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A PROOF OF THE INSTABILITY OF BACKWARD-DIFFERENCE MULTISTEP METHODS FOR THE NUMERICAL INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS

by

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1. Introduction

In this paper we consider the "backward-difference multistep method"

$$\sum_{m=1}^{q} \frac{1}{m} \nabla^m y_p = h f_p, \qquad (1.1)$$

for the numerical integration of the scalar ordinary differential equation $y'(x) = f(x, y(x))$. Here, q is a positive integer, h is the stepsize, and ∇ is the backward difference operator. It is shown by Henrici [12, p. 207] that if $y(x)$ is being approximated at the points $x_i = x_0 + ih$, then (1.1) is the method which results from interpolating $y(x)$ at the points $x_p, x_{p-1}, \ldots, x_{p-q}$ and then evaluating the derivative of the interpolating polynomial at the point x_p.

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With (1.1) we associate the polynomial

$$\rho(\zeta;q) = \sum_{m=1}^{q} \frac{1}{m} \zeta^{q-m} (\zeta - 1)^m, \tag{1.2}$$

and say that (1.1) satisfies the root condition if (i) the zeros of ρ lie inside or on the unit circle in the complex ζ-plane and (ii) ρ has only simple zeros on the unit circle. It is well-known (Henrici [12, p. 287]) that (1.1) is stable in the sense of Dahlquist if and only if (1.1) satisfies the root condition.

It is the purpose of the present paper to establish

Theorem 1.1

The backward-difference multistep method (1.1) satisfies the root condition iff $1 \le q \le 6$.

Theorem 1.1 has been believed for a long time. Mitchell and Craggs [15] computed the roots of $\rho(\zeta;q)$ numerically using Graeffe's root-squaring method: for $1 \le q \le 6$ they found that ρ had one root at $\zeta = 1$ and $q - 1$ roots inside the unit circle; for $q = 7$ they found that ρ had a root at $\zeta = 1$ and, approximately, at $\zeta = \pm i$; while for $q \ge 8$ they asserted that ρ had at least two roots lying outside the unit circle. Henrici [12, p. 220] conjectured that Theorem 1.1 held. Elsewhere, Henrici [12, p. 207] was less explicit, and this seems to have given rise to the belief that Theorem 1.1 had been proved (see, for example, Gear [11, p. 192]).

The method (1.1) has been little used because of its undesirable stability properties. Recently, however, (1.1) has assumed new importance since Gear [11] has shown that (1.1) is "stiffly stable" for $1 \le q \le 6$. See also Curtiss and Hirschfelder [8], Dill [9], Moretti [16], and van der Houwen [19].

2. Preliminaries

Let

$$p(z;q) = (1-z)^q \, \rho\left(\frac{1+z}{1-z}\right)/2z, \tag{2.1}$$

so that

$$p(z;q) = \sum_{m=1}^{q} \frac{1}{m} (1+z)^{q-m} (2z)^{m-1}, \tag{2.2}$$

$$= \sum_{i=0}^{q-1} a_i(q) z^{q-1-i}, \text{ say.} \tag{2.3}$$

The mapping $\zeta = (1+z)/(1-z)$ maps the unit disk in the ζ-plane into the left half plane in the z-plane. Noting that $p(\zeta;q)$ has a simple zero at $\zeta = 1$, we obtain

Lemma 2.1

The method (1.1) satisfies the root condition iff (i) the roots of $p(z;q)$ have non-positive real part and (ii) the imaginary roots of $p(z;q)$ are simple.

To analyse the roots of $p(z;q)$ we apply the Routh-Hurwitz theorem (Obreschkoff [18, p. 108], Marden [14, p. 180], to obtain

Lemma 2.2

Let $a_0(q) > 0$. Then the zeros of $p(z;q)$ have strictly negative real parts iff $D_k(q) > 0$ for $1 \le k \le q-1$. Here, $D_k(q)$ is the determinant of order k,

$$D_k(q) = \begin{vmatrix} a_1(q) & a_0(q) & 0 & 0 & 0 & 0 & \cdots \\ a_3(q) & a_2(q) & a_1(q) & a_0(q) & 0 & 0 & \cdots \\ a_5(q) & a_4(q) & a_3(q) & a_2(q) & a_1(q) & a_0(q) & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ a_{2k-1}(q) & a_{2k-2}(q) & \cdot & & \cdot & \cdot & a_k(q) \end{vmatrix}, \quad (2.4)$$

where $a_i(q) = 0$ if $i \ge q$.

The following lemma is an easy consequence of Lemmas 2.1 and 2.2:

Lemma 2.3

If $D_\ell(q) < 0$ for some ℓ, $1 \le \ell \le q-1$, then (1.1) does not satisfy the root condition.

Proof: For real ϵ set

$$p_\epsilon(z;q) = p(z+\epsilon;q) = \sum_{i=0}^{q-1} \tilde{a}_i(\epsilon;q) z^{q-1-i}, \quad \text{say.}$$

Let $D_\ell(\epsilon;q)$ be the determinant of the form (2.4) corresponding to $p_\epsilon(z;q)$.

Clearly, $a_0(0;q) = a_0(q) > 0$, while, by assumption, $D_\ell(0;q) = D_\ell(q) < 0$. Hence, by continuity, there exists $\delta > 0$ such that $a_0(\epsilon;q) > 0$ and $D_\ell(\epsilon;q) < 0$ if $|\epsilon| \leq \delta$. Applying Lemma 2.2 to $p_\epsilon(z;q)$, it follows that at least one zero of $p_\epsilon(z;q)$ has non-negative real part if $|\epsilon| \leq \delta$. Since the zeros of $p(z;q)$ are the translates by ϵ of the zeros of $p_\epsilon(z;q)$, $p(z;q)$ must have at least one root with strictly positive real part. From Lemma 2.1 we conclude that (1.1) does not satisfy the root condition.

Lemma 2.3 provides a possible approach to proving Theorem 1.1 but it is by no means obvious that this approach is feasible. (In this connection see the remarks at the end of this section.) An early step in the present investigation was, therefore, to compute the signs of the determinants $D_k(q)$ for a range of values of q.

The computations were performed using SAC-1 (system for Symbolic and Algebraic Calculations, version 1), on the UNIVAC 1108 computer at the University of Wisconsin. SAC-1 is a list processing system which, among other features, can manipulate polynomials with rational coefficients.

In SAC-1, polynomials with rational coefficients are represented as lists of appropriate size; for example, on the UNIVAC 1108, a polynomial of degree 100 with coefficients each of which is the ratio of two onehundred-decimal-digit integers would be represented as a list occupying approximately 4000 storage locations. SAC-1 is based on FORTRAN and is essentially machine independent; it has been implemented on several computers other than the UNIVAC 1108. A brief description of SAC-1, and further references, are given by Collins [3].

In SAC-1, polynomial manipulations are performed exactly unless the limits of storage or time are exceeded, in which case an error message is generated. However, there remains the possibility of computer errors and programming errors. Consequently, all SAC-1 output which was used in the proof of Theorem 1.1, was checked by hand.

Since it is computationally inefficient to use the representation (2.4), the signs of the determinants $D_k(q)$ were found by computing the associated Sturm sequences (Marden [14, p. 171 and p. 174]); for details see the Appendix. It was found that (i) $D_k(q) > 0$ for $1 \le k \le q - 1$ and $1 \le q \le 6$; (ii) $D_4(7) < 0$; (iii) $D_3(q) < 0$ for $8 \le q \le 11$; and (iv) $D_2(q) < 0$ for $12 \le q \le 35$. These results suggested that Theorem 1.1 could be proved by (a) showing analytically that $D_2(q) < 0$ for $q \ge q_0$ where q_0 is some not-too-large constant and (b) using brute force to treat the cases $q < q_0$. Such a proof is given in the next section with $q_0 = 36$.

The SAC-1 computations yielded not only the signs of the $D_k(q)$ but also the location of the zeros of the polynomials $p(z;q)$ with respect to the imaginary axis. These results are given in Table 2.1.

	Number of zeros whose real part is strictly			Number of zeros whose real part is strictly	
q	negative	positive	q	negative	positive
2	1	0	19	14	4
3	2	0	20	13	6
4	3	0	21	14	6
5	4	0	22	15	6
6	5	0	23	16	6
7	4	2	24	17	6
8	5	2	25	18	6
9	6	2	26	17	8
10	7	2	27	18	8
11	8	2	28	19	8
12	9	2	29	20	8
13	10	2	30	21	8
14	9	4	31	22	8
15	10	4	32	21	10
16	11	4	32	22	10
17	12	4	34	23	10
18	13	4	35	24	10

<u>Table 2.1</u>

Location of zeros of $p(z;q)$.

The results of Table 2.1 suggest that $p(z;q)$ always has an even number of zeros in the right half plane; this is proved in the following Lemma.

Lemma 2.4

(a) The real zeros of $\rho(\zeta;q)$ lie in the interval $[0,1]$.

