said to be symmetric with respect to $y = x$.

<u>Proof</u>: This is perhaps best seen by an example: ∎

<u>Example 1</u>. $\pi_{32} = 25341$; $\pi_{32}^{-1} = \pi_{91} = 51342$.

Given $\pi = 25341$, $P = 32$ is calculated using (14):

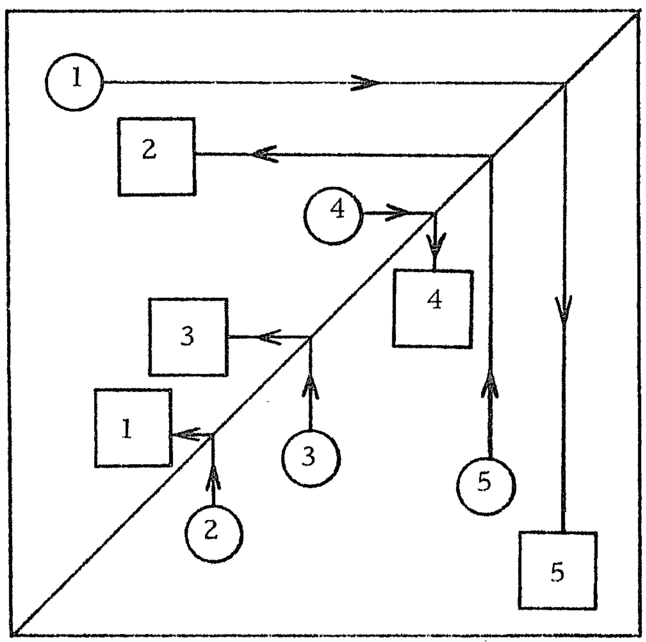
25341 2341 231 21

$2 + 5\times 2 + 20\times 1 + 60\times 0 = 32$. Then π^{-1}_{32} is found using (15) and Q = 91 is calculated using (14):

51342 1342 132 12

$1 + 5\times 2 + 20\times 1 + 60\times 1 = 91$; (15) can be justified by examining Fig. 11.

Fig. 11. P_5 of Example 1 and Its Reflection About $y = x$; ○ for π, □ for π^{-1}.



A sufficient condition for $D(P_N) = 1/N$ is now established. Obs. 5 serves as the basis of the algorithm offered in V, although Th. 1, Lems. 2,5,6, Prop. 4 and Obs. 4 are also used to facilitate the computation.

Observation 5. For $N \geq 4$, $D(P_N) = 1/N$ if P_N satisfies Obs. 2(a) and Th. 3(abc).

Proof: As mentioned in the proof of Prop. 2, given any P_N, $D(P_N)$ depends only on the values of $\Delta \equiv |\nu(P_N, p)/N - xy|$ in the SW corner and at the NE corner(s) of each of the ν-plateaus (see Def. 3). It is shown that if the constraints of Obs. 2(a) and Th. 3(abc) are satisfied, then $\Delta \leq 1/N$ at all the SW and NE corners. Whence, $D(P_N) = 1/N$ by (3) and Prop. 2. There are several types of SW and NE corners to consider. The trivial SW corner at (0,0) and NE corner at (1,1) are of no consequence, since $\Delta = 0$ at these points regardless of P_N.* Every pair of points specified in Obs. 2(a) defines one SW and one NE corner at $x_j y_i$; $\Delta = 1/N$ by construction at these corners if Obs. 2(a) is satisfied. The x(y) coordinate of each point of P_N defines a NE corner at the $y = 1 (x = 1)$ boundary of S (see Fig. 9); $\Delta \leq 1/N$ at these corners if Th. 3(ab) holds. The coordinates of point $1 \leq i \leq N$ of P_N define one SW corner (but no NE corner) at $x_i y_i$; $\Delta \leq 1/N$ at these corners provided Th. 3(c) holds. [Note that no two points of P_N share the same coordinate, by hypothesis of Th. 3(abc).] This covers all possible SW and NE corners for any given P_N. ∎

*Recall that the points of P_N must lie on the interior of S, by definition.

Most of the foregoing theoretical development for $N \geq 4$ is incorporated in a few-hundred statement FORTRAN program. The algorithm is described informally below in eight basic steps that include several explanatory remarks. Using the program the author claims to have found all optimal distributions (see Def. 4) for $N = 4$, 5 and 6 ($D_N = 1/N$), and also to have shown that $D_N > 1/N$ for $N = 7$ and 8. The latter result is deduced from the lack of output when running the program with $N = 7$ and 8. Apparently, this is experimental evidence in support of the conjecture that $D_N > 1/N$ for $N \geq 7$.

The UNIVAC 1108 total run times and the number of optimal distributions obtained with $D_N = 1/N$ are summarized in Table 4. The optimal distributions are all listed with accompanying sketches in VI.

Table 4. Some Data On the Computed Results.

N	N!	CPU Time(sec.)	Number of Optimal Distributions	
			Symmetric	Total
4	24	7.591	8	20
5	120	9.055	6	52
6	720	9.339	4	40
7	5040	13.262	0	0
8	40,320	85.388	0	0

Algorithm Description

1. Initialization. Select $4 \leq N \leq 8$. (An N larger than 8 was not tested.) Choose a tolerance (TOL) for safety in making decisions involving inequalities. (TOL = 10^{-3} was used.) Calculate N! and the weights $N!/i!$, $2 \leq i \leq N-1$, of Prop. 4 for use in coding the P^{th} permutation π_P with the formula of (14). Set $P = 1$. (All N! possible permutations are considered, at least briefly.)

2. Consider the P^{th} permutation $\pi_P = n_1 \ldots n_N$ of the ordinates (see Def. 2) of the N points. Call a subroutine for obtaining π_P from P. Form the inverse permutation $\pi_Q = \pi_P^{-1}$ using (15) (see Obs. 4) and calculate Q using (14). If $Q < P$, increment P and repeat step 2 if $P \leq N!$ (In this case the reflection π_Q about $y = x$ of π_P has already been considered. If $Q = P$, then $\pi_Q = \pi_P$ and P_N is symmetric.)

3. Test for the 4-point pattern of Th. 1. If such a pattern appears at all in P_N, it must occur in the lower left-hand corner of the unit square. (No more than one such 4-point pattern may occur.) If $N \geq 7$, this 4-point pattern may not occur (see Lem. 5). Increment P and go to step 2 if $P \leq N!$ and these rules are violated. (Step 3 quickly eliminates many permutations from further consideration.)

4. Establish the upper and lower bounds of Th. 3(abc). (Without loss of generality, it can be assumed that initially, no two points of P_N share the same x or y coordinate.) Test the consistency of Th. 3(ab) and Th. 3 (c), e.g., increment P and go to step 2 if $P \le N!$ and the product of the upper bounds of (ab) plus TOL is strictly less than the lower bound of (c).

