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NOISELIKE TRANSFORMS OF ω-EVENTS

by

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LIST OF SYMBOLS

ω	first infinite cardinal number
$\Sigma = \{a, b ---\}$	a finite alphabet
$0^*, \Sigma^*$	sets of all finite tapes formed from $\{0\}$, Σ
$x, y, y_1, y_i, z_1, r_2, w_j$	finite tapes
$\alpha, \beta, R, W, \alpha_1, \alpha_i, \beta_1, \beta_n, \beta_i$	sets of tapes
$\beta^{\omega}, 1^{\omega}, R^{\omega}, \Sigma^{\omega}$	sets of all infinite tapes formed from β, $\{1\}$, R, Σ
M, M', M"	sequential machines
δ, δ'	transition functions
ζ	set of states of sequential machine
s_0	starting state of machine
$s, s', s_i, s_j, s_k, s_{i_1}, s_{i_2}$	machine states
σ	sequence of states
u_1, u_m, u, v	designated sets of states of sequential machine
F,G	sets of designated state sets
$\sim$	static loss designator
$\cup, \in, \subseteq$	set-theoretic union, membership, inclusion

ABSTRACT

The present paper considers the effects of several types of noiselike transforms on regular sets of infinite tapes. These transforms can be interpreted as message distortions resulting from factors such as static, deletion, interference, and errors occurring during transmission or reception. We show that distortions of this kind do not destroy regularity.

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1. INTRODUCTION

While some physical systems or machines, on being activated, pass through a finite sequence of states and then terminate in an equilibrium condition, others have the property that under certain conditions no equilibrium is reached, and the system passes from state to state indefinitely, to be halted only by breakdown or human intervention. Buchi and Landweber (1967), Muller (1963), and McNaughton (1966) have studied the extension of the concept of regularity to sets of infinite sequences, which provide mathematical models of certain of these systems.

In this paper the effects of several types of noiselike transformations on regular sets of infinite tapes are examined. The finite analogues of some of these transforms were studied by Stearns and Hartmanis (1963). The transformations considered can be interpreted intuitively as message distortions and losses due to factors such as static, deletion, interference, and errors during transmission or reception. It will be shown that these distortions do not destroy regularity, even if an infinite number of occurrences of the given type of distortion exist.

2. ω-EVENTS

The reader is presumed to be familiar with the content of McNaughton (1966). We briefly restate the needed results and definitions from this paper:

An ω-event is a set of infinite sequences from some finite input alphabet Σ; i.e. a set of sequences of ordinality ω, where ω is the first infinite cardinal number.

Regular expressions (in the sense of McNaughton and Yamada (19 0)) can be extended to describe ω-events by introducing a new operator; thus, if α is a non-empty event (set of finite words) not containing the null word, α^{ω} is the set of all infinite sequences formed by concatenating countably infinitely many members of α. Thus $(0 \cup 1)^{\omega}$ is the set of all infinite sequences (or ordinality ω) of 0's and 1's.

If α is an event and β is an ω-event, then $\alpha\beta$ is an ω-event. Note that in general $\beta\alpha$ may have ordinality greater than ω, and need not be an ω-event. For example, $0^{*}1^{\omega}$ is an ω-event, but $1^{\omega}0^{*}$ is not.

Definition: An ω-event R^{ω} is regular if there exist regular events $\alpha_1, ---, \alpha_n, \beta_1, ---, \beta_n$ such that $R^{\omega} = \alpha_1\beta_1^{\omega} \cup \cdots \cup \alpha_n\beta_n^{\omega}$. (Clearly since R^{ω} is an ω-event no β_i can be allowed to contain the null word.)