(b) The zeros of $\rho(\zeta;q)$ outside the unit circle occur in conjugate pairs.

(c) The zeros of $p(z;q)$ in the right half plane occur in conjugate pairs.

Proof: Let ζ be real. From (1.2) we see that if $\zeta > 1$ then $\rho(\zeta;q) > 0$ while if $\zeta < 0$ then $(-1)^q \rho(\zeta;q) > 0$. Part (a) of the lemma follows. Parts (b) and (c) are immediate consequences of Part (a).

To conclude this section we give an example where the use of SAC-1 indicated that a possible method of proving Theorem 1.1 was infeasible. The mapping $z = 1/w$ maps the left half plane the z-plane onto the left half plane of the w-plane. Consequently, Lemmas 2.1 through 2.3 remain valid if $p(z;q)$ is replaced by

$$\tilde{p}(z;q) = z^{q-1} p(\tfrac{1}{z};q) = \sum_{m=1}^{q} \frac{1}{m} (1 + z)^{q-m} \, 2^{m-1}, \tag{2.5}$$

$$= \sum_{i=0}^{q-1} \tilde{a}_i(q) z^{q-1-i}, \text{ say,} \tag{2.6}$$

and $D_k(q)$ is replaced by $\tilde{D}_k(q)$. It can be seen from (2.2), (2.3), (2.5) and (2.6), that, for small i, $a_i(q)$ is much more complicated than $\tilde{a}_i(q)$. Therefore, if it were true that $\tilde{D}_2(q) < 0$ for $q \geq q_0$, we would have a much simpler method of proving Theorem 1.1. Unfortunately, this is far from being the case. For example, computations showed that $\tilde{D}_k(22) > 0$ for $1 \leq k \leq 13$.

3. Proof of Theorem 1.1.

Theorem 1.1 follows from Lemmas 2.1, 2.2, 2.3, 3.12 and 3.13.

We will use the "Eulerian integral of the first kind" namely (Whittaker and Watson [21, p. 253]),

$$B(u,v) = B(v,u) = \frac{\Gamma(u)\ \Gamma(v)}{\Gamma(u+v)} = \int_0^1 (1-x)^{u-1} x^{v-1} dx, \tag{3.1}$$

or, equivalently (Whittaker and Watson [21, p. 255]),

$$B(u,v) = B(v,u) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} = 2^{1-u-v} \int_{-1}^{+1} (1-x)^{u-1}(1+x)^{v-1}\, dx. \tag{3.2}$$

Here, $B(u,v)$ denotes the Beta-function while $\Gamma(u)$ denotes the Gamma-function. We will also use the functions

$$E(u,v) = \int_0^1 \frac{(1-x)^u(1+x)^v - (1+x)^u(1-x)^v}{2x}\, dx, \tag{3.3}$$

$$F(u,v) = \int_0^1 \frac{(1-x)^u[(1+x)^v - (1-x)^v]}{2x}\,dx, \tag{3.4}$$

and

$$G(u,v,\alpha) = \int_0^\alpha \frac{(1-x)^u[(1+x)^v - (1-x)^v]}{2x}\,dx. \tag{3.5}$$

In (3.1) through (3.4) the parameters u and v will always be such that the integrands are continuous. Finally, we will need the asymptotic expansion for the Gamma-function, namely (Whittaker and Watson [21, p. 253]),

$$1 \le \frac{\Gamma(x)}{(x)^{x-\frac{1}{2}}\, e^{-x}\, (2\pi)^{\frac{1}{2}}} \le e^{\frac{1}{12x}}, \quad x > 0, \tag{3.6}$$

or, equivalently,

$$0 \le [\log \Gamma(x)] - [(x - \tfrac{1}{2}) \log x - x + \tfrac{1}{2}\log 2\pi] \le \frac{1}{12x}, \quad x > 0. \tag{3.7}$$

Lemma 3.1.

$$a_i(q) = \sum_{m=1}^{q-i} \frac{2^{m-1}}{m} \binom{q-m}{i}, \qquad 0 \le i \le q-1,$$

where $\binom{q-m}{i}$ is the binomial coefficient $(q - m)! / (q - m - i)!\, i!$.

Proof: Follows immediately from (2.2) and (2.3).

Lemma 3.2

$$a_i(q) = \binom{q}{i} F(i, q-i), \quad 0 \le i \le q - 1.$$

Proof: Using the binomial expansion,

$$(1 - x)^i \{(1 + x)^{q-i} - (1 - x)^{q-i}\}$$

$$= (1 - x)^i \{(2x + [1 - x])^{q-i} - (1 - x)^{q-i}\},$$

$$= (1 - x)^i \sum_{m=1}^{q-i} \binom{q-i}{m} (2x)^m (1 - x)^{q - i - m},$$

$$= \sum_{m=1}^{q-i} \binom{q-i}{m} (2x)^m (1 - x)^{q-m}.$$

Hence, noting (3.1) and (3.4),

$$\binom{q}{i} F(i,q-i)$$

$$= \binom{q}{i} \int_0^1 \frac{(1-x)^i [(1+x)^{q-i} - (1 - x)^{q-i}]}{2x} dx,$$

$$= \binom{q}{i} \sum_{m=1}^{q-i} \binom{q-i}{m} \int_0^1 (2x)^{m-1} (1 - x)^{q-m} dx,$$

$$= \sum_{m=1}^{q-i} \binom{q}{i} \binom{q-i}{m} B(m, q-m+1)\, 2^{m-1},$$

$$= \sum_{m=1}^{q-i} \frac{q!}{i!\,(q-i)!} \cdot \frac{(q-i)!}{m!\,(q-i-m)!} \cdot \frac{(m-1)!\,(q-m)!}{q!} \cdot 2^{m-1},$$

$$= \sum_{m=1}^{q-i} \frac{2^{m-1}}{m} \binom{q-m}{i}.$$

Using Lemma 3.1, the lemma follows.

Lemma 3.3

$$E(u,v) = \int_{-1}^{+1} \frac{(1-x)^u (1+x)^v - 1}{2x}\, dx.$$

Proof: From (3.3),

$$E(u,v) = \int_0^1 \frac{(1-x)^u (1+x)^v - (1+x)^u (1-x)^v}{2x}\, dx,$$

$$= \int_0^1 \frac{(1-x)^u (1+x)^v - 1}{2x}\, dx + \int_0^1 \frac{1 - (1+x)^u (1-x)^v}{2x}\, dx.$$

Substituting $-y$ for x in the second integral,

$$E(u,v) = \int_0^1 \frac{(1-x)^u (1+x)^v - 1}{2x}\, dx + \int_{-1}^0 \frac{(1-y)^u (1+y)^v - 1}{2y}\, dy.$$

The lemma follows.

Lemma 3.4

(a) $E(u,v) = F(u,v) - F(v,u)$;

(b) $F(u,v) \geq 0$;

(c) $F(u + w, v - w) \leq F(u,v)$, if $0 \leq w \leq v$;

(d) $E(u,v) \geq 0$ if $v \geq u$.

Proof: Follows immediately from (3.3) and (3.4)

Lemma 3.5

Let w be an integer such that $1 \le w \le v$. Then

$$E(u,v) = E(u+w,\ v-w) + \sum_{k=1}^{w} 2^{u+v} B(u+k,\ v+1-k).$$

Proof: It follows from Lemma 3.3 that

$$E(u,v) - E(u+w,\ v-w)$$

$$= \int_{-1}^{+1} \frac{(1-x)^u (1+x)^v - (1-x)^{u+w} (1+x)^{v-w}}{2x}\, dx.$$

But,

$$(1-x)^u (1+x)^v - (1-x)^{u+w} (1+x)^{v-w}$$

$$= (1-x)^u (1+x)^{v-w} [(1+x)^w - (1-x)^w],$$

$$= (1-x)^u (1+x)^{v-w} (2x) \sum_{k=1}^{w} (1-x)^{k-1} (1+x)^{w-k},$$

$$= 2x \sum_{k=1}^{w} (1-x)^{u+k-1} (1+x)^{v-k}.$$

Hence, applying (3.2),

$$E(u,v) - E(u+w,\ v-w)$$

$$= \sum_{k=1}^{w} \int_{-1}^{+1} (1-x)^{u+k-1} (1+x)^{v-k}\, dx,$$

$$= \sum_{k=1}^{w} 2^{u+v} B(u+k,\ v+1-k),$$

which proves the lemma.