5. Compute the equalities of Obs. 2 (a) and test their consistency with the bounds of Th. 3 (ab). If an equality $x_j y_i$ is strictly outside the maximum possible range afforded by Th. 3 (ab) with TOL used to slightly widen the range, then go to step 2 after resetting any equalities and incrementing P, if $P \le N!$ (The equalities can be stored in the above (below) diagonal portion of an N×N matrix, for example.)

6. Compute the equalities of Lem. 2 and (10) and test their consistency with the bounds of Th. 3 (c). If an equality falls outside the permissible range of Th. 3 (c), even with TOL, then increment P, reset equalities and go to step 2, if $P \le N!$ If $N \ge 7$, test for the 3-point patterns of Lem. 6. If such a pattern is found, then increment P, and if $P \le N!$ reset (to zero, say) all equalities and go to step 2, unless the pattern is an exception satisfying the inequality in A and N of Lem. 6.

7. Consider each new (most recent) upper bound $x_j(\max)$ on an x_j that is involved in an equality $x_j y_i = E$ of Obs. 2 (a). If $E/x_j(\max) - TOL > y_i(\min)$, the old lower bound to y_i, then $y_i(\min)$ is replaced by the ratio $E/x_j(\max)$ and this new lower bound is flagged, unless $y_i(\max)$, the old upper bound to y_i, is exceeded in which case the equality E cannot (or can no longer) be supported, so P is then incremented, and if $P \le N!$ all equalities are reset and a return is made to step 2. A similar procedure is followed for every $x_j y_j$ equality of Lem. 2. And analogous tests are performed for the most recent upper bounds on the y_i's and the new lower bounds on the x_j's and y_i's. In the same way, using Th. 3(c) search for new bounds on the coordinates of those points of P_N that are not involved in an $x_j y_j$ equality of Lem. 2. (The first time step 7 is executed for a given P, every variable involved in an equality has "new" bounds, by definition. These bounds are implicit in the equality and the size of S.) If there are any newly-flagged bounds as yet not considered for their effect on other bounds, then repeat step 7. (The E's and the bounds are always rational numbers because N is finite. Consequently, the iterative process of step 7 eventually terminates.)

8. Termination. Output the optimal distribution of π_P (see Def. 4) consisting of the equalities $\{x_j y_i = E\}$ and $\{x_j y_j = E\}$ and the various ranges $\{L \leq x_j, y_i \leq U\}$ and $\{L \leq x_j y_j \leq U\}$ that must hold. ($D_N = 1/N$ for any set P_N satisfying these constraints.) If $P < N!$, increment P, reset all equalities and go to step 2. Otherwise, halt.

VI. ALL OPTIMAL DISTRIBUTIONS FOR N = 4, 5, AND 6 ($D_N = 1/N$)

On subsequent pages there are a total of 14, 29 and 22 blocks of data and accompanying sketches of optimal distributions for N = 4,5 and 6, respectively. The author believes that this tabulation is correct and complete.* As an illustration of how to read the table, consider the "fleshing-out" of the first block of data for N = 4:

				$x_jy_i = ?/4$		
				$j = 2$	$j = 3$	$j = 4$
$P = 3$	$\pi_P = 3241$	$Q = 9$	$\pi_Q = 4213$	$i = 1$	$i = 1,2$	$i = 1,2,3$
				$? = 1$	$? = 2,1$	$? = 3,*,*$

$$0 < x_1 \leq 1/4 \qquad 0 < x_1y_1 \leq 1/4 \qquad 3/4 \leq y_1 < 4/4$$
$$1/4 \leq x_2 \leq 1/3 \qquad x_2y_2 = 1/8 \qquad 3/8 \leq y_2 \leq 2/4$$
$$2/4 \leq x_3 \leq 2/3 \qquad 0 < x_3y_3 \leq 1/4 \qquad 0 < y_3 \leq 1/4$$
$$3/4 \leq x_4 < 4/4 \qquad 2/4 \leq x_4y_4 \leq 3/4 \qquad 2/4 \leq y_4 \leq 3/4 .$$

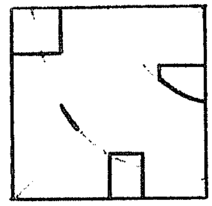
Note that Q and π_Q are absent for symmetric distributions (see Obs. 4), and that only the numerator of fractions expressible with N as the denominator is listed. Asterisks are used to indicate the absence of an x_jy_i equality constraint (see Obs. 2(a)). The occurrence of a fixed fraction rather than an interval of real numbers is indicated by centering the fraction between appropriate columns,

*The information in the table is derived with the algorithm described in V. Every distribution obtained by hand methods, including all the 4-point permutations and several others, agrees with the computed result.

as with the "1/8" in the first block of data. The sketch beside each block of data depicts (approximately to scale) the ranges (regions, arcs or points) in S of the N points for π_P. The distribution for π_Q is obtained by reflection about the line $y = x$ (see Obs. 4). For convenience, this line and the hyperbolic arcs corresponding to the non-zero lower bounds to $x_j y_j$ are faintly drawn in the sketch.

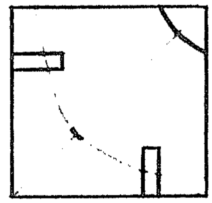
3 3241 9 4213 1 21 3**

0	1	0		1	3	4
1	1/3		1/8		3/8	2
2	2/3	0		1	0	1
3	4	2		3	2	3



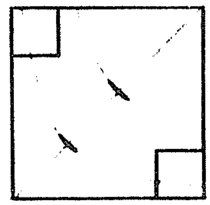
4 3214 1 21 ***

0	1	0		1	2/3	3
1/3	3/8		1/8		1/3	3/8
2/3	3	0		1	0	1
3	4	3		4	3	4



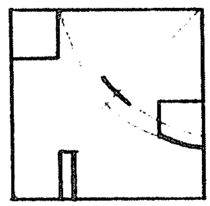
5 4231 1 2* 312

0	1	0		1	3	4
1	1/3		1/12		1	1/3
2	2/3		1/3		2	2/3
3	4	0		1	0	1



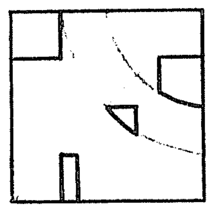
6 2431 17 4132 1 2* 3*2

0	1	0		1	3	4
1	1/3	0		1	0	1
2	2/3		1/3		2	2/3
3	4	1		2	1	2



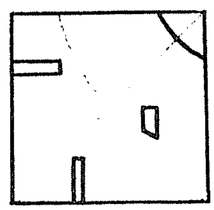
7 2341 21 4123 1 2* 3**

0	1	0	1	3	4
1	1/3	0	1	0	1
2	2/3	1	2	3/8	2
3	4	2	3	2	3



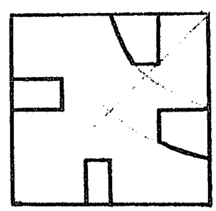
8 2314 16 3124 1 2* ***

0	1	0	1	2/3	3
1/3	3/8	0	1	0	1
2/3	3	1	2	1/3	2
3	4	3	4	3	4



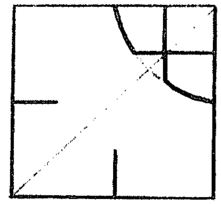
10 2413 15 3142 1 ** 2*3

0	1	0	1	2	2/3
3/8	2	0	1	0	1
2	3	2	3	3	4
3	4	1	2	1	2



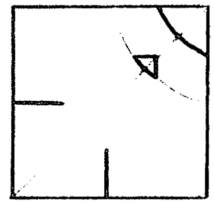
```
11 2143                          1 ** **3

     0        1        0        1          2
          2            0        1        0        1
     2        3        2        3        3        4
     3        4        2        3        2        3
```