Definition: An ω-event R^{ω} is finite-state if there is a finite (deterministic) automaton $(\zeta, \delta, s_0, \mathfrak{Y})$, where ζ is the state set, δ the transition function, $s_0 \in \zeta$ the start state, and $\mathfrak{Y}$ is a subclass $\{u_1, ---, u_m\}$ of the class of all nonempty subsets of states of the automaton such that, for any infinite sequence x whose terms are from alphabet Σ, x is in R^{ω} iff the precise set of states that the automaton assumes infinitely often when given x as input sequence (starting from the initial state) is one of the sets $u_1, ---, u_m$.

The above definition is easily extended to the case where the automaton is non-deterministic by stipulating that x be in R^{ω} iff it is possible for the automaton to take on precisely the states of one of the $u_1, ---, u_m$ infinitely often under the input x.

Theorem: (McNaughton) An ω-event is regular iff it is finite-state.

It is also the case that given the ω-event characterized in one way, the other kind of characterization is effectively determined.

The following corollary follows easily from this theorem, and will be used implicitly throughout:

Corollary: If an ω-event R^{ω} is the set of tapes accepted by a non-deterministic automaton, then there exists a deterministic automaton accepting R^{ω}.

3. NOISELIKE TRANSFORMS

We consider sets of infinite tapes which are obtained from regular ω-events through noiselike changes corresponding to static, signal loss, interference, and errors. The first type of transform considered is the case of deletions of finite segments from the members of an ω-events, with the locations of the deletions known.

Definition: Let $R^{\omega} \subseteq \Sigma^{\omega}$. Then $E(R^{\omega}) = \{x \in \Sigma^{\omega} : x = y_1 \sim y_2 \sim --- \sim y_i \sim ---,\ y_i \in \Sigma^*$ for each i, such that there exist $z\ ,---,z_i,---, \in \Sigma^*$ for which $y_1 z_1 y_2 z_2 --- y_i z_i --- \in R^{\omega}\}$. We will say an event is ω-regular iff it is a regular ω-event.

$E(R^{\omega})$ is simply the set obtained from R^{ω} by replacing blocks of consecutive symbols of members of R^{ω} by tildas, with no restrictions on the length or number of these blocks. The tildas may be thought of as "static" occurring at those points on the tape.

Note that any of the y_i and z_i above might be the empty word. In the case that all the y_i are empty, we obtain the tape $(\sim)^{\omega}$, since each $\sim$ can replace only finitely many symbols.

Theorem 1: If R^{ω} is ω-regular, then so is $E(R^{\omega})$.

Proof: Let $M = (\zeta, \delta, s., \mathfrak{F})$ be a deterministic automaton recognizing R^{ω}. Let $M' = (\zeta, \delta', \{s_0\}, \mathfrak{L})$ be the following non-deterministic automaton: $\delta'(s,a) = \{\delta(s,a)\}$, for all $s \in \zeta$,

$a \in \Sigma$; $\delta'(s,\sim) = \{s' \in \zeta : \text{exists } x \in \Sigma^{*} \text{ such that } \delta(s,x) = s'\}$. Let $G = \{v\text{:exists } u \in \mathfrak{F} \text{ such that } v \subseteq u\}$. Then we claim that M' recognizes $E(R^{\omega})$. Let x be in $E(R^{\omega})$. Then $x = y_1 \sim --- \sim y_i \sim ---$, where for some $z_1,---z_i,---$, $y_1z_1---y_iz_i--- \in R^{\omega}$. Suppose that $\delta(s_0,y_1) = s_j$; then $\delta'(s_0,y_1) = \{s_j\}$. If $\delta(s_j,z_1) = s_k$, then $s_k \in \delta'(s_j,\sim)$, hence M' can go to state s_k on receiving input $y_1 \sim$. Now suppose that for $i > 1$, if $\delta(s_0,y_1,z_. --y_i) = s_i$, then $s_i \in \delta'(s_0,y_1 \sim -- \sim y_i)$. Then as before, if $\delta(s_0,y_1,z_1--y_iz_i) = \delta(s_i,z_i) = s_{\ell}$, then $s_{\ell} \in \delta'(s_i,\sim)$, and thus $s_{\ell} \in \delta'(s_0,y_1\sim -- \sim y_i\sim)$. Hence if $x \in E(R^{\omega})$, then M' can accept x.