Lemma 3.6

Let $v+1 = ru$. Let $r \geq 1$ and $u \geq 2$. Then

$$\log B(u,v+1) \geq -u \log[(r+1)e] - \frac{1}{2} \log u + \frac{1}{2} \log \frac{r+1}{r} + \frac{1}{2} \log 6.$$

Proof: Using (3.1) and (3.7),

$$\begin{aligned}
&\log B(u,v+1) \\
&= \log \Gamma(u) + \log \Gamma(v+1) - \log \Gamma(u+v+1), \\
&\geq [(u - \tfrac{1}{2}) \log u - u + \tfrac{1}{2} \log 2\pi] + \\
&\quad + [(v + \tfrac{1}{2}) \log (v + 1) - (v + 1) + \tfrac{1}{2} \log 2\pi] - \\
&\quad - [(u + v + \tfrac{1}{2}) \log (u + v + 1) - (u + v + 1) + \tfrac{1}{2} \log 2\pi + \frac{1}{12(u + v + 1)}, \\
&= \{(u - \tfrac{1}{2}) \log u\} + \{(ru - \tfrac{1}{2}) \log (ru)\} - \\
&\quad - \{[(r+1)u - \tfrac{1}{2}] \log [(r+1)u]\} + \tfrac{1}{2} \log 2\pi - \frac{1}{[12(r+1)u]}, \\
&= \{u \log u - \tfrac{1}{2} \log u\} + \\
&\quad + \{ru \log u + ru \log r - \tfrac{1}{2} \log u - \tfrac{1}{2} \log r\} - \\
&\quad - \{(r+1)u \log u + (r + 1) u \log (r + 1) - \tfrac{1}{2} \log u - \tfrac{1}{2} \log (r + 1)\} + \\
&\quad + \tfrac{1}{2} \log 2\pi - \frac{1}{[12(r+1)u]}, \\
&= u\, [r \log r - (r + 1) \log (r + 1)] - \tfrac{1}{2} \log u + \\
&\quad + \tfrac{1}{2} \log (\tfrac{r+1}{r}) + \tfrac{1}{2} \log 2\pi - \frac{1}{[12(r+1)u]},
\end{aligned}$$

$$= - u[\log(r+1) + r \log (\frac{r+1}{r})] - \frac{1}{2} \log u +$$

$$+ \frac{1}{2} \log (\frac{r+1}{r}) + \frac{1}{2} \log 6 + \frac{1}{2} \log (\frac{\pi}{3}) - \frac{1}{[12(r+1)u]} .$$

Since

$$1 \le (1 + \frac{1}{r})^r \le e, \text{ if } r \ge 1,$$

it follows that

$$r \log (\frac{1+r}{r}) \le \log e, \text{ if } r \ge 1.$$

Using standard tables, for example Abramowitz and Stegun [1, p. 3], we find that

$$\frac{1}{2} \log (\frac{\pi}{3}) > \frac{1}{2} [1.1447 - 1.0987] = .023.$$

By assumption, $r \ge 1$ and $u \ge 2$ so that

$$\frac{1}{12(r+1)u} \le \frac{1}{48} < .023 < \frac{1}{2} \log (\frac{\pi}{3}).$$

The lemma follows.

Lemma 3.7

Let $v + 1 = ru$ with $r \ge 5$ and $u = 5$ or $u = 6$. Then

$$G(u,v,\frac{1}{4}) \le \frac{1}{2} 2^{u+v} B(u,v+1).$$

Proof: Using the binomial expansion we have

$$\frac{(1+x)^v - (1-x)^v}{2x} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{v} \binom{v}{k} x^{k-1} ,$$

from which we see that the function $[(1+x)^v - (1-x)^v]/2x$ is a monotone increasing function of x for $x \geq 0$.

Hence,

$G(u, v, \alpha)$

$$= \int_0^\alpha \frac{(1-x)^u[(1+x)^v - (1-x)^v]}{2x}\, dx,$$

$$\leq \frac{(1+\alpha)^v - (1-\alpha)^v}{2\alpha} \int_0^\alpha (1-x)^u \, dx,$$

$$\leq \frac{(1+\alpha)^v}{2\alpha(u+1)} \, ,$$

$$= \frac{(1+\alpha)^{v+1}}{2} \cdot \frac{1}{\alpha(u+1)(1+\alpha)} \, .$$

Taking logs and using Lemma 3.6,

$\log[G(u, v, \alpha)/(2^{u+v} B(u, v+1))]$

$$\leq (v+1) \log (1+\alpha) - \log[\alpha(u+1)(1+\alpha)] -$$

$$- (u+v+1) \log 2 - \log B(u, v+1),$$

$$\leq r\, u \log(1+\alpha) - \log[\alpha(u+1)(1+\alpha)]$$

$$- (r+1)u \log 2 -$$

$$- \{-u \log[(r+1)e] - \tfrac{1}{2} \log u + \tfrac{1}{2} \log (\tfrac{r+1}{r}) + \tfrac{1}{2} \log 6\},$$

$$\leq u\{\log[(r+1)e] - r \log[2/(1+\alpha)] - \log 2\} -$$

$$- \log[\alpha(u+1)(1+\alpha)] - \tfrac{1}{2} \log (6/u).$$

Since $(\frac{8}{5})^r \geq \frac{10}{6}(r+1)$, if $r \geq 5$,

we have

$$r \log (8/5) \geq \log [10(r+1)/6], \text{ if } r \geq 5.$$

Hence, setting $\alpha = \frac{1}{4}$, and remembering that, by assumption, $r \geq 5$ and $u = 5$ or $u = 6$,

$$\log[G(u,v,\tfrac{1}{4})/(2^{u+v} B(u,v+1)]$$
$$\leq u\{\log[(r+1)e] - r \log (8/5) - \log 2\} -$$
$$- \log [5(u+1)/16],$$
$$\leq u\{\log[3(r+1)] - \log [10(r+1)/6] - \log 2\}$$
$$- \log (30/16),$$
$$= u \log (18/20) - \log (15/8),$$
$$\leq 5 \log (9/10) - \log (15/8),$$
$$= -[5 \log (10/9) + \log (15/8)],$$
$$< - \log 2.$$

The lemma follows.

Lemma 3.8

If $v+1 = ru$ where $r \geq 5$ and $u = 5$ or $u = 6$ then

$$F(u,v) \leq 2^{u+v+1} B(u,v+1).$$

<u>Proof</u>: It follows immediately from (3.4) and (3.5) that

$$F(u,v) = G(u,v,\alpha) + I,$$

where

$$I = \int_{\alpha}^{1} \frac{(1-x)^{u}[(1+x)^{v} - (1-x)^{v}]}{2x} \, dx,$$

and α is a constant in (0,1) which will be chosen later.

Since $(1-x)/2x$ is a monotone decreasing function of x on $(0,1]$, it follows, using (3.2), that

$$I \le \int_{\alpha}^{1} \frac{(1-x)^{u}(1+x)^{v}}{2x} \, dx,$$

$$= \int_{\alpha}^{1} (1-x)^{u-1}(1+x)^{v} \cdot \frac{1-x}{2x} \, dx,$$

$$\le \frac{1-\alpha}{2\alpha} \int_{\alpha}^{+1} (1-x)^{u-1}(1+x)^{v} \, dx,$$

$$\le \frac{1-\alpha}{2\alpha} \int_{-1}^{+1} (1-x)^{u-1}(1+x)^{v} \, dx,$$

$$= \frac{1-\alpha}{2\alpha} \, 2^{u+v} \, B(u,v+1).$$

Hence,

$$F(u,v) \le \frac{1-\alpha}{2\alpha} \, 2^{u+v} \, B(u,v+1) + G(u,v,\alpha).$$

Setting $\alpha = \frac{1}{4}$ and using Lemma 3.7, the lemma follows.

Lemma 3.9

For $0 \le i \le q-4$ let

$$A_i(q;\beta) = \binom{q}{i} \{\beta\ B(i+4,\ q-i-3) + \sum_{k=1}^{4} B(i+k,\ q+1-i-k)\},$$

$$H(q;\beta) = \frac{(q-1)!\,(q-2)!}{(q-5)!\,(q-7)!} [A_0(q;0)\,A_3(q;0) - A_1(q;\beta)\,A_2(q;\beta)].$$

Then,

$$H(q;\beta) = T_0 - \beta T_1 - \beta^2 T_2,$$

where

$$T_0 = 8q[(q-6)(q^3 - 4q^2 + 31q - 20) + 280],$$

$$T_1 = 12q(q-6)(7q^2 - 53q + 154),$$

$$T_2 = 24.60.\ (q-6).$$

Proof:

$$A_i(q;\beta)$$

$$= \binom{q}{i} \{\beta \frac{(i+3)!\,(q-i-4)!}{q!} + \sum_{k=1}^{4} \frac{(i+k-1)!\,(q-i-k)!}{q!}\},$$

$$= \frac{q!}{i!(q-i)!} \quad \frac{i!(q-i-4)!}{q!} \quad \{\beta \frac{(i+3)!}{i!} +$$

$$+ \sum_{k=1}^{4} \frac{(i+k-1)!}{i!} \quad \frac{(q-i-k)!}{(q-i-4)!}\},$$

$$= \frac{(q-i-4)!}{(q-i)!} \{\beta \frac{(i+3)!}{i!} + \sum_{k=1}^{4} \frac{(i+k-1)!}{i!} \quad \frac{(q-i-k)!}{(q-i-4)!}\}.$$