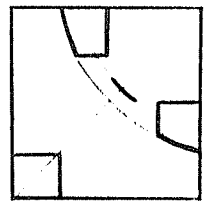
```
12 2134                          1 ** ***

     0        1        0        1          2
          2            0        1        0        1
    2/3       3        2        3       2/3       3
     3        4        3        4        3        4
```



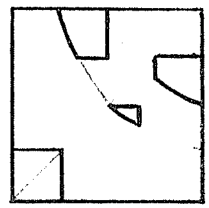
```
18 1432                          * *2 *32

     0        1        0        1        0        1
     1        2        1        2        3        4
     2       2/3           1/3          2       2/3
     3        4        1        2        1        2
```



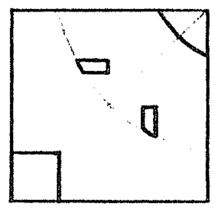
```
19 1342       22 1423            * *2 *3*

     0        1        0        1        0        1
     1        2        1        2        3        4
     2       2/3       1        2       3/8       2
     3        4        2        3        2        3
```



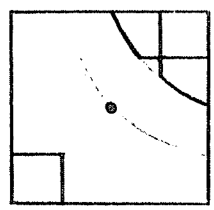
```
20 1324                          * *2 ***

     0        1        0        1        0        1
    1/3       2        1        2       2/3       3
    2/3       3        1        2       1/3       2
     3        4        3        4        3        4
```



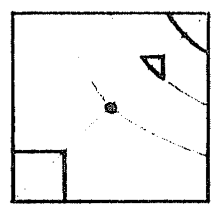
```
23 1243                          * ** **3

     0        1        0        1        0        1
          2            1        2           2
     2        3        2        3        3        4
     3        4        2        3        2        3
```



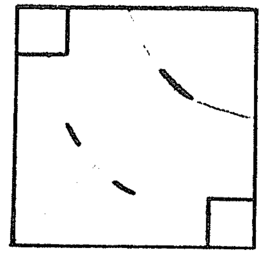
```
24 1234                          * ** ***

     0        1        0        1        0        1
          2            1        2           2
    2/3       3        2        3       2/3       3
     3        4        3        4        3        4
```



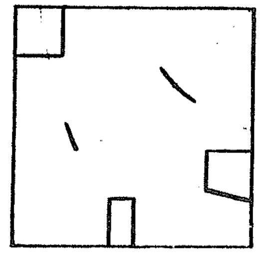
11 53241 1 21 3** 4213

0	1	0	1	4	5
1	1/4	1/10		2	1/2
2	1/2	1/10		1	1/4
3	3/4	9/20		3	3/4
4	5	0	1	0	1



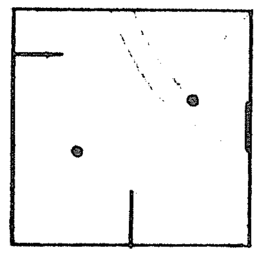
12 35241 71 53142 1 21 3** 42*3

0	1	0	1	4	5
1	1/4	1/10		2	1/2
2	1/2	0	1	0	1
3	3/4	9/20		3	3/4
4	5	1	2	1	2



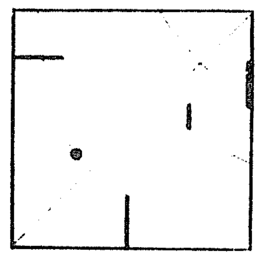
13 32541 51 52143 1 21 3** 4**3

0	1	0	1	4	
1/4		1/10		2	
1/2		0	1	0	1
3/4		9/20		3	
5		2	3	2	3



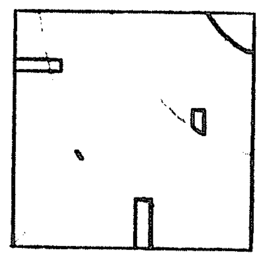
14 32451 56 52134 1 21 3** 4***

0	1	0	1	4	
1/4		1/10		2	
1/2		0	1	0	1
3/4		2	3	8/15	3
5		3	4	3	4



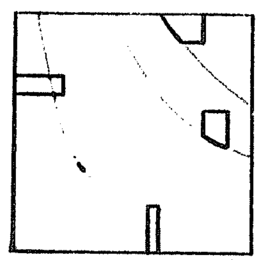
15 32415 45 42135 1 21 3** ****

0	1	0	1	3/4	4
1/4	4/15	1/10		3/8	2
1/2	8/15	0	1	0	1
3/4	4	2	3	1/2	3
4	5	4	5	4	5



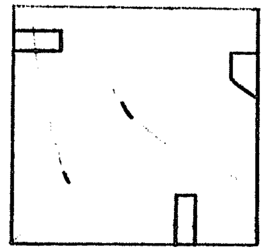
18 32514 44 42153 1 21 *** 3**4

0	1	0	1	2/3	3/4
4/15	3/10	1/10		1/3	3/8
8/15	3	0	1	0	1
3	4	3	4	8/9	5
4	9/10	2	3	4/9	3



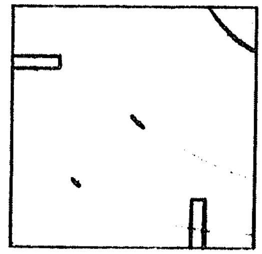
24 42351   36 52314   1 2* 312 4***

0	1	0	1	4	9/10
2/9	1/4	1/15		4/15	3/10
4/9	1/2	4/15		8/15	3
2/3	3/4	0	1	0	1
8/9	5	3	4	3	4



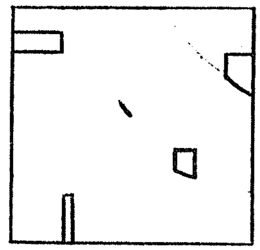
25 42315   1 2* 312 ****

0	1	0	1	3/4	4
1/4	4/15	1/15		1/4	4/15
1/2	8/15	4/15		1/2	8/15
3/4	4	0	1	0	1
4	5	4	5	4	5