We now show that if M' can accept x, then $x \in E(R^{\omega})$. Suppose $x = y_1\sim --- \sim y_i \sim ---$ is accepted by M'. Then there exists $v \subseteq u \in \mathfrak{F}$ and a state sequence $\sigma = s_0 s_{i_1} s_{i_2} ---$ such that the set of states occurring infinitely often in the sequence σ is v, and σ is a possible state sequence of M', given input x. Let the state (in this sequence) after input $y_1 \sim --- \sim y_i$ be s, and the state after input $y_1 \sim --- \sim y_i \sim$ be s', i.e. $s' \in \delta'(s,\sim)$. Then there exists a tape z_i such that $\delta(s,z_i) = s'$; moreover, both s and s' are in $v \subseteq u$. Thus, since u is strongly connected, z_i can be chosen so that M takes on every state of u at least once under input z_i, starting in state s. Thus, replacing the i'th ~ with the z_i described, we obtain the tape y_1z_1---y_iz_i---, and by construction, the set of states assumed infinitely often by M under this input is u. Hence $x \in E(R^{\omega})$, as desired, and $E(R^{\omega})$ is ω-regular if R^{ω} is.

Thus ω-regularity is preserved under "static". The following theorem states that ω-regularity is also preserved if the locations of the deletions from the tapes are unknown.

Definition: For $R^{\omega} \subseteq \Sigma^{\omega}$, let $D(R^{\omega}) = \{x = y_1y_2$---$y_i$---$\in \Sigma^{\omega}$: $y_j \in \Sigma^*$, and there exist z_1,---z_i,--- $\in \Sigma^*$, such that y_1z_1---y_iz_i--- $\in R^{\omega}\}$.

Theorem 2: If R^{ω} is ω-regular, then so is $D(R^{\omega})$. The proof is similar to that of Theorem 1.

A third type of noiselike transform is message interference in the form of insertions of members of one event into members of another. Before considering such transforms for ω-events, we note that they are regularity-preserving for finite-tape events:

Definition: Let $R, W \subseteq \Sigma^*$; we define the insertion $I_W(R)$ of W into R as follows: $I_W(R) = \{r_1w_1r_2w_2$---$w_{n-1}r_n$: w_1,---,$w_{n-1} \in W$, r_1,---,$r_n \in R\}$. Any of the w_i, r_i may be empty.

Theorem 3: If R and W are regular, then so is $I_W(R)$. The reader may easily verify Theorem 3 by means of non-deterministic automata. We extend the definition to ω-events:

Definition: Let $R^{\omega} \subseteq \Sigma^{\omega}$, $W \subseteq \Sigma^{*}$. Let $I_W(R^{\omega}) = \{r_1w_1 --- r_iw_i ---: r_1r_2 --- r_i --- \in R^{\omega}, w_1, ---, w_i, --- \in W\}$.

Theorem 4: If W is regular and R^{ω} is ω-regular, then $I_W(R^{\omega})$ is ω-regular.

Proof: Let $R^{\omega} = \bigcup_{i=1}^{n} \alpha_i\beta_i^{\omega}$; we have $I_W(R^{\omega}) = I_W(\bigcup_{i=1}^{n} \alpha_i\beta_i^{\omega}) = \bigcup_{i=1}^{n} I_W(\alpha_i\beta_i^{\omega})$. Also, since there are no deletion involved, and hence no possible loss of tape junctures, it is clear that it is immaterial whether the insertions involved were performed before or after assembly of the infinite tape from tapes of α and β; insertions occurring at junctures between tapes of α and β or β and β can be counted with either the preceding or the succeeding tape at will. For example, given $aw_0b_1w_1b_2w_2 --- b_iw_i ---$, where $a \in \alpha$, $b_i \in \beta$, $w_i \in W$ for all i, this tape might have been assembled either from the tapes a, w_0b_1, w_1b_2, ---, w_ib_{i+1}, ---, or from aw_0, b_1w_1, ---, b_iw_i, ---, etc., each set of tapes belonging to $I_W(\alpha)$ and $I_W(\beta)$ respectively. Thus we have $I_W(\alpha_i\beta_i^{\omega}) = I_W(\alpha_i)(I_W(\beta_i))^{\omega}$. Since for each i $I_W(\alpha_i)$ and $I_W(\beta_i)$ are regular, by McNaughton's result (see