Since

$$\sum_{k=1}^{4} \frac{(i+k-1)!}{i!} \frac{(q-i-k)!}{(q-i-4)!}$$

$$= [(q-i-1)(q-i-2)(q-i-3) +$$

$$+ (i+1)(q-i-2)(q-i-3)] + (i+1)(i+2)(q-i-3) +$$

$$+ (i+1)(i+2)(i+3),$$

$$= q(q-i-2)(q-i-3) + (i+1)(i+2)(q-i-3) +$$

$$+ (i+1)(i+2)(i+3),$$

$$= q[(q-i-2)(q-i-3) + (i+1)(i+2)],$$

we have that

$$A_i(q;\beta)$$

$$= \frac{(q-i-4)!}{(q-i)!} \{\beta(i+1)(i+2)(i+3) +$$

$$+ q[(q-i-2)(q-i-3) + (i+1)(i+2)]\}.$$

In particular,

$$A_0(q;0) = \frac{(q-4)!}{q!} \{q[(q-2)(q-3) + 2]\},$$

$$A_3(q;0) = \frac{(q-7)!}{(q-3)!} \{q[(q-5)(q-6) + 20]\},$$

$$A_1(q;\beta) = \frac{(q-5)!}{(q-1)!} \{24\beta + q[(q-3)(q-4) + 6]\},$$

$$A_2(q;\beta) = \frac{(q-6)!}{(q-2)!} \{60\beta + q[(q-4)(q-5) + 12]\}.$$

Since

$$\frac{(q-4)!}{q!} \cdot \frac{(q-7)!}{(q-3)!} = \frac{(q-2)(q-4)}{q} \cdot \frac{(q-5)!(q-7)!}{(q-1)!(q-2)!},$$

and

$$\frac{(q-5)!}{(q-1)!} \cdot \frac{(q-6)!}{(q-2)!} = (q-6) \cdot \frac{(q-5)!(q-7)!}{(q-1)!(q-2)!} ,$$

we have that

$H(q;\beta)$

$$= (q-2)(q-4)\, q\, \{(q-2)(q-3) + 2\} \cdot \{(q-5)(q-6) + 20\}$$

$$- (q-6)\, \{24\beta + q[(q-3)(q-4) + 6]\} \,.$$

$$\cdot \{60\beta + q[(q-4)(q-5) + 12]\} ,$$

$$= T_0 - \beta T_1 - \beta^2 T_2, \text{ say.}$$

Here,

T_0/q

$$= (q^2 - 6q + 8)(q^2 - 5q + 8)(q^2 - 11q + 50) -$$

$$- (q^2 - 6q)(q^2 - 7q + 18)(q^2 - 9q + 32),$$

$$= 8(q^2 - 5q + 8)(q^2 - 11q + 50) +$$

$$+ (q^2 - 6q)\, [(q^2 -5q + 8)(q^2 - 11q + 50) -$$

$$- (q^2 - 7q + 18)(q^2 - 9q + 32)].$$

But,

$$(q^2 - 5q + 8)(q^2 - 11q + 50) - (q^2 - 7q + 18)(q^2 - 9q + 32)$$

$$= 8(q^2 - 5q + 8) + (q^2 - 5q + 8)(q^2 - 11q + 42) -$$

$$- (q^2 - 7q + 18)(q^2 - 9q + 32),$$

$$= 8(q^2 - 5q + 8) + [(q^2 - 8q + 25) + (3q - 17)].$$

$$\cdot [(q^2 - 8q + 25) - (3q - 17)]$$

$$- [(q^2 - 8q + 25) + (q - 7)][(q^2 - 8q + 25) - (q - 7)],$$

$$= 8(q^2 - 5q + 8) + (q^2 - 8q + 25)^2 - (3q - 17)^2$$

$$- (q^2 - 8q + 25)^2 + (q - 7)^2,$$

$$= 8(q^2 - 5q + 8) - [4q - 24][2q - 10],$$

$$= 8(q^2 - 5q + 8) - 8(q - 6)(q - 5),$$

$$= 8[q^2 - 5q + 8 - (q^2 - 11q + 30)],$$

$$= 8[6q - 22].$$

Hence,

$T_0/8q$

$$= (q^2 - 5q + 8)(q^2 - 11q + 50) +$$

$$+ (q^2 - 6q)(6q - 22).$$

However,

$(q^2 - 5q + 8)(q^2 - 11q + 50)$

$$= (q^2 - 6q + q + 8)(q^2 - 11q + 50),$$

$$= (q^2 - 6q)(q^2 - 11q + 50) + (q - 6 + 14)(q^2 - 11q + 50),$$

$$= (q - 6)[(q^3 - 11q^2 + 50q) + (q^2 - 11q + 50)] +$$

$$+ 14 [(q - 6)(q - 5) + 20],$$

$$= (q - 6)[q^3 - 10q^2 + 39q + 50 + 14(q - 5)] + 280,$$

$$= (q - 6) [q^3 - 10q^2 + 53q - 20] + 280.$$

Hence,

$T_0/8q$

$$= (q - 6)[q^3 - 10q^2 + 53q - 20 + 6q^2 - 22q] + 280,$$

$$= (q - 6)[q^3 - 4q^2 + 31q - 20] + 280.$$

Also,

$T_1 \;/\; 12q(q - 6)$

$$= 2(q^2 - 9q + 32) + 5(q^2 - 7q + 18),$$

$$= 7q^2 - 53q + 154.$$

Finally,

$$T_2 = 24.60.\ (q - 6).$$

Combining the above results, the lemma follows.

<u>Lemma 3.10</u>

If $q \geq 15$ then $H(q;2) > 0$.

<u>Proof</u>: From Lemma 3.9,

$H(q;2)/8(q-6)$

$$= [T_0 - 2T_1 - 4T_2]/8(q - 6),$$

$$= q(q^3 - 4q^2 + 31q - 20) + 280q/(q - 6) -$$

$$- 3q(7q^2 - 53q + 154) - 720,$$

$$= q[q^3 - 25q^2 + 190q - 482] - 720 + 280q/(q - 6),$$

$$= q[q(q - 13)^2 + (q^2 + 21q - 482)] - 720 + 280q/(q - 6).$$

For $q \geq 14$,

$q^2 + 21q - 482 \geq 196 + 294 - 482 = 8$. Hence, for $q \geq 15$,

$$H(q;2)/8(q-6)$$
$$\geq q^2(q-13)^2 + 8q - 720 + 280q/(q-6),$$
$$\geq 15^2 \cdot 2^2 + 8.15 - 720 = 300.$$

The lemma follows.

Lemma 3.11

(a) $a_i(q) \geq 2^q A_i(q;0)$, if $0 \leq i \leq 3$ and $q \geq 14$;

(b) $a_i(q) \leq 2^q A_i(q;2)$, if $i = 1,2$ and $q \geq 36$;

(c) If $q \geq 36$ then

$$D_2(q) \leq -2^{2q} \frac{(q-5)!\,(q-7)!}{(q-1)!(q-2)!} H(q;2).$$

Proof: From Lemmas 3.2, 3.4, 3.5, and 3.9,

$$a_i(q)$$
$$= \binom{q}{i} F(i,q-i),$$
$$= \binom{q}{i} \{E(i,q-i) + F(q-i,i)\},$$
$$= \binom{q}{i} \{E(i+4,q-i-4) + F(q-i,i) +$$
$$+ 2^q \sum_{k=1}^{4} B(i+k,\ q-i+1-k)\},$$
$$= 2^q A_i(q;0) + \binom{q}{i} [E(i+4,q-i-4) + F(q-i,i)].$$

If $0 \le i \le 3$ and $q \ge 14$, then $q - i - 4 \ge i + 4$. Hence, from Lemma 3.4 (parts (b) and (d)),

$$E(i + 4, q - i - 4) + F(q - i, i) \ge 0$$

Part (a) of the lemma follows.

If $1 \le i \le 2$ and $q \ge 36$, then $i + 4 = 5$ or 6 and $q - i - 4 \ge 5(i+4)$. Hence, from Lemmas 3.4 and 3.8,

$$\begin{aligned} &E(i + 4, q - i - 4) + F(q - i, i) \\ &\quad = F(i + 4, q - i - 4) - F(q - i - 4, i + 4) + F(q - i, i), \\ &\quad \le F(i + 4, q - i - 4), \\ &\quad \le 2^{q+1} B(i + 4, q - i - 3). \end{aligned}$$

Hence,

$$\begin{aligned} a_i(q) &\le 2^q [A_i(q;0) + 2 \tbinom{q}{i} B(i + 4, q - i - 3)], \\ &= 2^q A_i(q;2), \end{aligned}$$

so that part (b) of the lemma has been proved.

Using parts (a) and (b) we have,

$$\begin{aligned} &D_2(q) \\ &\quad = a_1(q)\, a_2(q) - a_0(q)\, a_3(q), \\ &\quad \le 2^{2q} [A_1(q;2)\, A_2(q;2) - A_0(q;0)\, A_3(q;0)], \\ &\quad = -2^{2q} \frac{(q - 5)!\,(q - 7)!}{(q - 1)!\,(q - 2)!} H(q;2); \end{aligned}$$

which proves part (c) of the lemma.