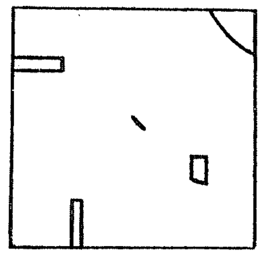
29 24351   96 51324   1 2* 3*2 4***

0	1	0	1	4	9/10
2/9	1/4	0	1	0	1
4/9	1/2	4/15		8/15	3
2/3	3/4	1	2	4/15	2
8/9	5	3	4	3	4



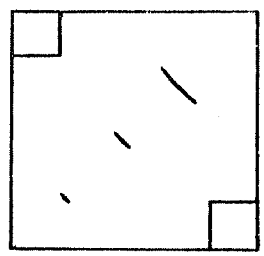
30 24315   85 41325   1 2* 3*2 ****

0	1	0	1	3/4	4
1/4	4/15	0	1	0	1
1/2	8/15	4/15		1/2	8/15
3/4	4	1	2	1/4	2
4	5	4	5	4	5



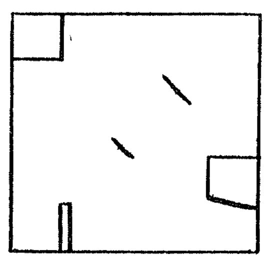
31 52341   1 2* 3** 4123

0	1	0	1	4	5
1	1/4	1/20		1	1/4
2	1/2	1		2	1/2
3	3/4	9/20		3	3/4
4	5	0	1	0	1



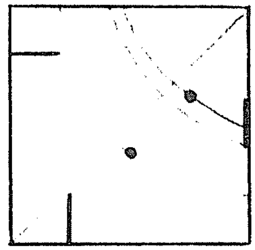
32 25341   91 51342   1 2* 3** 4*23

0	1	0	1	4	5
1	1/4	0	1	0	1
2	1/2	1		2	1/2
3	3/4	9/20		3	3/4
4	5	1	2	1	2



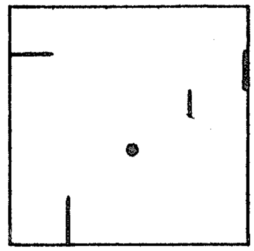
```
33 23541  111 51243   1 2* 3** 4**3

    0     1      0     1          4
       1/4       0     1      0     1
       1/2       1     2          2
       3/4          9/20          3
         5       2     3      2     3
```



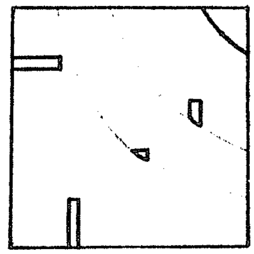
```
34 23451  116 51234   1 2* 3** 4***

    0     1      0     1          4
       1/4       0     1      0     1
       1/2       1     2          2
       3/4       2     3    8/15    3
         5       3     4      3     4
```



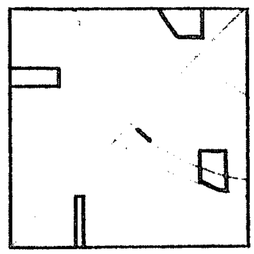
```
35 23415  105 41235   1 2* 3** ****

    0      1      0     1    3/4    4
  1/4    4/15     0     1     0     1
  1/2    8/15     1     2    3/8    2
  3/4      4      2     3    1/2    3
    4      5      4     5     4     5
```



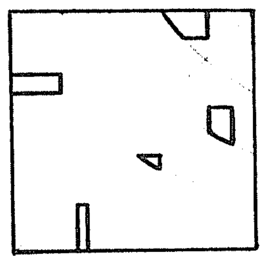
```
37 25314   84 41352   1 2* *** 3*24

    0      1      0     1    2/3   3/4
  4/15   3/10     0     1     0     1
  8/15     3        4/15     4/9   1/2
    3      4      3     4    8/9    5
    4    9/10     1     2    2/9    2
```



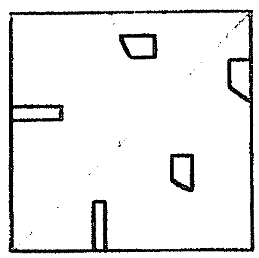
```
38 23514  104 41253   1 2* *** 3**4

    0      1      0     1    2/3   3/4
  4/15   3/10     0     1     0     1
  8/15     3      1     2    1/3    2
    3      4      3     4    8/9    5
    4    9/10     2     3    4/9    3
```



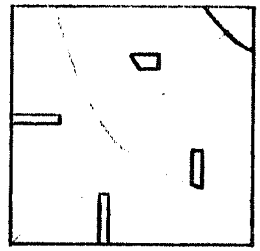
```
49 24153   78 31524   1 ** 2*3 **4*

    0      1      0     1    8/15   3
   1/3    3/8     0     1     0     1
   4/9     3      2     3     4   9/10
   2/3    3/4     1     2    4/15   2
   8/9     5      3     4     3     4
```



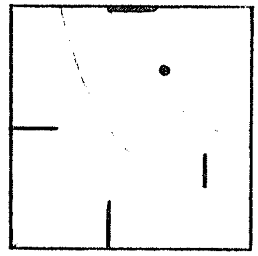
50 24135   75 31425   1 ** 2*3 ****

0		1	0		1	1/2		8/15
3/8		2	0		1	0		1
1/2		3	2		3	3/4		4
3/4		4	1		2	1/4		2
4		5	4		5	4		5



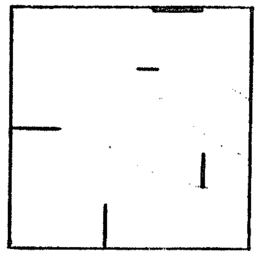
52 25143   73 31542   1 ** **3 2*43

0		1	0		1		1/2	
	2		0		1	0		1
2		3	2		3		5	
	3			9/20			3/4	
	4		1		2	1/4		2



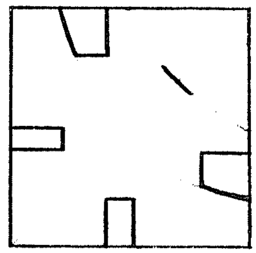
57 25134   74 31452   1 ** *** 2*34

0		1	0		1		1/2	
	2		0		1	0		1
8/15		3	2		3		3/4	
3		4	3		4		5	
	4		1		2	1/4		2



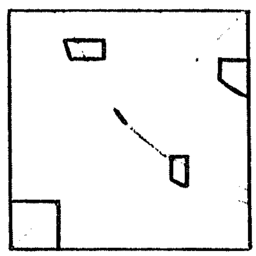
72 35142   * 12 *3* 24*3

0		1	0		1	2		1/2
1		2	1		2	4		5
2		1/2	0		1	0		1
3		3/4		9/20		3		3/4
4		5	1		2	1		2



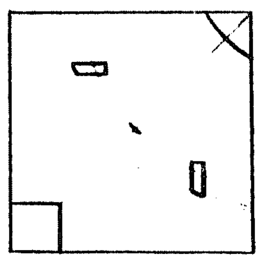
89 14352   97 15324   * *2 *32 *4**

0		1	0		1	0		1
2/9		2	1		2	4		9/10
4/9		1/2		4/15		8/15		3
2/3		3/4	1		2	4/15		2
8/9		5	3		4	3		4