Section 2) so is $\bigcup_{i=1}^{n} I_W(\alpha_i)(I_W(\beta_i))^{\omega} = \bigcup_{i=1}^{n} I_W(\alpha_i\beta_i^{\omega}) = I_W(R^{\omega})$, as claimed.

Finally, we consider distortions which may be interpreted as the effects of errors introduced while reading or transmitting tape symbols.

Definition: Let $R^{\omega} \subseteq \Sigma^{\omega}$. Then $F(R^{\omega}) = \{x \in \Sigma^{\omega}$: exists $y \in R^{\omega}$ such that x and y differ in only finitely many places$\}$.

Theorem 5: If R^{ω} is ω-regular, then so is $F(R^{\omega})$.

Proof: We describe informally an automaton which accepts $F(R^{\omega})$. Let $M = (\zeta, \delta, s_0, \mathfrak{F})$ be a deterministic automaton accepting R^{ω}. Define the non-deterministic automaton $M' = (\zeta, \delta', \{s_0\}, \mathfrak{F})$ as follows: for each $s \in \zeta$, $a \in \Sigma$, let $\delta'(s,a) = \{s' \in \zeta$: exists $b \in \Sigma$ such that $\delta(s,b) = s'\}$. Note that $\delta(s,a)$ is always in $\delta'(s,a)$. Let M'' be another finite-state automaton monitoring the input and state changes of M', and suppose that M'' has a red light (in the manner of the machine described by McNaughton (1966)) which flashes whenever M' makes a state change differing from the one M would have made, i.e. the light flashes iff M' changes from state s to state s' on an input a, and $\delta(s,a) \neq s'$. Clearly the system composed of M' and M'' is a finite-state automaton. We stipulate that the system accepts a tape iff with this tape as input the red light flashes only finitely often, and the exact set of states taken on infinitely often by M' is a member of $\mathfrak{F}$. It is easy to verify that this system recognizes $F(R^{\omega})$.

4. CONCLUSIONS

A large number of noiselike transforms besides the ones discussed can be defined and shown to be ω-regularity preserving. In particular, if R^{ω} is ω-regular, then so is the set of tapes with at most k errors, the set of tapes with at most k errors per any m consecutive symbols $(k < m)$, the set of tapes with infinitely many errors, etc. However, the transforms discussed are the ones that suggest themselves most naturally from considerations of actual systems, and the methods of proof employed are typical.

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REFERENCES

Büchi, J. R., and Landweber, L. H. (1967), Solving Sequential Conditions by Finite-state Strategies, Purdue Report CSD TR 14.

McNaughton, R. (1966), Testing and Generating Infinite Sequences by a finite Automaton. Information and Control 9, pp. 521-530.

Muller, D. E. (1963), Infinite sequences and finite machines. Switching Circuit Theory and Logical Design: Proc. Fourth Ann. Symp., pp. 3-16 (Inst. of Electrical and Electronic Engineers, New York).

Rabin, M. O., and Scott, D. (1959), Finite automata and their decision problems. IBM Journal of Research and Development, Vol. 3, No. 2, pp. 114-125.

Stearns, R. E., and Hartmanis, J. (1963), Regularity Preserving Modifications of Regular Expressions. Information and Control 6, pp. 55-69.