Lemma 3.12

(a) $D_4(7) < 0$,

(b) $D_3(q) < 0, \quad 8 \le q \le 11$,

(c) $D_2(q) < 0, \quad 12 \le q < 36$,

(d) $D_2(q) < 0, \quad q \ge 36$.

Proof: Using the recurrence relation

$$p(z;q) = (z+1)p(z;q-1) + (2z)^{q-1}/q,$$

the polynomials $p(z;q)$ can be easily computed for small values of q. It is found that

$$p(z;7)$$

$$= (2416 z^6 + 3577 z^5 + 4431 z^4 + 3920 z^3 + 2240 z^2 + 735 z + 105)/105,$$

$$p(z;8)$$

$$= (4096 z^7 + 5993 z^6 + 8008 z^5 + 8351 z^4 + 6160 z^3 + 2975 z^2 + 840 z + 105)/105,$$

p(z;9)

$$= (21248 z^8 + 30267 z^7 + 42003 z^6 + 49077 z^5 + 43533 z^4 + 27405 z^3 + 11445 z^2 + 2835 z + 315)/315,$$

p(z;10)

$$= (37376 z^9 + 51515 z^8 + 72270 z^7 + 91080 z^6 + 92610 z^5 + 70938 z^4 + 38850 z^3 + 14280 z^2 + 3150 z + 315)/315,$$

p(z;11)

$$= (733696 z^{10} + 977801 z^9 + 1361635 z^8 + 1796850 z^7 + 2020590 z^6 + 1799028 z^5 + 1207668 z^4 + 584430 z^3 + 191730 z^2 + 38115 z + 3465)/3465.$$

The determinants $D_i(q)$ mentioned in parts (a) and (b) of the lemma can now be computed.

$D_4(7)/(105)^4$

$$= \begin{vmatrix} 3577 & 2416 & 0 & 0 \\ 3920 & 4431 & 3577 & 2416 \\ 735 & 2240 & 3920 & 4431 \\ 0 & 105 & 735 & 2240 \end{vmatrix}$$

Dividing out common factors in the first column and last row,

$D_4(7)/(105)^4$

$$= (35)(49) \begin{vmatrix} 73 & 2416 & 0 & 0 \\ 80 & 4431 & 3577 & 2416 \\ 15 & 2240 & 3920 & 4431 \\ 0 & 3 & 21 & 64 \end{vmatrix}$$

Subtracting 33 times the first column from the second column, multiplying the second column by 73, subtracting 7 times the first column from the second column, and expanding by the first row,

$D_4(7)/(105)^4$

$$= (35)(49)(73) \begin{vmatrix} 130{,}183 & 3577 & 2416 \\ 127{,}280 & 3920 & 4431 \\ 219 & 21 & 64 \end{vmatrix}$$

Expanding by the last row,

$D_4(7)/(105)^4$

$$= (35)(49)(73) \{(219)(6378967) -$$

$$- (21)(269332393) + (64)(55036800)\}$$

$$= (35)(49)(73) \{-736631280\}$$

$$< 0.$$

$D_3(8)/(105)^3$

$$= \begin{vmatrix} 5993 & 4096 & 0 \\ 8351 & 8008 & 5993 \\ 2975 & 6160 & 8351 \end{vmatrix}$$

$= (5993)\ (29957928) - (4096)\ (51910026),$

$< [(6.0)\ (3.0) - (4.0)\ (5.1)]\ 10^{10},$

$= -2.4\ 10^{10} < 0.$

$D_3(9)/(315)^3$

$$= \begin{vmatrix} 30267 & 21248 & 0 \\ 49077 & 42003 & 30267 \\ 27405 & 43533 & 49077 \end{vmatrix}$$

$= (30267)\ (743767920) - (21248)\ (1579084794),$

$< [(3.1)(7.5) - (2.1)(15.7)]\ 10^{12},$

$= -9.72\ 10^{12} < 0.$

$D_3(10)/(315)^3$

$$= \begin{vmatrix} 51515 & 37376 & 0 \\ 91080 & 72270 & 51515 \\ 70938 & 92610 & 91080 \end{vmatrix}$$

$= (51515)\ (1811547450) - (37376)\ (4641195330),$

$< [(5.2)(1.9) - (3.7)(4.6)]\ 10^{13},$

$= -7.14\ 10^{13} < 0.$

$D_3(11)/(3465)^3$

$$= \begin{vmatrix} 977801 & 733696 & 0 \\ 1796850 & 1361635 & 977801 \\ 1799028 & 2020590 & 1796850 \end{vmatrix}$$

$= (977801)\ (470918927160) -$

$-(733696)\ (1469578545072)$

$< [(9.8)(4.8) - (7.3)(14.6)]\ 10^{16}$

$= -59.54\ 10^{16} < 0.$

Thus, parts (a) and (b) of the lemma have been proved.

Using the same recurrence relation, the coefficients $a_i(q)$ were computed for $0 \le i \le 3$ and $12 \le q \le 35$. The results are given in Table 3.1, $a_i(q)$ being the rational number $\tilde{a}_i(q)/d(q)$. These results were first computed using SAC-1 and then laboriously checked by hand using a desk calculator. Table 3.1 also gives upper bounds for $D_2(q)[d(q)]^2$, which were obtained by approximating $\tilde{a}_i(q)$ to three decimals, rounding up $\tilde{a}_1(q)$ and $\tilde{a}_2(q)$ and rounding down $\tilde{a}_0(q)$ and $\tilde{a}_3(q)$. For example,

$D_2(12)[d(12)]^2$

$= \tilde{a}_1(12)\ \tilde{a}_2(12) - \tilde{a}_0(12)\ \tilde{a}_3(12),$

$\le (1.72 \times 2.34 - 1.32 \times 3.15)\ 10^{12},$

$= -.1332\ 10^{12}.$

Since these upper bounds are all negative, part (c) of the lemma follows.

Finally, part (d) of the lemma follows from Lemmas 3.10 and 3.11.

q	$\tilde{a}_0(q)$	$\tilde{a}_1(q)$	$\tilde{a}_2(q)$	$\tilde{a}_3(q)$	d(q)	$[d(q)]^2 D_2(q) <$
12	13,25056	17,11497	23,39436	31,58485	3465	$-0.1332\ 10^{12}$
13	314,18368	394,75189	526,62129	714,72973	45045	$-1.6031\ 10^{14}$
14	577,76128	708,93557	921,37318	1241,35102	45045	$-6.3068\ 10^{14}$
15	1069,77280	1286,69685	1630,30875	2162,72420	45045	$-0.1576\ 10^{16}$
16	1992,29440	2356,46965	2917,00560	3793,03295	45045	$-0.6509\ 10^{16}$
17	63389,69600	73928,98885	89649,07925	1,14070,65535	7,65765	$-5.7840\ 10^{18}$
18	1,19151,00160	1,37318,68485	1,63578,06810	2,03719,73460	7,65765	$-0.1525\ 10^{20}$
19	42,71276,03200	48,72924,04255	57,17038,30605	69,78658,25130	145,49535	$-1.8483\ 10^{22}$
20	80,85349,33504	91,44200,07455	105,89962,34860	126,95696,55735	145,49535	$-0.4818\ 10^{23}$
21	153,50250,86464	172,29549,40959	197,34162,42315	232,85658,90595	145,49535	$-0.1242\ 10^{24}$
22	292,19608,33024	325,79800,27423	369,63711,83274	430,19821,32910	145,49535	$-0.4940\ 10^{24}$
23	12823,02720,08192	14213,86397,90281	15995,00778,46031	18396,21262,72232	3346,39305	$-0.0544\ 10^{28}$
24	24519,51866,34752	27036,89117,98473	30208,87176,36312	34391,22041,18263	3346,39305	$-0.1922\ 10^{28}$
25	2,34883,91135,88736	2,57782,04921,66125	2,86228,81471,73925	3,23000,46087,72875	16731,96525	$-0.1536\ 10^{30}$
26	4,50819,13836,17536	4,92665,96057,54861	5,44010,86393,40050	6,09229,27559,46800	16731,96525	$-0.5365\ 10^{30}$
27	26,00083,17110,19008	28,30455,29681,17191	31,10030,47352,84733	34,59720,41858,60550	50195,89575	$-0.1092\ 10^{32}$
28	50,06218,55770,54208	54,30538,46791,36199	59,40485,77034,01924	65,69750,89211,45283	50195,89575	$-0.4320\ 10^{32}$
29	2799,23919,82325,43232	3026,65953,74295,21803	3297,59702,90936,05567	3627,96863,21118,69003	14,55680,97675	$-0.1008\ 10^{36}$
30	5404,28177,67952,87552	5825,89873,56620,65035	6324,25656,65231,27370	6925,56566,12054,74570	14,55680,97675	$-0.4641\ 10^{36}$
31	3,23835,28979,44185,73312	3,48135,59588,61779,30197	3,76654,81436,77409,64555	4,10744,48905,95866,60140	451,26110,27925	$-0.0857\ 10^{40}$
32	6,26671,48955,23376,00512	6,71970,88568,05965,03509	7,24790,41025,39188,94752	7,87399,30342,73276,24695	451,26110,27925	$-0.5462\ 10^{40}$
33	12,13990,17999,19381,38112	12,98642,37523,29341,04021	13,96761,29593,45153,98261	15,12189,71368,12465,19447	451,26110,27925	$-0.0071\ 10^{42}$
34	23,54079,40260,99862,40512	25,12632,55522,48722,42133	26,95403,67116,74495,02282	29,08951,00961,57619,17708	451,26110,27925	$-0.0110\ 10^{42}$
35	45,69109,89226,79082,68032	48,66711,95783,48584,82645	52,08036,22639,23217,44415	56,04354,68078,32114,19990	451,26110,27925	$-0.1633\ 10^{42}$

Table 3.1

The coefficients $a_i(q)$ of $p(z;q)$.