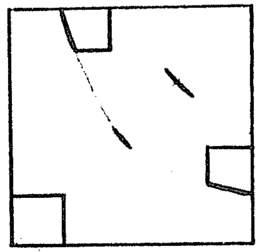
90 14325   * *2 *32 ****

0		1	0		1	0		1
1/4		2	1		2	3/4		4
1/2		8/15		4/15		1/2		8/15
3/4		4	1		2	1/4		2
4		5	4		5	4		5



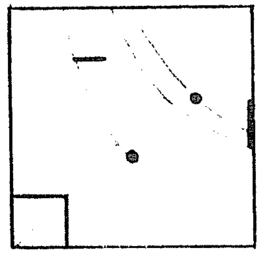
92 15342 * *2 *3* *423

0	1	0	1	0	1
1	2	1	2	4	5
2	1/2	1		2	1/2
3	3/4	9/20		3	3/4
4	5	1	2	1	2



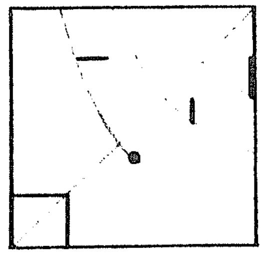
93 13542 112 15243 * *2 *3* *4*3

0	1	0	1	0	1
1/4	2	1	2	4	
1/2		1	2	2	
3/4		9/20		3	
5		2	3	2	3



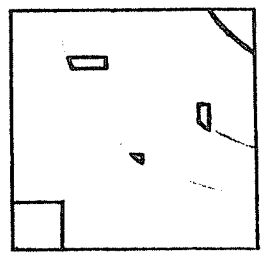
94 13452 117 15234 * *2 *3* *4**

0	1	0	1	0	1
1/4	2	1	2	4	
1/2		1	2	2	
3/4		2	3	8/15	3
5		3	4	3	4



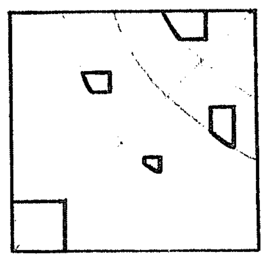
95 13425 110 14235 * *2 *3* ****

0	1	0	1	0	1
1/4	2	1	2	3/4	4
1/2	8/15	1	2	3/8	2
3/4	4	2	3	1/2	3
4	5	4	5	4	5



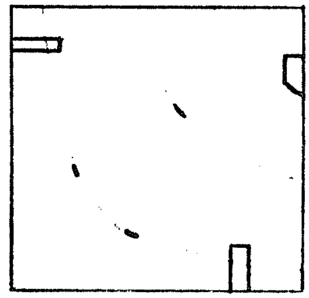
98 13524 109 14253 * *2 *** *3*4

0	1	0	1	0	1
4/15	2	1	2	2/3	3/4
8/15	3	1	2	1/3	2
3	4	3	4	8/9	5
4	9/10	2	3	4/9	3



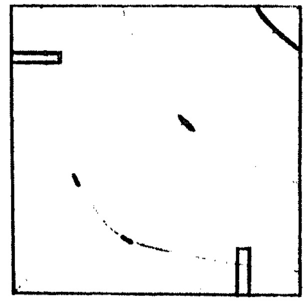
65 532461 85 632415 1 21 3** 4213 5****

0	1	0		1	5	8/9
3/16	1/5		1/12		5/12	4/9
3/8	2/5		1/12		5/24	2/9
9/16	3/5		3/8		5/8	4
3/4	4/5	0		1	0	1
15/16	6	4		5	4	5



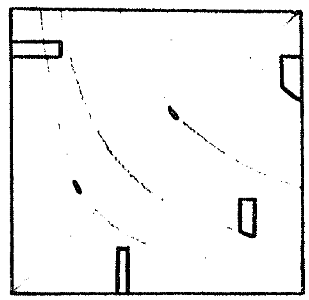
66 532416 1 21 3** 4213 *****

0	1	0		1	4/5	5
1/5	5/24		1/12		2/5	5/12
2/5	5/12		1/12		1/5	5/24
3/5	5/8		3/8		3/5	5/8
4/5	5	0		1	0	1
5	6	5		6	5	6



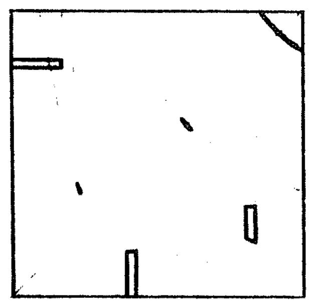
71 352461 445 631425 1 21 3** 42*3 5****

0	1	0		1	5	8/9
3/16	1/5		1/12		5/12	4/9
3/8	2/5	0		1	0	1
9/16	3/5		3/8		5/8	4
3/4	4/5	1		2	5/24	2
15/16	6	4		5	4	5



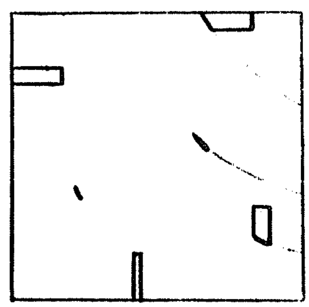
72 352416 426 531426 1 21 3** 42*3 *****

0	1	0		1	4/5	5
1/5	5/24		1/12		2/5	5/12
2/5	5/12	0		1	0	1
3/5	5/8		3/8		3/5	5/8
4/5	5	1		2	1/5	2
5	6	5		6	5	6



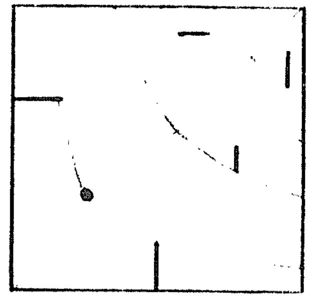
86 362415 425 531462 1 21 3** **** 42*35

0	1	0		1	3/4	4/5
5/24	2/9		1/12		3/8	2/5
5/12	4/9	0		1	0	1
5/8	4		3/8		9/16	3/5
4	5	4		5	15/16	6
5	8/9	1		2	3/16	2



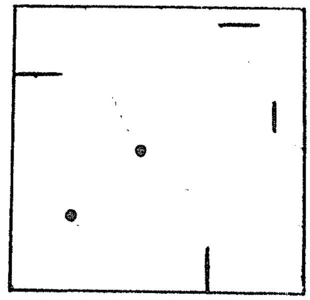
```
107 325164 268 421635   1 21 *** 3**4 ***5*

     0      1        0      1           4
       1/4             1/12             2
         3           0      1      0          1
   9/16     4        3      4          8/9
       3/4           2      3       4/9       3
     15/16           4      5     32/45       5
```



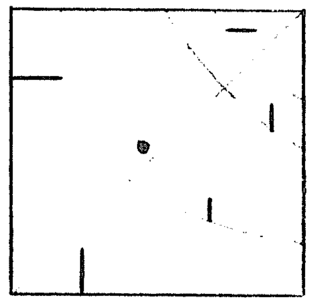
```
148 423615 215 523164   1 2* 312 **** 4***5

     0      1        0      1          3/4
       2/9             1/18            1/4
       4/9             2/9              3
         4           0      1      0          1
  32/45     5        4      5         15/16
         8/9         3      4      9/16       4
```



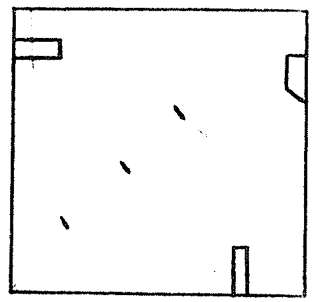
```
178 243615 575 513264   1 2* 3*2 **** 4***5

     0      1        0      1          3/4
       2/9           0      1      0          1
       4/9             2/9              3
         4           1      2      1/4        2
  32/45     5        4      5         15/16
         8/9         3      4      9/16       4
```



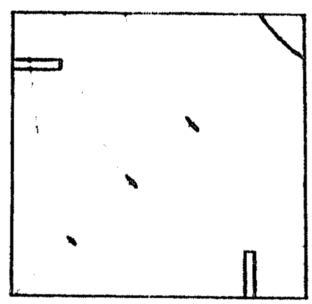
```
185 523461 205 623415   1 2* 3** 4123 5****

     0       1       0      1        5      8/9
    3/16    1/5        1/24        5/24     2/9
    3/8     2/5         1          5/12     4/9
    9/16    3/5        3/8         5/8       4
    3/4     4/5      0      1        0       1
   15/16     6       4      5        4       5
```



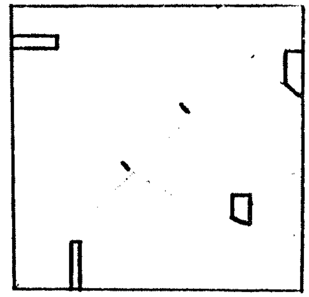
```
186 523416                1 2* 3** 4123 *****

     0       1       0      1       4/5      5
    1/5     5/24       1/24         1/5     5/24
    2/5     5/12        1           2/5     5/12
    3/5     5/8        3/8          3/5     5/8
    4/5      5       0      1        0       1
     5       6       5      6        5       6
```



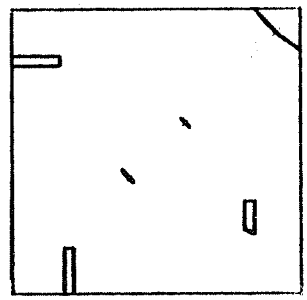
191 253461 565 613425 1 2* 3** 4*23 5****

0	1	0	1	5	8/9
3/16	1/5	0	1	0	1
3/8	2/5	1		5/12	4/9
9/16	3/5	3/8		5/8	4
3/4	4/5	1	2	5/24	2
15/16	6	4	5	4	5



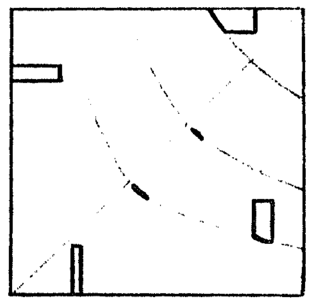
192 253416 546 513426 1 2* 3** 4*23 *****

0	1	0	1	4/5	5
1/5	5/24	0	1	0	1
2/5	5/12	1		2/5	5/12
3/5	5/8	3/8		3/5	5/8
4/5	5	1	2	1/5	2
5	6	5	6	5	6



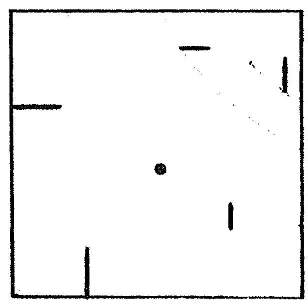
206 263415 545 513462 1 2* 3** **** 4*235

0	1	0	1	3/4	4/5
5/24	2/9	0	1	0	1
5/12	4/9	1		3/8	2/5
5/8	4	3/8		9/16	3/5
4	5	4	5	15/16	6
5	8/9	1	2	3/16	2



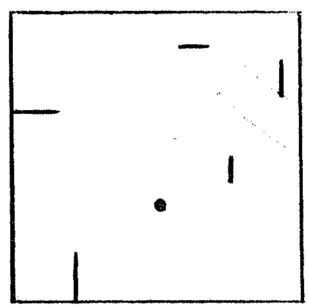
221 253164 508 413625 1 2* *** 3*24 ***5*

0	1	0	1	4	
1/4		0	1	0	1
3		2/9		4/9	
9/16	4	3	4	8/9	
3/4		1	2	2/9	2
15/16		4	5	32/45	5



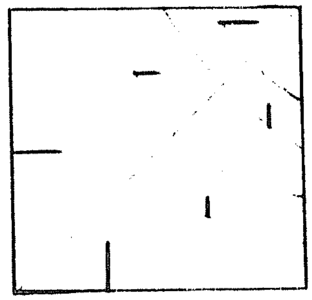
227 235164 628 412635 1 2* *** 3**4 ***5*

0	1	0	1	4	
1/4		0	1	0	1
3		1	2	2	
9/16	4	3	4	8/9	
3/4		2	3	4/9	3
15/16		4	5	32/45	5



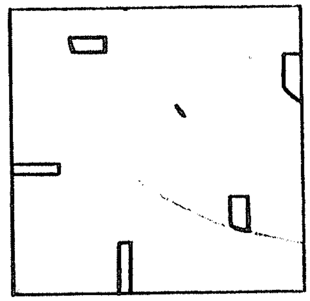
```
298 241635 467 315264   1 ** 2*3 **** **4*5

     0      1        0      1          3
        2            0      1      0       1
   4/9      3        2      3         3/4
        4            1      2     1/4      2
  32/45     5        4      5        15/16
        8/9          3      4     9/16     4
```



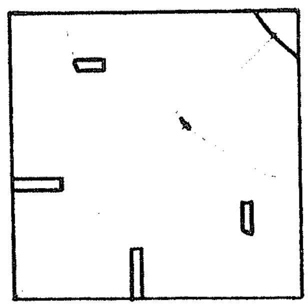
```
431 351462 446 361425   * 12 *3* 24*3 *5***

     0      1        0      1     5/12   4/9
   3/16     2        1      2       5    8/9
   3/8     2/5       0      1       0     1
   9/16    3/5          3/8       5/8     4
   3/4     4/5       1      2     5/24    2
  15/16     6        4      5       4     5
```