Lemma 3.13

The method (1.1) satisfies the root condition for $1 \le q \le 6$.

Proof: By direct computation,

$$p(z;1) = 1,$$

$$p(z;2) = 2z + 1,$$

$$p(z;3) = (10z^2 + 9z + 3)/3,$$

$$p(z;4) = (16z^3 + 19z^2 + 12z + 3)/3,$$

$$p(z;5) = (128z^4 + 175z^3 + 155z^2 + 75z + 15)/15,$$

$$p(z;6) = (208z^5 + 303z^4 + 330z^3 + 230z^2 + 90z + 15)/15.$$

By inspection we see that $p(z;q)$ has roots with strictly negative real part for $q = 1,2$, and 3.

For $q = 4,5$, and 6, we could compute the determinants $D_k(q)$. However, by hand it is easier to use the essentially equivalent algorithm of Routh. We follow Obreschkoff [18,p. 107] except that (i) we do not divide by the leading coefficient and (ii) we divide out common factors (these are indicated in brackets); this simplifies the computations without affecting the result. We obtain the following arrays:

q = 4

4	3	(4)
19	3	(1)
1		(45)
1		(3)

$\underline{q = 5}$

128	155	15	(1)
7	3	0	(25)
701	105		(1)
1			(1368)
1			(105)

$\underline{q = 6}$

104	165	45	(2)
303	230	15	(1)
149	69		(175)
13363	2235		(1)
1			(589032)
1			(2235)

Since the first columns of the arrays are positive, the polynomials $p(z;q)$ have roots with strictly negative real part for $q = 4, 5$, and 6.

Appealing to Lemma 2.1, the proof of the lemma is completed.

4. Remarks on the proof of Theorem 1.1

The proof of Theorem 1.1 given in the last section is unpleasantly long, but has resisted efforts by the author to shorten it. Here, we mention several approaches, which might, in other hands, lead to a shorter proof:

1. The bound for $G(u,v,\frac{1}{4})$ in Lemma 3.7 could be sharpened.
2. Values of α and β other than $\alpha = \frac{1}{4}$ and $\beta = 2$ could be used.
3. Duffin [10] gives several algorithms for analysing stability problems.
4. Using the expression for $a_i(q)$ given in Lemma 3.1, noting that $2^m = 2^{m+1} - 2^m$, and "summing by parts", it can be shown that

$$a_i(q) = \frac{q-i+1}{i} a_{i-1}(q) - \frac{1}{i} \alpha_{i-1}$$

where

$$\alpha_i = \sum_{m=1}^{q} 2^m \binom{q-m}{i} = 2^q - \sum_{k=0}^{i} \binom{q}{k}.$$

5. Using Lemma 3.2, $D_2(q)$ can be expressed as double integral, namely

$D_2(q)$

$$= -\int_0^1\int_0^1 \left[q \binom{q}{3} \frac{[(1+x)^q - (1-x)^q](1-y)^3[(1+y)^{q-3} - (1-y)^{q-3}]}{(2x)(2y)}\right.$$

$$\left. - \binom{q}{1}\binom{q}{2} \frac{(1+x)[(1+x)^{q-1} - (1-x)^{q-1}](1-y)^2[(1+y)^{q-2} - (1-y)^{q-2}]}{(2x)(2y)}\right] dxdy.$$

6. There is an algorithm due to Schur and Cohn (Marden [14, p. 198]) which enables one to determine how many zeros of a polynomial $\tilde{\rho}(\zeta)$ of degree $q-1$ lie inside the unit circle. More precisely, a sequence of determinants Δ_k, $1 \le k \le q-1$, is computed, and a necessary condition for all the zeros of $\tilde{\rho}$ to lie inside the unit circle is that $(-1)^k \Delta_k > 0$ for $1 \le k \le q-1$. For the polynomial

$$\tilde{\rho}(\zeta;q) = \rho(\zeta;q)/(\zeta-1),$$

where $\rho(\zeta;q)$ is given by (1.2), we find that

$$\Delta_1 = \left(\frac{1}{q}\right)^2 - \left(\sum_{m=1}^{q} \frac{1}{m}\right)^2$$

so that $(-1)^1 \Delta_1 > 0$. This means that the Schur-Cohn approach does not provide a trivial proof of Theorem 1.1.

Regarding the Schur-Cohn approach we may make the following observations:

(i) The coefficients of the polynomial $\rho(\zeta;q)$ of (1.2) are no less complicated than the coefficients of the polynomial $p(z;q)$ of (2.1). However, the coefficients of $\rho(\zeta;q)$ can be expressed in terms of the psi function

$$\psi(q+1) = \sum_{m=1}^{q} \frac{1}{m} - \gamma$$

(where γ is Euler's constant), whose properties have been studied (Nörlund [17, p. 99], Abramowitz and Stegun [1, p. 258]).

(ii) Duffin [10] has demonstrated a connection between the Schur-Cohn algorithm and the Routh-Hurwitz algorithm.

(iii) At the time of writing this report the author is somewhat confused about the Schur-Cohn algorithm. From the descriptions of Marden [14] and Householder [13] it appears that the Schur-Cohn algorithm provides only necessary conditions and that awkward special cases arise when one or more of the determinants Δ_k are zero. However, Wilf [22] uses a version of the Schur method described by Wall [20, p. 298] to obtain necessary and sufficient conditions for a polynomial to have all its zeros inside the unit circle.

APPENDIX

The calculation of the location, with respect to the imaginary axis, of the zeros of a real polynomial with integer coefficients.

In this appendix we describe an algorithm for determining the location, with respect to the imaginary axis, of the zeros of a real polynomial with integer coefficients. Using SAC-1, the algorithm has been implemented as a subroutine, ROUTH, a listing of which is given at the end of this appendix; the SAC-1 subroutines used by ROUTH are described by Collins [3-6] and Collins and Heindel [7]. The results of Tables 2.1 and 3.1, as well as the signs of $D_k(q)$ for $1 \le q \le 35$, were obtained by using ROUTH.

The input to ROUTH is a polynomial u in canonical SAC-1 form. That is,

$$u(z) = \sum_{k=1}^{n} \alpha_k z^{\beta_k}, \tag{A.1}$$

where the exponents β_k form a decreasing sequence of integers,

$$\beta_1 > \beta_2 > \ldots > \beta_n \ge 0; \tag{A.2}$$

and the coefficients α_k are nonzero integers.

The output from ROUTH is a list SEQ. If u has degree less than 1, then SEQ is the empty list. Otherwise

$$SEQ = (s_1, s_2, \ldots, s_{m+3}) \tag{A.3}$$

where

s_1 = number of zeros of u with strictly negative real part,

s_2 = number of imaginary zeros of u,

s_3 = number of zeros of u with strictly positive real part,

s_{j+3} = number of imaginary zeros of u with multiplicity j,

and $m \geq 0$ is the maximum of the multiplicities of the imaginary zeros of u.

It will be assumed, without loss of generality, that the leading coefficient of u, α_1, is positive. Let $i = \sqrt{-1}$ and let

$$w(z) = i^{\beta_1} \; u(z/i) = \sum_{k=1}^{n} \alpha_k \, z^{\beta_k} \, i^{\gamma_k}$$

where $\gamma_k = \beta_1 - \beta_k$. Let $\gamma_k = 2\epsilon_k + \tau_k$ where ϵ_k and τ_k are non-negative integers and $\tau_k = 0$ or $\tau_k = 1$. Then

$$w(z) = w_1(z) + i \, w_2(z)$$

where

$$w_1(z) = \sum_{\substack{k=1 \\ \tau_k=0}}^{n} \alpha_k \, z^{\beta_k} (-1)^{\epsilon_k},$$

$$w_2(z) = \sum_{\substack{k=1 \\ \tau_k=1}}^{n} \alpha_k \, z^{\beta_k} (-1)^{\epsilon_k}.$$

It suffices to determine the location of the zeros of w with respect to the real axis. For, as is easily seen, zeros of w (i) on the real axis (ii) in the upper half plane and (iii) in the lower half plane, correspond to zeros of u (i) on the imaginary axis (ii) in the right half plane and (iii) in the left half plane.