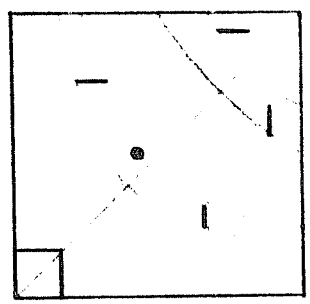
```
432 351426                  * 12 *3* 24*3 *****

     0      1        0      1      2/5    5/12
    1/5     2        1      2      4/5     5
    2/5    5/12      0      1       0      1
    3/5    5/8          3/8        3/5    5/8
    4/5     5        1      2      1/5     2
     5      6        5      6       5      6
```



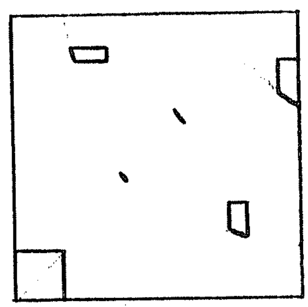
```
538 143625 581 153264   * *2 *32 **** *4**5

     0      1        0      1        0      1
    2/9     2        1      2           3/4
        4/9             2/9              3
         4           1      2       1/4     2
  32/45     5        4      5         15/16
        8/9          3      4       9/16    4
```



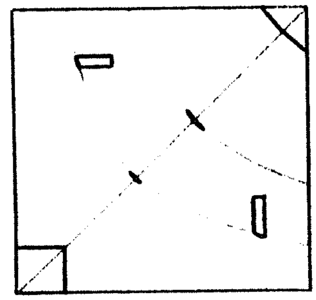
```
551 153462 566 163425   * *2 *3* *423 *5***

     0      1        0      1        0     1
   3/16     2        1      2        5    8/9
   3/8     2/5          1          5/12   4/9
   9/16    3/5         3/8         5/8     4
   3/4     4/5       1      2      5/24    2
  15/16     6        4      5        4     5
```



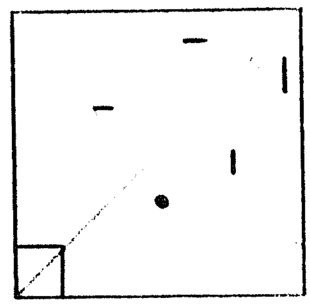
```
552 153426                   * *2 *3* *423 *****

     0        1         0        1         0        1
    1/5       2         1        2        4/5       5
    2/5     5/12            1             2/5     5/12
    3/5      5/8          3/8             3/5      5/8
    4/5       5         1        2        1/5       2
     5        6         5        6         5        6
```



```
587 135264 658 142635   * *2 *** *3*4 ***5*

     0        1         0        1         0        1
    1/4       2         1        2              4
          3             1        2              2
    9/16      4         3        4             8/9
        3/4             2        3        4/9       3
       15/16            4        5      32/45       5
```



APPENDIX

Calculation of optimal distributions for $N = 1, 2$ and 3.

From (2), (3), (5) and (6) one deduces that D_N is attained by solving the equalities

$$\text{(A1)} \qquad x_i - \frac{i-1}{N} = y_i - \frac{i-1}{N} = \frac{i}{N} - x_i y_i ,$$

where $1 \le i \le N$ is chosen so that these expressions are maximum. Clearly, $x_i = y_i$ for such an i; this yields:

$$\text{(A2)} \qquad x_i = \frac{\sqrt{1 + 4(2i-1)/N} - 1}{2}$$

$$= \frac{\sqrt{5}-1}{2}, \quad \text{if } i = N = 1;$$

$$= \left\{ \begin{array}{ll} \frac{\sqrt{3}-1}{2}, & \text{if } i = 1 \\ \frac{\sqrt{7}-1}{2}, & \text{if } i = 2 \end{array} \right\} \text{ and } N = 2;$$

$$= \left\{ \begin{array}{ll} \frac{\sqrt{7/3}-1}{2}, & \text{if } i = 1 \\ \frac{\sqrt{5}-1}{2}, & \text{if } i = 2 \\ \frac{\sqrt{23/3}-1}{2}, & \text{if } i = 3 \end{array} \right\} \text{ and } N = 3.$$

Thus, the optimal solution for $N = 1$ is $x_1 = y_1 = D_1 = (\sqrt{5} - 1)/2$. For $N = 2$, substituting the results of (A2) into (A1) reveals that $i = 1$ is the "critical index," i.e., $(\sqrt{3} - 1)/2 - 0 > (\sqrt{7} - 1)/2 - 1/2$, so

$x_1 = y_1 = D_2 = (\sqrt{3} - 1)/2$. This D_2 imposes limits on p_2: $x_2 - 1/2$, $y_2 - 1/2$, $1 - x_2 y_2 \leq (\sqrt{3} - 1)/2$, which imply that $x_2, y_2 \leq \sqrt{3}/2$ and $x_2 y_2 \geq (3 - \sqrt{3})/2$. Similarly, for $N = 3$ the critical index is $i = 2$, since $(\sqrt{7/3} - 1)/2 - 0$, $(\sqrt{23/3} - 1)/2 - 2/3 < (\sqrt{5} - 1)/2 - 1/3$. Hence, $x_2 = y_2 = D_1 = (\sqrt{5} - 1)/2$ and $D_3 = (3\sqrt{5} - 5)/6$. This D_3 imposes limits on p_1 and p_3: x_1, y_1, $1/3 - x_1 y_1$, $x_3 - 2/3$, $y_3 - 2/3$, $1 - x_3 y_3 \leq (3\sqrt{5} - 5)/6$ which imply that $x_1, y_1 \leq D_3$, $x_1 y_1 \geq (7 - 3\sqrt{5})/6$; $x_3, y_3 \leq (3\sqrt{5} - 1)/6$, $x_3 y_3 \geq (11 - 3\sqrt{5})/6$. See Table 1 and Fig. 2 for a summary of these results.

<u>Proof of Lemma 4</u>.

Clearly, $C = 0$ is feasible. Suppose $C > 0$, and that the first ℓ points of C in lexicographical order (see Def. 2) satisfy $y_{s_1} < \ldots < y_{s_\ell}$, where $s_1 < \ldots < s_\ell$, i.e., the i^{th} point of C in the first ℓ of the ordering is above and to the right of the $(i - 1)^{st}$ point of C, $1 < i \leq \ell$. There are two cases to consider for $C = 1$ and one case for $C > 1$.

Suppose $C > 1$ (and $B = D = 0$). Then Th. 2 implies that

$$\text{(A3a)} \qquad x_{s_\ell} y_{s_\ell} = \frac{(A + \ell + 1)^2}{(A + C + 2)N},$$

while by Th. 3(c) (which is not dependent on Lem. 4), (A3b) $$\frac{A + \ell}{N} \le x_{s_\ell} y_{s_\ell} \le \frac{A + \ell + 1}{N}.$$

The question is for what values of C does (A3a) fall in the range of (A3b)? Since $\ell \le C$,

$$x_{s_\ell} y_{s_\ell} = \frac{(A + \ell + 1)^2}{(A + C + 2)N} \le \frac{(A + \ell + 1)^2}{(A + \ell + 2)N} < \frac{(A + \ell + 2)}{(A + \ell + 2)} \frac{(A + \ell + 1)}{N} = \frac{A + \ell + 1}{N},$$

and $x_{s_\ell} y_{s_\ell}$ is strictly less than the larger end of the range.

However,

$$\frac{A + \ell}{N} \overset{?}{\le} x_{s_\ell} y_{s_\ell} = \frac{(A + \ell + 1)^2}{(A + C + 2)N},$$

with simplification yields $C \overset{?}{\le} \dfrac{\ell^2 + A\ell + 1}{A + \ell}$;

$\ell = 1$ implies that $x_{s_\ell} y_{s_\ell}$ is not less than the smaller end of the range iff $C = 2$ and $A = 0$, since $C > 1$ was assumed. By Lemma 3, these two points of C must satisfy $x_j < x_{s_1} < x_{s_2}$ and $y_j < y_{s_1} < y_{s_2}$ (see Fig. 8).

Now consider the case $C = 1$ with $B = D = 0$. By Th. 2

(A4a) $$x_{s_1} y_{s_1} = \frac{(A + 2)^2}{(A + 3)N};$$

Again, by Th. 3(c), the required range becomes

(A4b) $$\frac{A + 1}{N} \le x_{s_1} y_{s_1} \le \frac{A + 2}{N};$$

it is seen that (A4a) and (A4b) are compatible. If C = 1 and B + D = 1, then Th. 2 and (10) (evaluated with "A" = A + 2, "B" = "D" = "C" = 0) yield

(A5a) $$x_{s_1} y_{s_1} = \frac{(A + 3)^2}{(A + 4)N}$$

while Th. 3(c) implies that

(A5b) $$\frac{A + 2}{N} \leq x_{s_1} y_{s_1} \leq \frac{A + 3}{N} \;;$$

(A5a) and (A5b) are also compatible.

Finally, by Lem. 3 the only feasible point configuration for C > 2 where the ordering assumed at the beginning of this proof does not hold is that shown in Fig. 8 for C = 3.

Proof of Lemma 6.

Assuming that $D_N = 1/N$ is possible, consider the case of Cor. 2 with B = C = D = 0. Referring to Fig. A1,

(A6a) $$x_j y_j = \frac{(A + 1)^2}{(A + 2)N} \;,$$

by Cor. 2. There can be no point t of P_N with $x_t \leq x_j$ and $y_i < y_t$ or $x_k < x_t$ and $y_t \leq y_j$ by Lem. 5; these forbidden regions are indicated by X's in Fig. A1. Hence, from Th. 3(ab) and Fig. 9 with "B" = "D" = 1

(A6b) $$x_j, y_j \leq \frac{A + 2}{N} \;;$$

(A6ab) imply that

$$\frac{(A + 1)^2}{A + 2} \overset{?}{\leq} \frac{(A + 2)^2}{N}$$

must hold for $D_N = 1/N$, which occurs only for

(A6c) $$N \leq \frac{(A + 2)^3}{(A + 1)^2} .$$

If $A = 1$, (A6c) yields $N \leq 6$. Thus, only $A \neq 1$ is of interest for $B = C = D = 0$ and $N \geq 7$ in Lem. 6. By Obs. 2(a) and Th. 3(a)

(A6d) $$x_k y_j = \frac{A + 1}{N}; \quad x_k \leq \frac{k}{N} ;$$

(A6bd) imply that

(A7a) $$k \overset{?}{\geq} \frac{A + 1}{A + 2} N$$

must hold for $D_N = 1/N$; by symmetry,

(A7b) $$\ell \overset{?}{\geq} \frac{A + 1}{A + 2} N,$$

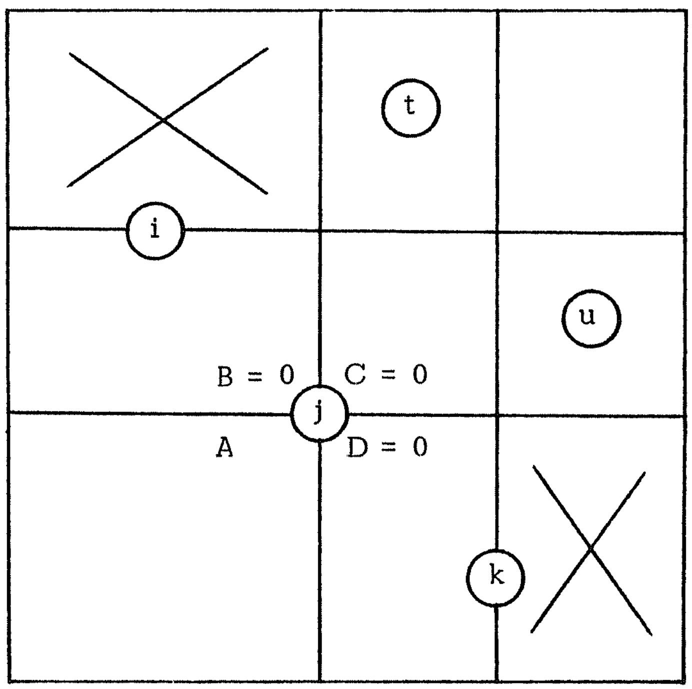
where $(\ell - 1)/N \leq y_i \leq \ell/N$ by Th. 3(b), and ℓ is the number of points in P_N with y coordinates $\leq y_i$. Now consider under what conditions it is necessary for the existence of two points t and u of P_N satisfying $x_j < x_t \leq x_k$ and $y_i < y_t$, and $x_k < x_u$ and $y_j < y_u \leq y_i$ (see Fig. A1). Both such points must exist if

(A7c) $$\frac{A + 1}{A + 2} N \overset{?}{>} A + 3$$

holds, for otherwise, (A7ab) could not hold, and $k = \ell = A + 3$ is determined by A and the three points i,j,k alone (see Def. 2). But the existence of the points t and u implies a configuration of four points i,t,k,u forbidden by Lem. 5. Since (A7c) holds if N is larger than $(A + 3)(A + 2)/(A + 1)$, Lem. 6 follows for $B = C = D = 0$.

It remains to examine the other cases of Cor. 2. Suppose $C = 1$ and $A = B = D = 0$. This is equivalent to the case just considered with

Fig. A1. Only Possible Point Configuration with the Hypotheses of Lem. 6.



A = 1; as seen above, this situation is not of interest for $N \geq 7$ in Lem. 6 and can be eliminated. The case with C = A = 1 and B = D = 0 is also equivalent to the first case with A = 2. This situation can be eliminated because (A7c) holds for $N \geq 7$ and A = 2. The cases of Cor. 2 with A = 0, 1, 2 and C = B + D = 1 (see Th. 2 for the position of point r) can be eliminated by Lem. 5. Finally, the cases C = 2,3 with A = B = D = 0 are also eliminated by Lem. 5 (see Fig. 8 for the relative positions of the points).

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