Before proceeding further, we recall a useful property of greatest common divisors. (Birkhoff and Maclane [2, p. 74]). Let $I[z]$ be the set of polynomials in the variable z with integer coefficients, and let $R[z]$ be the set of polynomials in the variable z with real coefficients. If $q_1, q_2 \in I[z]$, let $\gcd_{I[z]}\{q_1,q_2\}$ be the greatest common divisor of q_1 and q_2 in $I[z]$; that is,

(i) if $r, r_1, r_2 \in I[z]$, $q_1 = r\,r_1$, and $q_2 = r\,r_2$, then there exists $r_3 \in I[z]$ such that $\gcd_{I[z]}\{q_1,q_2\} = r\,r_3$,

(ii) $\gcd_{I[z]}\{q_1,q_2\}$ has a non-negative leading coefficient,

(iii) $\gcd_{I[z]}\{0,0\} = 0$.

The property referred to above is that if $q_1, q_2 \in I[z]$ and $r, r_1, r_2 \in R[z]$ are such that $q_1 = r r_1$, $q_2 = r r_2$, then there exists $r_3 \in R[z]$ such that $\gcd_{I[z]}\{q_1,q_2\} = r r_3$. In other words, if $q_1, q_2 \in I[z]$ then $\gcd_{I[z]}\{q_1,q_2\}$ is a greatest common divisor of q_1 and q_2 when these are regarded as polynomials in $R[z]$.

We now return to the problem of locating the zeros of w. Let

$$v = \gcd_{I[z]}\{w_1, w_2\},$$

$$p_1 = w_1/v,$$

$$p_2 = w_2/v.$$

Then

$$w = vp$$

where

$$p = p_1 + ip_2;$$

so that we must locate the zeros of v and p. It will be of importance that, because p_1 and p_2 have no common factors, p has no real zeros.

The zeros of v may be located as follows. Let g_j, $j \geq 1$, and h_j, $j \geq 0$, be the polynomials with integer coefficients defined by

$$h_0 = v,$$

$$h_{j+1} = \gcd_{I[z]}[h_j, h_j'], \; j \geq 0$$

$$g_{j+1} = h_j/h_{j+1}, \; j \geq 0;$$

where h_j' denotes the derivative of h_j. Let $m \geq 0$ be the smallest integer such that h_m is a constant polynomial. Then

$$v = \begin{cases} h_m \prod_{j=1}^{m} g_j, & m \geq 1, \\ h_0, & m = 0. \end{cases}$$

If $(z-z_0)^r$, $r \geq 2$, is a factor of h_j, then $(z-z_0)^{r-1}$ is a factor of h_j', so that g_j has only simple zeros. Furthermore, the zeros of g_j are the zeros of v of multiplicity not less than j. Let n_j denote the number of real zeros of g_j. Remembering that p has no real zeros, we have that

$$s_2 = \sum_{j=1}^{m} n_j,$$

$$s_{i+3} = \begin{cases} n_i - n_{i+1}, & 1 \leq i < m. \\ n_m, & i = m, \end{cases}$$

where the s_k are elements of the list (A.3).

Since v is a real polynomial, its non-real zeros occur in conjugate pairs, so that the number of zeros of v in the upper half plane is equal to the number of zeros of v in the lower half plane namely

$$[\deg(v) - s_2]/2,$$

where deg(v) denotes the degree of v.

The above computations to locate the zeros of v are easily carried out using SAC-1, since there are SAC-1 subroutines for computing the derivative of a polynomial (PDERIV), the gcd of two polynomials (PGCD), and the number of real roots of a polynomial with simple roots (NROOTS).

The zeros of p can be located as follows. Let $\deg(p_1)$ denote the degree of p_1. By construction, $\deg(p_2) < \deg(p_1)$. Thus, if $\deg(p_1) = 0$ then p has no zeros. If $\deg(p_1) > 0$ we can find nonzero integers f_k, g_k, and polynomials with real coefficients p_k, q_k such that

(i) $f_k\, p_k = q_{k+1}\, p_{k+1} - g_k\, p_{k+2}$, $1 \le k \le \mu$,

(ii) $f_k\, g_k > 0$, $1 \le k \le \mu$

(iii) $\deg(p_{k+2}) < \deg(p_{k+1})$, $1 \le k \le \mu$,

(iv) $p_{\mu+1} \not\equiv 0$, $p_{\mu+2} = 0$.

Dividing by f_k we have

$$p_k = \tilde{q}_{k+1}\, p_{k+1} - \left(\frac{g_k}{f_k}\right) p_{k+2}, \quad 1 \le k \le \mu,$$

where $\tilde{q}_k = q_k / f_{k-1}$. Let b_k and c_k be, respectively, the signs of the leading coefficients of p_k and q_k. Let m_k and n_k be, respectively, the degrees of p_k and q_k. Then,

$$c_k = b_{k-1}/b_k, \quad 2 \le k \le \mu,$$

$$n_k = m_{k-1} - m_k, \quad 2 \le k \le \mu .$$

Set

$$e_k = (-1)^{n_k}\, c_k,$$

and

$$\Delta V = \mathscr{N}(c_1, \ldots, c_\mu) - \mathscr{N}(e_1, \ldots, e_\mu);$$

where

$$\mathcal{N}(\lambda_1, \ldots, \lambda_\mu)$$

denotes the number of negative terms in the sequence $\lambda_1, \ldots, \lambda_\mu$. Then it follows from a result quoted by Marden [14, p. 172] that the polynomial p has

$$[\deg(p_1) + \Delta V]/2$$

zeros in the upper half plane and

$$[\deg(p_1) - \Delta V]/2$$

zeros in the lower half plane. Recalling the previous results concerning v, the terms s_1 and s_3 in the list SEQ are given by

$$s_1 = [\deg(v) - s_2]/2 + [\deg(p_1) + \Delta V]/2,$$

$$s_3 = [\deg v - s_2]/2 + [(\deg(p_1) - \Delta V]/2.$$

The above computations to locate the zeros of p are easily carried out using SAC-1. There is a SAC-1 subroutine (ISTURM) which, given p_1 and p_2, computes the sequence $p_3, p_4, \ldots, p_{\mu+1}$. However, ISTURM stores all these polynomials, which could lead to storage problems if p_1 is of high degree. Hence, a modification of ISTURM has been incorporated into ROUTH.

The signs of the determinants $D_k(q)$ of (2.4) can be found once the corresponding quantities c_k are known. For, using the results and notation of Marden [14, p. 175 - p. 180] we find that

$$\text{sign}\,(D_{k-2}(q)\ \ D_k(q))$$

$$= \text{sign}\,(\delta_{k-2}\,\delta_k),$$

$$= \text{sign}\,(\Delta_{k-1}\,\Delta_k),$$

$$= \text{sign}\,(b_{n-k+1,n-k+1}\ b_{n-k,n-k})$$

$$= \text{sign}\,(b_{k-1}\,b_k)$$

$$= \text{sign}\,(b_{k-1})/\text{sign}\,(b_k)$$

$$= \text{sign}\,(c_k).$$

We give below a listing of the subroutine ROUTH. A program, MITCHEL, was written which generated the polynomials $p(z;q)$ of (2.1) using the recurrence relation

$$p(z;q) = (z+1)p(z;q-1) + \frac{(2z)^{q-1}}{q},$$

and then used ROUTH to locate the zeros of $p(z;q)$. A listing of MITCHEL is also given.

Tables 2.1 and 3.1 contain the results obtained for $2 \le q \le 35$. These computations required about 40 minutes on the 1108. It may be remarked that the time required increased very rapidly with q. For example, it took less than ten minutes to compute the results for $2 \le q \le 25$ but about 15 minutes to compute the results for $q = 35$.

```
C     PROGRAM MITCHEL
C     PROGRAM TO GENERATE POLYNOMIALS P(Z,Q) CORRESPONDING TO
C     BACKWARD DIFFERENCE MULTISTEP METHODS OF ORDER Q
C     *****************************************************
      COMMON /TR1/ AVAIL,STAK,RECORD(72)
      COMMON /TR2/ SYMLST
      COMMON /TR3/ BETA
      COMMON /TR4/ PRIME,PEXP
      IMPLICIT INTEGER (A-Z)
      DIMENSION SPACE(20000),PA(1000)
C     *****************************************************
C     INITIALIZE SAC1
C     LISTS STORED IN ARRAY SPACE
C     PA ARRAY USED TO GENERATE PRIME LIST
C     *****************************************************
      IN = 5
      OUT = 6
      CALL BEGIN(SPACE,20000)
      BETA=2**33
      CONS=4000000001
      PRIME=GENPR(PA,1000,CONS)
      CONS=PFA(CONS,0)
      CALL ELPOF2(CONS,PEXP,DUM)
      CALL ERLA(CONS)
      SYMLST=0
C     *****************************************************
C     QMAX IS MAXIMUM ORDER
C     *****************************************************
      QMAX=35
      PRINT 10,QMAX
10    FORMAT( '1 QMAX= ' , I5)
C     *****************************************************
C     SET UP BASIC RATIONAL FUNCTIONS
C     RZ0    =+1Z**0
C     RZ1    =+1Z**1
C     RZ1PZ0=+1Z**1+1Z**0
C     R2Z1   =+2Z**1
C     *****************************************************
      ZL=PFA(35,0)
      Z=PROSYM(ZL)
      CALL ERASE(ZL)
      ONE=PFA(1,0)
      PZ1=PFA(1,0)
      PZ1=PFL(ONE,PZ1)
      PZ1=PFL(Z,PZ1)
      RZ1=RPOLY(PZ1)
      CALL PERASE(PZ1)
      RZ0=RQ(RZ1,RZ1)
      RZ1PZ0=RSUM(RZ1,RZ0)
      R2Z1=RSUM(RZ1,RZ1)
C     *****************************************************
C     GENERATE POLYNOMIALS USING RECURRENCE RELATION
C     P(Z,Q)= (Z+1)P(Z,Q-1)+( (2Z)**(Q-1) )/Q
C     *****************************************************
      RZ=0
      RZTOQ=BORROW(RZ0)
```

```
      RQZ=0
      DO 500 Q=1,QMAX
C
      RQZT=RSUM(RQZ,RZ0)
      CALL RERASE(RQZ)
      RQZ=RQZT
C
      RZT1=RPROD(RZ,RZ1PZ0)
      RZT2=RQ(RZTOQ,RQZ)
      CALL RERASE(RZ)
      RZ=RSUM(RZT1,RZT2)
      CALL RERASE(RZT1)
      CALL RERASE(RZT2)
C
      RTEMP=RPROD(RZTOQ,R2Z1)
      CALL RERASE(RZTOQ)
      RZTOQ=RTEMP
C
      PRINT 50,Q
50    FORMAT('1Q,RZ= ' ,I5)
      CALL RWRITE(OUT,RZ)
      PZ=BORROW(FIRST(RZ))
      CALL ROUTH(PZ,SEQ)
      I=1
      TEMP1=SEQ
400   IF (TEMP1 .EQ. 0 ) GO TO 450
      CALL ADV(TEMP2,TEMP1)
      PRINT 420,I,TEMP2
420   FORMAT( '   SEQ(' ,I5 , ')= ' , I5 )
      I=I+1
      GO TO 400
450   CONTINUE
      CALL ERASE(SEQ)
      CALL PERASE(PZ)
500   CONTINUE
C     ****************************************************
C     TIDY UP
C     ****************************************************
      CALL RERASE(RZ)
      CALL RERASE(RZTOQ)
      CALL RERASE(RQZ)
      CALL RERASE(RZ0)
      CALL RERASE(RZ1)
      CALL RERASE(RZ1PZ0)
      CALL RERASE(R2Z1)
      CALL ERASE(PRIME)
      CALL ERASE(SYMLST)
C     ****************************************************
C     CHECK THAT ALL LISTS HAVE BEEN DELETED
C     LENGTH OF AVAIL SHOULD EQUAL HALF SIZE OF ARRAY SPACE
C     ****************************************************
      L=LENGTH(AVAIL)
      PRINT 1000,L
1000  FORMAT( ' LENGTH OF AVAIL = ' , I10)
      STOP
      END
```

```
      SUBROUTINE ROUTH(U,SEQ)
C     ****************************************************
C     INPUT IS THE POLYNOMIAL U
C     OUTPUT IS THE LIST SEQ
C     SEQ=0 IF U HAS DEGREE ZERO
C     SEQ(1)= NO OF ZEROS OF U WITH STRICTLY NEGATIVE REAL PART
C     SEQ(2)= NO OF IMAGINARY ZEROS OF U
C     SEQ(3)= NO OF ZEROS OF U WITH STRICTLY POSITIVE REAL PART
C     SEQ(I+3) IS THE NO OF IMAGINARY ZEROS WITH MULTIPLICITY I
C     ****************************************************
      IMPLICIT INTEGER (A-Z)
      DIMENSION      W(2)
      IN=5
      OUT=6
C     ****************************************************
C     PRINT MESSAGE ON ENTRY TO SUBROUTINE
C     ****************************************************
      PRINT 10
10    FORMAT('0 SUBROUTINE ROUTH     ' )
C     ****************************************************
C     TEST WHETHER U HAS DEGREE ZERO
C     ****************************************************
      IF( PDEG(U).GT. 0) GO TO 20
      SEQ=0
      RETURN
20    CONTINUE
C     ****************************************************
C     T=U IF U HAS POSITIVE LEADING COEFFICIENT, T=-U OTHERWISE
C     ****************************************************
      T=PABS(U)
C     ****************************************************
C     SET UP W(1) AND W(2)
C     ****************************************************
      W(1)=0
      W(2)=0
      NT=PDEG(T)
100   T1=PRED(T)
      T2=PDIF(T,T1)
      NT2=PDEG(T2)
      GAMMA=NT-NT2
      EPS=GAMMA/2
      TAU=GAMMA-2*EPS
      S=(-1)**EPS
      J=TAU+1
      IF(S.EQ.(+1)) WT =PSUM(W(J),T2)
      IF(S.EQ.(-1)) WT =PDIF(W(J),T2)
      CALL PERASE(W(J))
      W(J)=WT
      CALL PERASE(T)
      CALL PERASE(T2)
      T=T1
      IF(T.NE.0) GO TO 100
C     ****************************************************
C     COMPUTE V,P1,P2
C     ****************************************************
      V=PGCD(W(1),W(2))
```

```
      P1   =PQ(W(1),V)
      P2   =PQ(W(2),V)
      CALL PERASE(W(1))
      CALL PERASE(W(2))
C     ***************************************************
C     PRINT V,P1,P2
C     ***************************************************
      PRINT 150
150   FORMAT( '0V,P1  ,P2  = ')
      CALL PWRITE(OUT,V)
      CALL PWRITE(OUT,P1  )
      CALL PWRITE(OUT,P2  )
C     ***************************************************
C     LOCATE ZEROS OF V
C     ***************************************************
      NV=PDEG(V)
      NI=0
      SEQ =0
      NOLD=-1
      Z=FIRST(V)
      H=V
200   NH=PDEG(H)
      IF(NH.EQ.0) GO TO 300
      HD=PDERIV(H,Z)
      HT=PGCD(H,HD)
      G =PQ(H,HT)
      N=NROOTS(G )
      ND=NOLD-N
      IF(NOLD.GE.0) SEQ=PFA(ND,SEQ)
      NOLD=N
      NI=NI+N
      CALL PERASE(H)
      H=HT
      CALL PERASE(HD)
      CALL PERASE(G )
      GO TO 200
300   CALL PERASE(H)
      IF(NOLD.GE.0) SEQ=PFA(NOLD,SEQ)
      SEQ=INV(SEQ)
      NP=(NV-NI)/2
      NN=NP
C     ***************************************************
C     LOCATE ZEROS OF P
C     ***************************************************
C     COMPUTE AND PRINT STURM SEQUENCE
C     ************************
      PRINT 400
400   FORMAT( '0 STURM SEQUENCE ')
      CALL PWRITE(OUT,P1)
      NP1=PDEG(P1)
      IF ( NP1.EQ.0 ) GO TO 500
      DELV=0
      B1=PSIGN(P1)
      M1=PDEG(P1)
401   B2=PSIGN(P2)
      M2=PDEG(P2)
```

```
      CALL PWRITE(OUT,P2)
      N=M1-M2
      C=B1/B2
      E =C*(-1)**N
      IF( C .LT.0 ) DELV=DELV+1
      IF ( E .LT.0 ) DELV=DELV-1
      B1=B2
      M1=M2
      DUM=PSREM(P1,P2)
      IF ( DUM.EQ.0) GO TO 404
      CALL PERASE(P1)
      P1=P2
      DUM1=PCONT(DUM)
      P2=PSQ(DUM,DUM1)
      CALL PERASE(DUM)
      CALL ERLA(DUM1)
      IF( (B2.EQ.-1) .AND. (B2**N.EQ.1))GO TO 401
      DUM=P2
      P2=PNEG(P2)
      CALL PERASE(DUM)
      GO TO 401
404   CONTINUE
      NP=NP+(NP1+DELV)/2
      NN=NN+(NP1-DELV)/2
500   CONTINUE
      CALL PERASE(P1)
      CALL PERASE(P2)
C     ****************************************************
C     STORE ANSWERS IN SEQ
C     ****************************************************
      SEQ=PFA(NP,SEQ)
      SEQ=PFA(NI,SEQ)
      SEQ=PFA(NN,SEQ)
      RETURN
      END
```

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